Using Geometric Optics to Infer Properties of Pulse Solutions of a Semilinear Hyperbolic Boundary Value Problem

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1 Introduction

Hyperbolic systems of partial differential equations have many physical applications, including in equations of general relativity, Maxwell's equations for electromagnetic waves, and Euler equations for compressible fluids such as gases. Solutions to such systems are "wave-like." A pulse-valued system of PDEs uses data that depends on a parameter in which the data has polynomial decay. Usually, ϵ signifies wavelength, and is located in the denominator of this parameter. These systems are relevant when one must consider variations in wavelength.

In many cases, solutions to hyperbolic systems are quite difficult to construct or study. However, applied mathematicians studying nonlinear geometric optics have developed centuries of expertise in developing approximate solutions. The systems that these approximate solutions solve are much simpler than the original systems, allowing one to readily "read off" their properties. Properties include what the pulses u^ϵ look like, how they decay, what phases they depend on, what their amplitudes are, what their propagation velocity is, how their support changes with time, how they reflect off the boundary and interact with each other, etc.

Joly, Metivier and Rauch pioneered the approach of using these approximate solutions to infer properties of actual solutions. Doing so involves improving upon the approximate solution by adding nontrivial "corrector" terms, and then showing the improved approximate solution is sufficiently close to the actual solution in an appropriate norm that it inherits some of its properties. The norm used is similar but weaker than the standard Sobolev norm, yet still suffices to imply such properties. Also, this technique only works for small ϵ, or in other words, for systems with small wavelength.

This so-called "WKB method" has been used primarily in situations where the "phase" is linear with respect to the time and spatial variables. The phase is a function derived from the system around whose zero set the function's support is concentrated. Joly, Metivier, and Rauch showed that nonlinear phases give rise to behavior that prevents the use of the WKB method to study wavetrain systems. Wavetrain-valued systems are like pulse-valued systems, except the data is periodic in the parameter rather than exhibiting polynomial decay. Since pulses behave differently, we believe we can obtain results about pulse-valued problems with nonlinear phases.

The "first problem" of the project is a pulse-valued system in free space with only one nonlinear phase. We construct an approximate solution, use a proposition by Metivier to construct the actual solution, and then proceed with error analysis and implication of properties. A novel aspect of our approach is using Metivier's proposition in a unique way. Whereas Metivier's proposition allowed him to sidestep shortcomings of classical theory while working with discontinuous solutions, we use it to obtain a uniform time of existence for all epsilon sufficiently small.

As for the second phase of the project, we plan to consider systems with more than one nonlinear pulse and study the interaction between the pulses. This usually requires constructing a more sophisticated "corrector" term, in order to do error analysis.

2 A First Problem

2.1 Construction of Approximate Solution

The first problem we aim to investigate is the $N \times N$ semilinear system in $\mathbb{R}^{n+1}$:

$$
\begin{aligned}
&(a)\ \mathcal{L}(t,x,\partial_{t,x})u^\epsilon := \partial_t u^\epsilon + \sum_{j=1}^{n} A_j(t,x)\partial_j u^\epsilon = f(t,x,u^\epsilon) + g(t,x,\theta)|_{\theta=\frac{\psi_k}{\epsilon}} \\
&(b)\ u^\epsilon = 0 \text{ in } t<0
\end{aligned} \tag{1}
$$

where $\mathcal{L}(t,x,\partial_{t,x})$ is assumed to be strictly hyperbolic. $A_j(t,x)$ are $N \times N$ matrices. We assume $f(t,x,0) = 0$. Thus, if g is identically zero, then $u^\epsilon = 0$ solves the system. $g(t,x,\theta)$ is a pulse, that is, a function that is parameterized in terms of θ and evaluated somewhere, with polynomial decay in θ. Actually, in our case, we make further assumptions on g: $g(t,x,\theta) \in C^\infty$ has compact support in (x,θ) and $g = 0$ in $t < 0$. ψ_k is a phase of the system, constructed from the operator in section 2.1.2.

To investigate the problem, we construct an approximate pulse solution u_a of the form

$$
\begin{aligned}
&u_a(t,x) = [U_0(t,x,\theta) + \epsilon U_1^\epsilon(t,x,\theta)]\,|_{\theta=\frac{\psi_k}{\epsilon}}, \text{ where} \\
&U_0 = \sigma_k(t,x,\theta) r_k(t,x) \\
&U_1^\epsilon = V(t,x,\theta)
\end{aligned} \tag{2}
$$

Here, σ_k is a scalar function. σ_k and V will be defined later. r_k is defined as follows. The goal is to show the approximate pulse solutions are sufficiently close to proper pulse solutions such that we can infer qualitative information about the proper solutions.

2.1.1 Definition of r_k

Using $\eta = (\eta_0, \eta') = (\eta_0, \eta'', \eta_n)$, we define

$$\mathcal{A}(t, x, \eta') := \sum_{j=1}^{n} A_j(t, x)\eta_j$$

By strict hyperbolicity, this matrix has N distinct real eigenvalues when (t, x) is in a particular open neighborhood $\mathcal{O}$ of $(0, 0)$. We define $R_k(t, x, \eta')$ and $L_k(t, x, \eta')$ as the right and left eigenvectors of this matrix respectively, and $\lambda_k(t, x, \eta')$ as the corresponding eigenvalues. We define in such a way that $L_i R_j = \delta_{ij}$. To see that this is possible, note that $L_k \mathcal{A} R_j = \lambda_k L_k R_j = \lambda_j L_k R_j$, so $j \neq k \implies L_k R_j = 0$. Thus, fixing j, we note that $L_j R_j \neq 0$, since the contrary would imply that R_j is orthogonal to a full basis and is thus the zero vector, which is not allowed of eigenvectors. Accordingly, when $j = k$ we scale R_k such that $L_k R_j = 1$.

