Using Algebraic Graph Theory To Find Equiangular Lines in $\mathbb{R}^n$

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Directed Reading Program
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The Problem

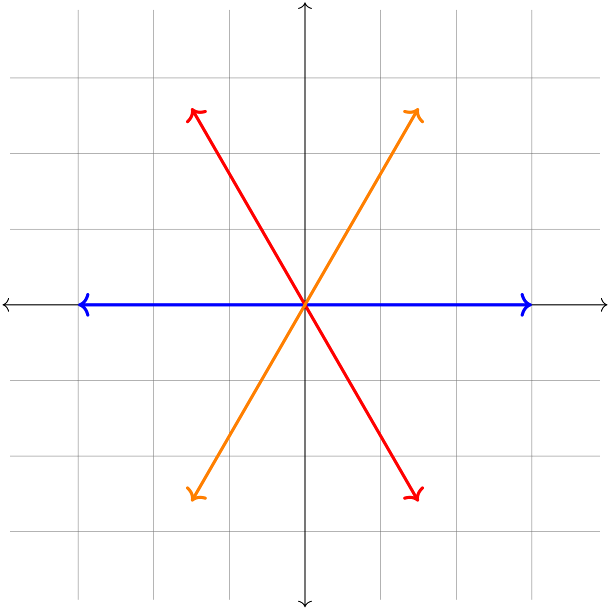
How many equiangular lines can we fit in $\mathbb{R}^n$?

What are Equiangular Lines?

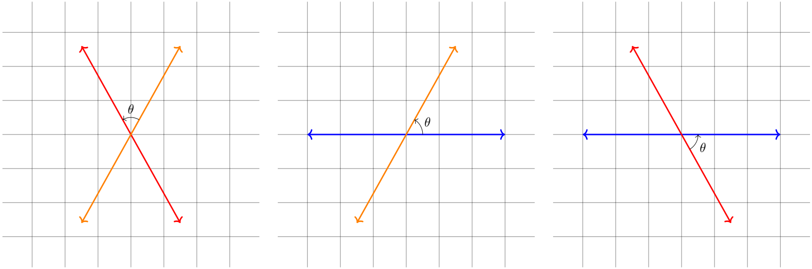
Definition

A set of lines in Euclidean space is equiangular if every pair of lines has the same angle.

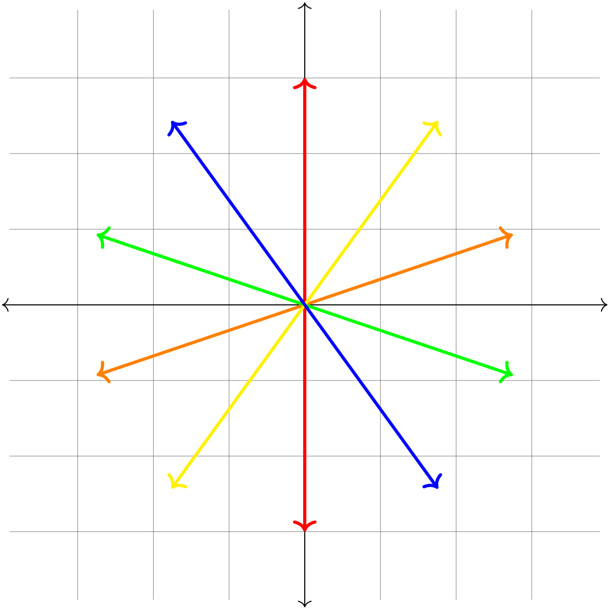
Example



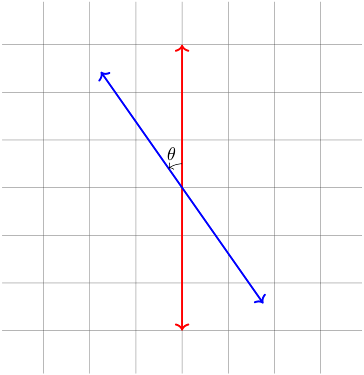
Example (cont'd)

