Hilbert's Nullstellensatz: Computation and Proof

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Directed Reading Program
UNC Department of Mathematics

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About the Talk

- Why Give This Talk?
  - Expose undergraduates to an area of modern research
  - Gain an appreciation for algebra's applications

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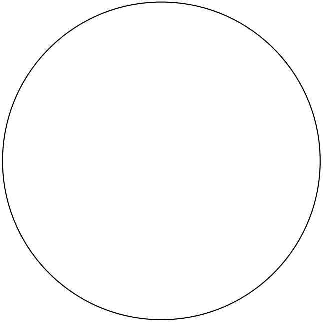
- Agenda
  - Motivation / Basic Terms (V and I)
  - Computing I(P)
  - Proof of the Nullstellensatz

Motivation



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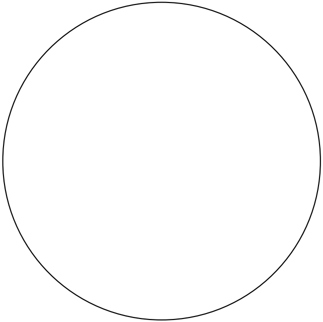

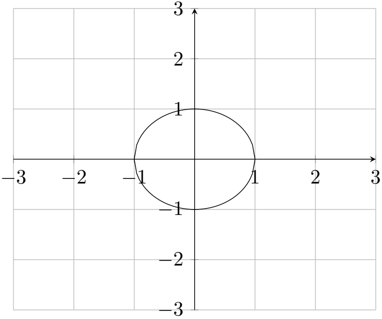

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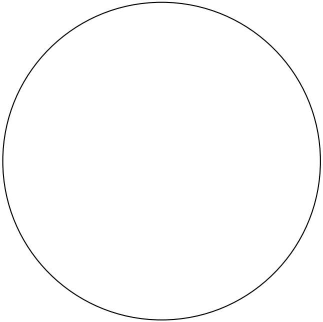

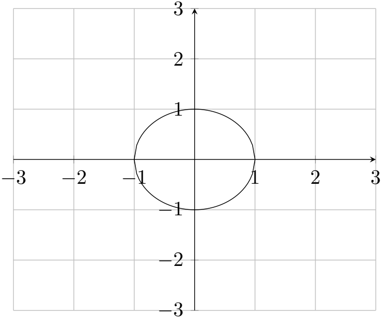

$x^2 + y^2 = 1$, or

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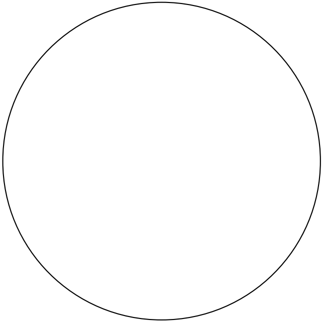
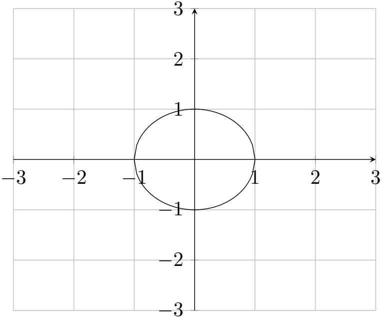
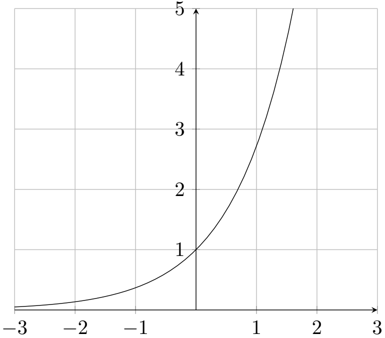




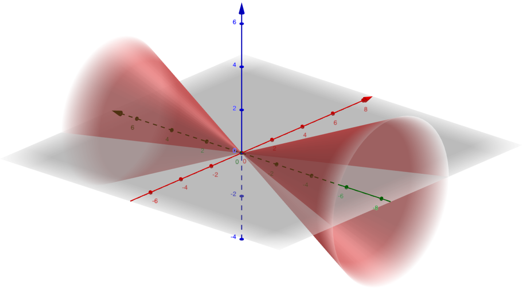
$x^2 + y^2 = 1$, or
$x^2 + y^2 - 1 = 0$

Motivation



$$y - e^x = 0$$

Motivation



$$z^2 + x^2 - y^2 = 0$$

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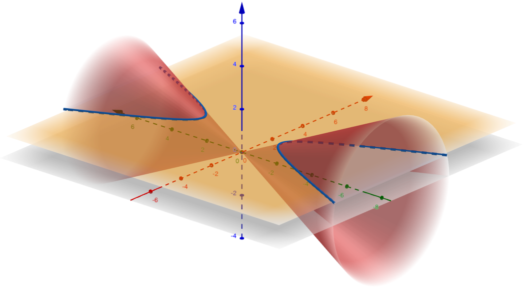