$l_k(t, x), r_k(t, x)$ are then constructed, respectively, as $R(t, x, \psi_k), L(t, x, \psi_k)$, where ψ_k is defined as follows.

2.1.2 Construction of ψ_k

Fix $k \in \{1, ..., N\}$. Now, we will construct ψ_k using an Eikonal equation. We will use ψ_k to construct a vector field X_k that we use to construct σ_k, the not-yet-defined portion of U_0.

Solve the following Eikonal equation using the method of characteristics:

$$\begin{aligned} \partial_t \psi_k &= -\lambda_k(t, x, d_x \psi_k) \\ \psi_k|_{t=0} &= x_n \end{aligned} \tag{3}$$

We use this initial condition so that at $t = 0$, $d_x \psi_k = (0, ..., 0, 1)$, and thus $\mathcal{A}$ has distinct eigenvalues near 0.

2.1.3 Setting Conditions for V, σ_k

We want $\mathcal{L}(t, x, \partial_{t,x})u_a - f(t, x, u_a) - g(t, x, \theta)|_{\theta = \frac{\psi_k}{\epsilon}}$ to be sufficiently small. We do some calculations:

(4)

$$\begin{aligned}
\mathcal{L}(t,x,\partial_{t,x})U_0(t,x,\frac{\psi}{\epsilon}) &= \Big[\frac{1}{\epsilon}\mathcal{L}(t,x,d\psi)(\partial_\theta\sigma_k(t,x,\theta))r_k + (\mathcal{L}(t,x,\partial_{t,x})\sigma_k)r_k \\
&+ \sigma_k\mathcal{L}(t,x,\partial_{t,x})r_k\Big]|_{\theta=\frac{\psi_k}{\epsilon}} \\
&= \Big[\frac{1}{\epsilon}(\partial_\theta\sigma_k(t,x,\theta))(\mathcal{A}(t,x,d_x\psi)r_k - \lambda_k r_k) + (\mathcal{L}(t,x,\partial_{t,x})\sigma_k)r_k \\
&+ \sigma_k\mathcal{L}(t,x,\partial_{t,x})r_k\Big]|_{\theta=\frac{\psi_k}{\epsilon}} \\
&= \Big[(\mathcal{L}(t,x,\partial_{t,x})\sigma_k)r_k + \sigma_k\mathcal{L}(t,x,\partial_{t,x})r_k\Big]|_{\theta=\frac{\psi_k}{\epsilon}} \\
&= \Big[\mathcal{L}(t,x,\partial_{t,x})U_0(t,x,\theta)\Big]|_{\theta=\frac{\psi_k}{\epsilon}}
\end{aligned}$$

Here, applying the operator $\mathcal{L}(t,x,\partial_{t,x})$ to a function of t,x,θ denotes differentiation with respect to t,x but not θ. This convention will be used throughout the paper. Notice that the $1/\epsilon$ term canceled out using the Eikonal equation. This is one advantage to constructing ψ_k using the Eikonal equation.

Continuing with our calculations, we have

(5)

$$\mathcal{L}(t,x,\partial_{t,x})V(t,x,\frac{\psi_k}{\epsilon}) = \Big[\mathcal{L}(t,x,\partial_{t,x})V(t,x,\theta) + \frac{1}{\epsilon}\mathcal{L}(t,x,d\psi_k)\partial_\theta V(t,x,\theta)\Big]|_{\theta=\frac{\psi_k}{\epsilon}}$$

By the Fundamental Theorem of Calculus, we have that, for fixed $\theta = \frac{\psi_k}{\epsilon}$,

$$\begin{aligned}
f(t,x,u_a) &= \Big(\int_0^1 f_u(t,x,U_0+s\epsilon U_1)ds\Big)(\epsilon U_1) + f(t,x,U_0) \\
&= K_1(t,x,U_0,\epsilon U_1)\epsilon U_1 + f(t,x,U_0)
\end{aligned} \tag{6}$$

Combining all these calculations, we have that

(7)

$$\mathcal{L}(t,x,\partial_{t,x})u_a - f(t,x,u_a) - g(t,x,\theta)|_{\theta=\frac{\psi_k}{\epsilon}} = \frac{1}{\epsilon}\mathcal{E}_{-1} + \mathcal{E}_0 + \epsilon\mathcal{E}_1, \text{ where}$$

$$\mathcal{E}_{-1} = 0$$

$$\mathcal{E}_0 = [\mathcal{L}(t,x,\partial_{t,x})U_0(t,x,\theta) + \mathcal{L}(t,x,d\psi_k)\partial_\theta V(t,x,\theta) - f(t,x,U_0) - g(t,x,\theta)]\,|_{\theta=\frac{\psi_k}{\epsilon}}$$

$$\mathcal{E}_1 = [\mathcal{L}(t,x,\partial_{t,x})V(t,x,\theta) - K_1(t,x,U_0,\epsilon U_1)U_1]\,|_{\theta=\frac{\psi_k}{\epsilon}}$$

We want $\mathcal{E}_0$ to be small, so we construct V, σ_k accordingly.

2.1.4 The Projection Operator

Definition 2.1. *Define* $\Pi_m(t,x,\eta')$ *as the projection on span* $R_m(t,x,\eta')$ *in the decomposition*

$$\mathbb{C}^N = \oplus_{l=1}^N span\ R_l(t,x,\eta').$$

That is, given $v \in \mathbb{C}^N$, $\Pi_m(t,x,\eta')v = (L_m(t,x,\eta')x)R_m(t,x,\eta')$.

We define $l_m(t,x) := L_m(t,x,d_x\psi_k)$ *and* $r_m(t,x) := R_m(t,x,d_x\psi_k)$. *Then define* $\pi_m(t,x) := \Pi_m(t,x,d_x\psi_k)$.

