Nonlinear Geometric Optics for Pulses with Nonlinear Phases

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1 Introduction

1.1 General presentation

This article deals with the use of nonlinear geometric optics techniques to study the behavior of pulse solutions of semilinear first-order hyperbolic systems of partial differential equations with nonlinear phases. Hyperbolic systems have many physical applications, including in equations of general relativity, Maxwell's equations for electromagnetic waves, and Euler equations for compressible fluids such as gases. Solutions to hyperbolic systems are "wavelike," that is, they exhibit features that propagate in space at finite speeds. Both pulses and wavetrains are waves which solve hyperbolic problems. A pulse can be pictured as a very localized high-frequency oscillation that travels through space. A wavetrain is also a high-frequency oscillation that travels through space, but its oscillations are much more spread out. In geometric optics we describe both pulses and wavetrains using profiles $\sigma_j(t,x,\theta)|_{\theta=\psi_j/\epsilon}$. Here, $\psi_j(t,x)$ is a function whose level surfaces are surfaces of constant phase and ϵ is the wavelength. In the pulse case, the profiles σ_j decay in θ as $|\theta|$ gets large. In the wavetrain case, the profiles σ_j are periodic in θ.

In many cases, solutions to hyperbolic systems are quite difficult to construct or study. However, applied mathematicians have developed expertise over many years in finding approximate solutions [1] [2]. Since the systems that these approximate solutions solve are much simpler than the original systems, one can readily identify properties of an approximate solution by observing the structure of the system it solves. Properties include what the solutions look like, how they decay, what phases they depend on, what their amplitudes are, what their propagation velocity is, how their support changes with time, how they reflect off the boundary and interact with each other, etc.

Joly, Metivier and Rauch are pioneers in proving these approximate solutions are close to exact solutions [3]. Their approach involves first improving upon the approximate solution by adding nontrivial "corrector" terms. Then, one shows that for small wavelength, the improved approximate solution is sufficiently close to the exact solution in an appropriate norm. Closeness in the norm implies that the exact solution inherits properties of the approximate solution. The choice of norm can vary depending on the problem in question. For the purposes of this article, the norm used is stronger than the L^∞ norm and similar to the standard Sobolev norm.

This so-called "WKB method" has been used primarily in situations where the characteristic phase is linear with respect to the time and spatial variables [4]. As for the nonlinear case, Joly, Metivier, and Rauch showed that problematic behavior sometimes prevents the use of the WKB method to study wavetrain solutions [5]. Since pulses interact less strongly than wavetrains, we believe it should be easier to obtain results about pulse solution with nonlinear phases.

Firstly, this article utilizes the WKB method to characterize a pulse solution of a problem in free space with only one nonlinear phase. That is, we construct an approximate solution and show its closeness to the exact solution in an appropriate norm for small wavelength. A novel aspect of our approach is the way in which a proposition of Metivier is used. Whereas Metivier's proposition allowed him to sidestep shortcomings of classical theory while working with discontinuous solutions [6], we use it to obtain a time of existence independent of ϵ on which smooth exact solutions exist and are regular. Note that established hyperbolic theory shows the existence and regularity of smooth exact solutions (where the data is smooth), where the time interval can vary with ϵ [3]. The novelty of our approach comes in obtaining solutions on a fixed time interval independent of ϵ.

Next, this article considers the form of the approximate solution to a problem with two nonlinear phases in free space. The problem consists of "launching" two pulses towards each other such that they collide at time $t = 0$. For more details, see (1.2). In an $N \times N$ system, the pulses propagate along 2 of the system's N characteristic surfaces, all of which intersect transversally at $t = x_n = 0$. Given the nonlinearity of the phases, it is unknown whether the nonlinear interaction of the pulses during collision at $t = 0$ will give rise to only two pulses in $t > 0$, or as many as N pulses. Following the WKB method, we propose an approximate solution to the problem that suggests the solution still consists of only two pulses beyond $t = 0$. A term of our approximate solution which takes account of the interaction of pulses is left unconstructed, although a condition is stated that it should satisfy in order to allow us to do error analysis as done for the single-phase problem. The condition is stated in (3.6c). We leave the construction of this "interaction term" and the error analysis of the approximate solution as another project, which could confirm the appropriateness of our approximate solution. We expect the approximate solution to be close to the exact solution for small wavelength in a norm similar to the norm used in the analysis of the single-phase problem.

1.2 The equations and main assumptions

We define the following two $N \times N$ semilinear systems for $u^\epsilon(t, x) \in \mathbb{R}^N$ with space-time variables $(t, x) \in \mathbb{R}^{1+n}$. The first is the single-phase problem, the second the multiphase problem.

(1.1)

$$(a)\ L(t, x, \partial_{t,x})u^\epsilon := \partial_t u^\epsilon + \sum_{j=1}^{n} A_j(t, x)\partial_j u^\epsilon = f(t, x, u^\epsilon) + g(t, x, \theta)|_{\theta=\frac{\psi_k}{\epsilon}}$$

$$(b)\ u^\epsilon = 0 \text{ in } t < 0.$$

(1.2)

$$(a) L(t, x, \partial_{t,x})u^\epsilon := \partial_t u^\epsilon + \sum_{j=1}^{n} A_j(t, x)\partial_j u^\epsilon = f(t, x, u^\epsilon)$$

$$(b) u^\epsilon(t, x)|_{t<-T'} = \sigma_1(t, x, \psi_1(t, x)/\epsilon)r_1(t, x) + \sigma_2(t, x, \psi_2(t, x)/\epsilon)r_2(t, x) + O(\epsilon)$$

Here, $\epsilon > 0$ is a parameter. $\{\psi_j\}_{j=1}^N$ are the (nonlinear) characteristic phases of the operator L defined in Definition 1.7, and ψ_k is the k-th characteristic phase for fixed $k \in \{1, ..., N\}$. In (1.2), $\sigma_j(t, x, \theta_j)$ are given smooth scalar profiles with compact support in θ_j, r_j are appropriate eigenvectors derived from the operator L and defined in Definition 2.2, and $O(\epsilon)$ is a term that has size $O(\epsilon)$ in an appropriate norm that is discussed below Definition 1.9. The surfaces Σ_1 and Σ_2 around which σ_1 and σ_2 have their respective support intersect transversally at $t = x_n = 0$ as discussed below Definition 1.8. In particular, the pulses "collide" at $t = 0$. It is a possibility for (1.2) to have a solution that exists long enough for the interaction of the pulses after collision to be analyzed, as shown in section 3.2.

We make the following assumptions on these equations.

Assumption 1.1. *$A_j(t,x) \in C^\infty(\mathbb{R}^{1+n})$ are $N \times N$ real matrices for $j = 1, ..., n$.*

Assumption 1.2. *$f \in C^\infty(\mathbb{R}^{1+n} \times \mathbb{R}^N, \mathbb{R}^N)$ and $f(t,x,0) = 0$.*

Note that the assumption that $f(t,x,0) = 0$ implies that if g is identically zero, then $u^\epsilon = 0$ solves (1.1) Thus, g is named the "forcing term."

Assumption 1.3. *$g(t,x,\theta) \in C^\infty(\mathbb{R}^{1+n} \times \mathbb{R}, \mathbb{R}^N)$ and $g = 0$ in $t < 0$. g has compact support in (x,θ).*

The assumption that g has compact support in θ goes further than the usual assumption for pulses made on g (which is decay in θ).

$g(t,x,\theta)$ is known as the "pulse profile" of (1.1) and $g(t,x,\psi_k/\epsilon)$ as the "pulse data" of (1.1). A solution to a problem that includes pulse data is known as a "pulse solution," or sometimes colloquially as a "pulse."

Assumption 1.4 (Strict hyperbolicity)**.** *Let $(\tau,\xi) \in \mathbb{R}_\tau \times \mathbb{R}^n_\xi$ denote variables dual to (t,x). The operator L is strictly hyperbolic with respect to t on a neighborhood $\mathcal{O}$ of $(t,x) = 0$. That is, there exist functions $\tau_i(t,x,\xi)$: $C^\infty(\mathcal{O} \times (\mathbb{R}^n_\xi \setminus 0), \mathbb{R})$, $i = 1, ..., N$, positively homogeneous of degree one in ξ, such that*

$$\tau_1 < \tau_2 < ... < \tau_N \text{ on } \mathcal{O} \times (\mathbb{R}^n_\xi \setminus 0)$$

$$\det\left(\tau I + \sum_{j=1}^n A_j(t,x)\xi_j\right) = \prod_{i=1}^N (\tau - \tau_i(t,x,\xi)) . \tag{1.3}$$

Using Assumption 1.4, we construct characteristic phases ψ_j of the operator L that are used to pose the problems (1.1) and (1.2).

Definition 1.5. *Define:*

$$\mathcal{A}(t,x,\xi) := \sum_{j=1}^n A_j(t,x)\xi_j$$

By Assumption 1.4, $\mathcal{A}(t,x,\xi)$ has N distinct real eigenvalues when $(t,x) \in \mathcal{O}$ and when $\xi \neq 0$.

Definition 1.6. *Define* $\{\lambda_j(t,x,\xi)\}_{j=1}^N$ *as the eigenvalues of* $\mathcal{A}(t,x,\xi)$*, where* $\lambda_1 < ... < \lambda_N$ *on* $\mathcal{O} \times (\mathbb{R}^n_\xi \setminus 0)$ *as in (1.3).*

Definition 1.7. *For* $j \in \{1,...,N\}$*, define the characteristic phase* $\psi_j(t,x)$ *as the solution to the following system:*

$$\begin{aligned} \partial_t \psi_j &= -\lambda_j(t,x,d_x\psi_j) \\ \psi_j|_{t=0} &= x_n. \end{aligned} \tag{1.4}$$

Using the method of characteristics, one obtains a smooth solution on a (possibly small) neighborhood of the origin. See Theorem 2 in Chapter 3.2 of Evans for more details on this solution method [7].

1.3 Main results

The results take place in a small domain Ω_T near the origin. We define the domain as follows:

$$\begin{aligned} \Omega &:= \{(t,x) : -T_0 < t < T_0 - \alpha|x|\} \\ \Omega_T &:= \Omega \cap \{t < T\} \end{aligned} \tag{1.5}$$

We require that Ω_T satisfy certain constraints for our results to hold. α and T_0 are chosen such that Ω_T is a domain of determinacy. In particular, we require that α and T_0 satisfy the conditions necessary for Lemma 4.4 to hold (see proof of lemma). We also choose T sufficiently small such that Proposition 2.5 holds.

The results also use a norm that is defined in reference to a surface constructed from characteristic surfaces.

Definition 1.8. *Define*

$$\Sigma_j := \{(t,x) : \psi_j(t,x) = 0\} \text{ for } j = 1,...,N,$$
$$\Sigma := \begin{cases} \Sigma_k & \text{in the single-phase problem} \\ \bigcup_{i=1}^N \Sigma_i & \text{in the multiphase problem} \end{cases}$$

Note that for $i \neq j$, Σ_i and Σ_j intersect transversally at $t = x_n = 0$ since $\nabla\psi_i = (-\lambda_i, 0, ..., 0, 1)^T$ and $\nabla\psi_j = (-\lambda_j, 0, ..., 0, 1)^T$ are linearly independent at $t = 0$ by (1.4). These characteristic surfaces play a significant role in the geometry of the problem since they are the surfaces along which pulses propagate. To see why, observe that the vector field X (resp. X_i) used in (2.11) (resp. (3.13)) the system that constructs σ_k (resp. s_i) is tangent to Σ (resp. Σ_i) by Lemma 2.4. Thus, as the pulses collide at $t = 0$ along the intersection $t = x_n = 0$ of these surfaces, it is unknown whether this interaction could cause pulses to propagate along any of the remaining $N-2$ pulses. Proposition 1.11 addresses this question.

In the case of problem (1.1), we define the norm in the manner of Metivier [6]:

Definition 1.9. *$N^m(\Omega_T)$ is the space of functions $u \in L^2(\Omega_T)$ such that*

$$M_1 ... M_l u \in L^2(\Omega_T) \tag{1.6}$$

for any sequence $M_1, ..., M_l$ of $l \leq m$ vector fields (with C^∞ coefficients on Ω_T) tangent to Σ.

In the case of problem (1.2), the N^m norm consists of functions conormal with respect to $\Sigma = \cup_i \Sigma_i$ in an appropriate sense that we will not specify here. For a precise definition, see [6]. The $O(\epsilon)$ term of (1.2) is $O(\epsilon)$ in the norm $L^\infty \cap N^m(\Omega_{-T'})$, where $-T' > T_0$.

