Constraints on n-lattice simplices of volume $\frac{1}{n!}$ in n-cube, unique under symmetry

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Mathematics REU

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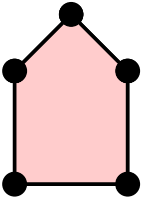
Introduction



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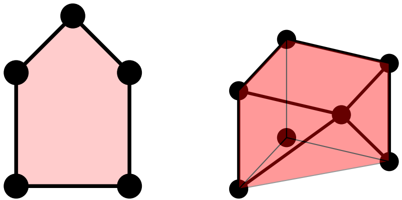

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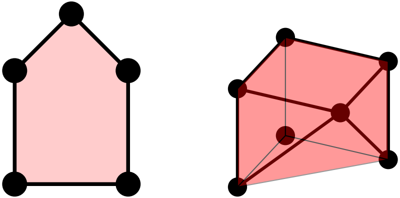

Definition

A **(convex) polytope** in $\mathbb{R}^n$ is the convex hull of finitely many points in $\mathbb{R}^n$. That is,

$$\text{conv}\{v_1, ..., v_d\} = \left\{ \sum_{i=1}^{d} \lambda_i v_i : \sum_{i=1}^{d} \lambda_i = 1, \lambda_i \geq 0 \right\}$$

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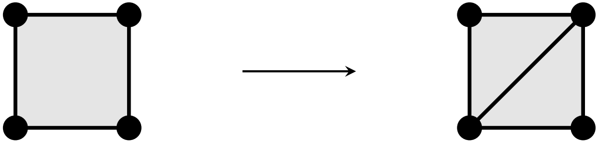

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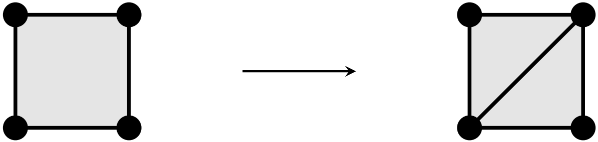

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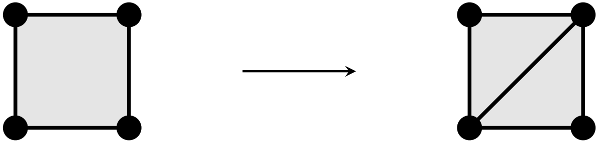

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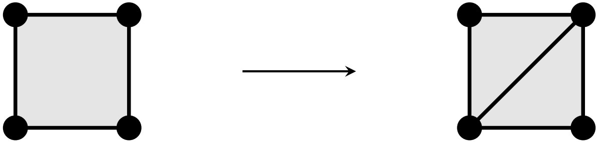

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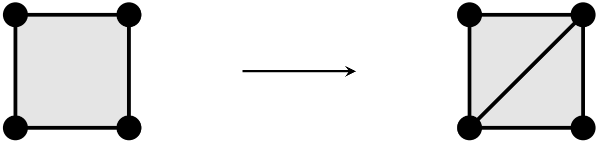

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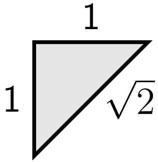

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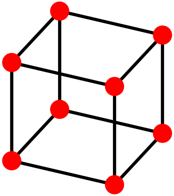
$\{v_1, ..., v_{d+1}\}$ are **affinely independent** if $\{v_1 - v_{d+1}, ..., v_d - v_{d+1}\}$ is linearly indepedent.

Definition

A **triangulation** $\mathcal{T}$ of a full dimensional polytope $\mathcal{P}$ in $\mathbb{R}^n$ is a finite collection of n-simplices whose vertices are vertices of $\mathcal{P}$ and that satisfies the following two properties:

$$\bigcup_{\Delta \in \mathcal{T}} \Delta = \mathcal{P} \tag{UP}$$

Given $\Delta_1, \Delta_2 \in \mathcal{T}, \Delta_1 \cap \Delta_2$ is a common face (possibly empty) (IP)



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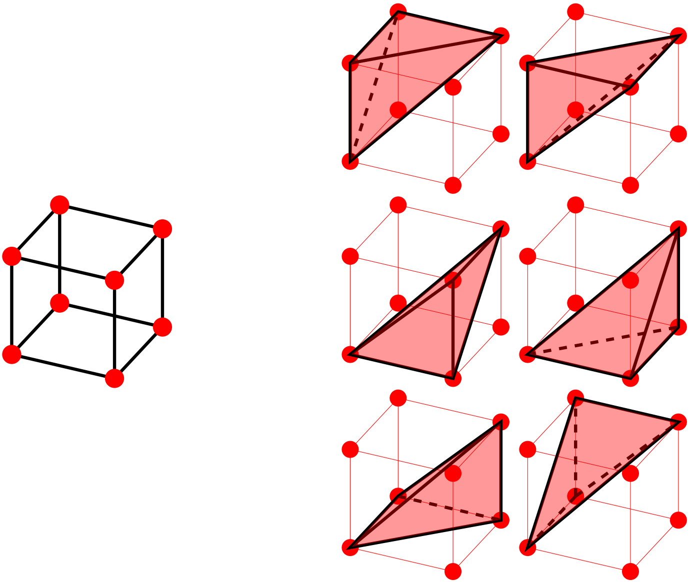