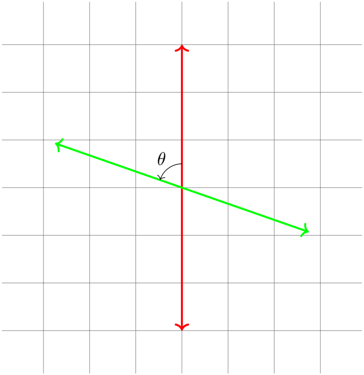


Non-Example

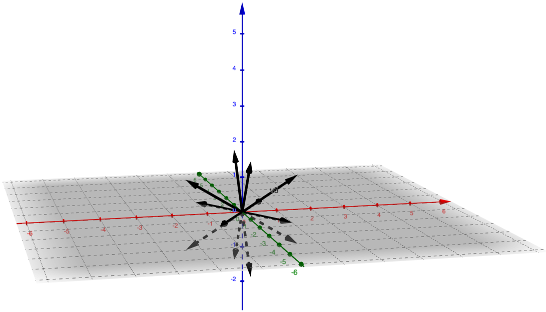


Non-Example (cont'd)





In $\mathbb{R}^3$



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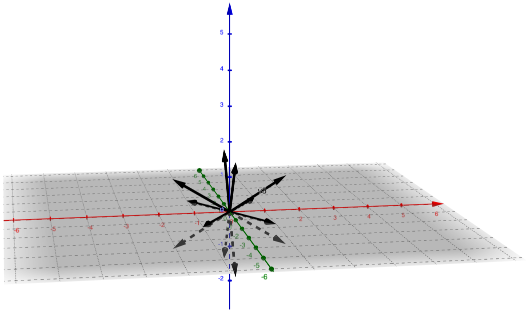

What does it mean to "fit"?

Definition

We say k equiangular lines "fit" in $\mathbb{R}^n$ if one can construct k distinct, equiangular lines through the origin of $\mathbb{R}^n$.

What we know (or don't)

- For each $n \in \mathbb{N}$, we know an upper bound for the lines that fit in $\mathbb{R}^n$.
- For some but not all $n \in \mathbb{N}$, we know the *maximal* amount of lines that fit in $\mathbb{R}^n$.
- We know how to translate this problem into linear algebra and graph theory.

The Translation

- Equiangular Lines
- Equiangular Tight Frames
- Grammian
- Seidel Adjacency Matrix
- Regular Two Graph

Connection 1

There is a relationship between equiangular lines and equiangular tight frames.

Frames

Definition

Let $\mathcal{V}$ be a vector space and $\{e_i\}_{i=1}^k$ be a set of vectors in $\mathcal{V}$.
$\{e_i\}_{i=1}^k$ is a frame if $\exists A, B > 0$ such that $0 < A \leq B < \infty$
and
for each $v \in \mathcal{V}$:

$$A||v||^2 \leq \sum_{i=1}^{k} |\langle v, e_i \rangle|^2 \leq B||v||^2.$$

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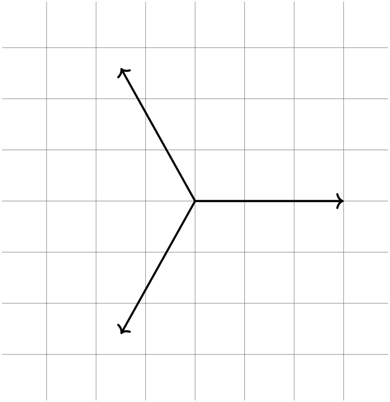
Note: A frame is "tight" if $A = B$ (cf. orthonormal bases).

Frames

Additional Property:
$\{e_i\}_{i=1}^k$ is a frame in $\mathbb{R}^n$ if and only if it is a spanning set for $\mathbb{R}^n$.

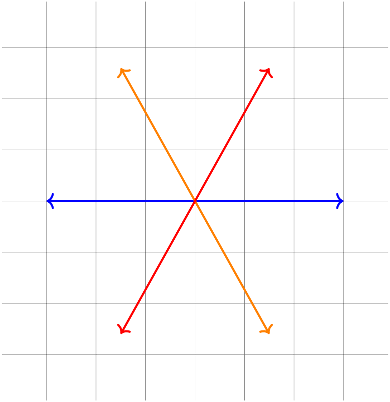
Check your understanding: Given a spanning set in $\mathbb{R}^n$, what must it satisfy to be an equiangular tight frame?