2.1.5 Construction of σ_k

Let

$$\mathcal{F} := \mathcal{L}(t,x,\partial_{t,x})U_0(t,x,\theta) - f(t,x,U_0) - g(t,x,\theta). \tag{8}$$

We see that $\mathcal{E}_0 = [\pi_k(t,x)\mathcal{F} + (1-\pi_k(t,x))\mathcal{F} + \mathcal{L}(t,x,d\psi_k)\partial_\theta V(t,x,\theta)]\,|_{\theta=\frac{\psi_k}{\epsilon}}$.
We will construct σ_k such that $\pi_k\mathcal{F} = 0$ and V such that $(1-\pi_k)\mathcal{F}+\mathcal{L}(t,x,d\psi_k)\partial_\theta V(t,x,\theta) = 0$.

In our construction of σ_k, we use the following lemma.

Lemma 2.2. *For* $j = 1, ..., n$*:*

$$L_m(t,x,\eta')A_j(t,x)R_m(t,x,\eta') = \partial_{\eta_j}\lambda_m(t,x,\eta') \tag{9}$$

Proof. Differentiate the following equation with respect to η_j and then multiply by $L_m(t,x,\eta')$ on the left:

$$0 = \Big[-\lambda_m(t,x,\eta')I + \sum_{j=1}^{n} A_j(t,x)\eta_j\Big]R_m(t,x,\eta') = \mathcal{L}(t,x,-\lambda_m,\eta')R_m.$$

□

The lemma implies $l_k(t,x)A_j(t,x)r_k(t,x) = \partial_{\eta_j}\lambda_k(t,x,d_x\psi_k)$.

Using the identity derived from the above lemma, we observe that

$$\begin{aligned}
\pi_k\mathcal{L}(t,x,\partial_{t,x})U_0(t,x,\theta) =& (X(t,x,\partial_{t,x})\sigma_k(t,x,\theta) + c_k(t,x)\sigma_k)r_k, \text{ where}\\
X(t,x,\partial_{t,x}) :=& \partial_t + \sum_{j=1}^{n} \partial_{\eta_j}\lambda_k(t,x,d_x\psi_k)\partial_j \\
c_k(t,x) =& l_k\mathcal{L}(t,x,\partial_{t,x})r_k
\end{aligned} \tag{10}$$

Accordingly, $\pi_k\mathcal{F} = (X(t,x,\partial_{t,x})\sigma_k + c_k\sigma_k - l_kf(t,x,U_0) - l_kg(t,x,\theta))r_k = 0$. This equation is solved using the solution of the following equation:

$$\begin{aligned}
&(a)X(t,x,\partial_{t,x})\sigma_k + c_k\sigma_k - h(t,x,\sigma_k) - l_kg(t,x,\theta) = 0\\
&(b)\sigma_k|_{t<0} = 0,\\
&\quad\text{where } h(t,x,\sigma_k) := l_kf(t,x,U_0) = l_kf(t,x,\sigma_kr_k)
\end{aligned} \tag{11}$$

The choice of initial condition is suitable for σ_k, since it makes $U_0 = \sigma_kr_k$ close to U, which is identically 0 in $\{t<0\}$. One can solve for $\sigma_k(t,x,\theta)$ using the method of characteristics. Thus, using such σ_k, we have that $\pi_k\mathcal{F} = 0$.

2.1.6 Construction of $U_1 = V$

Next, we construct V such that $(1-\pi_k)\mathcal{F} + \mathcal{L}(t,x,d\psi_k)\partial_\theta V(t,x,\theta) = 0$. To do so, we define a partial inverse $Q(t,x)$ for $\mathcal{L}(t,x,d\psi_k)$ that satisfies

$$\mathcal{L}(t,x,d\psi_k)Q(t,x) = Q(t,x)\mathcal{L}(t,x,d\psi_k) = 1-\pi_k(t,x). \tag{12}$$

First, observe that $\mathcal{A}(t,x,d_x\psi_k) = \sum_{j=1}^N \lambda_j(t,x,d_x\psi_k)\Pi_j(t,x,d_x\psi_k)$. Accordingly, $\mathcal{L}(t,x,d\psi_k) = \sum_{j=1}^N(\lambda_j(t,x,d_x\psi_k) - \lambda_k(t,x,d_x\psi_k))\Pi_j(t,x,d_x\psi_k)$. Thus, $Q(t,x) := \sum_{j\neq k}(\lambda_j(t,x,d_x\psi_k) - \lambda_k(t,x,d_x\psi_k))^{-1}\Pi_j(t,x,d_x\psi_k)$ satisfies the above property.

Using this partial inverse, V satisfies the desired property when it is defined:

$$\begin{aligned} V(t,x,\theta) &:= -\int_{-\infty}^{\theta} Q(t,x)H(t,x,s)ds, \text{ where} \\ H(t,x,\theta) &= (1-\pi_k)\mathcal{F}. \end{aligned} \tag{13}$$

Note that $V=0$ when $t<0$. Moreover, this integral is well defined because the integrand has compact support in θ. This is due to the properties of the components of the sum that defines $\mathcal{F}$. g has compact support in θ, and choosing $\overline{\theta}$ such that $g=0$, $\sigma_k=0$ solves (11), leaving us with $\mathcal{F}=0$. Furthermore, V is bounded in θ, so as $\epsilon\to 0$, $\epsilon\mathcal{E}_1 \to 0$.

Thus, with σ_k and V defined as above, we have that $\mathcal{E}_0 = 0$.

2.2 Error Analysis

Now that we have an approximate solution, we aim to show it is sufficiently close to an actual solution. We aim to do so by examining the error term, defined as

$$w := u - u_a,$$

where u_a is the defined approximate solution, and u is a proper solution.

2.2.1 Defining a Problem for w

Using (1) and (7), we observe that w satisfies

$$\mathcal{L}(t,x,\partial_{t,x})w - (f(t,x,u) - f(t,x,u_a)) = -\epsilon\mathcal{E}_1, \tag{14}$$

where $\mathcal{E}_1$ is defined as in (7).

