Dynamic Fair Division with Indivisible Goods

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Introduction

We study online fair division of indivisible items.

- n agents, 1 item arrives each step
- Allocation at every step must be fair

Background and Definitions

- **Proportional** : For any agent A, A's value for their allocation is at least $\frac{1}{n}$ of their value for all items.
- **EF** : For any pair A and B, A does not envy B.
- **EF1** : For any pair A and B, A envies B by at most one of B's items.

Settings

Since fairness is often impossible to achieve in the purely online settings, we consider the following related settings:

- **Disruptions** (Changing the Past) : Items arrive online and we are allowed d disruptions per step. A disruption consists of moving one previously assigned item between two agents.
- **Prefix Fairness** (Knowing the Future) : We are informed that T items will arrive and are given the values of each item for each agent.
- **Cache** : Items arrive online and we are allowed to keep a cache of up to d items that can be allocated fractionally and don't need to be allocated permanently.

Algorithms

We present two algorithms. The first, envy-cycle elimination, is modified from an algorithm given by Lipton et al [1]. The second, fractional item allocation, is based on the observation that there always exists a proportional allocation using at most $n-1$ fractional items.

Algorithm 1 (Modified Envy-Cycle Elimination):

- Let t be the last time allocation was envy-free. Ignore all items before time t.
- When a new item arrives, assign it to an arbitrary unenvied agent.
- As long as an envy-cycle exists, swap allocations to eliminate the cycle.

Algorithm 2 (Fractional Item Elimination):

- Consider the bipartite graph $G = (U, V, E)$ where U is the set of agents, V is the set of items, and $(u, v) \in E$ when agent u is allocated a non-zero portion of item v.
- When a new item arrives, give $\frac{1}{n}$ to each agent.
- Shift around the allocation of the fractional items to remove cycles without decreasing agent's value for their allocations. This results in at most $n-1$ fractionally allocated items.

Main Results

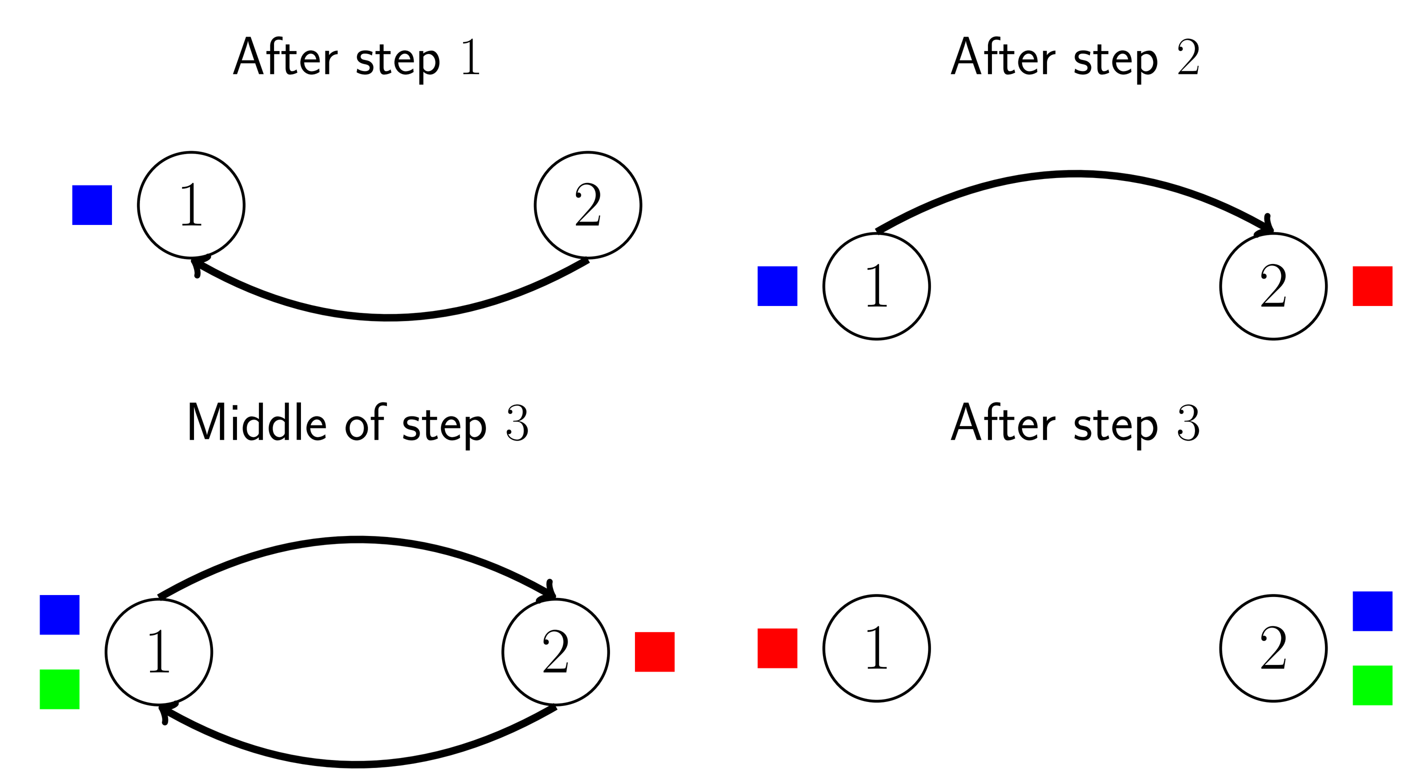
We obtain the following results.
For $n = 2$, we have

