The Extra Perfect Square Puzzle

16, 49, 169, 256, and 361 are squares that also result in squares after removing the last digit. What is next one? Try not to write code.

Lets form the square in two different ways. First, let the square that results from chopping off the last digit as x^2, and the last digit as d. Then one way to express the number is $n = 10 \times x^2 + d$. Next, to try and find some relationship with the last digit d , also express the square as the square of some number with a possibly different last digit m, so also $n = (10 \times y + m)^2 = 100 \times y^2 + 20 \times y \times m + m^2$.

Putting these together:

$$10 \times x^2 + d = (10 \times y + m)^2 = 100 \times y^2 + 20 \times y \times m + m^2$$

We know both sides of this have the same last digit, so expand m^2 into a decimal representation $m^2 = 10 \times q + d$. We know that m^2 has its last digit d because both sides are equal and all other terms have 10 as a factor. So $q = \left\lfloor \frac{m^2}{10} \right\rfloor$, and the equation expands to

$$10 \times x^2 + d = (10 \times y + m)^2 = 100 \times y^2 + 20 \times y \times m + 10 \times q + d$$

After subtracting d from both sides, both sides are also multiples of 10, so divide and get the following:

$$x^2 = 10 \times y^2 + 2 \times y \times m + q$$

Since $q = \left\lfloor \frac{m^2}{10} \right\rfloor$ there is a hidden relationship as well. However, there are only 10 distinct values of m, so we can write the family of 10 Diophantine equations

$$x^2 - 10 \times y^2 + 2 \times y \times m + q = 0$$

with m, q given in the table:

m	q
0	0
1	0
2	0
3	0
4	1
5	2
6	3
7	4
8	6
9	8

Each equation can be expressed as the two variable quadratic Diophantine equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

with $a = 1, b = 0, c = -10, d = 0, e = -2 \times m, f = q$.

This can be solved here:

https://math.stackexchange.com/questions/580491/general-quadratic-diophantine-equation

and you said not to write code, but you didn't say not to use calculators I found on the internet, of which there are two, and one here, that shows you the manual steps.

https://www.alpertron.com.ar/QUAD.HTM

Each solution has one or more values for (x_0, y_0) and iterators for (x_{n+1}, y_{n+1}). There are no solutions for $m = 0$ or $m = 5$, which can readily be seen by writing those specific equations. The $m = 0$ case reduces to $x^2 = 10 \times y^2$ so obviously can't have a solution since 10 is not a square. For $m = 5$,

$$x^2 = 10 \times y^2 + 10 \times y + 2 = 2 \times (5 \times y^2 + 5 \times y + 1)$$

Therefore x must be even, but the term $(5 \times y^2 + 5 \times y + 1)$ is always odd, so the right hand side has exactly one factor of two, so cannot be a square.

The initial solutions are as follows

m	
1	(0,0)
2	(0,0)
3	(4,1) (2,-1) (-4,1) (-2,1) (0,0)
4	(1,0) 5,-2) (-1,0) (-5,2)
6	(-5,1) (-1,-1) (5,1) (1,-1)
7	(-2,0) (-4,2) (4,2) (4,-2) (0,-1)
8	(0,-1)
9	(0,-1)

