P Set #4

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1. We take the ansatz that $f(x,y,t)$ is seperable, namely f is of the form

$$f(x,y,t) = X(x)Y(y)T(t) \Rightarrow \begin{cases} f_{xx} = X''(x)Y(y)T(t) \\ f_{yy} = X(x)Y''(y)T(t) \\ f_t = X(x)Y(y)T'(t) \end{cases}$$

Substituting into our PDE, we obtain:

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{T'(t)}{T(t)}$$

As these are functions of independent variables, we may fix two of them to obtain an ODE in terms of the third. For example

$$\frac{X''(x)}{X(x)} + \frac{Y''(y_0)}{Y(y_0)} = \frac{T'(t_0)}{T(t_0)} \Rightarrow X''(x) = X(x)\left[\frac{T'(t_0)}{T(t_0)} - \frac{Y''(y_0)}{Y(y_0)}\right] = \alpha X(x)$$

Thus, we obtain by analogy the system of homogenous ODEs:

$$\begin{cases} X''(x) - \alpha X(x) = 0 & X'(0) = X'(L_x) = 0 \\ Y''(y) - \beta Y(y) = 0 & Y'(0) = Y'(L_y) = 0 \\ T'(t) - \gamma T(t) = 0 \ ; & \text{by Neumann BCs} \end{cases}$$

We proceed case-wise and WLOG consider the X part explicitly

- $\underline{\alpha = 0}$: $X''(x) = 0 \ \& \ X'(0) = X'(L_x) = 0 \Rightarrow X(x) = a, \ a \in \mathbb{R}$

- $\underline{\alpha > 0}$: $X''(x) - \alpha X(x) = 0 \ \& \ X'(0) = X'(L_x) = 0$

  Which has general solution of the form

  $$X(x) = c_1 e^{\sqrt{\alpha}x} + c_2 e^{-\sqrt{\alpha}x} \quad \begin{matrix} X'(0) = 0 \Rightarrow c_1 = c_2 \\ ; X'(L_x) = 0 \Rightarrow c_1 = 0 \end{matrix} \Rightarrow X(x) = a, \ a \in \mathbb{R}$$

- $\underline{\alpha < 0}$: $X''(x) - \alpha X(x) = 0 \ \& \ X'(0) = X'(L_x) = 0$

  Which has general solution of the form (taking $\alpha = |\alpha|$ below)

  $$X(x) = c_1 e^{i\sqrt{\alpha}x} + c_2 e^{-i\sqrt{\alpha}x} = c_1 \cos(\sqrt{\alpha}x) + i c_2 \sin(\sqrt{\alpha}x)$$

  $$X'(x) = -c_1\sqrt{\alpha}\sin(\sqrt{\alpha}x) + i c_2 \sqrt{\alpha}\cos(\sqrt{\alpha}x)$$

  $$X'(0) = i c_2 \sqrt{\alpha} = 0 \Rightarrow \boxed{c_2 = 0}$$

$$X'(L_x) = -c_1\sqrt{\alpha}\sin(\sqrt{\alpha}L_x) = 0 \text{ , with } c_1 \& \alpha \neq 0$$

$$\Rightarrow \boxed{\alpha_n = \left(\frac{\pi n}{L_x}\right)^2}, \ n \in \mathbb{N}$$

$$\therefore X(x) = c_1 \cos\left(\frac{\pi n}{L_x}x\right)$$

And by analogy:

$$\Rightarrow \boxed{\beta_m = \left(\frac{\pi m}{L_y}\right)^2}, \ m \in \mathbb{N}$$

$$\therefore Y(y) = d_1 \cos\left(\frac{\pi m}{L_y}y\right)$$

Thus, $\phi_{nm} = k\cos\left(\frac{\pi n}{L_x}x\right)\cos\left(\frac{\pi m}{L_y}y\right)$ & $\lambda_{nm} = \left(\frac{\pi n}{L_x}\right)^2 + \left(\frac{\pi m}{L_y}\right)^2$ ✓

In $T(t)$ we have:

$$T'(t) = T(t)\left[\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}\right] = T(t)\left[-\lambda_{nm}\right]$$

$\Rightarrow T(t) = e^{-\lambda_{nm}t}$; therefore smallest λ_{nm} dominate solution

For $L_x = \frac{1}{2}L_y$, the 6 leading eigenvectors are

$n=1$, $m=1$: $\phi_{11}e^{-\lambda_{11}t} = \cos\left(\frac{\pi}{L_x}x\right)\cos\left(\frac{\pi}{2L_x}y\right)e^{-1.25\left(\frac{\pi}{L_x}\right)^2 t}$

$n=1$, $m=2$: $\phi_{12}e^{-\lambda_{12}t} = \cos\left(\frac{\pi}{L_x}x\right)\cos\left(\frac{\pi}{L_x}y\right)e^{-2\left(\frac{\pi}{L_x}\right)^2 t}$

$n=1$, $m=3$: $\phi_{13}e^{-\lambda_{13}t} = \cos\left(\frac{\pi}{L_x}x\right)\cos\left(\frac{3\pi}{2L_x}y\right)e^{-3.25\left(\frac{\pi}{L_x}\right)^2 t}$

$n=2$, $m=1$: $\phi_{21}e^{-\lambda_{21}t} = \cos\left(\frac{2\pi}{L_x}x\right)\cos\left(\frac{\pi}{2L_x}y\right)e^{-4.25\left(\frac{\pi}{L_x}\right)^2 t}$

$n=2$, $m=2$: $\phi_{22}e^{-\lambda_{22}t} = \cos\left(\frac{2\pi}{L_x}x\right)\cos\left(\frac{\pi}{L_x}y\right)e^{-5\left(\frac{\pi}{L_x}\right)^2 t}$

$n=2$, $m=3$: $\phi_{23}e^{-\lambda_{23}t} = \cos\left(\frac{2\pi}{L_x}x\right)\cos\left(\frac{3\pi}{2L_x}y\right)e^{-6.25\left(\frac{\pi}{L_x}\right)^2 t}$