$$z^2 + x^2 - y^2 = 0$$
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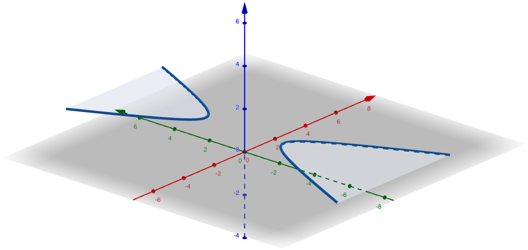

$$\{z^2 + x^2 - y^2 = 0\} \cap \{z - 1 = 0\}$$

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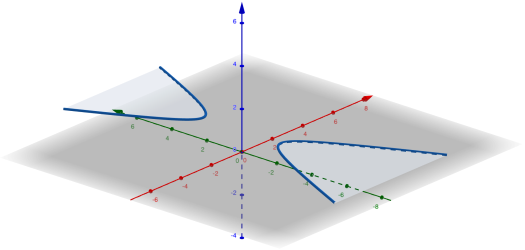

$$V(z^2 + x^2 - y^2, z - 1)$$

V and I

For k a field, $k[X_1, ..., X_n]$ is a ring.

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$p \in \mathbb{R}[X, Y] \implies p(x, y) = \sum_{i,j} a_{ij} x^i y^j$ (finite sum)

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Eg. $(x^2 + y^2 - 1, x - y) \subset \mathbb{R}[X, Y]$

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- $\{(x^2 + y^2 - 1)r_1 + (x - y)r_2 : r_1, r_2 \in \mathbb{R}[X, Y]\}$

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Let I be an ideal in $k[X_1, ..., X_n] =: R$.

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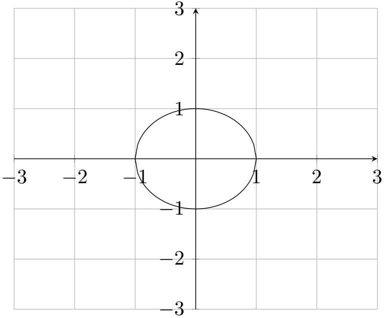
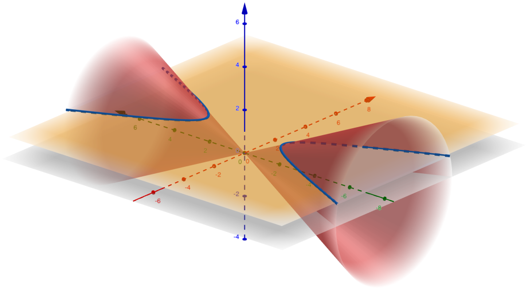
Define $V(I) := \{x \in k^n : f(x) = 0, \forall f \in I\}$

V and I



$$V(x^2 + y^2 - 1)$$

V and I



$$V(z^2 + x^2 - y^2)$$
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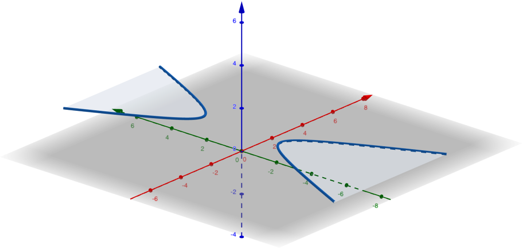

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V and I

Does V have an inverse?

V and I

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Let $C \subset k^n$ and let $R = k[X_1, ..., X_n]$.

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Let $f_1, f_2 \in I(C)$.
Then $\forall x \in C$, $(f_1 + f_2)(x) = 0$. So $f_1 + f_2 \in I(C)$.

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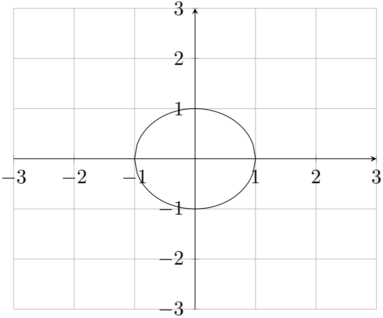
Then $\forall x \in C$, $(f_1 + f_2)(x) = 0$. So $f_1 + f_2 \in I(C)$.
Let $f \in I(C)$, $r \in R$.
Then $(fr)(x) := f(x)r(x) = 0, \forall x \in I(C)$. So $fr \in I(C)$.

