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P-Set #3

1 Disconnected $\Rightarrow$ Reducible: For a disconnected graph $G=(V,E)$ with adjacency matrix W, WLOG G has 2 connected components $A,B \subsetneq V(G)$. with $|A|=m$ & $|B|=n$.

Thus, $\forall a \in A$ & $\forall b \in B$, $\{a,b\} \notin E(G) \Rightarrow W_{ij} = \begin{cases} 0, & 1\le i\le m,\ 1\le j\le m \\ 0, & m+1\le i\le m+n,\ m+1\le j\le m+n \\ 0 \text{ or } 1, & \text{otherwise} \end{cases}$

$\Rightarrow W = \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}$ So W is block upper triangular, Therefore W is reducible. ■

Disconnected $\Leftarrow$ Reducible: For a graph with reducible adjacency matrix W, by definition of reducible, $\exists P$ s.t. $PWP^T = T$, where T is block upper triangular.

$T = \begin{bmatrix} 0 & B' \\ A' & 0 \end{bmatrix}$ So no $a \in A'$ is adjacent to any $b \in B'$ $\Rightarrow$ no edge with an endpoint in both A' & B'.

Therefore the relabeled graph corresponding to adjacency matrix T is disconnected, with connected components A' & B'. As P is a permutation matrix, the relabeled graph defined by T is isomorphic to the original graph defined by W. ■

(2) (A) $W=\begin{bmatrix} 0 & A \\ A^T & 0\end{bmatrix}$, $\sum_i \lambda_i = Tr(W) = 0$

$\Rightarrow$ either $\lambda_i = 0, \forall i$ **OR** $\exists \lambda_i > 0 \wedge \lambda_j < 0$, for at least one i & j
Since all elements of W are non-negative, by Perron-Frobenius Theorem
$\exists \lambda_i > 0$ for some i $\therefore \exists \lambda_j < 0$ for some j. ■

(B) $W=\begin{bmatrix} 0 & B \\ A & 0\end{bmatrix}$ for a bipartite graph

Take $\lambda_i > 0$ & $v_i = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ to be its corresponding eigenvector s.t.

$$Wv_i = \begin{bmatrix} Bv_2 \\ Av_1 \end{bmatrix} = \begin{bmatrix} \lambda_i v_1 \\ \lambda_i v_2 \end{bmatrix} = \lambda_i v_i$$

Now consider, $v_j = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix}$

$$Wv_j = \begin{bmatrix} Bv_2 \\ -Av_1 \end{bmatrix} = \begin{bmatrix} \lambda_i v_1 \\ -\lambda_i v_2 \end{bmatrix} = -\lambda_i \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} = -\lambda_i v_j$$

So v_j is an eigenvector corresponding to eigenvalue $-\lambda_i$ ■

(3) Let $L=D-W$ be the graph Laplacian of a graph with a single connected component, and take x to be an eigenvector corresponding to eigenvalue 0. Thus we have:

$$Lx=(D-W)x=0x=0$$

$$\Rightarrow \sum_{j=1}^{n} \deg(i)x_i - W_{ij}x_j = \sum_{j=1}^{n} W_{ij}x_i - W_{ij}x_j = 0$$

$$= \sum_{j=1}^{n} W_{ij}(x_i-x_j) = 0$$

$$x^TLx = x^T(D-W)x = x^T0x$$

$$\Rightarrow \sum_{i=1}^{n} x_i\left(\sum_{j=1}^{n} W_{ij}(x_i-x_j)\right) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} W_{ij}(2x_i^2-2x_ix_j)=0$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} W_{ij}\underbrace{(x_i^2-2x_ix_j+x_j^2)}_{\text{Follows by Symmetry of Multiplication}} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\underbrace{W_{ij}}_{\geq 0}\underbrace{(x_i-x_j)^2}_{\geq 0}=0$$

Fix an i. Then by connectivity we know,

$$\exists j \text{ s.t. } W_{ij}>0 \Rightarrow x_i=x_j \;\forall i \;\therefore\; x \propto \vec{1}$$

So we have shown the geometric multiplicity of $\lambda_1=0$ to be 1. Furthermore, as L is symmetric PSD, it has a spectral decomposition. Therefore, its eigenbasis is full rank, with n linearly independent eigenvectors, which correspond to n unique eigenvalues.
So we have shown the algebraic multiplicity of $\lambda_1=0$ is also 1, for a graph with a single connected component.