The result for the single-phase problem is as follows:

Theorem 1.10. *Given Assumptions 1.1, 1.2, 1.3, and 1.4, for $m > (n+5)/2$, for appropriately chosen T_0, α, and for small T, there exists u^ϵ a solution and u_a^ϵ an approximate solution to (1.1) both in $L^\infty \cap N^m(\Omega_T)$, such that*

$$|u^\epsilon - u_a^\epsilon|_{L^\infty \cap N^m(\Omega_T)} = O(\epsilon) \text{ as } \epsilon \to 0, \tag{1.7}$$

and u_a^ϵ has the form:

$$\begin{aligned} &u_a^\epsilon(t,x) = [U_0(t,x,\theta) + \epsilon U_1(t,x,\theta)]\,|_{\theta = \frac{\psi_k}{\epsilon}}, \text{ where} \\ &U_0 = \sigma_k(t,x,\theta) r_k(t,x) \\ &U_1 = V(t,x,\theta). \end{aligned} \tag{1.8}$$

Here, σ_k is a scalar function with compact support in θ defined in (2.11), V is defined in (2.13) and r_k is the k-th eigenvector associated to $\mathcal{A}(t, x, d_x\psi_k)$ defined in Definition 2.2.

For the multiphase problem, we have the following partial result:

Proposition 1.11. *Given Assumptions 1.1, 1.2, and 1.4, one can construct an approximate solution to (1.2) of the following form. We use the notation* $\theta = (\theta_1, ..., \theta_N)$ *and* $\psi = (\psi_1, ..., \psi_N)$.

$$\begin{aligned} &u_a^\epsilon := U_0(t,x,\theta)|_{\theta=\psi/\epsilon} + \epsilon U_1^\epsilon(t,x), \text{ where} \\ &U_0(t,x,\theta) := s_1\left(t,x,\theta_1\right) r_1(t,x) + s_2\left(t,x,\theta_2\right) r_2(t,x), \\ &U_1^\epsilon(t,x) := V(t,x,\psi/\epsilon) + W(t,x,\tau,\theta_0)|_{\tau=t/\epsilon, \theta_0 = x_n/\epsilon}, \\ &V(t,x,\theta) = V_1(t,x,\theta_1) + V_2(t,x,\theta_2), \end{aligned} \tag{1.9}$$

where s_1 *and* s_2 *are scalar functions constructed in (3.13) that satisfy* $s_i = \sigma_i$ *in* $t < -T'$ *for* $i = 1, 2$, V_1 *and* V_2 *are vector functions constructed in (3.18), and* r_i *is the* i*-th eigenvector associated to* $\mathcal{A}(t, x, d_x\psi_i)$ *defined in Definition 2.2. The vector function* W *is not constructed, but it must satisfy (3.6c).*

Moreover, for $m > (n+5)/2$ and for some $T > 0$ such that $T < T_0$, we expect that the approximate solution u_a^ϵ and exact solution u^ϵ to the multiphase problem (1.2) satisfy

$$|u^\epsilon - u_a^\epsilon|_{L^\infty \cap N^m(\Omega_T)} = O(\epsilon) \text{ as } \epsilon \to 0 \tag{1.10}$$

when u^ϵ exists on this domain, and when $N^m(\Omega_T)$ is appropriately defined for the multiphase problem as discussed above.

The U_0 terms of (1.8) and (1.9) are modeled after traditional approximate solutions to nonlinear geometric optics problems. The U_1 terms in (1.8) are the appropriately constructed corrector terms that provide the closeness conditions (1.7) and (1.10).

2 The Single-Phase Problem

In this section we prove Theorem 1.10. The paper [9] also gives a rigorous construction of a single pulse, but there the pulse is launched from data on an initial surface rather than from interior forcing data. Our construction of the corrector term V in section 2.1.5 is somewhat simpler than in [9]. Also, our error analysis in section 2.3 is more streamlined, since we make use of the $L^\infty \cap N^m$ estimates of Metivier [6], which were proved originally for discontinuous progressing waves. The paper [9] does not consider the multiphase problem.

2.1 Construction of approximate solution

We start with a definition of the eigenvectors r_j, given that r_k is a component of the approximate solution.

2.1.1 Definition of r_j

As noted, $\mathcal{A}(t,x,\xi)$ has N distinct real eigenvalues defined in Definition 1.6 as $\{\lambda_j\}_{j=1}^N$ when $(t,x) \in \mathcal{O}$ and when $\xi \neq 0$.

Definition 2.1. *Define $L_j(t,x,\xi)$ and $R_j(t,x,\xi)$ respectively as the left and right eigenvectors of $\mathcal{A}(t,x,\xi)$ corresponding to λ_j for $j = 1, ..., N$. We scale $\{R_j(t,x,\xi)\}_{j=1}^N$ such that when $(t,x) \in \mathcal{O}$ and $\xi \neq 0$, $L_i R_j = \delta_{ij}$.*

To see that it is possible for $L_i R_j$ to be scaled to equal δ_{ij} on the designated domain, note that $L_i \mathcal{A} R_j = \lambda_i L_i R_j = \lambda_j L_i R_j$. So on the domain, the distinctness of eigenvalues given by Assumption 1.4 implies that $j \neq i \implies L_i R_j = 0$. Thus, $L_j R_j \neq 0$ for all j, since the contrary would imply that R_j is orthogonal to a full basis. Then we scale R_j appropriately.

Definition 2.2. *For $j \in \{1, ..., N\}$, define*

$$\begin{aligned} l_j(t,x) &:= L(t,x,d_x\psi_j) \\ r_j(t,x) &:= R(t,x,d_x\psi_j). \end{aligned}$$

Using the fixed k from earlier, we obtain r_k.

2.1.2 Setting conditions for approximate solution

Before defining the rest of our terms, we identify conditions that we want them to hold. For the approximate solution to be close to u^ϵ, we want $L(t,x,\partial_{t,x})u_a^\epsilon - f(t,x,u_a) - g(t,x,\theta)|_{\theta=\frac{\psi_k}{\epsilon}}$ to be sufficiently small. We do some calculations:

(2.1)

$$L(t,x,\partial_{t,x})U_0(t,x,\frac{\psi}{\epsilon}) = \left[\frac{1}{\epsilon}L(t,x,d\psi)(\partial_\theta\sigma_k(t,x,\theta))r_k + (L(t,x,\partial_{t,x})\sigma_k)r_k + \sigma_k L(t,x,\partial_{t,x})r_k\right]|_{\theta=\frac{\psi_k}{\epsilon}}$$
$$= \left[\frac{1}{\epsilon}(\partial_\theta\sigma_k(t,x,\theta))(\mathcal{A}(t,x,d_x\psi)r_k - \lambda_k r_k) + (L(t,x,\partial_{t,x})\sigma_k)r_k + \sigma_k L(t,x,\partial_{t,x})r_k\right]|_{\theta=\frac{\psi_k}{\epsilon}}$$
$$= \left[(L(t,x,\partial_{t,x})\sigma_k)r_k + \sigma_k L(t,x,\partial_{t,x})r_k\right]|_{\theta=\frac{\psi_k}{\epsilon}}$$
$$= \left[L(t,x,\partial_{t,x})U_0(t,x,\theta)\right]|_{\theta=\frac{\psi_k}{\epsilon}}$$

Here, applying the operator $L(t,x,\partial_{t,x})$ to a function of t,x,θ denotes differentiation with respect to t,x but not θ. This convention will be used throughout the paper. Notice that the $1/\epsilon$ term canceled out using the Eikonal equation.

Continuing with our calculations, we have

(2.2)

$$L(t,x,\partial_{t,x})V(t,x,\frac{\psi_k}{\epsilon}) = \left[L(t,x,\partial_{t,x})V(t,x,\theta) + \frac{1}{\epsilon}L(t,x,d\psi_k)\partial_\theta V(t,x,\theta)\right]|_{\theta=\frac{\psi_k}{\epsilon}}$$

By the Fundamental Theorem of Calculus, we have that, for fixed $\theta = \frac{\psi_k}{\epsilon}$,

$$\begin{aligned} f(t,x,u_a^\epsilon) &= \left(\int_0^1 f_u(t,x,U_0+s\epsilon U_1)ds\right)(\epsilon U_1) + f(t,x,U_0) \\ &=: K_1(t,x,U_0,\epsilon U_1)\epsilon U_1 + f(t,x,U_0) \end{aligned} \tag{2.3}$$

Combining all these calculations, we have that

(2.4)

$$L(t,x,\partial_{t,x})u_a^\epsilon - f(t,x,u_a^\epsilon) - g(t,x,\theta)|_{\theta=\frac{\psi_k}{\epsilon}} = \frac{1}{\epsilon}\mathcal{E}_{-1} + \mathcal{E}_0 + \epsilon\mathcal{E}_1, \text{ where}$$

$$\mathcal{E}_{-1} = 0$$
$$\mathcal{E}_0 = [L(t,x,\partial_{t,x})U_0(t,x,\theta) + L(t,x,d\psi_k)\partial_\theta V(t,x,\theta) - f(t,x,U_0) - g(t,x,\theta)]\,|_{\theta=\frac{\psi_k}{\epsilon}}$$
$$\mathcal{E}_1 = [L(t,x,\partial_{t,x})V(t,x,\theta) - K_1(t,x,U_0,\epsilon U_1)U_1]\,|_{\theta=\frac{\psi_k}{\epsilon}}$$

We will construct V, σ_k such that $\mathcal{E}_0 = 0$.

2.1.3 The projection operator

To identify separate conditions for σ_k and V, we separate $\mathcal{E}_0$ using the projection operator π_k defined below, taking inspiration from [4], [9].

Definition 2.3. *Define $\Pi_m(t,x,\xi)$ as the projection on span $R_m(t,x,\xi)$ in the decomposition*

$$\mathbb{C}^N = \oplus_{l=1}^N span\ R_l(t,x,\xi). \tag{2.5}$$

That is, given $v \in \mathbb{C}^N$, $\Pi_m(t,x,\xi)v = (L_m(t,x,\xi)v)R_m(t,x,\xi)$.
Define $\pi_m(t,x) := \Pi_m(t,x,d_x\psi_m)$.

Let

$$\mathcal{F} := L(t,x,\partial_{t,x})U_0(t,x,\theta) - f(t,x,U_0) - g(t,x,\theta). \tag{2.6}$$

We see that $\mathcal{E}_0 = [\pi_k(t,x)\mathcal{F} + (1-\pi_k(t,x))\mathcal{F} + L(t,x,d\psi_k)\partial_\theta V(t,x,\theta)]\,|_{\theta=\frac{\psi_k}{\epsilon}}$. We will construct σ_k and V such that

$$\begin{aligned} &(a)\ \pi_k\mathcal{F} = 0 \\ &(b)\ (1-\pi_k)\mathcal{F} + L(t,x,d\psi_k)\partial_\theta V(t,x,\theta) = 0 \end{aligned} \tag{2.7}$$

In particular, σ_k solves away the projected parts of $\mathcal{F}$, and V the unprojected parts.

2.1.4 Construction of σ_k

In our construction of σ_k, we use the following lemma attributed to Lax [8].

Lemma 2.4. *For $i,j = 1,...,n$:*

$$L_i(t,x,\xi)A_j(t,x)R_i(t,x,\xi) = \partial_{\xi_j}\lambda_i(t,x,\xi) \tag{2.8}$$

Proof. Differentiate the following equation with respect to ξ_j and then multiply by $L_i(t,x,\xi)$ on the left:

$$0 = \Big[-\lambda_i(t,x,\xi)I + \sum_{l=1}^n A_l(t,x)\xi_l\Big]R_i(t,x,\xi) = L(t,x,-\lambda_i,\xi)R_i. \tag{2.9}$$

$\square$

The lemma implies $l_k(t,x)A_j(t,x)r_k(t,x) = \partial_{\xi_j}\lambda_k(t,x,d_x\psi_k)$.