- **Disruptions** : Modify Algorithm 2 by rounding the fractional item to the agent with a larger portion. The resulting allocation is **EF1** and uses 1 disruption per step.
- **Prefix Fairness** : Run Algorithm 1 and use the final allocation, which is prefix **EF1**.

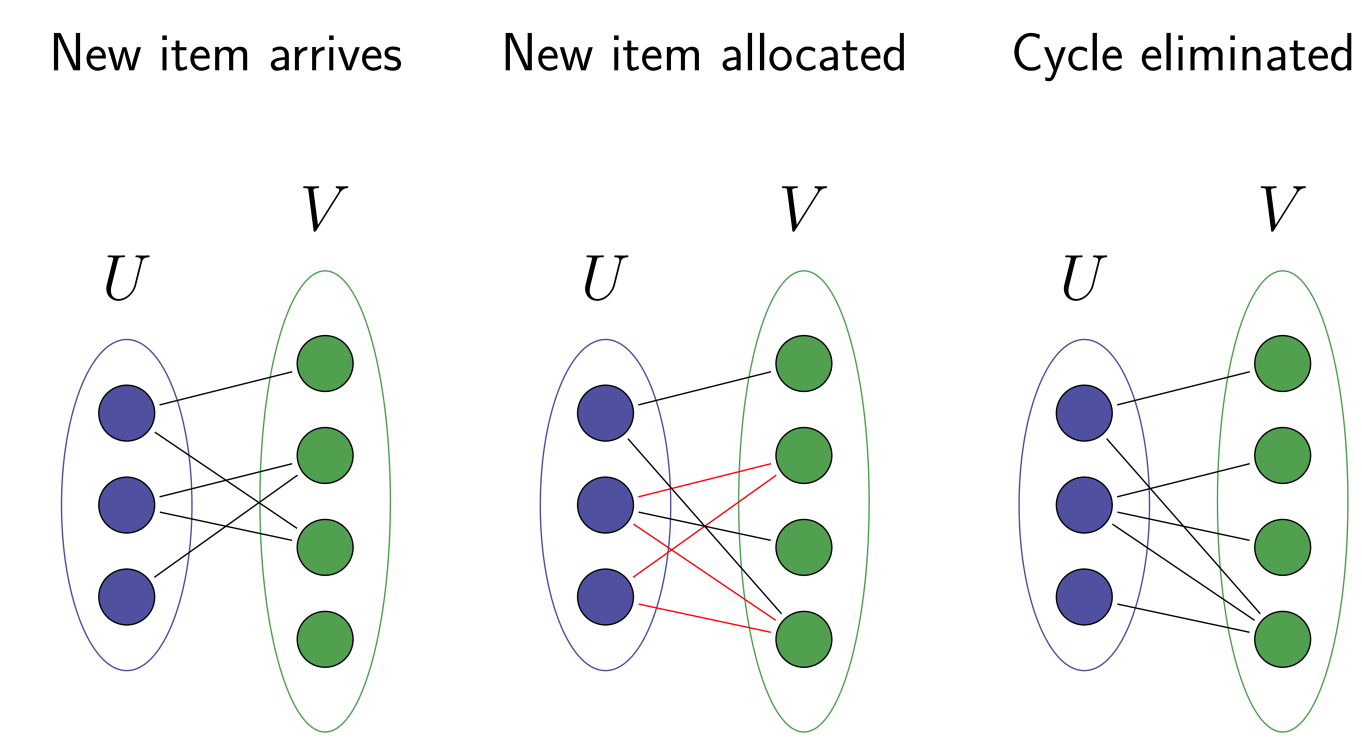
For general n, we also have:

- **Cache** : Use Algorithm 2 directly. The resulting allocation is proportional and uses a cache of $n-1$ fractional items.

Envy Cycle Elimination



Fractional Item Elimination



Future Directions

- Envy-freeness results when $n > 2$

References

[1] R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi.
On approximately fair allocations of indivisible goods.
In *Proceedings of the 5th ACM Conference on Electronic Commerce*, EC '04, pages 125–131, 2004.