Dynamic Fair Division with Indivisible Goods

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Introduction

We study online fair division of indivisible items.

- n agents, 1 item arrives each step
- Allocation at every step must be fair

Background and Definitions

- **EF** : For any pair A and B, A does not envy B. Formally, A's value for their own allocation is at least A's value for B's allocation.
- **EF1 (Envy-free up to one good)** : For any pair A and B, there exists an item that A can remove from B's allocation such that after the removal, A does not envy B.
- **Round Robin** : EF1 protocol for the offline setting. The protocol fixes an arbitrary ordering and each agent takes turns picking their most preferred item.

Settings

Since fairness criteria (such as EF1) are often impossible to achieve in purely online settings, we consider the following elements in our model.

- **Disruptions** : We consider when we can use some number of disruptions each step. A disruption consists of moving one previously assigned item between two agents.
- **Knowing the Future** : We are informed that T items will arrive and are given the values of each item for each agent.
- **Restricted Values** : We also consider when the support of each agent's item values is at most size c.

Summary of Results

We present results for both the online setting and knowing the future (informed algorithm) setting. Results are either per step or total number of disruptions after T steps.

Result	n = 2	n > 2
Online Upper Bound	$O(T)$ total	$c \cdot n$ per step $O(T^{3/2})$ total
Online Lower Bound	$\Omega(T)$ total	$\Omega(T)$ total
Informed Upper Bound	No disruptions	$c \cdot n$ per step $O(T^{3/2})$ total
Informed Lower Bound	No disruptions	$\Omega(T)$ total

Upper Bound Details

Positive results for $n > 2$ in the online setting:

Algorithm 1 (Reverse Round Robin):

- We maintain a main pile and side pile as items arrive. Items are initially added to the side pile.
- Items in the main pile and side pile are both allocated with round robin, but with reversed orderings.
- Items in the side pile are merged into the main pile every $\sqrt{T}$ steps.
- We show that this uses $O(T^{3/2})$ disruptions over T steps.

Algorithm 2 (Online Round Robin):

- We run round robin but with the following tie-breaking criteria.
- Agents prefer items they chose the previous time step and also prefer items in the order they chose them in the previous time step.
- We show that when agents have at most c distinct item values, this requires $c \cdot n$ disruptions per step.

Lower Bound Details

Lower bound for $n = 3$ in the informed algorithm setting:

We start with the following 20 item gadget:

- Assume no disruptions are used.
- Define an allocation as satisfying $A > B > C$ if A does not envy B and C and B does not envy C.
- We start with 7 items such that the final allocation cannot satisfy $A > C > B$.
- We add 6 items such that without using any disruptions, the final allocation cannot satisfy $C > A > B$ or $A > C > B$.
- We can then show B must be envied by either A or C and use the remaining items to force an allocation that results in large envy.

We can then repeat this gadget an arbitrary number of times by significantly increasing item values to show an $\Omega(T)$ lower bound.

Future Directions

- Closing the gap between upper and lower bounds for $n > 2$ for both the online and informed setting.

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