Using the identity derived from the above lemma, we observe that

(2.10)
$$\begin{aligned}\pi_k\left(L(t,x,\partial_{t,x})U_0(t,x,\theta)\right) =& (X(t,x,\partial_{t,x})\sigma_k(t,x,\theta) + c_k(t,x)\sigma_k)r_k, \text{ where}\\ X(t,x,\partial_{t,x}) :=& \partial_t + \sum_{j=1}^{n}\partial_{\xi_j}\lambda_k(t,x,d_x\psi_k)\partial_j\\ c_k(t,x) =& l_k L(t,x,\partial_{t,x})r_k\end{aligned}$$

Accordingly, we want σ to satisfy $\pi_k\mathcal{F} = (X(t,x,\partial_{t,x})\sigma_k + c_k\sigma_k - l_k f(t,x,U_0) - l_k g(t,x,\theta))r_k = 0$. We define σ_k as the solution of the following system:

$$\begin{aligned}&(a)X(t,x,\partial_{t,x})\sigma_k + c_k\sigma_k - h(t,x,\sigma_k) - l_k g(t,x,\theta) = 0\\ &(b)\sigma_k|_{t<0} = 0,\\ &\quad \text{where } h(t,x,\sigma_k) := l_k f(t,x,U_0) = l_k f(t,x,\sigma_k r_k)\end{aligned} \tag{2.11}$$

This system can be solved using the method of characteristics. The choice of initial condition is suitable for σ_k, since it makes $U_0|_{\theta=\frac{\psi_k}{\epsilon}} = \sigma_k r_k|_{\theta=\frac{\psi_k}{\epsilon}}$ equal to u^ϵ in $\{t<0\}$.

Thus, using such σ_k, we have that $\pi_k\mathcal{F} = 0$.

2.1.5 Construction of $U_1 = V$

Next, we construct V such that $(1-\pi_k)\mathcal{F} + L(t,x,d\psi_k)\partial_\theta V(t,x,\theta) = 0$. To do so, we define a partial inverse $Q(t,x)$ for $L(t,x,d\psi_k)$ that satisfies

$$L(t,x,d\psi_k)Q(t,x) = Q(t,x)L(t,x,d\psi_k) = 1 - \pi_k(t,x). \tag{2.12}$$

First, observe that $\mathcal{A}(t,x,d_x\psi_k) = \sum_{j=1}^N \lambda_j(t,x,d_x\psi_k)\Pi_j(t,x,d_x\psi_k)$. Accordingly, $L(t,x,d\psi_k) = \sum_{j=1}^N(\lambda_j(t,x,d_x\psi_k) - \lambda_k(t,x,d_x\psi_k))\Pi_j(t,x,d_x\psi_k)$. Thus, $Q(t,x) := \sum_{j\neq k}(\lambda_j(t,x,d_x\psi_k) - \lambda_k(t,x,d_x\psi_k))^{-1}\Pi_j(t,x,d_x\psi_k)$ satisfies the above property.

Using this partial inverse, V satisfies the desired property when it is defined:

$$\begin{aligned}V(t,x,\theta) :=& -\int_{-\infty}^{\theta} Q(t,x)H(t,x,s)ds, \text{ where}\\ H(t,x,\theta) =& \mathcal{F}.\end{aligned} \tag{2.13}$$

Note that $V = 0$ when $t < 0$. Moreover, this integral is well defined because the integrand has compact support in θ. This is due to the properties of the components of the sum that defines $\mathcal{F}$. g has compact support in θ, and choosing $\overline{\theta}$ such that $g = 0$, $\sigma_k = 0$ solves (2.11), leaving us with $\mathcal{F} = 0$. Furthermore, V is bounded in θ, so as $\epsilon \to 0$, $\epsilon\mathcal{E}_1 \to 0$.

Thus, with σ_k and V defined as above, we have that $\mathcal{E}_0 = 0$.

2.2 Construction of exact solution

In this section, we prove the following proposition:

Proposition 2.5. *There exists a $T > 0$ such that for all $\epsilon > 0$, the problem stated in (1.1) has a solution $u^\epsilon \in L^\infty \cap N^m(\Omega_T)$.*

We do so by modifying an argument used by Metivier in [6] to prove a proposition cited as Proposition 2.13 in this article. In order to apply his argument, we cite some of his estimates and obtain a bound on our pulse data. To facilitate easy calculations, we also make a change of variables.

2.2.1 Reduction and Definitions

To proceed, we reduce to the case where $\Sigma = \{(t, x) : x_n = 0\}$. Section 4.1 in the Appendix justifies this reduction. With this reduction, calculations involving the N^m norm become much easier.

Definition 2.6. *We define as $C_b^\infty(X)$ the set of bounded $C^\infty(X)$ functions h such that $\partial^\alpha h$ is bounded on X for every α.*

Definition 2.7. *We define as $\mathcal{M}$ the set of vector fields defined on Ω_T with coefficients in C_b^∞ that are tangent to $\Sigma = \{x_n = 0\}$.*

Proposition 2.8. *$\mathcal{M}$ is generated as a module over $C_b^\infty(\Omega^T)$ by*

$$\partial_t, \partial_{x_1}, ..., \partial_{x_{n-1}}, x_n\partial_{x_n}. \tag{2.14}$$

Proof. For simplicity we label $t = x_0$ and denote $x = (x_0, x'', x_n)$. We must show $X \in \mathcal{M}$ iff $X = \sum_{j=0}^{n-1} a_j\partial_j + a_n x_n \partial_{x_n}$, where $a_j \in C_b^\infty(\Omega^T)$ for $j = 0, ..., n$. The reverse direction is easy to see, since $X \cdot \nabla x_n = a_n x_n = 0$ at $x_n = 0$.

For the forward inclusion, we use the Fundamental Theorem of Calculus. Let $X \in \mathcal{M}$. Then,

$$X = \sum_{j=0}^{n} a_n \partial_{x_j}, \text{ where } a_j \in C_b^\infty(\Omega^T), \text{ and } \begin{pmatrix} a_0 \\ ... \\ a_n \end{pmatrix} \cdot \nabla x_n = 0 \text{ at } x_n = 0. \tag{2.15}$$

Thus, we see that $a_n(x)|_{x_n=0} = 0$. Using the Fundamental Theorem of Calculus, we factor out x_n:

$$\begin{aligned} a_n(x_0, x'', x_n) - a_n(x_0, x'', 0) &= x_n \left[\int_0^1 \partial_{x_n} a_n(x_0, x'', t x_n) dt \right] \\ &= x_n \tilde{a}_n(x_0, x'', x_n). \end{aligned} \tag{2.16}$$

And $\tilde{a}_n \in C^\infty(\Omega_T)$ by differentiation under the integral sign. □

Definition 2.9. *(a) Let $(V_1, ..., V_{n+1}) =: \tilde{V}$ denote the generating set (2.14) and let $\alpha = (\alpha_1, ..., \alpha_{n+1})$ be a multi-index. Define a norm on $N^m(\Omega_T)$ by*

$$|u|_m^2 = \sum_{|\alpha| \le m} |\tilde{V}^\alpha u|^2_{L^2(\Omega_T)}. \tag{2.17}$$

(b) Let $u^\epsilon \in L^2(\Omega_T)$ be a function that depends on the small parameter ϵ. We say $u^\epsilon \in N^m(\Omega_T)$ if there exists $\epsilon_0 > 0$ such that $\sup_{\epsilon \in (0, \epsilon_0]} |u^\epsilon|_m < \infty$.

2.2.2 Uniform $N^m(\Omega_T)$ bounds for pulse data and approximate solution

Firstly, we define:

Definition 2.10.

$$\begin{aligned} g^\epsilon(t,x) &:= g\left(t, x, \frac{x_n}{\epsilon}\right) \\ u_a^\epsilon(t,x) &:= U_0\left(t, x, \frac{x_n}{\epsilon}\right) + \epsilon U_1\left(t, x, \frac{x_n}{\epsilon}\right) \end{aligned}$$

In order to apply Metivier's argument, we must first find an $N^m(\Omega_T)$ bound for g^ϵ that is uniform with respect to ϵ. It will also be necessary for section 2.3 to find the same type of bound for u_a^ϵ, so we analyze both terms here.

Proposition 2.11. *For all $m \in \mathbb{N}$, g^ϵ and u_a^ϵ have estimates in $N^m(\Omega_T)$ that are uniform with respect to ϵ.*

In the proof we use the following convention to distinguish between partial derivatives:

Notation 2.12.

$$\partial_{y_n} g\left(t, x, \frac{x_n}{\epsilon}\right) := [\partial_{x_n} g(t, x, \theta)]_{\theta = \frac{x_n}{\epsilon}}$$
$$\partial_{\theta} g\left(t, x, \frac{x_n}{\epsilon}\right) := [\partial_{\theta}\left(g(t, x, \theta)\right)]_{\theta = \frac{x_n}{\epsilon}}$$

In this notation, we have that

$$\partial_{x_n}\left(g\left(t, x, \frac{x_n}{\epsilon}\right)\right) = \partial_{y_n} g\left(t, x, \frac{x_n}{\epsilon}\right) + \frac{1}{\epsilon}\partial_\theta g\left(t, x, \frac{x_n}{\epsilon}\right).$$

Proof. We handle V separately from both g and $U_0 = \sigma_k r_k$.
We claim that, for all multi-indeces α, $\tilde{V}^\alpha\left(g(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}}\right)$ and $\tilde{V}^\alpha\left(\sigma_k(t,x,\theta) r_k(t,x)|_{\theta=\frac{x_n}{\epsilon}}\right)$ both have the form:

$$h(t, x, \theta)|_{\theta = \frac{x_n}{\epsilon}}, \tag{2.18}$$

where h is an arbitrary smooth function that has compact support in θ. As for the base case, this trivially holds for g. As for $\sigma_k r_k$, picking θ outside the compact support of g implies that, for that θ, $\sigma_k = 0$ solves (2.11), the equation that constructs σ_k.

Inductive step: The following argument suffices for g, $\sigma_k r_k$, so we use g without loss of generality. Assume $\tilde{V}^\alpha\left(g(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}}\right)$ has that form for all $|\alpha| \le k$. Then for $j < n$, $\partial_j\left(\tilde{V}^\alpha\left(g(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}}\right)\right) = \partial_j h(t, x, \frac{x_n}{\epsilon}) = \partial_j h(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}}$. And we have

$$\begin{aligned} x_n\partial_{x_n}\left(\tilde{V}^\alpha\left(g(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}}\right)\right) &= x_n \partial_{x_n}\left(h\left(t, x, \frac{x_n}{\epsilon}\right)\right) \\ &= [x_n \partial_{x_n} h(t,x,\theta) + \theta\partial_\theta h(t,x,\theta)]_{\theta=\frac{x_n}{\epsilon}}, \end{aligned} \tag{2.19}$$

which has the form of (2.18). Thus, $\tilde{V}^\alpha\left(g(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}}\right)$ has the form of (2.18) for all $|\alpha| \le k+1$, so by induction, the same form holds for all α.

Using (2.18), we get that for $|\alpha| \leq m$, $\epsilon < 1$, and on Ω_T,

$$|\tilde{V}^{\alpha}\left(g|_{\theta=\frac{x_n}{\epsilon}}\right)| \leq \sup_{\substack{(t,x)\in\Omega_T \\ |\theta|\leq K_\theta}} |h(t,x,\theta)|. \tag{2.20}$$

Since $\overline{\Omega}_T$ is compact, $\tilde{V}^{\alpha}\left(g|_{\theta=\frac{x_n}{\epsilon}}\right)$ is uniformly bounded (with respect to ϵ) in $L^2(\Omega_T)$, so $g|_{\theta=\frac{x_n}{\epsilon}}$ has a uniform bound in $N^m(\Omega_T)$.

We use a similar argument for V. We claim that $\tilde{V}^{\alpha}\left(V(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}}\right)$ has the form

$$\left[\theta f(t,x,\theta) + \int_{-\infty}^{\theta} g(t,x,s)ds\right]_{\theta=\frac{x_n}{\epsilon}}, \tag{2.21}$$

where f, g are arbitrary smooth functions that have compact support in θ, s respectively.

Base step: $V(t,x,\theta) = \int_{-\infty}^{\theta} Q(t,x)H(t,x,s)ds$. Its integrand QH has compact support in s since $g(\cdot,s) = 0 \implies \sigma_k = 0$ (where $\theta = s$) and $H(t,x,s) = (1-\pi_k)\mathcal{F} = (1-\pi_k)\left(L(t,x,\partial_{t,x})U_0(t,x,s) - f(t,x,U_0) - g(t,x,s)\right) = 0$. Here, s acts as a variable dual to θ.