In search of an initial condition, we observe that in $t<0$, $U_0 = 0$ by (11), and $U_1 = V = 0$ by observing that H in (13) is identically 0 in $t<0$. Thus, $w = u-u_a$ is identically 0 in $t<0$.

Using the Fundamental Theorem of Calculus, we observe that

$$\begin{aligned} f(t,x,u) - f(t,x,u_a) &= \left(\int_0^1 f_{u_a}(t,x,u_a+s(u-u_a))ds\right)w \\ &= K_2(t,x,u,u_a)w. \end{aligned}$$

Thus, w satisfies the problem

$$
\begin{aligned}
&(a)\ \mathcal{L}(t,x,\partial_{t,x})w - K_2(t,x,u,u_a)w = -\epsilon\mathcal{E}_1\\
&(b)\ w|_{t<0} = 0.
\end{aligned} \tag{15}
$$

2.2.2 Construction of Exact Solution

Using a proposition of Metivier [1], we can construct an exact solution to the problem (1).

We define certain domains of interest:

$$
\begin{aligned}
&\Omega := \{(t,x) : -T_0 < t < T_0 - \alpha|x|\} \cap \{x_n \geq 0\}\\
&b\Omega := \{(t,x') : (t,x',0) \in \Omega \cap \{x_n = 0\}\}\\
&\Omega_T := \Omega \cap \{t < T\} \text{ and } \Omega_T^+ := \Omega_T \cap \{x_n > 0\}
\end{aligned} \tag{16}
$$

Definition 2.3. *We define as C_b^∞ the set of bounded C^∞ functions h such that $\partial^\alpha h$ is bounded for every α.*

Definition 2.4. *We define as $\mathcal{M}$ the set of vector fields with coefficients in C_b^∞ that are tangent to $\Sigma = \{x_n = 0\}$.*

Proposition 2.4.1. *$\mathcal{M}$ is generated as a module over $C_b^\infty(\Omega^T)$ by*

$$
\partial_t, \partial_{x_1}, ..., \partial_{x_{n-1}}, x_n\partial_{x_n}. \tag{17}
$$

Proof. For simplicity we label $t = x_0$. We must show $X \in \mathcal{M}$ iff $X = \sum_{j=0}^{n-1} a_j\partial_j + a_n x_n \partial_{x_n}$, where $a_j \in C_b^\infty(\Omega^T)$ for $j = 0,...,n$. The reverse direction is easy to see, since $X \cdot \nabla x_n = a_n x_n = 0$ at $x_n = 0$.

For the forward inclusion, we use the Fundamental Theorem of Calculus. Let $X \in \mathcal{M}$. Then,

$$
X = \sum_{j=0}^{n} a_n \partial_{x_j}, \text{ where } a_j \in C_b^\infty(\Omega^T), \text{ and } \begin{pmatrix} a_0 \\ ... \\ a_n \end{pmatrix} \cdot \nabla x_n = 0 \text{ at } x_n = 0.
$$

Thus, we see that $a_n(x)|_{x_n=0} = 0$. Using the Fundamental Theorem of Calculus, we factor out x_n:

$$
\begin{aligned}
a_n(x_0,x'',x_n) - a_n(x_0,x'',0) &= x_n\left[\int_0^1 \partial_{x_n} a_n(x_0,x'',tx_n)dt\right]\\
&= x_n\tilde{a}_n(x_0,x'',x_n).
\end{aligned}
$$

And $\tilde{a}_n \in C^\infty(\Omega_T)$ by differentiation under the integral sign. $\square$

Definition 2.5. *(a) Given $m \in \mathbb{N}\cup\{0\}$, define the set of conormal distributions of order m with respect to Σ on Ω_T as*

(18)
$N^m(\Omega_T) := \{u \in L^2(\Omega_T) : M_1...M_l u \in L^2(\Omega_T)$ *for any* $M_j \in \mathcal{M},\ j = 1,...,l,\ l \leq m\}$.

(b) Let $(V_1, ..., V_{n+1}) =: \tilde{V}$ denote the generating set (17) and let $\alpha = (\alpha_1, ..., \alpha_{n+1})$ be a multi-index. Define a norm on $N^m(\Omega_T)$ by

$$|u|_m^2 = \sum_{|\alpha| \leq m} |\tilde{V}^\alpha u|^2_{L^2(\Omega_T)}. \tag{19}$$

c) Let $u^\epsilon \in L^2(\Omega_T)$ be a function that depends on the small parameter ϵ. We say $u^\epsilon \in N^m(\Omega_T)$ if there exists $\epsilon_0 > 0$ such that $\sup_{\epsilon\in(0,\epsilon_0]} |u^\epsilon|_m < \infty$.

f

Definition 2.6. *We define:*

$$\begin{aligned} g^\epsilon(t,x) &:= g\left(t, x, \frac{x_n}{\epsilon}\right) \\ u_a^\epsilon(t,x) &:= U_0\left(t, x, \frac{x_n}{\epsilon}\right) + \epsilon U_1\left(t, x, \frac{x_n}{\epsilon}\right) \end{aligned}$$

Theorem 2.7. *For all $m \in \mathbb{N}$, g^ϵ and u_a^ϵ have estimates in $N^m(\Omega_T)$ that are uniform with respect to ϵ.*

In the proof we use the following convention to distinguish between partial derivatives:

Notation 2.8.

$$\begin{aligned} \partial_{y_n} g\left(t, x, \frac{x_n}{\epsilon}\right) &:= [\partial_{x_n} g(t,x,\theta)]_{\theta=\frac{x_n}{\epsilon}} \\ \partial_\theta g\left(t, x, \frac{x_n}{\epsilon}\right) &:= [\partial_\theta\left(g(t,x,\theta)\right)]_{\theta=\frac{x_n}{\epsilon}} \end{aligned}$$