and the iterators all have identical form

$x_{n+1} = -19 \times x_n - 60 \times y_n - 6 \times m$

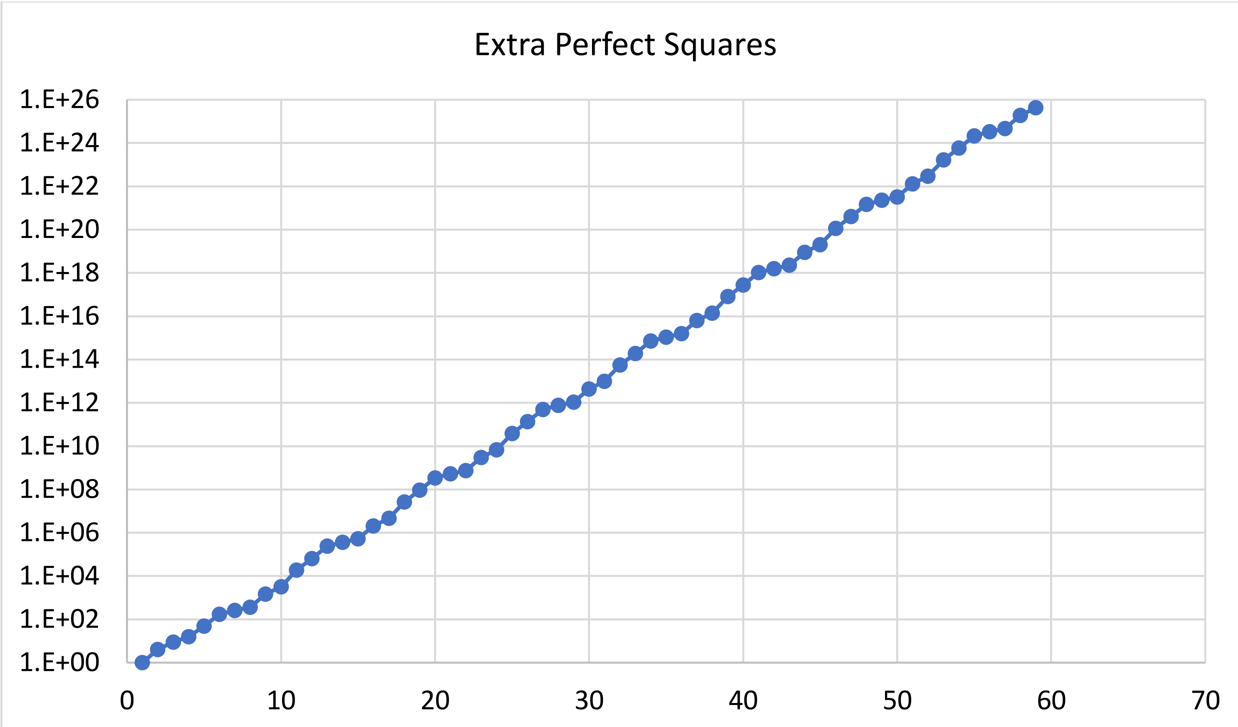
$y_{n+1} = -6 \times x_n - 19 \times y_n - 2 \times m$

So we generate all possibilities using excel, see the spreadsheet, and here are the first few, although they include ones where the first digit is 0.

1
4
9
16
49
169
256
361

1444
3249
18496
64009
237169
364816
519841
2079364
4678569
26666896
92294449
341991049
526060096
749609641
2998438564
6746486769
38453641216
133088524969
493150849009
758578289296
1080936581761
4323746327044
9728429235849
55450123962256
191913560704369
711123182273449

And here is a chart:



Note the period of 7 behaviour. We look at the ratios of the $n+1^{st}$ number to nth, and the $n+7^{th}$ to nth.

	n[i+7]/n[i]	n[i+1]/n[i]
1		
4		4
9		2.25
16		1.777778
49		3.0625
169		3.44898
256		1.514793
361	361	1.410156
1444	361	4
3249	361	2.25
18496	1156	5.692829
64009	1306.306122	3.460694
237169	1403.366864	3.705245
364816	1425.0625	1.538211
519841	1440.00277	1.42494
2079364	1440.00277	4
4678569	1440.00277	2.25
26666896	1441.765571	5.699798
92294449	1441.897999	3.461012
341991049	1441.971965	3.705435

526060096	1441.987457	1.538228
749609641	1441.997921	1.424951
2998438564	1441.997921	4
6746486769	1441.997921	2.25
38453641216	1441.999144	5.699802
133088524969	1441.999236	3.461012
493150849009	1441.999288	3.705435
758578289296	1441.999298	1.538228
1080936581761	1441.999306	1.424951
4323746327044	1441.999306	4
9728429235849	1441.999306	2.25
55450123962256	1441.999306	5.699802
191913560704369	1441.999306	3.461012
711123182273449	1441.999307	3.705435
1093869367100420	1441.999307	1.538228
1558709801289000	1441.999307	1.424951

They each converge quickly to some value. I have no idea what this means.

As for the extra perfect square, I don’t think there is one, but can’t prove it. This method generates the perfect squares exactly out to 15 digits, when floating point roundoff might be a problem, but there aren’t any other extra perfect squares so far.