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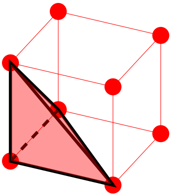

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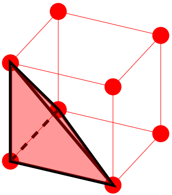

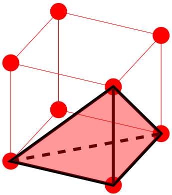

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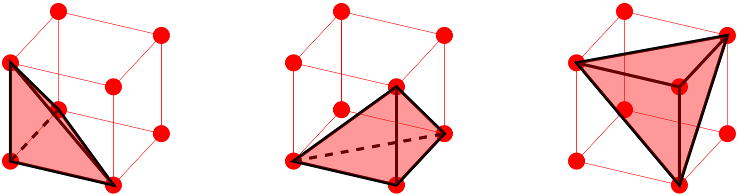

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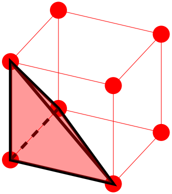

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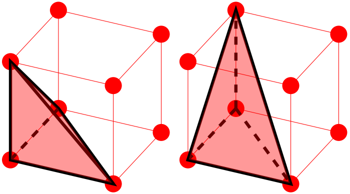

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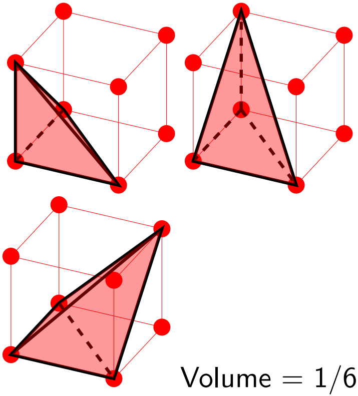
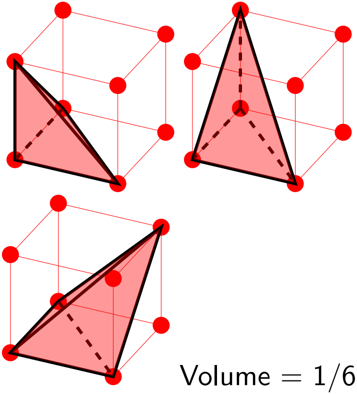


Volume $= 1/6$



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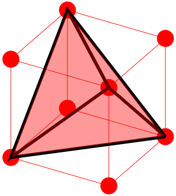

Volume $= 1/3$

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Definition

A triangulation of an n-dimensional polytope in $\mathbb{R}^n$ is **unimodular** if all its n-simplices are translations of linear transformations of the standard simplex conv$\{0, e_1, ..., e_n\}$, where the linear transformation's components are integers, and its determinant is ± 1.

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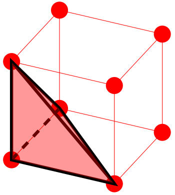
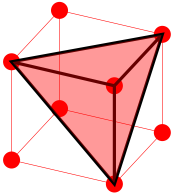
Why ask this question?

- Applications in Linear Algebra [1]
- Unimodular triangulations not well understood.



$$\left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$



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- $\implies$ The important information is the distance between vertices.

The Representation

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Let x, y be elements of $[0,1]^n \cap \mathbb{Z}^n$. Define the **taxi-cab distance** of x, y, denoted $\delta(x,y)$, as the minimum number of standard basis vectors that must be added to or subtracted from x to get y.

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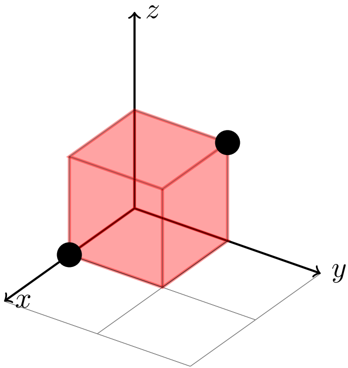

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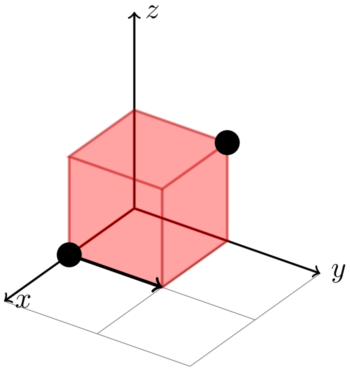