Inductive step: Assume $\tilde{V}^{\alpha}\left(V|_{\theta=\frac{x_n}{\epsilon}}\right)$ has the form above for all $|\alpha| \leq k$. Then clearly when $j < n$,

$$\partial_j\left(\tilde{V}^{\alpha}\left(V(t,x,\theta)|_{\theta=\frac{x_n}{\epsilon}}\right)\right) = \left[\theta\partial_j f(t,x,\theta) + \int_{-\infty}^{\theta} \partial_j g(t,x,s)ds\right]_{\theta=\frac{x_n}{\epsilon}}, \tag{2.22}$$

which has the desired form. And,

$$\begin{aligned} x_n\partial_{x_n}\left(\tilde{V}^{\alpha}\left(V|_{\theta=\frac{x_n}{\epsilon}}\right)\right) &= x_n\Bigg[\frac{1}{\epsilon}f(t,x,\frac{x_n}{\epsilon}) + \frac{x_n}{\epsilon}\left(\partial_{y_n}f(t,x,\frac{x_n}{\epsilon}) + \frac{1}{\epsilon}\partial_\theta f(t,x,\frac{x_n}{\epsilon})\right) \\ &\quad + \frac{1}{\epsilon}g(t,x,\frac{x_n}{\epsilon}) + \int_{-\infty}^{\frac{x_n}{\epsilon}} \partial_{y_n}g(t,x,s)ds\Bigg] \\ &= \left[\theta h(t,x,\theta) + \int_{-\infty}^{\theta} x_n\partial_{y_n}g(t,x,s)ds\right]_{\theta=\frac{x_n}{\epsilon}}, \text{ where} \\ & h(t,x,\theta) = f(t,x,\theta) + x_n\partial_{y_n}f(t,x,\theta) + \theta\partial_\theta f(t,x,\theta) + g(t,x,\theta). \end{aligned} \tag{2.23}$$

So $x_n\partial_{x_n}\left(\tilde{V}^\alpha\left(V|_{\theta=\frac{x_n}{\epsilon}}\right)\right)$ has the form of (2.21). Thus, $\tilde{V}^\alpha\left(V|_{\theta=\frac{x_n}{\epsilon}}\right)$ has the form of (2.21) for $|\alpha|\leq k+1$, so by induction, the form holds for all α.

Using (2.21), we see that, for $|\alpha|\leq m$, one can choose a constant K' sufficiently large such that, for $\epsilon<1$ and on Ω_T,

$$|\tilde{V}^\alpha\left(V|_{\theta=\frac{x_n}{\epsilon}}\right)|\leq K'\sup_{\substack{(t,x)\in\Omega_T\\|\theta|\leq K'}}|f(t,x,\theta)|+2K'\sup_{\substack{(t,x)\in\Omega_T\\|s|\leq K'}}|g(t,x,s)|\,. \tag{2.24}$$

Since $\overline{\Omega}_T$ is compact, $\tilde{V}^\alpha\left(V|_{\theta=\frac{x_n}{\epsilon}}\right)$ is uniformly bounded (with respect to ϵ) in $L^2(\Omega_T)$, so $V|_{\theta=\frac{x_n}{\epsilon}}$ has a uniform bound in $N^m(\Omega_T)$. $\square$

2.2.3 Metivier's proposition for time of existence

Using Proposition 2.11 (specifically the estimate for g^ϵ), we can modify the proof of the following proposition of Metivier to construct u^ϵ with a time of existence that holds for all $\epsilon>0$. The proposition of Metivier that we use is Proposition 6.1.1 in [6], stated in our article as Proposition 2.13. Here, T_0 and α denote the parameters defined in (1.5) of this paper.

Proposition 2.13. *Let T_0 and α be chosen such that Ω_T is a domain of determinacy, and fix $0<T_1<T_0$. Let $G(t,x,\nu,u)\in C^\infty(\overline{\Omega}_T\times\mathbb{R}^M\times\mathbb{R}^N)$, where ν is an $\mathbb{R}^M$-valued function in $L^\infty\cap N^m(\Omega_{T_1})$, $m>(n+5)/2$, and $G(t,x,\nu(t,x),0)=0$ when $t<0$. Consider the problem:*

$$\begin{aligned}&L(t,x,\partial_{t,x})u(t,x)=G(t,x,\nu(t,x),u(t,x))\\&u|_{t<0}=0\end{aligned} \tag{2.25}$$

Then, there is $T:0<T<T_1$ and $u\in L^\infty\cap N^m(\Omega_T)$ a solution to (2.25).

2.2.4 Metivier's estimates

Metivier's argument involves the following estimates proven in [6]. The first set of estimates (Proposition 2.14) is nonlinear, while the other two (Propositions 2.15 and 2.16) are linear. An exposition of Proposition 2.15 can be found in section 4.2 in the appendix. Ω_T must be a domain of determinacy for these estimates to hold.

Proposition 2.14. *(a) For $M\in\mathbb{N}$, let $F(y,Z)\in C^\infty(\mathbb{R}_+^{1+n}\times\mathbb{R}^M,\mathbb{R}^N)$. For any $0<T_1<T_0$ and any $m\in\mathbb{N}_0$, $\exists$ a constant $C>0$ and an increasing* *function* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for any* $T \in [-T_1, T_1]$*, if* $Z \in L^\infty \cap N^m(\Omega_T)$*, then* $f(y) := F(y, Z(y))$ *belongs to* $L^\infty \cap N^m(\Omega_T)$ *and satisfies*

$$(i)\ |f|_{L^\infty(\Omega_T)} \leq h\left(|Z|_{L^\infty(\Omega_T)}\right)$$
$$(ii)\ |f|_{N^m(\Omega_T)} \leq C + h\left(|Z|_{L^\infty(\Omega_T)}\right)|Z|_{N^m(\Omega_T)}.$$

(b) For T_1 *as above and any* $n \in \mathbb{N}_0$*,* $\exists C > 0$ *such that for any* $T \in [-T_1, T_1]$*, if* u, v *are real valued functions in* $L^\infty \cap N^m(\Omega_T)$*, then*

$$|uv|_{L^\infty \cap N^m(\Omega_T)} \leq C|u|_{L^\infty \cap N^m(\Omega_T)}|v|_{L^\infty \cap N^m(\Omega_T)}.$$

Proposition 2.15. *Let* $m > 0$ *and* $0 < T_1 < T_0$*. Then* $\exists C$ *such that for any* $0 < T < T_1$*, any* $f \in N^m(\Omega_T)$ *which vanishes for* $t < 0$*,* $\exists$ *on* Ω_T *a unique solution* $u \in N^m(\Omega_T)$ *of the linear problem*

$$\begin{aligned} &L(t,x,\partial_{t,x})u = f \\ &u = 0 \ in\ t < 0 \end{aligned} \tag{2.26}$$

which satisfies

$$|u|_{N^m(\Omega_T)} \leq CT|f|_{N^m(\Omega_T)}$$

Proposition 2.16. *For* $m > (n+5)/2$ *and* $0 < T_1 < T_0$*, there is* C *such that, for any* T *such that* $0 < T < T_1$*, any solution* $u \in N^m(\Omega_T)$ *of the linear problem (2.26), where* $f \in L^\infty \cap N^m(\Omega_T)$*, is bounded and satisfies:*

$$|u|_{L^\infty(\Omega_T)} \leq CT\left[|f|_{L^\infty(\Omega_T)} + |f|_{N^m(\Omega_T)}\right].$$

2.2.5 Proof of uniform time of existence

Following Metivier's approach to proving Proposition 2.13, we construct the solution using Picard iteration, except with an important difference. Instead of constructing u_n iteratively, we construct u_n^ϵ iteratively. We define $u_0^\epsilon = 0$ and define u_n^ϵ:

(2.27)
$$\begin{aligned} &L(t,x,\partial_{t,x})u_{n+1}^\epsilon = G(t,x,\nu^\epsilon(t,x),u_n^\epsilon(t,x)) = f(t,x,u_n^\epsilon(t,x)) + g^\epsilon(t,x) \\ &u_{n+1}^\epsilon|_{t<0} = 0 \end{aligned}$$

Note that, in the language of Proposition 2.13, our g^ϵ plays the role of his ν, and our $f(t,x,u^\epsilon) + g^\epsilon(t,x)$ plays the role of his $G(t,x,\nu(t,x),u(t,x))$.

We will show that for sufficiently small T and for all ϵ, the sequence $(u_n^\epsilon)_{n\in\mathbb{N}}$ is Cauchy in $L^\infty \cap N^m(\Omega_T)$. To prove this, we prove the two estimates in the following lemma.

Lemma 2.17. $\exists C_0, C_1$ *such that:*

(2.28)

$$(a)\ |f(t,x,u_n^\epsilon) - f(t,x,u_{n-1}^\epsilon)|_{L^\infty\cap N^m(\Omega_T)} \leq C_0|u_n^\epsilon - u_{n-1}^\epsilon|_{L^\infty\cap N^m(\Omega_T)}$$
$$(b)\ |u_{n+1}^\epsilon - u_n^\epsilon|_{L^\infty\cap N^m(\Omega_T)} \leq C_1 T|f(t,x,u_n^\epsilon) - f(t,x,u_{n-1}^\epsilon)|_{L^\infty\cap N^m(\Omega_T)}$$

Once we proof Lemma 2.17, the proposition follows by a brief argument.

Proof (of Lemma 2.17). Our proof relies on the fact that there exist $M > 0, T > 0$ such that for all ϵ and for all $n \in \mathbb{N}$, $|u_n^\epsilon|_{L^\infty\cap N^m(\Omega_T)} \leq M$, which we prove below.

Using Propositions 2.14, 2.15, and 2.16, we observe that there $\exists C_2, C_3$ constants and an increasing function $G^* : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $0 < T < T_1$ and for all $n \in \mathbb{N}$:

$$\begin{aligned} |u_{n+1}^\epsilon|_{N^m(\Omega_T)} &\leq T\left[C_2 + G^*\left(|u_n^\epsilon|_{L^\infty(\Omega_T)}\right)|u_n^\epsilon|_{N^m(\Omega_T)}\right] \\ |u_{n+1}^\epsilon|_{L^\infty(\Omega_T)} &\leq T\left[C_3 + (1 + |u_n^\epsilon|_{N^m(\Omega_T)})G^*(|u_n^\epsilon|_{L^\infty(\Omega_T)})\right] \end{aligned} \tag{2.29}$$

These estimates hold for all ϵ and n. Indeed, C_2, C_3, and h depend on m, T_1, and $f(t,x,\cdot)$, and C_2 depends additionally on $|g^\epsilon(t,x)|_{N^m(\Omega_T)}$, C_3 additionally on $|g^\epsilon(t,x)|_{L^\infty\cap N^m(\Omega_T)}$. Since g has $L^\infty \cap N^m(\Omega_T)$ bounds that hold for all ϵ, and since there is no dependence on n, the estimates hold for all ϵ and n.

Given that $u_0^\epsilon = 0$ for all ϵ, use an induction argument to show that, for T sufficiently small, for all ϵ, and for all n,

$$\begin{aligned} |u_n^\epsilon|_{L^\infty(\Omega_T)} &\leq 1 \\ |u_n^\epsilon|_{N^m(\Omega_T)} &\leq 1 \end{aligned} \tag{2.30}$$

Thus, for such small T, $|u_n^\epsilon|_{L^\infty\cap N^m(\Omega_T)}$ has a bound that does not depend on ϵ or n.

We use this boundedness to prove the first estimate in Lemma 2.17. Observe that $(u_n^\epsilon, u_{n-1}^\epsilon)$ has a bound in $L^\infty \cap N^m(\Omega_T)$ that does not depend on n or ϵ. So by Proposition 2.14a, $H(t,x,u_n^\epsilon,u_{n-1}^\epsilon)$ has such a bound as well, where H is defined through

(2.31)

$$\begin{aligned} f(t,x,u_n^\epsilon) - f(t,x,u_{n-1}^\epsilon) &= \left(\int_0^1 f_u(t,x,u_{n-1}^\epsilon + s(u_n^\epsilon - u_{n-1}^\epsilon))ds\right)(u_n^\epsilon - u_{n-1}^\epsilon) \\ &= H(t,x,u_n^\epsilon,u_{n-1}^\epsilon)(u_n^\epsilon - u_{n-1}^\epsilon). \end{aligned}$$

Applying Proposition 2.14b to (2.31) gives us the first estimate in Lemma 2.17.