In this notation, we have that

$$\partial_{x_n}\left(g\left(t, x, \frac{x_n}{\epsilon}\right)\right) = \partial_{y_n} g\left(t, x, \frac{x_n}{\epsilon}\right) + \frac{1}{\epsilon}\partial_\theta g\left(t, x, \frac{x_n}{\epsilon}\right).$$

Proof. We handle V separately from g, $U_0 = \sigma_k r_k$.
We claim that, all multiindeces α, $\tilde{V}^\alpha\left(g(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}}\right)$ and $\tilde{V}^\alpha\left(\sigma_k(t,x,\theta)r_k(t,x)|_{\theta=\frac{x_n}{\epsilon}}\right)$ both have the form:

$$\begin{aligned} &h(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}}, \\ &\text{where } h \text{ is an arbitrary smooth function} \\ &\text{that has compact support in } \theta. \end{aligned} \tag{20}$$

As for the base case, this trivially holds for g. As for $\sigma_k r_k$, picking θ outside the compact support of g implies that, for that θ, $\sigma_k = 0$ solves (11), the equation that constructs σ_k.

Inductive step: The following argument suffices for g, $\sigma_k r_k$, so we use g without loss of generality. Assume $\tilde{V}^\alpha\left(g(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}}\right)$ has that form for all $|\alpha| \leq k$.

Then for $j < n$, $\partial_j \left(\tilde{V}^\alpha \left(g(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}} \right) \right) = \partial_j h(t,x,\frac{x_n}{\epsilon}) = \partial_j h(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}}$. And we have

$$\begin{aligned} x_n \partial_{x_n} \left(\tilde{V}^\alpha \left(g(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}} \right) \right) &= x_n \partial_{x_n} \left(h\left(t, x, \frac{x_n}{\epsilon} \right) \right) \\ &= \left[x_n \partial_{x_n} h(t,x,\theta) + \theta \partial_\theta h(t,x,\theta) \right]_{\theta=\frac{x_n}{\epsilon}}, \end{aligned}$$

which has the form of (20). Thus, $\tilde{V}^\alpha \left(g(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}} \right)$ has the form of (20) for all $|\alpha| \le k+1$, so by induction, the same form holds for all α.

Using (20), we get that for $|\alpha| \le m$, $\epsilon < 1$, and on Ω_T,

$$|\tilde{V}^\alpha \left(g|_{\theta=\frac{x_n}{\epsilon}} \right)| \le \sup_{\substack{(t,x)\in\Omega_T \\ |\theta| \le K_\theta}} |h(t,x,\theta)|.$$

Since $\overline{\Omega}_T$ is compact, $\tilde{V}^\alpha \left(g|_{\theta=\frac{x_n}{\epsilon}} \right)$ is uniformly bounded (with respect to ϵ) in $L^2(\Omega_T)$, so $g|_{\theta=\frac{x_n}{\epsilon}}$ has a uniform bound in $N^m(\Omega_T)$.

We use a similar argument for V. We claim that $\tilde{V}^\alpha \left(V(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}} \right)$ has the form

$$\left[\theta f(t,x,\theta) + \int_{-\infty}^{\theta} g(t,x,s) ds \right]_{\theta=\frac{x_n}{\epsilon}}, \tag{21}$$

where f, g are arbitrary smooth functions that have compact support in θ, s respectively.

Base step: $V(t,x,\theta) = \int_{-\infty}^{\theta} Q(t,x) H(t,x,s) ds$. Its integrand QH has compact support in s since $g(\cdot,s) = 0 \implies \sigma_k = 0$ (where $\theta = s$) and $H(t,x,s) = (1-\pi_k)\mathcal{F} = (1-\pi_k)\left(\mathcal{L}(t,x,\partial_{t,x}) U_0(t,x,s) - f(t,x,U_0) - g(t,x,s) \right) = 0$. Here, s acts as a variable dual to θ.

Inductive step: Assume $\tilde{V}^\alpha \left(V|_{\theta=\frac{x_n}{\epsilon}} \right)$ has the form above for all $|\alpha| \le k$. Then clearly when $j < n$, $\partial_j \left(\tilde{V}^\alpha \left(V(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}} \right) \right)$ is $\left[\theta \partial_j f(t,x,\theta) + \int_{-\infty}^{\theta} \partial_j g(t,x,s) ds \right]_{\theta=\frac{x_n}{\epsilon}}$, which has the desired form. And,

$$\begin{aligned} x_n \partial_{x_n} \left(\tilde{V}^\alpha \left(V|_{\theta=\frac{x_n}{\epsilon}} \right) \right) &= x_n \left[\frac{1}{\epsilon} f(t,x,\frac{x_n}{\epsilon}) + \frac{x_n}{\epsilon} \left(\partial_{y_n} f(t,x,\frac{x_n}{\epsilon}) + \frac{1}{\epsilon} \partial_\theta f(t,x,\frac{x_n}{\epsilon}) \right) \right. \\ &\quad \left. + \frac{1}{\epsilon} g(t,x,\frac{x_n}{\epsilon}) + \int_{-\infty}^{\frac{x_n}{\epsilon}} \partial_{y_n} g(t,x,s) ds \right] \\ &= \left[\theta h(t,x,\theta) + \int_{-\infty}^{\theta} x_n \partial_{y_n} g(t,x,s) ds \right]_{\theta=\frac{x_n}{\epsilon}}, \text{ where} \\ & h(t,x,\theta) = f(t,x,\theta) + x_n \partial_{y_n} f(t,x,\theta) + \theta \partial_\theta f(t,x,\theta) + g(t,x,\theta). \end{aligned}$$

So $x_n \partial_{x_n} \left(\tilde{V}^\alpha \left(V|_{\theta = \frac{x_n}{\epsilon}} \right) \right)$ has the form of (21). Thus, $\tilde{V}^\alpha \left(V|_{\theta = \frac{x_n}{\epsilon}} \right)$ has the form of (21) for $|\alpha| \leq k+1$, so by induction, the form holds for all α.