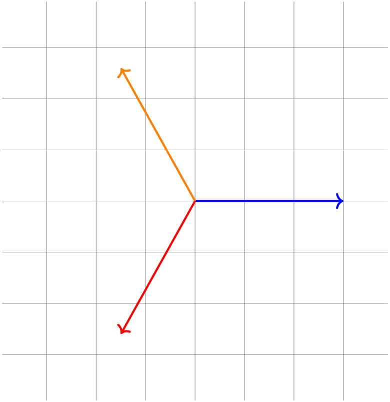
Example Frame



- This is the "Mercedes Benz Frame" in $\mathbb{R}^2$
- $\left\{ \sqrt{\frac{2}{3}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{-1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{-1}{2} \\ \frac{-\sqrt{3}}{2} \end{bmatrix} \right\}$
- It is an "equiangular tight frame."

Example Frame





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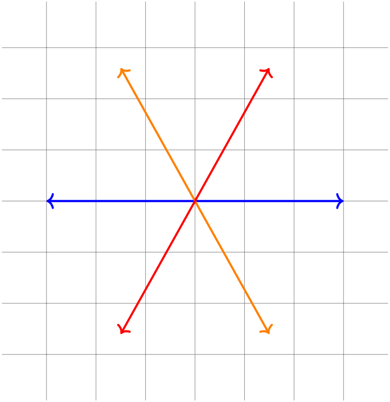
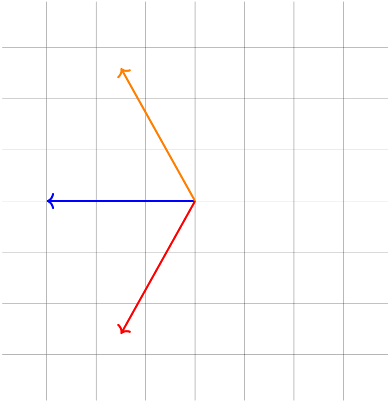
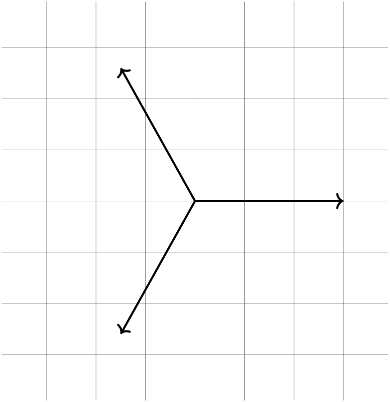




Frame to Grammian



$$\left\{ \sqrt{\frac{2}{3}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{-1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{-1}{2} \\ \frac{-\sqrt{3}}{2} \end{bmatrix} \right\}$$

Frame to Grammian

Definition

Let $\{e_i\}_{i=1}^{k}$ be a frame in $\mathbb{R}^n$.
Then the Grammian of this frame is given by $\Theta\Theta^T$, where

$$\Theta^T = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ e_1 & e_2 & ... & e_k \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}, \Theta = \begin{bmatrix} \leftarrow & {e_1}^T & \rightarrow \\ \leftarrow & {e_2}^T & \rightarrow \\ & ... & \\ \leftarrow & {e_k}^T & \rightarrow \end{bmatrix}.$$

Frame to Grammian (Mercedes-Benz Example)

Let $e_1 = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{-1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$, $e_3 = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{-1}{2} \\ \frac{-\sqrt{3}}{2} \end{bmatrix}$

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$$\Theta^T = \sqrt{\frac{2}{3}}\begin{bmatrix}1 & \frac{-1}{2} & \frac{-1}{2}\\0 & \frac{\sqrt{3}}{2} & \frac{-\sqrt{3}}{2}\end{bmatrix},\ \Theta = \sqrt{\frac{2}{3}}\begin{bmatrix}1 & 0\\\frac{-1}{2} & \frac{\sqrt{3}}{2}\\\frac{-1}{2} & \frac{-\sqrt{3}}{2}\end{bmatrix}$$

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$$\Theta\Theta^T = \begin{bmatrix}1 & \frac{-1}{2} & \frac{-1}{2}\\\frac{-1}{2} & 1 & \frac{-1}{2}\\\frac{-1}{2} & \frac{-1}{2} & 1\end{bmatrix}$$