We proceed by <u>induction on components.</u>

By Induction Hypothesis, graph G_1 has $n-1$ connected components, and the multiplicity of $\lambda=0$ of its graph Laplacian is $n-1$. Now, take $G'=G_1 \dot\cup G_2$ to be the disjoint union between G_1, and a connected graph G_2, so G' has n connected components. WLOG assume the vertices of G' are ordered according to the connected components they belong to; otherwise we may permute the vertex labels to obtain a graph isomorphic to G' whose adjacency matrix is block diagonal. This implies the graph Laplacian of G' is also block diagonal, and may be expressed as:

$$\mathbb{L}[G'] = \begin{bmatrix} \mathbb{L}[G_1] & 0 \\ 0 & \mathbb{L}[G_2] \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

Now consider a vector (above) with 1s along the dimension of of $\mathbb{L}[G_2]$ and 0s elsewhere. Embed the $n-1$ eigenvectors with $\lambda=0$ of $\mathbb{L}[G_1]$ (by IH) using 0s analogously and observe they are orthogonal to the first vector, which is the unique eigenvector with $\lambda=0$ for a single connected component by the base case. Thus we have found n linearly independent eigenvectors corresponding to $\lambda=0$ for $\mathbb{L}[G']$, so the geometric multiplicity is at least n.

Furthermore, the characteristic polynomial of $\mathbb{L}[G']$ is given by the product of the characteristic polynomials of $\mathbb{L}[G_1]$ & $\mathbb{L}[G_2]$

$$\chi(\mathbb{L}[G']) = \prod_{i=1}^{2} \chi(\mathbb{L}[G_i])$$

But we know the algebriac multiplicity of $\lambda=0$ for $\mathbb{L}[G_1]$ is $n-1$ by IH, and the algebraic multiplicity of $\lambda=0$ for $\mathbb{L}[G_2]$ is 1 by the base case, so the <u>algebraic multiplicity</u> of $\lambda=0$ for $\mathbb{L}[G']$ is n. Additionally, the algebraic multiplicity upper bounds the geometric multiplicity, thus the <u>geometric multiplicity</u> of $\lambda=0$ for $\mathbb{L}[G']$ is also n. ∎

(5) $\mathbb{P}\{X_{t+1} \in S^c \mid X_t \in S\} + \mathbb{P}\{X_{t+1} \in S \mid X_t \in S^c\}$

$$= \frac{\sum_{j \in S} \sum_{j \in S^c} \pi_i \mathbb{P}_{ij}}{\frac{Vol(S)}{Vol(G)}} + \frac{\sum_{i \in S^c} \sum_{j \in S} \pi_i \mathbb{P}_{ij}}{\frac{Vol(S^c)}{Vol(G)}}$$

$$= \frac{\sum_{i \in S} \sum_{j \in S^c} \left(\frac{\cancel{deg(i)}}{\cancel{Vol}(G)}\right)\left(\frac{W_{ij}}{\cancel{deg(i)}}\right)}{\frac{Vol(S)}{\cancel{Vol}(G)}} + \frac{\sum_{i \in S^c} \sum_{j \in S^c} \left(\frac{\cancel{deg(i)}}{\cancel{Vol}(G)}\right)\left(\frac{W_{ij}}{\cancel{deg}(i)}\right)}{\frac{Vol(S^c)}{\cancel{Vol}(G)}}$$

$$= \frac{\sum_{i \in S} \sum_{j \in S^c} W_{ij}}{Vol(S)} + \frac{\sum_{i \in S^c} \sum_{j \in S} W_{ij}}{Vol(S^c)}$$

$$= \frac{Cut(S)}{Vol(S)} + \frac{Cut(S^c)}{Vol(S^c)} = Ncut(S)$$

(6) In (3) we showed $x^T L x = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(x_i - x_j)^2$

$$y^T L y = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \underbrace{(y_i - y_j)}_{0}^2$$

If $\begin{cases} i,j \in S \\ i,j \in S^c \end{cases} \Rightarrow y_i = y_j$, so WLOG $i \in S$ & $j \in S^c$

$$\therefore = \frac{1}{2}\left(\sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \left(\left(\frac{Vol(S^c)}{Vol(S)\,Vol(G)}\right)^{1/2} + \left(\frac{Vol(S)}{Vol(S^c)\,Vol(G)}\right)^{1/2} \right)^2 \right)$$

$$= \frac{1}{2}\left(\frac{Vol(S^c)}{Vol(S)\,Vol(G)} + \frac{Vol(S)}{Vol(S^c)\,Vol(G)} + 2\left(\frac{\cancel{Vol(S^c)}}{\cancel{Vol(S)}\,Vol(G)} \cdot \frac{\cancel{Vol(S)}}{\cancel{Vol(S^c)}\,Vol(G)}\right)^{1/2}\right)\left(\sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}\right)$$

$$= \frac{1}{2}\left(\frac{1}{Vol(S)} + \frac{1}{Vol(S^c)}\right)\left(\sum_{i \in S} \sum_{j \in S^c} w_{ij} + \sum_{i \in S^c} \sum_{j \in S} w_{ij}\right)$$

$$= \frac{1}{2}\left(\frac{1}{Vol(S)} + \frac{1}{Vol(S^c)}\right)\left(Cut(S) + Cut(S^c)\right)$$

$$= \frac{1}{2}\left(\frac{2\,Cut(S)}{Vol(S)} + \frac{2\,Cut(S^c)}{Vol(S^c)}\right)$$

$$= \frac{Cut(S)}{Vol(S)} + \frac{Cut(S^c)}{Vol(S^c)} = Ncut(S)$$