To get the second estimate in Lemma 2.17, we subtract the problem (2.27) for n from the same problem for $n+1$ to get:

$$\begin{aligned} &L\left(u^\epsilon_{n+1} - u^\epsilon_n\right) = f(t,x,u^\epsilon_n) - f(t,x,u^\epsilon_{n-1}) \\ &u^\epsilon_{n+1} - u^\epsilon_n = 0 \text{ in } t < 0 \end{aligned} \tag{2.32}$$

Using the first estimate in Lemma 2.17 and the boundedness of u^ϵ_n in $L^\infty \cap N^m(\Omega_T)$, we observe that the RHS of the interior equation is in $L^\infty \cap N^m(\Omega_T)$. Thus, we use Propositions 2.15 and 2.16 to get the second estimate in Lemma 2.17. □

Proof (of Proposition 2.5). Combining the two estimates in Lemma 2.17, we observe that $\exists C$ such that for T sufficiently small and for all ϵ,

$$|u^\epsilon_{n+1} - u^\epsilon_n|_{L^\infty \cap N^m(\Omega_T)} \leq CT|u^\epsilon_n - u^\epsilon_{n-1}|_{L^\infty \cap N^m(\Omega_T)}. \tag{2.33}$$

Use (2.33) and pick $T = \frac{1}{2C}$. Then, u^ϵ_n is Cauchy for all ϵ on $L^\infty \cap N^m(\Omega_T)$. □

2.3 Error analysis

Now that we have the approximate and exact solutions, we show that they are close to each other for small wavelength. We do so by examining the error term, defined as

$$w^\epsilon := u^\epsilon - u^\epsilon_a, \tag{2.34}$$

where u^ϵ_a is the defined approximate solution, and u^ϵ is a exact solution.

2.3.1 Defining the error problem

Using (1.1) and (2.4), we observe that w^ϵ satisfies

$$L(t,x,\partial_{t,x})w^\epsilon - (f(t,x,u^\epsilon) - f(t,x,u^\epsilon_a)) = -\epsilon\mathcal{E}_1, \tag{2.35}$$

where $\mathcal{E}_1$ is defined as in (2.4).

In search of an initial condition, we observe that in $t < 0$, $U_0 = 0$ by (2.11), and $U_1 = V = 0$ by observing that H in (2.13) is identically 0 in $t < 0$. Thus, $w = u - u_a$ is identically 0 in $t < 0$.

Using the Fundamental Theorem of Calculus, we observe that

$$f(t,x,u^\epsilon) - f(t,x,u_a^\epsilon) = \left(\int_0^1 f_{u_a}(t,x,u_a^\epsilon + s(u^\epsilon - u_a^\epsilon))ds\right) w^\epsilon =: K_2(t,x,u^\epsilon,u_a^\epsilon)w^\epsilon. \tag{2.36}$$

Thus, w^ϵ satisfies the problem

$$\begin{aligned} &(a)\ L(t,x,\partial_{t,x})w^\epsilon - K_2(t,x,u^\epsilon,u_a^\epsilon)w^\epsilon = -\epsilon\mathcal{E}_1 \\ &(b)\ w^\epsilon|_{t<0} = 0. \end{aligned} \tag{2.37}$$

2.3.2 Analysis of error term

We analyze the error problem (2.37). We have

(2.38)

$$|K_2(t,x,u^\epsilon,u_a^\epsilon)w^\epsilon - \epsilon\mathcal{E}_1|_{L^\infty\cap N^m(\Omega_T)} \le |K_2(t,x,u^\epsilon,u_a^\epsilon)w^\epsilon|_{L^\infty\cap N^m(\Omega_T)} + |\epsilon\mathcal{E}_1|_{L^\infty\cap N^m(\Omega_T)}$$

Lemma 2.18. *$\epsilon\mathcal{E}_1$ is $O(\epsilon)$ in $L^\infty \cap N^m(\Omega_T)$ as $\epsilon \to 0$.*

Proof. Recall, by (2.4),

(2.39)

$$\begin{aligned} |\epsilon\mathcal{E}_1|_{L^\infty\cap N^m(\Omega_T)} &= \left|\epsilon\left[L(t,x,\partial_{t,x})V(t,x,\theta) - K_1(t,x,U_0,\epsilon U_1)U_1\right]_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty\cap N^m(\Omega_T)} \\ &\le \left|\left[\epsilon L(t,x,\partial_{t,x})V(t,x,\theta)\right]_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty\cap N^m(\Omega_T)} \\ &+ \left|\left[\epsilon K_1(t,x,U_0,\epsilon U_1)U_1\right]_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty\cap N^m(\Omega_T)} \end{aligned}$$

As for the first term,

(2.40)

$$\begin{aligned} &\left|\left[\epsilon L(t,x,\partial_{t,x})V(t,x,\theta)\right]_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty\cap N^m(\Omega_T)} = \epsilon\left|\int_{-\infty}^{\frac{x_n}{\epsilon}} L(t,x,\partial_{t,x})\left(QH(t,x,s)\right)ds\right|_{L^\infty\cap N^m(\Omega_T)} \\ &\le \epsilon\left(\left|\int_{-\infty}^{\frac{x_n}{\epsilon}} L(t,x,\partial_{t,x})\left(QH(t,x,s)\right)ds\right|_{L^\infty(\Omega_T)} + \left|\int_{-\infty}^{\frac{x_n}{\epsilon}} L(t,x,\partial_{t,x})\left(QH(t,x,s)\right)ds\right|_{N^m(\Omega_T)}\right) \\ &\le \epsilon\left(2K_s \sup_{\substack{(t,x)\in\Omega_T \\ |s|\le K_\theta}} \left|L(t,x,\partial_{t,x})\left(QH(t,x,s)\right)\right| + \left|\int_{-\infty}^{\frac{x_n}{\epsilon}} L(t,x,\partial_{t,x})\left(QH(t,x,s)\right)ds\right|_{N^m(\Omega_T)}\right). \end{aligned}$$

The $L^\infty(\Omega_T)$ bound comes from the compact support of the integrand in s. And the $N^m(\Omega_T)$ bound is a constant, shown by letting $L(t,x,\partial_{t,x})\left(QH(t,x,s)\right)$ play the role of $QH(t,x,s)$ and applying the same argument as used in Proposition 2.11 for V.

As for the second term,

(2.41)
$$\begin{aligned}
&\left|[\epsilon K_1(t,x,U_0,\epsilon U_1)U_1]_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty\cap N^m(\Omega_T)}\\
&=\left|\left[(\epsilon V)\int_0^1 f_u(t,x,U_0+s(u_a-U_0))ds\right]_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty\cap N^m(\Omega_T)}\\
&\preccurlyeq \epsilon\left|V|_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty\cap N^m(\Omega_T)}\left|\left[\int_0^1 f_u(t,x,U_0+s(u_a-U_0))ds\right]_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty\cap N^m(\Omega_T)}
\end{aligned}$$

where $\preccurlyeq$ denotes $\leq$ modulo multiplication by a constant. The $\preccurlyeq$ inequality is given by Proposition 2.14b.

The bound for $\left|V|_{\theta=\frac{x_n}{\epsilon}}\right|_{L^\infty\cap N^m(\Omega_T)}$ comes from Proposition 2.11. The L^∞ bound for the next term is given by recognizing the integral as a function $F(t,x,U_0,V)$, applying Proposition 2.14a, and using the uniform boundedness of (U_0,V) in $L^\infty(\Omega_T)$. The N^m bound for this term comes from differentiation under the integral sign and identifying similar L^∞ bounds, which give the appropriate $L^2(\Omega_T)$ bounds. □

Proposition 2.19. *For suitable T, w is $O(\epsilon)$ in $L^\infty\cap N^m(\Omega_T)$ as $\epsilon\to 0$.*

Proof. We use the Lemma to revise (2.38). We get:

(2.42)
$$\begin{aligned}
&|K_2(t,x,u,u_a)w-\epsilon\mathcal{E}_1|_{L^\infty\cap N^m(\Omega_T)}\leq |K_2(t,x,u,u_a)w|_{L^\infty\cap N^m(\Omega_T)}+O(\epsilon)\\
&\qquad\text{(Using Prop 6.1b)}\ \preccurlyeq |K_2(t,x,u,u_a)|_{L^\infty\cap N^m(\Omega_T)}|w|_{L^\infty\cap N^m(\Omega_T)}+O(\epsilon)
\end{aligned}$$

Using Proposition 2.14a and the uniform boundedness in $L^\infty\cap N^m(\Omega_T)$ of u,u_a, we get:

$$|K_2|_{L^\infty\cap N^m(\Omega_T)}\leq C+h\left(|(u,u_a)|_{L^\infty(\Omega_T)}\right)\left(1+|(u,u_a)|_{N^m(\Omega_T)}\right)\leq C' \tag{2.43}$$

for all ϵ. So we simplify (2.42) to get

$$|K_2(t,x,u,u_a)w-\epsilon\mathcal{E}_1|_{L^\infty\cap N^m(\Omega_T)}\preccurlyeq |w|_{L^\infty\cap N^m(\Omega_T)}+O(\epsilon) \tag{2.44}$$

Applying Propositions 2.15 and 2.16 to (2.37) gives us

$$|w|_{L^\infty \cap N^m(\Omega_T)} \preccurlyeq T\,|K_2 w - \epsilon \mathcal{E}_1| \tag{2.45}$$

Combining (2.44) and (2.45) and doing some algebra, we get

$$|w|_{L^\infty \cap N^m(\Omega_T)} \preccurlyeq \frac{O(\epsilon)}{1-T} \tag{2.46}$$

So for suitable T, w is $O(\epsilon)$.

$\square$

3 The Multiphase Problem

In this section, we prove Proposition 1.11. Our prediction that only two pulses emerge in $t > 0$ after two pulses launched in $t < 0$ interact at $t = 0$ is in strong contrast to what is known to happen in the wavetrain case. In that case up to N wavetrains, all of order 1 in amplitude, may emerge in $t > 0$ after interaction at $t = 0$ of two wavetrains that are launched in the past. Thus, our prediction is in keeping with the general expectation that in nonlinear problems, pulses should interact more weakly than wavetrains. This reflects the fact that the pulse profiles decay in θ, while wavetrain profiles are periodic in θ.

We start by constructing the approximate solution to (1.2). Then, in section 3.2, we show that the problem (1.2) can be posed to have a solution long enough for us to analyze interaction after collision.

3.1 Construction of approximate solution

3.1.1 The Ansatz

Before our calculations, we do not know how many pulses to expect after collision. Thus, we make an educated guess, or "Ansatz", of the form of the solution, that takes account of this ambiguity:

$$
\begin{aligned}
&u_a^\epsilon := U_0(t,x,\theta)|_{\theta=\psi/\epsilon} + \epsilon U_1^\epsilon(t,x), \text{ where}\\
&U_0(t,x,\theta) := \sum_{i=1}^N s_i\,(t,x,\theta_i)\,r_i(t,x),\\
&U_1^\epsilon(t,x) := V(t,x,\psi/\epsilon) + W(t,x,\tau,\theta_0)|_{\tau=t/\epsilon,\theta_0=x_n/\epsilon},\\
&V(t,x,\theta) = \sum_{i=1}^N V_i(t,x,\theta_i).
\end{aligned}
\tag{3.1}
$$

We assume that u_a^ϵ has this form throughout this section, and by our calculations, show that u_a^ϵ also has the form of (1.9). Namely, we show that s_i and $V_i = 0$ for $i = 3, ..., N$.

3.1.2 Setting conditions for the approximate solution

We want $L(t,x,\partial_{t,x})u_a^\epsilon - f(t,x,u_a^\epsilon)$ to be small. We define the following operators:

$$
\begin{aligned}
\mathcal{L}_1(t,x,d\psi,\partial_\theta) &:= \sum_{i=1}^N L(t,x,d\psi_i)\partial_{\theta_i}\\
\mathcal{L}_2(t,x,\partial_{\tau,\theta_0}) &:= I\partial_\tau + A_n\partial_{\theta_0}.
\end{aligned}
\tag{3.2}
$$

By some calculations similar to those in section 2.1.2, we obtain the following:

(3.3)

$$
\begin{aligned}
&L(t,x,\partial_{t,x})u_a^\epsilon - f(t,x,u_a^\epsilon) = \frac{1}{\epsilon}\mathcal{E}_{-1} + \mathcal{E} = \epsilon\mathcal{E}_1, \text{ where}\\
&\mathcal{E}_{-1} = 0\\
&\mathcal{E}_0 = \left[\sum_{i=1}^N [L\,(s_i r_i)] + \mathcal{L}_1(t,x,d\psi,\partial_\theta)V + \mathcal{L}_2(t,x,\partial_{\tau,\theta_0})W - f(t,x,U_0)\right]_{\theta=\frac{\psi}{\epsilon},(\tau,\theta_0)=\frac{1}{\epsilon}(t,x_n)}\\
&\mathcal{E}_1 = \left[\left(\sum_{i=1}^N LV_i(t,x,\theta_i)\right) - K_3(t,x,U_0,\epsilon U_1^\epsilon)U_1^\epsilon + LW(t,x,\tau,\theta_0)\right]_{\theta=\frac{\psi}{\epsilon},(\tau,\theta_0)=\frac{1}{\epsilon}(t,x_n)}
\end{aligned}
$$

Here, $K_3(t,x,U_0,\epsilon U_1^\epsilon) := \int_0^1 f_u(t,x,U_0 + m\epsilon U_1^\epsilon)dm$.

