Borel Measures and the Lebesgue-Stieltjes Integral

David Snider

UNC Chapel Hill
MATH 753 with Idris Assani

November 30, 2022

Outline

- History
- Borel Measures
  - Definition
  - Characterization of σ-finite Borel measures on $\mathbb{R}$.
- Application: Lebesgue-Stieltjes Integral
  - Lebesgue-Stieltjes Measure
  - Definition of Integral

History

- Émile Borel (1871-1956)
  - "Sur quelques points de la théorie des fonctions" in 1893.
- Henri Lebesgue (1875-1941)
  - Advised by Borel
  - Used Borel's theory to develop his integration theory
- Thomas Jan Stieltjes (1856-1894)
  - Integration of one function with respect to another

What is it?

Definition

Let X be a topological space. Its Borel sigma algebra $\mathcal{B}(X)$ is the sigma-algebra generated by the open sets of X. A Borel measure is a measure defined on $\mathcal{B}(X)$.

Examples

- $X = \mathbb{R}$. Use $\mu((a,b)) = b - a$ and Caratheodory.
- $X =$ unit sphere. Let $\mathcal{T} = \{$intersections of open balls in $\mathbb{R}^3$ and the sphere$\}$. $\mathcal{T}$ is a basis for the topology of X. For $U \in \mathcal{T}$, define $\mu(U)$ as its surface area. Use Caratheodory.
- $X = \mathbb{R}$. Set $F(x) = e^x$, and let $\mu\left((a,b)\right) = F(b) - F(a)$. Use Caratheodory.

Examples

- $X = \mathbb{R}$. Use $\mu((a, b)) = b - a$ and Caratheodory.
- $X =$ unit sphere. Let $\mathcal{T} = \{$intersections of open balls in $\mathbb{R}^3$ and the sphere$\}$. $\mathcal{T}$ is a basis for the topology of X. For $U \in \mathcal{T}$, define $\mu(U)$ as its surface area. Use Caratheodory.
- $X = \mathbb{R}$. Set $F(x) = e^x$, and let $\mu\left((a, b)\right) = F(b) - F(a)$. Use Caratheodory.

Guiding Questions:

- Relationship between functions and Borel measures.
- Does continuity play a role?

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is right-continuous if $x_n \to x$ from above implies $f(x_n) \to f(x)$.

In our notation, we say a function $f : \mathbb{R} \to \mathbb{R}$ is increasing if it is nondecreasing.

Characterization

Theorem

Given a σ-finite Borel measure, one obtains an increasing, right-continuous function via

$$F(x) = \begin{cases} \mu((0,x]) & x > 0 \\ 0 & x = 0 \\ -\mu((-x,0]) & x < 0 \end{cases}$$

Proof.

Let $x_n \to x$ from the right, where $x \geq 0$. Using continuity of measure from above,

$$F(x) = \mu((0, x]) = \mu\left(\bigcap_n (0, x_n]\right) = \lim_n \mu(0, x_n]) = \lim_n F(x_n)$$

For the case when $x < 0$, use continuity of measure from below. It is increasing by monotonicity. □

Characterization

Theorem

If $F : \mathbb{R} \to \mathbb{R}$ *is an increasing, right continuous function, there is a unique Borel measure* μ_F *on* $\mathbb{R}$ *such that* $\mu_F((a, b]) = F(b) - F(a)$ *for all* a, b*. If* G *is another function, then* $\mu_F = \mu_G$ *iff* $F = G$ *plus a constant.*

Characterization

Proof: Show F gives a σ-finite pre-measure μ_0. Let $\mathcal{A}$ be the algebra $\{\bigsqcup_{j=1}^{n}(a_j, b_j]\}$, where $\bigsqcup$ denotes a union of disjoint sets. We aim to show

$$\mu_0\left(\bigsqcup_{j=1}^{n}(a_j, b_j]\right) := \sum_{j=1}^{n}[F(b_j) - F(a_j)]$$

is a premeasure on $\mathcal{A}$, where $\mu_0(\emptyset) = 0$ by definition.

Proof (cont'd): Well-definedness of μ_0: Let $\bigsqcup_1^n (a_j, b_j] = (a, b]$.
After relabeling j, we have $a = a_1 < b_1 = a_2 < \ldots < b_n = b$.
Then, $\sum_1^n [F(b_j) - F(a_j)] = F(b) - F(a)$.
So, if $\bigsqcup_{i=1}^n I_i = \bigsqcup_{j=1}^m J_j$, then

$$\mu_0\left(\bigsqcup_{i=1}^n I_i\right) = \sum_{i=1}^n \mu_0(I_i) = \sum_{i,j} \mu_0(I_i \cap J_j) = \sum_{j=1}^m \mu_0(J_j) = \mu_0\left(\bigsqcup_{j=1}^m J_j\right)$$

Proof (cont'd): WTS $\bigsqcup_1^\infty I_i \in \mathcal{A} \implies \mu_0\left(\bigsqcup_1^\infty I_i\right) = \sum_1^\infty \mu_0(I_i)$. We have $\bigsqcup_1^\infty I_i = \bigsqcup_1^n (a_j, b_j]$. By finite additivity,

$$\mu_0\left(\bigsqcup_1^\infty I_i\right) = \sum_{j=1}^n \mu_0\left(\bigsqcup_1^\infty I_i \cap (a_j, b_j]\right),$$

If the identity holds for any summand in RHS, then it holds for $\bigsqcup_1^\infty I_i$. So we reduce to the case where $\bigsqcup_1^\infty I_i = (a, b]$.