Using (21), we see that, for $|\alpha| \leq m$, one can choose K sufficiently large such that, for $\epsilon < 1$ and on Ω_T,

$$|\tilde{V}^\alpha \left(V|_{\theta = \frac{x_n}{\epsilon}} \right)| \leq K \sup_{\substack{(t,x) \in \Omega_T \\ |\theta| \leq K}} |f(t,x,\theta)| + 2K \sup_{\substack{(t,x) \in \Omega_T \\ |s| \leq K}} |g(t,x,s)| \, .$$

Since $\overline{\Omega}_T$ is compact, $\tilde{V}^\alpha \left(V|_{\theta = \frac{x_n}{\epsilon}} \right)$ is uniformly bounded (with respect to ϵ) in $L^2(\Omega_T)$, so $V|_{\theta = \frac{x_n}{\epsilon}}$ has a uniform bound in $N^m(\Omega_T)$. □

Using Theorem 2.7 (specifically the estimate for g^ϵ), we can apply the following proposition of Metivier to construct u^ϵ. The original version is Proposition 6.1.1 in [1]. Here, T_0 and α denote the parameters defined in (16) in this paper.

Proposition 2.8.1. *Let T_0 and α be sufficiently small, and fix $0 < T_1 < T_0$. Let $G(t,x,\nu,u) \in C^\infty(\overline{\Omega}_T \times \mathbb{R}^M \times \mathbb{R}^N)$, where ν is an $\mathbb{R}^M$-valued function in $L^\infty \cap N^m(\Omega_{T_1})$, $m > (n+5)/2$, and $G(t,x,\nu(t,x),0) = 0$ when $t < 0$. Consider the problem:*

$$\begin{aligned} &Lu(t,x) = G(t,x,\nu(t,x),u(t,x)) \\ &u|_{t<0} = 0 \end{aligned} \tag{22}$$

Then, there is $T : 0 < T < T_1$ and $u \in L^\infty \cap N^m(\Omega_T)$ a solution to (22).

To apply this proposition, we let $\nu = \nu^\epsilon = g^\epsilon$, and our $f(t,x,u) + g^\epsilon(t,x)$ plays the role of Metivier's $G(t,x,\nu(t,x),u(t,x))$.

Theorem 2.9. *Using Theorem 2.7, Proposition 2.8.1 gives us a time of existence for u^ϵ that holds for all ϵ sufficiently small.*

Following the proof of Metivier's proposition, we construct the solution using Picard iteration, with an important difference. Instead of constructing u_n iteratively, we construct u_n^ϵ iteratively. We define $u_0^\epsilon = 0$ and define u_n^ϵ:

$$\begin{aligned} &Lu_{n+1}^\epsilon = G(t,x,\nu^\epsilon(t,x),u_n^\epsilon(t,x)) = f(t,x,u_n^\epsilon(t,x)) + g^\epsilon(t,x) \\ &u_{n+1}^\epsilon|_{t<0} = 0 \end{aligned} \tag{23}$$

We will show that for sufficiently small T and for all ϵ sufficiently small, the sequence $(u_n^\epsilon)_{n \in \mathbb{N}}$ is Cauchy in $L^\infty \cap N^m(\Omega_T)$. To do this, we prove the following stronger estimate: $\exists C$ such that for T sufficiently small and for all ϵ sufficiently small,

$$||u_{n+1}^\epsilon - u_n^\epsilon||_{L^\infty \cap N^m(\Omega_T)} \leq CT ||u_n^\epsilon - u_{n-1}^\epsilon||_{L^\infty \cap N^m(\Omega_T)}. \tag{24}$$

To do this, we prove the following estimates, which combined give us (24). The names of the constants agree with those in Metivier's proof.

(25)
(a) $||G(t, x, \nu^\epsilon, u_n^\epsilon) - G(t, x, \nu^\epsilon, u_{n-1}^\epsilon)||_{L^\infty \cap N^m(\Omega_T)} \leq C_3 ||u_n^\epsilon - u_{n-1}^\epsilon||_{L^\infty \cap N^m(\Omega_T)}$
(b) $||u_{n+1}^\epsilon - u_n^\epsilon||_{L^\infty \cap N^m(\Omega_T)} \leq C_4 T ||G(t, x, \nu^\epsilon, u_n^\epsilon) - G(t, x, \nu^\epsilon, u_{n-1}^\epsilon)||_{L^\infty \cap N^m(\Omega_T)}$

We use the following estimates of Metivier:

Proposition 2.9.1. *(a) For $M \in \mathbb{N}$, let $F(y, Z) \in C^\infty(\mathbb{R}_+^{1+n} \times \mathbb{R}^M, \mathbb{R}^N)$. For any $0 < T_1 < T_0$ and any $m \in \mathbb{N}_0$, $\exists$ a constant $C_0 > 0$ and an increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $T \in [-T_1, T_1]$, if $Z \in L^\infty \cap N^m(\Omega_T)$, then $f(y) := F(y, Z(y))$ belongs to $L^\infty \cap N^m(\Omega_T)$ and satisfies*

$$(i)\ |f|_{L^\infty(\Omega_T)} \leq h\left(|Z|_{L^\infty(\Omega_T)}\right)$$
$$(ii)\ |f|_{N^m(\Omega_T)} \leq C_0 + h\left(|Z|_{L^\infty(\Omega_T)}\right)|Z|_{N^m(\Omega_T)}.$$

(b) For T_1 as above and any $n \in \mathbb{N}_0$, $\exists C > 0$ such that for any $T \in [-T_1, T_1]$, if u, v are real valued functions in $L^\infty \cap N^m(\Omega_T)$, then

$$|uv|_{L^\infty \cap N^m(\Omega_T)} \leq C|u|_{L^\infty \cap N^m(\Omega_T)}|v|_{L^\infty \cap N^m(\Omega_T)}.$$

