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① a, Nech V a W sú vektorové priestory a $f: V \to W$ je lineárne zobrazenie.

Potom:
- jadrom lineárneho zobrazenia f je

$$Ker(f) = \{\vec{\alpha} \in V;\ f(\vec{\alpha}) = \vec{0}\}$$

- obrazom lineárneho zobrazenia f je

$$Im(f) = \{\vec{\beta} \in W;\ f(\vec{\alpha}) = \vec{\beta}\}.$$

b, $B: \mathbb{R}^3 \to \mathbb{R}^3 : (x_1, x_2, x_3) \mapsto (x_1 + x_2 + x_3,\ x_1 - x_2,\ -x_1 + x_2)$

$$Ker(B) = \{\vec{x} \in \mathbb{R}^3 : B(\vec{x}) = \vec{0}\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\underline{x_3 = t}$

$-2x_2 - t = 0$

$\underline{x_2 = -\frac{t}{2}}$

$$Ker(B) = \{(-\tfrac{t}{2}, -\tfrac{t}{2}, t) : t \in \mathbb{R}\}$$

$x_1 - \frac{t}{2} + t = 0$

$\underline{x_1 = -\frac{t}{2}}$

$\dim(Ker(B)) = 1$

$$Im(B) = \{B(\vec{x}) : \vec{x} \in \mathbb{R}^3\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 0 & b \\ -1 & 1 & 0 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & -1 & b-a \\ 0 & 2 & 1 & c+a \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & b+a \\ 0 & -2 & -1 & b-a \\ 0 & 2 & 1 & c+a \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & b+a \\ 0 & -2 & -1 & b-a \\ 0 & 0 & 0 & c+b \end{array}\right]$$

$c = -b$

$$Im(B) = \{(a, b, -b) : a, b \in \mathbb{R}\}$$

$\dim(Im(B)) = 2$

$\mathrm{Ker}(B) = \{(-\frac{t}{2}, -\frac{t}{2}, t) : t \in \mathbb{R}\}$ => ortonormálna

$\mathrm{Im}(B) = \{(a, b, -b) : a, b \in \mathbb{R}\}$ => ortonormálna

Štandardná báza $\mathbb{R}^{\cancel{3}n}$ je vzhľadom na geometrický skalárny súčin ortonormálna.

(2)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & \alpha & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

(1.) CHARAKTERISTICKÝ POLYNÓM:

$$\mathcal{X}(A) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & \alpha-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)(\alpha-\lambda)(1-\lambda) + \cancel{2\cdot 0\cdot 1} + \cancel{1\cdot 0\cdot 2} - 1\cdot(\alpha-\lambda)\cdot 1 - \cancel{2\cdot 0\cdot(1-\lambda)} - \cancel{(1-\lambda)\cdot 0\cdot 2}$$

$$= -\lambda^3 + \alpha\cdot\lambda^2 + 2\lambda^2 - 2\alpha\cdot\lambda =$$

$$= -\lambda^3 + (\alpha+2)\lambda^2 - 2\alpha\lambda =$$

$$= -\lambda(\lambda^2 - (\alpha+2)\lambda + 2\alpha) =$$

$$= -\lambda(\lambda-2)(\lambda-\alpha)$$

(2) VLASTNÉ ČÍSLA

$\lambda_1 = 0$

$\lambda_2 = 2$

$\lambda_3 = \alpha$

(III.) VLASTNÉ VEKTORY

pre $\lambda_1 = 0$

$$\mathrm{Ker}(A - \lambda_1 I): \begin{bmatrix} 1 & 2 & 1 \\ 0 & \alpha & 0 \\ 1 & 2 & 1 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + x_3 = 0 \\ x_2 = 0 \end{array}$$

$\alpha \neq 0$

$\lambda_2 = 2$

$$\mathrm{Ker}(A - \lambda_2 I): \left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & \alpha-2 & 0 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & \alpha-2 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 - x_3 = 0 \\ x_2 = 0 \end{array}$$

$\alpha \neq 2$

pre $\lambda_3 = \alpha$

$$\operatorname{Ker}(A - \lambda_3 I) = \left[\begin{array}{ccc|c} 1-\alpha & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 1-\alpha & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & \frac{-2}{\alpha-1} & \frac{-1}{\alpha-1} & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 1-\alpha & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{-2}{\alpha-1} & \frac{-1}{\alpha-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{2\alpha}{\alpha-1} & \frac{-\alpha^2+2\alpha}{\alpha-1} & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & \frac{-2}{\alpha-1} & \frac{-1}{\alpha-1} & 0 \\ 0 & \frac{2\alpha}{\alpha-1} & \frac{-\alpha^2+2\alpha}{\alpha-1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{-2}{\alpha-1} & \frac{-1}{\alpha-1} & 0 \\ 0 & 1 & \frac{-\alpha+2}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{-\alpha+2}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 - x_3 = 0 \\ x_2 - \frac{\alpha+2}{2} x_3 = 0 \end{array}$$

$x_1 = x_3$

$x_2 = \frac{\alpha-2}{2} x_3$ $\qquad \alpha \neq +1$

$x_3 = x_3$ $\qquad \alpha \neq 0$

Aby sme vedeli zostrojiť diagonálnu maticu, tak α musí byť $\Rightarrow \alpha \in \mathbb{R} \setminus \{0,2\}$

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③ $T^2 = 2T$, $\lambda \in \{0, 2\}$

zoberieme vektor $\vec{x}$, potom

$T^2 \vec{x} = 2T\vec{x}$ $\rightarrow$ $T \cdot T \cdot \vec{x} - 2T \cdot \vec{x} = T(\lambda\vec{x}) - 2(\lambda\vec{x}) =$

$= \lambda \cdot \lambda \cdot \vec{x} - 2 \cdot \lambda\vec{x}$

$\underbrace{(T^2 - 2T)\vec{x}}$ $= 0$

$(\lambda^2 - 2\lambda) = 0$

$\lambda(\lambda - 2) = 0 \iff \underline{\underline{\lambda = 0}} \lor \underline{\underline{\lambda = 2}}$

✓

(4) $\langle x, y \rangle_S := \langle Sx, S\cancel{y}x \rangle \geq 0$

$\hookrightarrow = 0 \Leftrightarrow x = 0 \Rightarrow Sx = 0$

⑤ a, Nech $M = (m_{ij})_{n \times n}$, $n \in N$ je štvorcová matica, ktorej prvky sú reálne čísla.

Číslo λ sa nazýva VLASTNÁ HODNOTA, ktoré sa počíta pomocou charakteristického polynómu ako $\det(M - \lambda I)$

Vektor sa nazýva VLASTNÝ VEKTOR, ktorý prislúcha k vlastnej hodnote λ.