We aim to construct s_i, V, and W such that $\mathcal{E}_0 = 0$.

3.1.3 The operator E

Let $\mathcal{F}$ be defined as

$$\mathcal{F} := \left(\sum_{i=1}^{N} L(s_i r_i)\right) - f(t,x,U_0). \tag{3.4}$$

Given an operator $E = P \circ S$ to be defined soon, we notice that

$$\begin{aligned}\mathcal{E}_0 &= \mathcal{F} + \mathcal{L}_1(t,x,d\psi,\partial_\theta)V + \mathcal{L}_2(t,x,\partial_{\tau,\theta_0})W \\ &= E\mathcal{F} + (1-E)S\mathcal{F} + (1-E)(1-S)\mathcal{F} + \mathcal{L}_1(t,x,d\psi,\partial_\theta)V + \mathcal{L}_2(t,x,\partial_{\tau,\theta_0})W\end{aligned} \tag{3.5}$$

Again, we take inspiration from [4] and [9] to identify conditions that we want our terms to satisfy. We would like to construct s_i, V and W such that

$$\begin{aligned}&\text{(a) } E\mathcal{F} = 0 \\ &\text{(b) } (1-E)S\mathcal{F} + \mathcal{L}_1(t,x,d\psi,\partial_\theta)V = 0 \\ &\text{(c) } (1-E)(1-S)\mathcal{F} + \mathcal{L}_2(t,x,\partial_{\tau,\theta_0})W = 0.\end{aligned} \tag{3.6}$$

We provide an overview of the action of $E = P \circ S$ on $\mathcal{F}$ before providing precise definitions. The use of this operator is inspired by [4], [9]. A similar operator–the projection operator π_k–was used in section 2.1.3 to identify conditions that σ_k and V must satisfy. Here, we use a similar projection operator P, however we also incorporate a selection operator S to account for the dependence of the approximate solution on more than one scalar profile s_i. The operator S acts on a function by "selection," selecting the terms that depend solely on one of the s_i and its derivatives and on none of the other s_i and their derivatives. The operator P acts on a function by "polarization," applying the operator π_i to any term that depends on one of the s_i and its derivatives and on none of the other s_i and their derivatives.

Now, we define these actions specifically. Observe that the function $f(t,x,U_0) = f(t,x,\sum_{i=1}^N s_i r_i)$ can be written equivalently as a function $\tilde{f}(t,x,s)$, where $s = (s_1,...,s_N)$, and where $\tilde{f}(t,x,s) = f(t,x,\sum_{i=1}^N s_i r_i)$. Next, we observe that

$$\tilde{f}(t,x,s) = \sum_{i=1}^{N} \tilde{f}(t,x,e_i s_i) + \left(\tilde{f}(t,x,s) - \sum_{i=1}^{N} \tilde{f}(t,x,e_i s_i)\right), \tag{3.7}$$

where e_i is the i-th standard basis vector.

We define $S\mathcal{F}$ through its actions on the terms in $\mathcal{F}$:

$$
\begin{aligned}
Sf(t,x,U_0) &:= \sum_{i=1}^{N} \tilde{f}(t,x,e_i s_i) \\
S\left(\sum_{i=1}^{N} L(s_i r_i)\right) &:= \left(\sum_{i=1}^{N} L(s_i r_i)\right) \\
S\mathcal{F} &:= \left(\sum_{i=1}^{N} L(s_i r_i)\right) + Sf(t,x,U_0)
\end{aligned}
\tag{3.8}
$$

Now for the projection operator. Firstly, we decompose $\sum_{i=1}^{N} L(s_i r_i)$ into two parts, one depending on s and the other on $\partial_{t,x} s$:

$$
\sum_{i=1}^{N} (Ls_i) r_i := p_1(t,x,\partial_{t,x} s) \text{ and } \sum_{i=1}^{N} s_i (Lr_i) := p_2(t,x,s), \tag{3.9}
$$

and now we define the actions of P as

$$
\begin{aligned}
Pf(t,x,U_0) &= f(t,x,U_0) \\
PSf(t,x,U_0) &= \sum_{i=1}^{N} \pi_i(t,x) \tilde{f}(t,x,e_i s_i) \\
Pp_1(t,x,\partial_{t,x} s) &:= \sum_{i=1}^{N} \pi_i(t,x) \left[(Ls_i(t,x,\theta_i)) r_i(t,x)\right] \\
Pp_2(t,x,s) &:= \sum_{i=1}^{N} \pi_i(t,x) \left[s_i(t,x,\theta)(Lr_i(t,x))\right] \\
P\left(\sum_{i=1}^{N} L(s_i r_i)\right) &:= Pp_1 + Pp_2
\end{aligned}
\tag{3.10}
$$

Using this operator, we say that U_0 solves away the single-phase-dependent parts of $\mathcal{F}$ that are projected onto an appropriate eigenspace, V solves away the single-phase dependent unprojected parts, and W solves away the parts that depend on more than one phase. In other words, the W term solves away the "interaction" terms. The construction of the W term is not considered in this paper, and it remains to be explored.

3.1.4 Construction of s_i

We construct s_i by solving equations derived from the equality $E\mathcal{F} = 0$:

$$
\begin{aligned}
E\mathcal{F} &= \sum_{i=1}^{N} \pi_i(t,x)\left[L(s_i r_i) - \tilde{f}(t,x,e_i s_i)\right] \\
&= \sum_{i=1}^{N} \left(l_i(t,x)\left[L(s_i r_i) - \tilde{f}(t,x,e_i s_i)\right]\right) r_i(t,x) = 0.
\end{aligned} \tag{3.11}
$$

To further develop this condition, we use the linear independence of $\{r_i\}$ near $t = 0$. This linear independence is given through the linear independence of $\{R_i(t,x,\xi)\}$ for fixed ξ and using the fact that $d_x\psi_i|_{t=0} = e_n$ for all $i = 1, ..., N$. Using this linear independence, we conclude that (3.11) is satisfied near $t = 0$ if and only if for each $i = 1, ..., N$,

$$
l_i(t,x)\left[L(s_i r_i) - \tilde{f}(t,x,e_i s_i)\right] = 0 \text{ near } t = 0. \tag{3.12}
$$

Through some calculations and Lemma 2.4, we use equation (3.12) to obtain an interior condition on s_i analogous to the interior condition for σ_k defined in equation (2.11a). The initial condition makes the solution close to the exact solution of (1.2):

$$
\begin{aligned}
&\text{(a) } X_i(t,x,\partial_{t,x})s_i + c_i s_i - h_i(t,x,s_i) = 0 \\
&\text{(b) } s_i = \begin{cases} \sigma_i & i = 1,2 \\ 0 & i = 3, ..., N \end{cases} \quad \text{in } t < -T',
\end{aligned} \tag{3.13}
$$

where X_i, c_i and h_i are given by:

$$
\begin{aligned}
&X_i(t,x,\partial_{t,x}) := \sum_{j=0}^{n} \partial_{\xi_j}\lambda_i(t,x,d_x\psi_i)\partial_j \\
&c_i(t,x) := l_i L r_i \\
&h_i(t,x,s_i) := l_i \tilde{f}(t,x,e_i s_i).
\end{aligned} \tag{3.14}
$$

As was done for σ_k's system (2.11), we solve for s_i using the method of characteristics. For $i = 3, ..., N$, $s_i = 0$ is the unique solution of the system.

3.1.5 Construction of V

Next, we construct V such that (3.6b) holds.

$$\begin{aligned}(1-E)S\mathcal{F}&+\mathcal{L}_1(t,x,d\psi,\partial_\theta)V\\ =&\sum_{i=1}^{N}\left[(1-\pi_i)\left(L(s_ir_i)-\tilde{f}(t,x,e_is_i)\right)+L(t,x,d\psi_i)\partial_{\theta_i}V_i(t,x,\theta_i)\right].\end{aligned} \tag{3.15}$$

Thus, in order for (3.6b) to hold, it suffices to construct V_i such that for $i=1,...,N$,

$$(1-\pi_i)\left(L(s_ir_i)-\tilde{f}(t,x,e_is_i)\right)+L(t,x,d\psi_i)\partial_{\theta_i}V_i(t,x,\theta_i)=0. \tag{3.16}$$

We construct Q_i such that

$$L(t,x,d\psi_i)Q_i=Q_iL(t,x,d\psi_i)=1-\pi_i \tag{3.17}$$

using the same procedure as in section 2.1.5, where i plays the role of k. Then V_i satisfies (3.16) when V_i is defined as

$$\begin{aligned}&V_i(t,x,\theta_i):=-\int_{-\infty}^{\theta_i}Q_i(t,x)H_i(t,x,m)dm,\text{ where}\\ &H_i(t,x,\theta_i):=L(s_ir_i)-\tilde{f}(t,x,e_is_i).\end{aligned} \tag{3.18}$$

This integral is well-defined, since the integrand has compact support in θ_i. We observe this by considering the equations that define s_i. For $j=1,2$, choose θ_j outside the compact support of σ_j. Then for that θ_j, $s_i=0$ solves (3.13) for $i=1,...,N$, which means the integrand is 0 there.

Note also that the fact that $s_i=0$ for $i=3,...,N$ implies that $V_i=0$ for $i=3,...,N$.

3.2 Feasibility of interaction analysis using (1.2)

For our analysis of the behavior of the solution to (1.2) after pulse collision to be meaningful, we must show that a problem can be posed that allows us to do such analysis. That is, we must show that there is a way to construct the problem such that the RHS of (1.2b) satisfies (1.2a) in $t<-T'$. Then, we must show that problem (1.2) has a unique solution that exists long enough after collision in order for the interaction of the pulses to be analyzed.

3.2.1 Construction of initial condition

Assume all the following work takes place in a domain of determinacy.

In $t < -T'$ and for each $i = 1, 2$, solve

$$\begin{aligned} Lu_i &= g_i\left(t, x, \frac{\psi_i}{\epsilon}\right) \\ u_i &= 0 \text{ in } t < -T'' - 1, \end{aligned} \tag{3.19}$$

where the supports of $h_i(t, x) := g_i(t, x, \psi_i/\epsilon)$ are compact, small, and contained in $t \in [-T'' - 1, -T'' + 1]$ for some $T'' > 0$ satisfying $-T_0 < -T'' - 1 < -T'' + 1 < -T'$. If needed, we can use small $\delta > 0$ instead of 1. We also require that the supports of h_i intersect with Σ_i in an appropriate location such that the two pulses u_i collide at $t = 0$.

We apply our results for the single-phase problem to get that for each $i = 1, 2$, $u_i = \sigma_i(t, x, \psi_i)r_i(t, x) + r_{i,\epsilon}(t, x)$ for $i = 1, 2$, where $r_{i,\epsilon}$ is $O(\epsilon)$ in $N^m \cap L^\infty(\Omega_T)$ as $\epsilon \to 0$.

Then, in $t < -T'$, we define $u = u_1 + u_2$ so that:

$$\begin{aligned} Lu &= g_1\left(t, x, \frac{\psi_1}{\epsilon}\right) + g_2\left(t, x, \frac{\psi_2}{\epsilon}\right) \\ u &= 0 \text{ in } t < -T'' - 1 \end{aligned} \tag{3.20}$$

Notice that in the domain $\Omega_T \cap [-T'' + 1, -T']$, u satisfies the equation

$$\begin{aligned} &Lu = \chi(t)f(t, x, u) \\ &u = \sigma_1 r_1 + \sigma_2 r_2 + r_{1,\epsilon} + r_{2,\epsilon} \text{ in } t < -T', \end{aligned} \tag{3.21}$$

where f is some nonlinear function satisfying $f(t, x, 0) = 0$, and χ is smooth and satisfies

$$\chi(t) = \begin{cases} 0 & t < -T' \\ 1 & t > -T'/2. \end{cases} \tag{3.22}$$

This is true because $\chi f = g_1(t, x, \psi_1/\epsilon) + g_2(t, x, \psi_2/\epsilon) = 0$ for $i = 1, 2$ on this domain. Thus, we have constructed a problem (3.21) of the form (1.2) such that (1.2b) satisfies (1.2a) in $t < -T'$.