Grammian to Seidel Adjacency Matrix (Example)

$$\frac{1}{\phi+2}\begin{bmatrix} \phi+2 & \phi & \phi & \phi & \phi & -\phi \\ \phi & \phi+2 & \phi & -\phi & \phi & \phi \\ \phi & \phi & \phi+2 & \phi & -\phi & \phi \\ \phi & -\phi & \phi & \phi+2 & -\phi & -\phi \\ \phi & \phi & -\phi & -\phi & \phi+2 & -\phi \\ -\phi & \phi & \phi & -\phi & -\phi & \phi+2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & -1 \\ -1 & 1 & 1 & -1 & -1 & 0 \end{bmatrix}$$

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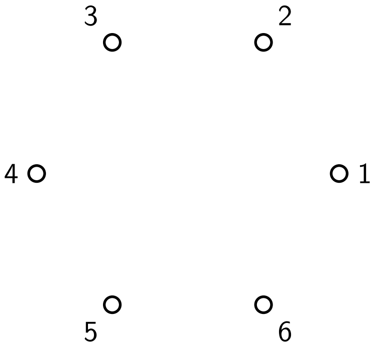
Seidel Adjacency Matrix to Graph (Example)

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & 0 & -1 & 1 & 1 \\ 1 & 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 1 & -1 & 0 \end{bmatrix}$$



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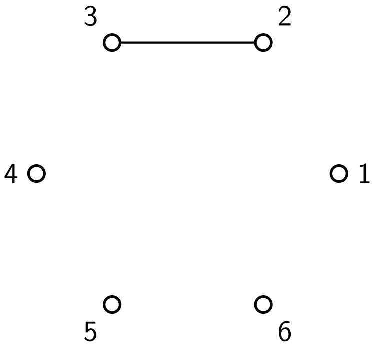

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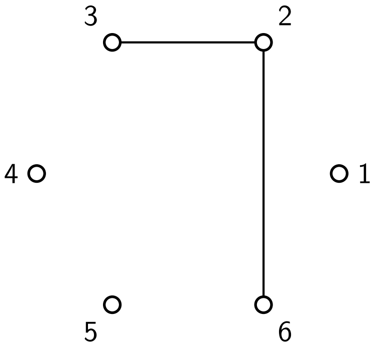

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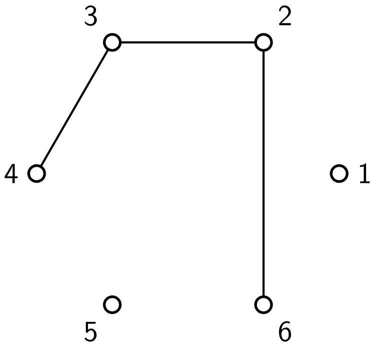

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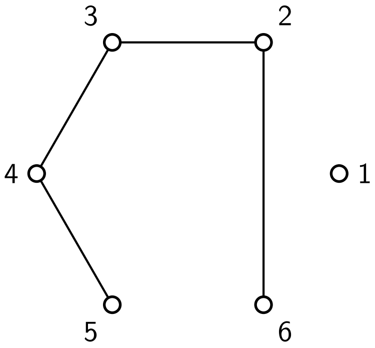

Seidel Adjacency Matrix to Graph (Example)

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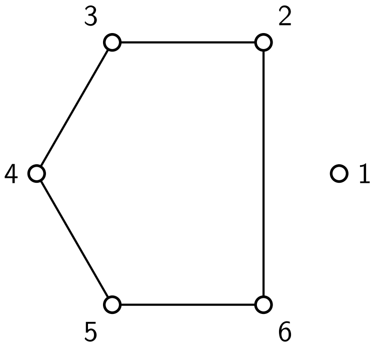
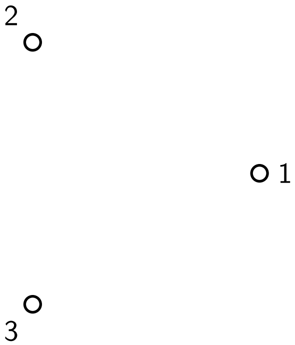


Seidel Adjacency Matrix to Graph (Example 2)

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$



Seidel Adjacency Matrix to Graph (Example 2)