Proposition 2.9.2. *Let $m > 0$ and $0 < T_1 < T_0$. Then $\exists C$ such that for any $0 < T < T_1$, any $f \in N^m(\Omega_T)$ which vanishes for $t < 0$, $\exists$ on Ω_T a unique solution $u \in N^m(\Omega_T)$ of the linear problem*

$$\begin{aligned} &Lu = f \\ &u = 0 \text{ in } t < 0 \end{aligned} \tag{26}$$

which satisfies

$$||u||_{N^m(\Omega_T)} \leq CT||f||_{N^m(\Omega_T)}$$

Proposition 2.9.3. *For $m > (n+5)/2$ and $0 < T_1 < T_0$, there is C such that, for any T such that $0 < T < T_1$, any solution $u \in N^m(\Omega_T)$ of the linear problem (26), where $f \in L^\infty \cap N^m(\Omega_T)$, is bounded and satisfies:*

$$||u||_{L^\infty(\Omega_T)} \leq CT\left[||f||_{L^\infty(\Omega_T)} + ||f||_{N^m(\Omega_T)}\right].$$

Boundedness: To show (25), we first show that $\exists M > 0, T > 0$ such that for all ϵ sufficiently small and for all $n \in \mathbb{N}$, $||u_n^\epsilon||_{L^\infty \cap N^m(\Omega_T)} \leq M$.

We use Metivier's conclusion from Propositions 2.9.1, 2.9.2, and 2.9.3 that $\exists C_0, C_1, C_2$ constants and an increasing function $G^* : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $0 < T < T_1$ and for all $n \in \mathbb{N}$:

$$\begin{aligned} &||u_{n+1}^\epsilon||_{N^m(\Omega_T)} \leq \sqrt{T}C_1 + T\left[C_0 + G^*\left(||u_n^\epsilon||_{L^\infty(\Omega_T)}\right)||u_n^\epsilon||_{N^m(\Omega_T)}\right] \\ &||u_{n+1}^\epsilon||_{L^\infty(\Omega_T)} \leq C_2 + T\left[C_0 + (1 + ||u_n^\epsilon||_{N^m(\Omega_T)})G^*(||u_n^\epsilon||_{L^\infty(\Omega_T)})\right] \end{aligned} \tag{27}$$

These estimates hold for all small ϵ. This is because the constant C_0 and function G^* depend only on $f(t,x,\cdot)$, m and T_1. (What about constants C_1 and C_2? Where do they come from? Are they even needed?) $u_0^\epsilon = 0$ for all ϵ. So we can use Metivier's same induction argument to show that for T sufficiently small, for all ϵ sufficiently small, and for all n,

$$
\begin{aligned}
||u_n^\epsilon||_{L^\infty(\Omega_T)} &\leq C_2 + 1 \\
||u_n^\epsilon||_{N^m(\Omega_T)} &\leq 1
\end{aligned}
$$

Thus, $||u_n^\epsilon||_{L^\infty \cap N^m(\Omega_T)}$ has a bound that does not depend on ϵ or n.

Estimate a: Using this result along with the result from Theorem 2.7 that $\nu^\epsilon = g^\epsilon$ has a uniform bound in $L^\infty \cap N^m(\Omega_T)$, we observe that $(\nu^\epsilon, u_n^\epsilon, u_{n-1}^\epsilon)$ has a bound in $L^\infty \cap N^m(\Omega_T)$ that does not depend on n or ϵ. So by Proposition 2.9.1a, $H(t,x,\nu^\epsilon, u_n^\epsilon, u_{n-1}^\epsilon)$ has such a bound as well, where H is defined through

(28)

$$
\begin{aligned}
G(t,x,\nu^\epsilon,u_n^\epsilon) - G(t,x,\nu^\epsilon,u_{n-1}^\epsilon) &= \left(\int_0^1 G_u(t,x,\nu^\epsilon(t,x), u_{n-1}^\epsilon + s(u_n^\epsilon - u_{n-1}^\epsilon))ds \right)(u_n^\epsilon - u_{n-1}^\epsilon) \\
&= H(t,x,\nu^\epsilon,u_n^\epsilon,u_{n-1}^\epsilon)(u_n^\epsilon - u_{n-1}^\epsilon).
\end{aligned}
$$

Applying Proposition 2.9.1b to (28) gives us estimate (25a).

Estimate b: To get estimate (25b), we define the problem:

$$
\begin{aligned}
&L\left(u_{n+1}^\epsilon - u_n^\epsilon\right) = G(t,x,\nu^\epsilon,u_n^\epsilon) - G(t,x,\nu^\epsilon,u_{n-1}^\epsilon) \\
&u_{n+1}^\epsilon - u_n^\epsilon = 0 \text{ in } t < 0
\end{aligned}
\tag{29}
$$

Using estimate (25a) and the boundedness of u_n^ϵ in $L^\infty \cap N^m(\Omega_T)$, we observe that the RHS of the interior equation is in $L^\infty \cap N^m(\Omega_T)$. Thus, we use Propositions 2.9.2 and 2.9.3 to get (25b).

Conclusion: Now that we have estimates (25a) and (25b), we have estimate (24). Use (24) and pick $T = \frac{1}{2C}$. Then u_n^ϵ is Cauchy for all sufficiently small ϵ.