3.2.2 Existence and uniqueness after collision

We extend the solution long enough to observe interaction using the following proposition from Metivier's paper [6].

Proposition 3.1. *Pick T_0, α such that Ω is a domain of determinacy. Consider the system given by $Lu = f(t,x,u)$ where $f(t,x,0) = 0$. Given any T_1 and T in $(-T_0, T_0)$ with $T_1 < T$, there is $\rho > 0$ such that if u is a solution to the system on Ω_{T_1} and $|u|_{N^m \cap L^\infty(\Omega_{T_1})} \leq \rho$, then u can be extended to solve the system on Ω_T.*

Here, our $-T'$ plays the role of Metivier's T_1, and we choose T large enough to observe the interaction of the pulses. We are given a ρ, and we scale g_i used in (3.19) and (3.20) such that the $|g_i(t,x,\psi_i/\epsilon)|_{N^m \cap L^\infty(\Omega_{-T'})}$ are sufficiently small to ensure that $|u|_{N^m \cap L^\infty(\Omega_{-T'})} \leq \rho$. Then, the proposition extends u to a solution on Ω_T.

Note that in order to fully justify our use of the proposition, we would need to show the uniform boundedness of $g_i(t,x,\psi_i/\epsilon)$ with respect to ϵ in the $L^\infty \cap N^m(\Omega_{T_1})$ norm. This would give us a time of existence independent of ϵ. Here, the norm used is left undefined in this paper. It is defined in [6], as noted below Definition 1.9.

4 Appendix

4.1 Change of coordinates

Without loss of generality, we can assume that the surface

$$\Sigma := \{(t,x) : \psi(t,x) = 0\}$$

is equal to $\{(t,x) : x_n = 0\}$ and that the matrix $A_n(t,x)$ is singular. We can make these assumptions by defining a change of coordinates as follows.

Denote $(t,x) = (t,x',x_n)$ and define $\Phi(t,x) := (t,x',\psi(t,x))$. Then at 0, we have

$$\Phi'(0) = \begin{pmatrix} I & 0 \\ \nabla_{t,x'}\psi_k & \partial_{x_n}\psi_k \end{pmatrix}, \tag{4.1}$$

which is invertible using the initial condition in the Eikonal equation (1.4). Accordingly, we have a locally defined diffeomorphism. We let $y = (y_0, y'', y_n)$

and use the notation $\Phi : \mathbb{R}^{n+1}_{t,x} \to \mathbb{R}^{n+1}_{y}$. Observe that $\{\psi = 0\} = \{y_n = 0\}$. Using the product rule, one verifies the following.

Proposition 4.1. *$u(t,x)$ satisfies the original system (1.1) if and only if $v(y) = u \circ \Phi^{-1}(y)$ satisfies the analogous system:*

$$\begin{aligned} \mathring{L}(y, \partial_y) v = \mathring{f}(y, v) + \mathring{g}(y, \theta)|_{\theta = y_n/\epsilon} \\ v = 0 \ \text{at}\ y_0 = 0, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \text{where } \mathring{L}(y, \partial_y) &= \sum_{i=0}^{n} \mathring{A}_i(y) \partial_{y_i}, \\ \mathring{A}_i(y) &= \begin{cases} A_i \circ \Phi^{-1}(y) & i < n \\ \mathcal{A}(\Phi^{-1}(y), d_x\psi \circ \Phi^{-1}(y)) + I \partial_t \psi(\Phi^{-1}(y)) & i = n \end{cases}, \\ \mathring{f}(y, v) &= f(\Phi^{-1}(y), v), \\ \mathring{g}(y, \theta) &= g(\Phi^{-1}(y), \theta). \end{aligned}$$

Proof. (1.1) is satisfied if and only if (1.1) with the substitution $u(t,x) = v \circ \Phi(t,x)$ is satisfied. Applying the product rule and combining like terms gives (4.2). □

Note that $\mathring{A}_n$ is singular by the Eikonal equation (1.4).

To proceed with our reduction, we must verify that this new system satisfies our assumptions.

Proposition 4.2. *(4.2) satisfies strict hyperbolicity with respect to y_0.*

Proof.

$$\begin{aligned} &\det\left(\mathring{\tau} I + \sum_{j=1}^{n} \mathring{A}_j(y) \mathring{\xi}_j\right) \\ &= \det\left((\mathring{\tau} + \partial_t \psi \mathring{\xi}_n) I + \left(\sum_{j=1}^{n-1} A_j(t,x)(\mathring{\xi}_j + \partial_j \psi \mathring{\xi}_n)\right) + A_n \partial_n \psi \mathring{\xi}_n\right) \end{aligned} \tag{4.3}$$

Let $(\mathring{\xi}_1, ..., \mathring{\xi}_n) \neq 0$. Then $(\mathring{\xi}_1 + \partial_1 \psi \mathring{\xi}_n, ..., \mathring{\xi}_{n-1} + \partial_{n-1} \psi \mathring{\xi}_n, \partial_n \psi \mathring{\xi}_n) \neq 0$. Indeed, if $\mathring{\xi}_n \neq 0$ then that last coordinate is nonzero by the initial condition in (1.4). And if $\mathring{\xi}_n = 0$, then $(\mathring{\xi}_1 + \partial_1 \psi \mathring{\xi}_n, ..., \mathring{\xi}_{n-1} + \partial_{n-1} \psi \mathring{\xi}_n, \partial_n \psi \mathring{\xi}_n) = (\mathring{\xi}_1, ..., \mathring{\xi}_{n-1}, 0) \neq$

0 by assumption. We let $\xi = (\mathring{\xi}_1 + \partial_1\psi\mathring{\xi}_n, ..., \mathring{\xi}_{n-1} + \partial_{n-1}\psi\mathring{\xi}_n, \partial_n\psi\mathring{\xi}_n)$ and $\tau = \mathring{\tau} + \partial_t\psi\mathring{\xi}_n$, and we apply (1.3) to get:

$$
\begin{aligned}
\det\left(\mathring{\tau}I + \sum_{j=1}^{n} \mathring{A}_j(y)\mathring{\xi}_j\right) &= \det\left(\tau I + \sum_{j=1}^{n} A_j(y)\xi_j\right) \\
&= \prod_{i=1}^{N} (\tau - \tau_i(t,x,\xi)) \\
&= \prod_{i=1}^{N} \left(\mathring{\tau} + (\partial_t\psi\mathring{\xi}_n - \tau_i(t,x,\xi))\right) \\
&= \prod_{i=1}^{N} \left(\mathring{\tau} - \mathring{\tau}_i(t,x,\mathring{\xi})\right).
\end{aligned} \tag{4.4}
$$

Since each $\mathring{\tau}_i$ shifts τ_i by the same amount, that is by $\partial_t\psi\mathring{\xi}_n$, we have that $\mathring{\tau}_1 < \mathring{\tau}_2 < ...\mathring{\tau}_N$. □

4.2 Proof of $N^m(\Omega_T)$ estimate

We prove Proposition 2.15 using the additional assumption that A_j are symmetric for all j, in the manner of Alterman and Rauch [9]. For a proof that relies solely on strict hyperbolicity, refer to [6]. Our proof involves first proving estimates on a weighted N^m norm, and then deriving the unweighted estimates. We define the weighted N^m norm as follows.

Definition 4.3.

$$
|u|_{N^m_\lambda(X)} := \left[\sum_{|\alpha|\le m} \left|e^{-\lambda t}(\partial_t, \partial_{x_1}, ..., x_n\partial_{x_n})^\alpha u\right|^2_{L^2(X)}\right]^{1/2}. \tag{4.5}
$$

Firstly, we prove the following lemmas:

Lemma 4.4. *There exists $\lambda_0 > 0$ such that for all $u \in C^1(\overline{\Omega}_T)$ with $u|_{t=0} = 0$, one has*

$$
\forall\lambda > \lambda_0,\ (e^{-\lambda t}L(t,x,\partial_{t,x})u, e^{-\lambda t}u) \ge (\lambda - \lambda_0)|e^{-\lambda t}u|^2, \tag{4.6}
$$

where the norm and inner product are from $L^2(\Omega_T)$.

Proof. We observe that

$$\left(e^{-\lambda t}Lu, e^{-\lambda t}u\right) = \left((L+\lambda I)e^{-\lambda t}u, e^{-\lambda t}u\right), \tag{4.7}$$

where parentheses with commas denote the $L^2(\Omega_T)$ inner product.

We examine the L term on the RHS with each of its summands viewed separately. Letting $v = e^{-\lambda t}$, we use $A_1 = A_1^*$ and Gauss-Green integration by parts to get

$$\begin{aligned}(A_1\partial_1 v, v) &= (\partial_1 v, A_1 v)\\ &= -\left(v, \partial_1(A_1 v)\right) + \int_{\partial\Omega_T} \nu_1 v\cdot A_1 v dS\\ &= -\left(v, \partial_1 A_1 v\right) - \left(v, A_1\partial_1 v\right) + \int_{\partial\Omega_T} \nu_1 v\cdot A_1 v dS\end{aligned} \tag{4.8}$$

By adding $(v, A_1\partial_1 v)$ to both sides, we get

$$2\left(A_1\partial_1 v, v\right) = -\left(v, \partial_1 A_1 v\right) + \int_{\partial\Omega_T} \nu_1 v\cdot A_1 v dS \tag{4.9}$$

As for the entire L term, this gives:

$$\begin{aligned}2\left(L(e^{-\lambda t}u), e^{-\lambda t}u\right) = &-\int_{\Omega_T}\left\langle \sum_{j=0,\ldots,n} \partial_j A_j e^{-\lambda t}u, e^{-\lambda t}u\right\rangle dy\\ &+\int_{\partial\Omega_T}\left\langle \sum_{j=0,\ldots,n} \nu_j A_j e^{-\lambda t}u, e^{-\lambda t}u\right\rangle dS,\end{aligned} \tag{4.10}$$

where brackets denote the Euclidean inner product.

Altogether, we have

$$\begin{aligned}\left(e^{-\lambda t}Lu, e^{-\lambda t}u\right) &= \left((\lambda I + L)e^{-\lambda t}u, e^{-\lambda t}u\right)\\ &= \lambda\left(e^{-\lambda t}u, e^{-\lambda t}u\right)\\ &+\frac{1}{2}\Bigg[-\int_{\Omega_T}\left\langle \sum_{j=0,\ldots,n} \partial_j A_j e^{-\lambda t}u, e^{-\lambda t}u\right\rangle dy\\ &+\int_{\partial\Omega_T}\left\langle \sum_{j=0,\ldots,n} \nu_j A_j e^{-\lambda t}u, e^{-\lambda t}u\right\rangle dS\Bigg]\\ &\geq (\lambda-\lambda_0)\left(e^{-\lambda t}u, e^{-\lambda t}u\right).\end{aligned} \tag{4.11}$$

The last inequality comes from the fact that $\partial_j A_j$ is uniformly bounded in the Euclidean norm on Ω_T, and from the assumption that α, which is used to define Ω_T in (1.5), is chosen sufficiently small such that the sign of the boundary integral is nonnegative.