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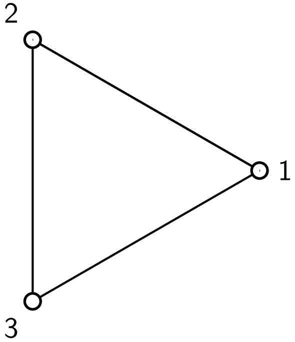
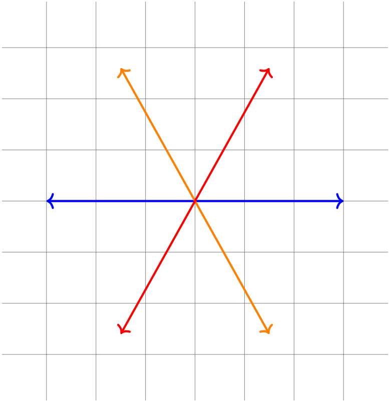
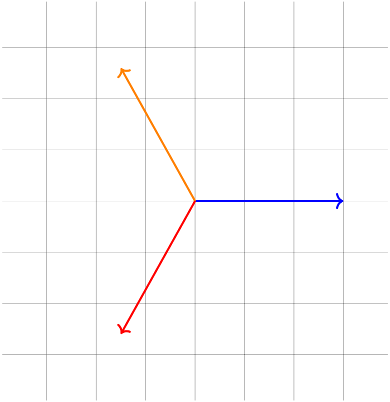


Switching





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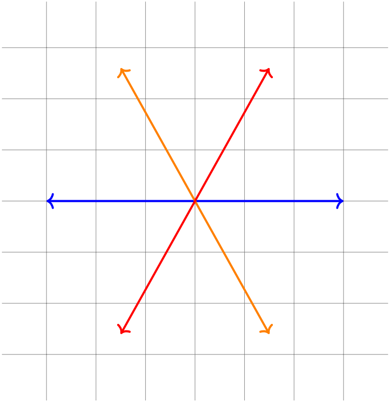

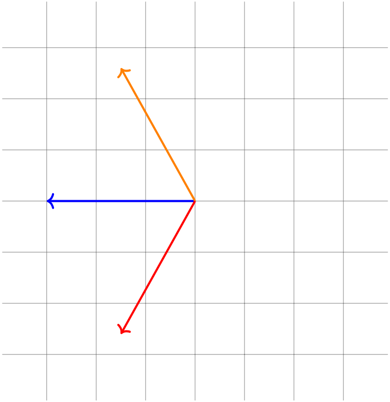

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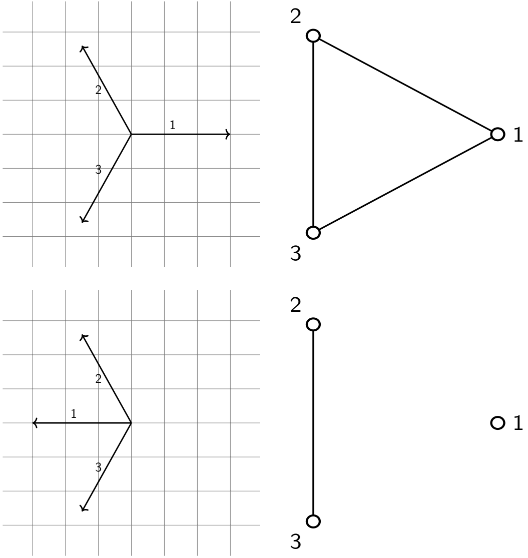
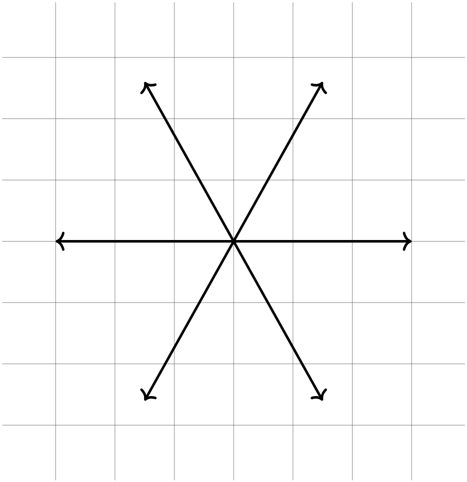
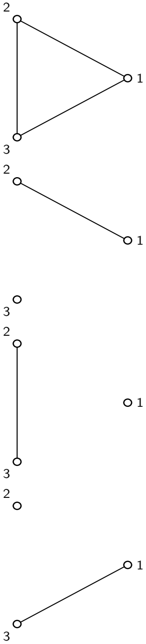


Two Graph





Thank You!

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