Proof (cont'd): Let $I = (a, b]$. We have

$$\mu_0(I) = \mu_0\left(\bigcup_1^n I_i\right) + \mu_0\left(I \setminus \bigcup_1^n I_i\right) \geq \mu_0\left(\bigcup_1^n I_i\right) = \sum_1^n \mu_0(I_i)$$

Let $n \to \infty$ to get $\mu_0\left(\bigsqcup_1^\infty I_i\right) \geq \sum_1^\infty \mu_0(I_i)$.

Characterization

For reverse inequality, first assume a and b are finite. Fix $\epsilon > 0$. Use right-continuity of F. $\exists\delta > 0$ such that $F(a+\delta) - F(a) < \epsilon$ and $\exists\delta_i > 0$ such that

- $F(b_i + \delta_i) - F(b_i) < \epsilon 2^{-i}$
- Under some relabeling, $\{(a_i, b_i + \delta_i)\}_{i=1}^{N}$ covers $[a+\delta, b]$
- $b_i + \delta_i \in (a_{i+1}, b_{i+1} + \delta_{i+1})$ for $i = 1, ..., N-1$

$$\begin{aligned}
\mu_0(I) &< F(b) - F(a+\delta) + \epsilon \\
&\leq F(b_N + \delta_N) - F(a_1) + \epsilon \\
&= F(b_N + \delta_N) - F(a_N) + \sum_1^{N-1}[F(a_{i+1}) - F(a_i)] + \epsilon \\
&\leq F(b_N + \delta_N) - F(a_N) + \sum_1^{N-1}[F(b_i + \delta_i) - F(a_i)] + \epsilon \\
&< \sum_1^{N}[F(b_i) + \epsilon 2^{-1} - F(a_i)] + \epsilon] < \sum_1^{\infty}\mu_0(I_i) + 2\epsilon
\end{aligned}$$

Proof (cont'd): When a or b is not finite: Let $a = -\infty$. We notice that for $M < \infty$, the intervals $(a_i, b_i + \delta_i)$ cover $[-M, b]$, so $F(b) - F(-M) \leq \sum_1^\infty \mu_0(I_i) + 2\epsilon$. If $b = \infty$, then for $P < \infty$ we obtain $F(P) - F(a) \leq \sum_1^\infty \mu_0(I_i) + 2\epsilon$. Let $\epsilon \to 0, M, P \to \infty$.

Characterization

Proof (cont'd):

- μ_0 is a pre-measure on $\mathcal{A}$
- μ_0 is σ-finite.
- Using Folland Theorem 1.14, $\exists$ a unique measure on $\mathcal{B}(\mathbb{R})$ that extends μ_0.
- $F - G = k$ iff F and G give the same premeasure.

Characterization

Notice the similarity between the following isomorphisms.

$$\frac{\{F : \mathbb{R} \to \mathbb{R} : F \nearrow, \text{right-continuous}\}}{\{\text{constant functions}\}} \cong \{\sigma\text{-finite Borel measures on } \mathbb{R}\}$$

$$\frac{C^1(\mathbb{R})}{\{\text{constant functions}\}} \cong C(\mathbb{R})$$

Characterization

Questions:

- Why are the conditions on F so strict?
- What if we consider Borel signed measures?

Questions:

- Why are the conditions on F so strict?
- What if we consider Borel signed measures?

Example

Let $F(x) = \sin x$. What is $\mu_F((0, \infty))$?

Lebesgue-Stieltjes Integral

Definition

Given interval $[a, b] \subset \mathbb{R}$, with non-negative, Borel measurable f, and increasing, right-continuous g, we define

$$\int_a^b f dg(x) := \int_a^b f d\mu_g,$$

where μ_g is the Lebesgue-Stieltjes measure given by g.

Using the theory of bounded variation, we extend the definition.

Lebesgue-Stieltjes Integral

Using the theory of bounded variation, we extend the definition.

Definition

Given interval $[a, b] \subset \mathbb{R}$, with bounded, Borel measurable f, and g of bounded variation in $[a, b]$ and right continuous. We decompose g into the difference of two increasing functions $g_1 - g_2$ and define

$$\int_a^b f dg(x) := \int_a^b f dg_1 - \int_a^b f dg_2,$$

Question: Why can't we just define the integral using the Radon-Nikodym derivative?

$$\int_a^b f dg(x) := \int_a^b f \frac{d\mu_g}{d\mu} d\mu?$$

Question: Why can't we just define the integral using the Radon-Nikodym derivative?

$$\int_a^b f dg(x) := \int_a^b f \frac{d\mu_g}{d\mu} d\mu?$$

Answer: The Lebesgue-Stieltjes integral is a generalization of the above integral.

Example

Let δ_0 be the Dirac measure on $\mathbb{R}$ and Δ_0 its CDF. Δ_0 is increasing and right continuous, but we do not have that $\mu_{\Delta_0} << \mu$, the Lebesgue measure. So the Lebesgue-Stieltjes Integral with respect to Δ_0 cannot be defined using a Radon-Nikodym derivative.

An Application of the Lebesgue-Stieltjes Integral

The Lebesgue-Stieltjes Integral allows for a definition of probabilistic concepts when the probability measure is not absolutely continuous with respect to Lebesgue measure.

An Application of the Lebesgue-Stieltjes Integral

The Lebesgue-Stieltjes Integral allows for a definition of probabilistic concepts when the probability measure is not absolutely continuous with respect to Lebesgue measure.

$$E[f(x)] := \int_{-\infty}^{\infty} f(x) d\mu_G(x).$$

References

- "Real Analysis: Modern Techniques and Their Applications." Gerald B. Folland.
- MacTutor History of Mathematics Archive
- Abstract from "The Lebesgue-Stieltjes Integral" by M. Carter and B. van Brunt.