2.3 Error Analysis (in Progress)

We have

(30)

$$
|K(t,x,u,u_a)w - \epsilon\mathcal{E}_1|_{L^\infty \cap N^m(\Omega_T)} \leq |K(t,x,u,u_a)w|_{L^\infty \cap N^m(\Omega_T)} + |\epsilon\mathcal{E}_1|_{L^\infty \cap N^m(\Omega_T)}
$$

Lemma 2.10. *$\epsilon\mathcal{E}_1$ is $O(\epsilon)$ in $L^\infty \cap N^m(\Omega_T)$ as $\epsilon \to 0$.*

Proof. Recall, by (7),

$$
\begin{aligned}
|\epsilon\mathcal{E}_1|_{L^\infty \cap N^m(\Omega_T)} &= \left| \epsilon \left[\mathcal{L}(t,x,\partial_{t,x})V(t,x,\theta) - K_1(t,x,U_0,\epsilon U_1)U_1\right]_{\theta = \frac{x_n}{\epsilon}} \right|_{L^\infty \cap N^m(\Omega_T)} \\
&\leq \left| \left[\epsilon\mathcal{L}(t,x,\partial_{t,x})V(t,x,\theta)\right]_{\theta=\frac{x_n}{\epsilon}} \right|_{L^\infty \cap N^m(\Omega_T)} + \left| \left[\epsilon K_1(t,x,U_0,\epsilon U_1)U_1\right]_{\theta=\frac{x_n}{\epsilon}} \right|_{L^\infty \cap N^m(\Omega_T)}
\end{aligned}
$$

As for the first term,

$$\begin{aligned}
\left|[\epsilon\mathcal{L}(t,x,\partial_{t,x})V(t,x,\theta)]_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty\cap N^m(\Omega_T)} &= \epsilon\left|\int_{-\infty}^{\frac{x_n}{\epsilon}} \mathcal{L}(t,x,\partial_{t,x})\left(HQ(t,x,s)\right)ds\right|_{L^\infty\cap N^m(\Omega_T)} \\
&\le \epsilon\left(2K_\theta \sup_{\substack{(t,x)\in\Omega_T\\|s|\le K_\theta}}\left(\mathcal{L}(t,x,\partial_{t,x})\left(HQ(t,x,s)\right)\right)\right) \\
&\quad + \epsilon\left|\int_{-\infty}^{\frac{x_n}{\epsilon}} \mathcal{L}(t,x,\partial_{t,x})\left(HQ(t,x,s)\right)ds\right|_{N^m(\Omega_T)},
\end{aligned}$$

where the $N^m(\Omega_T)$ bound is a constant, shown by letting $\mathcal{L}(t,x,\partial_{t,x})\left(HQ(t,x,s)\right)$ play the role of $HQ(t,x,s)$ and applying the same argument as used in Theorem 2.7 for V.

As for the second term,

$$\begin{aligned}
\left|[(f(t,x,u_a)-f(t,x,U_0))]_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty\cap N^m(\Omega_T)} &= \left|\left[(u_a-U_0)\int_0^1 f_u(t,x,U_0+s(u_a-U_0))ds\right]_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty\cap N^m(\Omega_T)} \\
&= \left|\left[(\epsilon V)\int_0^1 f_u(t,x,U_0+s(u_a-U_0))ds\right]_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty\cap N^m(\Omega_T)} \\
\text{(Using Prop 6.1b)}\ &\le \epsilon C\,|V|_{L^\infty\cap N^m(\Omega_T)}\left|\left[\int_0^1 f_u(t,x,U_0+s(u_a-U_0))ds\right]_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty}
\end{aligned}$$

The bound for $|V|_{L^\infty\cap N^m(\Omega_T)}$ comes from Theorem 2.7 and the bound for the next term comes from the fact that the integrand is bounded in $\Omega_T\cup\{0\le s\le 1\}$, $\theta\in\mathbb{R}_\theta$. $\square$

So revising (30), we have

(31)

$$\begin{aligned}
|K(t,x,u,u_a)w-\epsilon\mathcal{E}_1|_{L^\infty\cap N^m(\Omega_T)} &\le |K(t,x,u,u_a)w|_{L^\infty\cap N^m(\Omega_T)}+O(\epsilon) \\
\text{(Using Prop 6.1b)}\ &\le C|K(t,x,u,u_a)|_{L^\infty\cap N^m(\Omega_T)}|w|_{L^\infty\cap N^m(\Omega_T)}+O(\epsilon)
\end{aligned}$$

Whether using a direct estimate on K or rather Proposition 6.1a from the notes, it appears the best we can get for K is showing it is $O(1)$.

Using this assumption, we modify (31) to get

$$|K(t,x,u,u_a)w-\epsilon\mathcal{E}_1|_{L^\infty\cap N^m(\Omega_T)}\le O(1)|w|_{L^\infty\cap N^m(\Omega_T)}+O(\epsilon) \tag{32}$$

Letting $f=K(t,x,u,u_a)w-\epsilon\mathcal{E}_1$ and using Cor 4.1.3 and Prop 5.1.1,

$$\begin{aligned}
|w|_{N^m(\Omega_T)}\le CT|f|_{N^m(\Omega_T)} &\le CT|f|_{L^\infty\cap N^m(\Omega_T)} \\
&\le CT\left(O(1)|w|_{L^\infty\cap N^m(\Omega_T)}+O(\epsilon)\right)
\end{aligned}$$

The same bound suffices for $|w|_{L^\infty(\Omega_T)}$, so

$$|w|_{L^\infty\cap N^m(\Omega_T)} \leq 2CT\left(O(1)|w|_{L^\infty\cap N^m(\Omega_T)} + O(\epsilon)\right)$$

Thus,

$$|w|_{L^\infty\cap N^m(\Omega_T)}(1 - 2CTO(1)) \leq 2CTO(\epsilon)$$

and

$$|w|_{L^\infty\cap N^m(\Omega_T)} \leq 2CTO(\epsilon)/(1 - 2CTO(1))$$

So for suitable T, w is $O(\epsilon)$.

To do: Add in change of coordinates discussion somewhere.

References

[1] Guy Metivier, *Propogation, Interaction and Reflection of Discontinuous Progressing Waves for Semilinear Hyperbolic Systems*, American Journal of Mathematics **111(2)** (1989), 239–287.