The Riddler 22-01-28 presents a truck owner who replaces 1/12 of his transmission fluid every month, by draining 1/12 of the contents, and adding 1/12 of new fluid. After a very long time, what fraction of the fluid is less than one year old?

Let x[i] be the fraction that is i months old, the instant after he replaces the fluid. We can see that x[0] = 1/12. For other values of i, note that 11/12 of the fluid that was i-1 months previously is kept and is now i months old, so $x[i]=x[i-1]\times\frac{11}{12}$. So, by induction $x[i]=\frac{\left(\frac{11}{12}\right)^i}{12}$. To find the total amount that is strictly less than one year old, we sum the series for 0 to 11 months old, $\sum_{i=0}^{11} x[i]=\frac{\left(1-\left(\frac{11}{12}\right)^{12}\right)}{12\times\left(1-\frac{11}{12}\right)}=1-\left(\frac{11}{12}\right)^{12}$. This can be evaluated as $\frac{12^{12}-11^{12}}{12^{12}}=\frac{5777672071535}{12^{12}}=\frac{5777672071535}{8916100448256}\approx 0.648004371985863\approx 1-e^{-1}$, where the last term corresponds to exponential decay for one time constant, as we expect intuitively.