This assumption can be made by the following argument. At the $t = 0$ boundary of Ω_T, $u = 0$, so the integrand over this boundary is 0. At the $t = T$ boundary, $\nu = (1, 0, ...)$, so the integrand $\langle \sum_{j=0,...,n} \nu_j A_j e^{-\lambda t} u, e^{-\lambda t} u \rangle = \langle A_0 e^{-\lambda t} u, e^{-\lambda t} u \rangle = |e^{-\lambda t} u|^2 \geq 0$. As for the lateral boundary of Ω_T, we let $(t, x) = (l(x), x)$ denote its graph. Then, $\nu = (1, -\nabla_x l)/|(1, -\nabla_x l)|$. For all $v \in \mathbb{R}^n$, we note that $\langle \nu_0 A_0 v, v \rangle = \langle \nu_0 I v, v \rangle = |v|^2/|(1, -\nabla_x l)|$. Thus, in order for $\langle \sum_{j=0,...,n} \nu_j A_j v, v \rangle \geq 0$ for all $v \in \mathbb{R}^n$, we must assume that $|\langle \sum_{j=1,...,n} \nu_j A_j v, v \rangle| = |\langle \sum_{j=1,...,n} \left(-\frac{\partial l}{\partial x_j} / |(1, -\nabla_x l)| \right) A_j v, v \rangle| \leq \epsilon |v|^2$ for a sufficiently small ϵ. Since the A_j are bounded, this can be achieved by choosing $l(x)$ such that $\left| \frac{\partial l}{\partial x_j} \right|$ are sufficiently small for $j = 1, ..., n$. To do this, we choose α sufficiently small. Then, the integrand of this boundary is nonnegative. □

Corollary 4.5. *There exist constants λ_0, K such that for all $u \in C^1(\overline{\Omega}_T)$ with $u|_{t=0} = 0$ and for $\lambda \geq \lambda_0$, one has*

$$|u|_{N^0_\lambda(\Omega_T)} \leq K |f|_{N^0_\lambda(\Omega_T)} / \lambda. \tag{4.12}$$

Proof. Using the Cauchy-Schwarz inequality and Lemma 4.4, we observe the following inequalities, which use the $L^2(\Omega_T)$ inner product and norm:

$$|e^{-\lambda t} L u| |e^{-\lambda t} u| \geq \left(e^{-\lambda t} L u, e^{-\lambda t} u \right) \geq (\lambda - \lambda_0) |e^{-\lambda t} u|^2 \tag{4.13}$$

Divide by $|e^{-\lambda t} u|$ on both sides. □

In order to proceed with the proof of the N^m_λ estimate, we define an operator similar to L such that its nth matrix is diagonal. By strict hyperbolicity, the matrix A_n from the operator L has m distinct, real eigenvalues near the origin. This implies that A_n is diagonalizable there. We denote $S^{-1} A_n S = D$, where D is diagonal.

$$
\begin{aligned}
L(t,x,\partial_{t,x})u &= \left(\sum_{j=0}^{n} A_j\partial_j\right) u = f(t,x) \\
S^{-1}Lu &= \left(\sum_{j=0}^{n} S^{-1}A_j\partial_j\right) u = S^{-1}f(t,x) \\
S^{-1}L(Sv) &= \left(\sum_{j=0}^{n}\left[\tilde{A}_j\partial_j + S^{-1}A_j\partial_j S\right]\right) v = S^{-1}f(t,x),
\end{aligned}
\tag{4.14}
$$

where $\tilde{A}_j = S^{-1}A_jS$. Note that we used a change of variables $u = Sv$ and that $\tilde{A}_n$ is diagonal. Moreover, $\tilde{A}_j$ is smooth since S and S^{-1} is smooth. To see that S is smooth, note that S is a matrix of eigenvectors of A_j. Since the eigenvalues are real and distinct, then near each $(t_0,x_0) \in \Omega_T$, each eigenvector $r_j(t,x)$ can be expressed as $r_j(t,x) = P_j(t,x)r_j(t_0,x_0)$ where P_j is smooth, as shown in Proposition 3.4 in Chapter 7 of [10]. To see that S^{-1} is smooth, use the cofactor matrix formula for the inverse matrix.

Definition 4.6. *Define* $\tilde{L}(t,x,\partial_{t,x}) := \sum_{j=0}^{n}\tilde{A}_j(t,x)\partial_j = \partial_t + \sum_{j=1}^{n}\tilde{A}_j(t,x)\partial_j$.

Using the last line of (4.14), we observe that

$$
\tilde{L}(t,x,\partial_{t,x})v = S^{-1}f(t,x) - B(t,x)v, \tag{4.15}
$$

where B is a zero-th order term.

Lemma 4.7. *The commutator* $[\tilde{L}(t,x,\partial_{t,x}),\mathcal{M}] \subset C_b^\infty\tilde{L}(t,x,\partial_{t,x}) + \mathcal{M}$.

Proof. We show this on the generators of $\mathcal{M}$. For $M = x_n\partial_n$, this is straightforward:

$$
\begin{aligned}
\tilde{L}M - M\tilde{L} &= \left(\sum_j \tilde{A}_j\partial_j\right)(x_n\partial_n) - x_n\partial_n\left(\sum_j \tilde{A}_j\partial_j\right) \\
&= \tilde{A}_n\partial_n - x_n\sum_j\left(\partial_n\tilde{A}_j\right)\partial_j \\
&= \tilde{L} - \sum_{j\neq n}\tilde{A}_j\partial_j - x_n\sum_j\left(\partial_n\tilde{A}_j\right)\partial_j \in C_b^\infty\tilde{L} + \mathcal{M}.
\end{aligned}
\tag{4.16}
$$

For $M = \partial_i$, $i < n$, we have:

$$\tilde{L}M - M\tilde{L} = -\left(\sum_{j\neq n} \partial_i \tilde{A}_j \partial_j\right) - \partial_i \tilde{A}_n \partial_n \tag{4.17}$$

The first term on the RHS is in $\mathcal{M}$, but the second term requires additional analysis. Note that we can assume that $\tilde{A}_n$ is singular because we can assume that A_n is singular, as discussed in section 4.1. Thus, at least one diagonal entry of $\tilde{A}_n$ is 0, and since the eigenvalues are distinct, at most one diagonal entry can be 0. So WLOG $\tilde{A}_n$ has the form $\mathrm{diag}(0, \lambda_2, ..., \lambda_m)$. Accordingly,

$$\partial_i \tilde{A}_n = \begin{pmatrix} 0 & & & \\ & \partial_i \lambda_2 & & \\ & & ... & \\ & & & \partial_i \lambda_m \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & \frac{\partial_i \lambda_2}{\lambda_2} & & \\ & & ... & \\ & & & \frac{\partial_i \lambda_m}{\lambda_m} \end{pmatrix} \tilde{A}_n =: C\tilde{A}_n. \tag{4.18}$$

The continuity of the eigenvalues guarantees that $j > 1$ implies $M \geq |\lambda_j| \geq \epsilon$ and $M_\partial \geq |\partial_i \lambda_j|$ for some $M, \epsilon, M_\partial > 0$ on some domain. Accordingly, $C \in C_b^\infty$, and our term $\partial_i \tilde{A}_n \partial_n = B\tilde{L} - N$, where $N \in \mathcal{M}$. □

Proposition 4.8. *For all $m \geq 0$, there exist constants K, λ_0 such that for $\lambda \geq \lambda_0$,*

$$|v|_{N_\lambda^m(\Omega_T)} \leq K|\tilde{L}v|_{N_\lambda^m(\Omega_T)}/\lambda.$$

Proof. We proceed by induction. For the base case, we offer a variation of Corollary 4.5 adapted for $\tilde{L}$.

We note that we can apply Lemma 4.4 in the case where the operator $\tilde{L}$ is used instead of L by the following argument. Since S diagonalizes a symmetric matrix, its columns are orthogonal eigenvectors. By choosing eigenvectors with appropriate norm, we guarantee that S is an orthogonal matrix. Then $S^{-1} = S^T$, so $\tilde{A}_j^T = (S^{-1}A_jS)^T = S^{-1}A_jS = \tilde{A}_j$. Since the $\tilde{A}_j$ are symmetric for all j, we can apply Lemma 4.4 to $\tilde{L}$. Using this result, we modify (4.13) to get:

$$|v|_{N_\lambda^0(\Omega_T)} \preccurlyeq |\tilde{L}v|_{N_\lambda^0(\Omega_T)}/\lambda, \tag{4.19}$$

where $\preccurlyeq$ denotes that the inequality $\leq$ holds when a certain $\lambda_0 > 0$ is subtracted from λ, when $\lambda \geq \lambda_0$, and when multiplication by a constant is allowed.

Now for the inductive step. Assume that (4.19) holds in the case where the $N^0_\lambda(\Omega_T)$ norm is replaced with the $N^m_\lambda(\Omega_T)$ norm for some $m \geq 0$. We seek to show that (4.19) holds in the case where the norm used is $N^{m+1}_\lambda(\Omega_T)$. If M_i is a generator for $\mathcal{M}$, then

$$\tilde{L}\,(M_i v) = M_i \tilde{L} v + [\tilde{L}, M_i] v. \tag{4.20}$$

We apply the inductive assumption to this equation to get the following estimate, where $C \in C^\infty_b(\Omega_T)$ and $V_i \in \mathcal{M}$. For the estimate in the second line, we use the boundedness of derivatives of $C \in C^\infty_b(\Omega_T)$ and of the $C^\infty_b(\Omega_T)$ coefficients of V_i.

$$\begin{gathered} |M_i v|_{N^m_\lambda(\Omega_T)} \preccurlyeq |M_i \tilde{L} v + C\tilde{L} v + V_i v|_{N^m_\lambda(\Omega_T)}/\lambda. \\ \text{Accordingly, } |M_i v|_{N^m_\lambda(\Omega_T)} \preccurlyeq \left(|\tilde{L} v|_{N^{m+1}_\lambda(\Omega_T)} + |v|_{N^{m+1}_\lambda(\Omega_T)} \right)/\lambda. \end{gathered} \tag{4.21}$$

We use the definition of the $N^m_\lambda(\Omega_T)$ norm, the inductive assumption, and estimate (4.21) to obtain the inequalities:

(4.22)

$$\begin{aligned} |v|^2_{N^{m+1}_\lambda(\Omega_T)} &= |v|^2_{N^m_\lambda(\Omega_T)} + \sum_{|\alpha|=m} \sum_{M\in\{\text{gens}\}} |e^{-\lambda t}(\partial_t, \partial_{x_1}, ..., x_n\partial_{x_n})^\alpha M v|^2_{L^2(\Omega_T)} \\ |v|^2_{N^{m+1}_\lambda(\Omega_T)} &\preccurlyeq \left(|\tilde{L} v|^2_{N^m_\lambda(\Omega_T)} \right)/\lambda^2 + \sum_{M\in\{\text{gens}\}} |Mv|^2_{N^m_\lambda(\Omega_T)} \\ |v|^2_{N^{m+1}_\lambda(\Omega_T)} &\preccurlyeq \left(|\tilde{L} v|_{N^{m+1}_\lambda(\Omega_T)} + |v|_{N^{m+1}_\lambda(\Omega_T)} \right)^2/\lambda^2 \end{aligned}$$

Take the square root of the last inequality, subtract $|v|_{N^{m+1}_\lambda(\Omega_T)}/\lambda$ from both sides, and let $\lambda \geq \lambda_0$ for some λ_0 sufficiently large. □

Corollary 4.9. *For all $m \geq 0$, there exist constants K, λ_0 such that for $\lambda \geq \lambda_0$,*

$$|u|_{N^m_\lambda(\Omega_T)} \leq K|f|_{N^m_\lambda(\Omega_T)}/\lambda.$$

Proof. By the above proposition, we know the estimate $|v|_{N^m_\lambda(\Omega_T)} \preccurlyeq |\tilde{L}v|_{N^m_\lambda(\Omega_T)}/\lambda$ holds for large λ and for all $m > 0$. For each $m > 0$, we derive the estimate $|u|_{N^m_\lambda(\Omega_T)} \preccurlyeq |f|_{N^m_\lambda(\Omega_T)}/\lambda$ using $\tilde{L}v = S^{-1}f - Bv$, $u = Sv$, and the fact that S, S^{-1}, and B are in $C^\infty_b(\Omega_T)$. B is in this function space because it is a smooth function of A_j, S, and derivatives of S, and S and S^{-1} are in this function space by the discussion following (4.14) and by the compactness of Ω_T. □

Proof (of Proposition 2.15). Proposition 2.15 is a corollary of Corollary 4.9. Letting $\lambda = \max\{\lambda_0, T^{-1}\}$, the following inequalities prove the proposition. We use $V = (\partial_t, \partial_{x_1}, ..., x_n\partial_{x_n})$.

(4.23)

$$
\begin{aligned}
|u|_{N^m(\Omega_T)} = \left[\sum_{|\alpha|\le m}\int_{\Omega_T}|V^\alpha u|^2\right]^{1/2} &= \left[\sum_{|\alpha|\le m}\int_{\Omega_T}\left|e^{\lambda t}e^{-\lambda t}V^\alpha u\right|^2\right]^{1/2} \\
&\le e^{\lambda T}|u|_{N^m_\lambda(\Omega_T)} \\
&\le e^{\lambda T}K|f|_{N^m_\lambda(\Omega_T)}/\lambda \\
&= e^{\lambda T}K\left[\sum_{|\alpha|\le m}\int_{\Omega_T}|e^{-\lambda t}V^\alpha f|^2\right]^{1/2}/\lambda \\
&\le e^{\lambda T}K\left[\sum_{|\alpha|\le m}\int_{\Omega_T}|V^\alpha f|^2\right]^{1/2}/\lambda \\
&= e^{\lambda T}K|f|_{N^m(\Omega_T)}/\lambda \\
&\le CT|f|_{N^m(\Omega_T)}.
\end{aligned}
$$

□

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