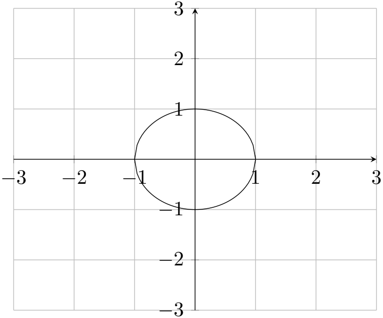
V and I



$$I(C) =$$

Let $C = \{x \in \mathbb{R}^2 : |x| = 1\}$.

V and I



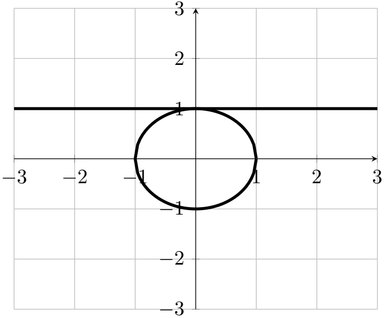
Let $C = \{x \in \mathbb{R}^2 : |x| = 1\}$.

$$I(C) = (x^2 + y^2 - 1).$$

V and I

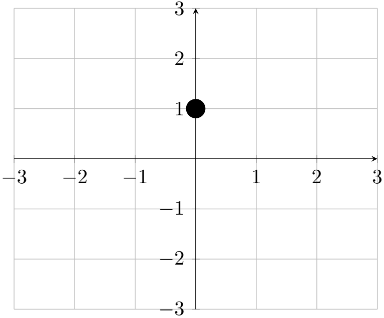
Are V and I inverses of each other?

A Computation



$$V(X^2 + Y^2 - 1), V(Y - 1)$$

A Computation



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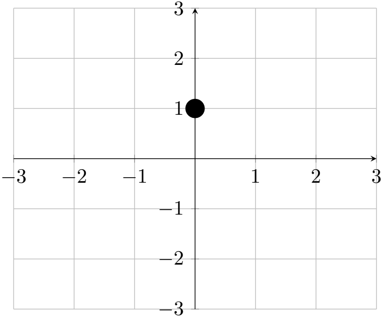

$$J = (X^2 + Y^2 - 1, Y - 1)$$
$$I(V(J)) = \text{?}$$

A Computation

$I(\{(0,1)\}) = ?$

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Then, $f = p_1 X + p_2(Y-1) + p_3$, where p_3 satisfies...

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So $p_3(X, 1) \neq 0$.

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So $p_3(X,1) \neq 0$.
Thus, $f(0,1) = 0 + 0 + p_3(0,1) \neq 0$. So $f \notin I(\{(0,1)\})$.

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I and V are not strict inverses of each other.

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Likewise, if $J = (X^2)$,
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$X \in (X)$ but $X \notin (X^2)$.

Nullstellensatz

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- b) Let $J \subset A$ be an ideal, $J \neq (1)$; then $V \neq \emptyset$.
  - Weak Nullstellensatz
- c) For any $f \in I(V(J))$, $\exists n \in \mathbb{N}$ such that $f^n \in J$.
  - Strong Nullstellensatz

Reid, p. 63

Nullstellensatz

In an algebraically closed field,

- If a polynomial is non-constant, then it has a zero (FTA).
- If the ideal generated by a set of polynomials is not the ideal of a constant, then the vanishing set of those polynomials is nonempty (Weak NSS).

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- Hard Fact: ϕ is an isomorphism.
- $X_i \mapsto b_i$. Let $a_i = \phi^{-1}(b_i)$. Then $X_i - a_i \in \ker f_2 = M$.
- Thus, $(X_1 - a_1, ..., X_n - a_n) \subset M$, and thus $= M$.

Nullstellensatz

(b) Let $J \subset A$ be an ideal, $J \neq (1)$; then $V \neq \emptyset$. (Weak Nullstellensatz)

- $J \neq R \implies \exists M$ maximal ideal w/ $J \subset M$.

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(b) Let $J \subset A$ be an ideal, $J \neq (1)$; then $V \neq \emptyset$. (Weak Nullstellensatz)

- $J \neq R \implies \exists M$ maximal ideal w/ $J \subset M$.
- $V(M) = \{P\}$, and $J \subset M$, so $V(J) \ni P$.

Nullstellensatz

(c) For any $J \subset A$, $I(V(J)) = \sqrt{J}$. (Strong Nullstellensatz)

- Rabinowitsch Trick
- Look it up on Wikipedia!

Thank You!

Acknowledgements:

- Hunter Dinkins, my DRP Mentor
- The DRP Committee

Citation

Miles Reid. *Undergraduate Algebraic Geometry*. Cambridge University Press, Cambridge, 1989.