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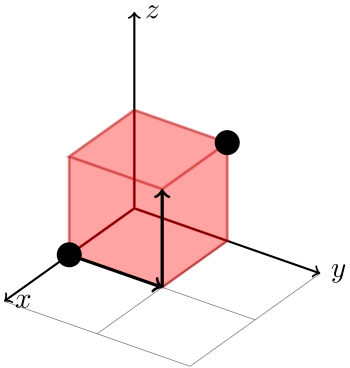

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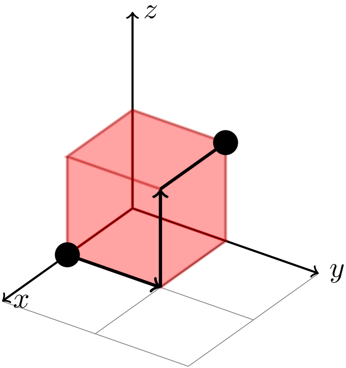

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Let Δ be an d-simplex in $\mathbb{Z}^n$. Its **adjacency matrix** is

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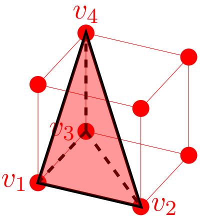
Define the multigraph $G = (V, E)$ of Δ as the graph of $d + 1$ vertices with a_{ij} edges between v_i, v_j.

The Representation



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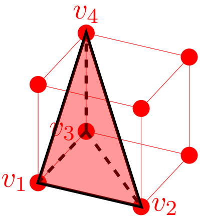

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 3 \\ 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

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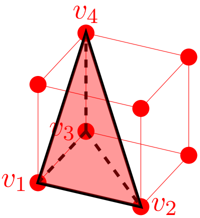

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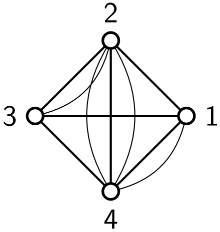

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- In search of more constraints, we encountered interesting properties.

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Let Δ *be an* n*-lattice simplex in* $[0,1]^n$ *with adjacency matrix* A *and characteristic polynomial* $(-1)^{n+1}\lambda^{n+1} + k_n\lambda^n + ... + k_1\lambda + k_0$.

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$$p_G(\lambda) = (-1)^{n+1}\lambda^{n+1} + k_n\lambda^n + ... + k_1\lambda + k_0$$
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$$\begin{aligned} k_1\lambda &= \sum_{k=1}^{n+1} \sum_{\sigma \in S_{n+1,k}} (-1)^\sigma a_{1\sigma(1)} ... a_{(n+1)\sigma(n+1)} \\ &= -\lambda \sum_{k=1}^{n+1} \sum_{\sigma \in S_{n+1,k}} (-1)^\sigma a_{1\sigma(1)} ... a_{(k-1)\sigma(k-1)} a_{(k+1)\sigma(k+1)} ... a_{(n+1)\sigma(n+} \\ &= -\lambda \sum_{k=1}^{n+1} \det A_{n-1,k}. \end{aligned}$$

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$$p_G(\lambda) = (-1)^{n+1}\lambda^{n+1} + k_n\lambda^n + ... + k_1\lambda + k_0$$
$$= \sum_{\sigma\in S_{n+1}} (-1)^\sigma a_{1\sigma(1)}...a_{(n+1)\sigma(n+1)}.$$

The coefficient k_1 is the sum of terms such that σ maps one and only one element to itself. Call the set of such permutations, where k is that element, $S_{n+1,k}$, and let $\{A_{n-1,k}\}_{k=1}^{n+1}$ be the adjacency matrices of the $(n-1)$-dimensional faces. Then,

$$k_1\lambda = \sum_{k=1}^{n+1} \sum_{\sigma\in S_{n+1,k}} (-1)^\sigma a_{1\sigma(1)}...a_{(n+1)\sigma(n+1)}$$
$$= -\lambda \sum_{k=1}^{n+1} \sum_{\sigma\in S_{n+1,k}} (-1)^\sigma a_{1\sigma(1)}...a_{(k-1)\sigma(k-1)}a_{(k+1)\sigma(k+1)}...a_{(n+1)\sigma(n+}$$
$$= -\lambda \sum_{k=1}^{n+1} \det A_{n-1,k}.$$

Theorem

Let Δ be an n-lattice simplex in $[0,1]^n$, and let $\mathcal{F} = \{f\}$ be the set of faces of Δ of dimension $\leq n-1$, including the empty set, which we assign the trivial dimension -1 and polynomial $p_\emptyset = 1$. Let A_Δ be the adjacency matrix of Δ. Then,

$$p_\Delta(\lambda) = \det A_\Delta - \sum_{f \in \mathcal{F}} \lambda^{n - \dim f} p_f(\lambda) \tag{2}$$

Proof is similar.

Theorems

Conjecture

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Proposition

To prove the conjecture, it suffices to show that if $A^{-1} = (b_{ij})$,

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Proof: Uses Cayley-Menger determinant and a block matrix identity from J.R. Silvester [3].

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