Sphere Method-7-3 Using No Matrix Inversions for Linear Programs(LPs)

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Abstract

Existing software implementations for solving Linear Programming (LP) models are all based on full matrix inversion operations involving every constraint in the model in every step. This **linear algebra component** in these systems makes it difficult to solve dense models even with moderate size, and it is also the source of accumulating roundoff errors affecting the accuracy of the output.

We present a new version of the Sphere method, SM- 7-3, for LP not using any pivot steps; and computational results on it.

Key words: Linear Programming (LP), Interior point methods (IPMs) , solving LPs by descent feasible methods without using matrix inversions.

1 Sphere Method, SM-7-3, for LP

In 2006, Sphere methods for LP, IPMs based on the properties of spheres (instead of ellipsoids like in other IPMs) were introduced in Murty [2006a, b]. The initial version of the sphere method also needed pivot steps for matrix inversions, but these pivot steps only involve a subset of constraints in the original LP. After some other versions, in this paper we describe SM-7-3, not involving any pivot steps.

SMs consider LPs in the form:

$$\begin{aligned} \text{min} \quad & z = cx \\ \text{subject to} \quad & Ax \geq b \end{aligned} \tag{1}$$

where A is an $m \times n$ data matrix; with a known interior feasible solution x (i.e., satisfying $Ax > b$). LPs in any other form can be directly transformed into this form, see [Murty 2009a, b], Murty, Oskoorouchi [2010]. Here is some basic notation that we will use.

- **Notation for rows and columns of A:** $A_{i.}, A_{.j}$ denote the i^{th} row, and j^{th} column of A. The index i has range $1 \leq i \leq m$, and j ranges in $1 \leq j \leq n$.

- **Feasible region and its interior:** K denotes the set of feasible solutions of (1), and $K^0 = \{x : Ax > b\}$ is its interior.

- **Facetal hyperplanes, and their half-spaces containing K:** $FH_i = \{x : A_{i.}x = b_i\}$, the i-th facetal hyperplance of K for $i = 1$ to m. Also, $FH_i^+ = \{x : A_{i.}x \geq b_i\}$ is the half-space of FH_i containing K.

- **IFS:** Interior feasible solution, a point $x \in K^0$

- $\delta(x)$: Defined for $x \in K$, it is the radius of the largest ball inside K with x as center. From Murty [2006a, b], we know that $\delta(x) = \text{minimum}\{\frac{A_{i.}x - b_i}{||A_{i.}||} : i = 1, ..., m\}$. For any point x on the boundary of K , i.e., satisfying at least one of the constraints in (1) as an equation, $\delta(x) = 0$ by this definition.

- **Largest ball inscribed in K with a given IFS x as center:** $B(x) = \{y : ||y - x|| \leq \delta(x)\}$ is that largest inscribed ball in K with x as its center.

- **Touching constraint index set at a given IFS:** $T(x)$ defined for $x \in K^0$, is the set of all indices i satisfying: $\frac{A_{i.}x - b_i}{||A_{i.}||} = \text{Minimum}\{\frac{A_{p.}x - b_p}{||A_{p.}||} : p = 1 \text{ to } m\} = \delta(x)$. The facetal hyperplane $FH_i = \{x : A_{i.}x = b_i\}$ is a tangent plane to $B(x)$ for each $i \in T(x)$, that's why $T(x)$ is called the **index set of touching constraints in (1) defining** K, at x.

- **Touching point** x^i **:** Defined for $x \in K^0$ and $i \in T(x)$, it is the nearest point on FH_i to x, it is the orthogonal projection $x - A_{i.}^T(A_{i.}x - b_i)/||A_{i.}||^2$ of x on FH_i. It is the point where the ball $B(x)$ touches FH_i for $i \in T(x)$.

- **NTP (Near Touching Point) corresponding to** $i \in T(x)$ **:** Defined for $x \in K^0$ and $i \in T(x)$, it is the point $(1-\epsilon)x^i + \epsilon x$; ϵ distance away from the touching point x^i on the line segment joining x^i to x, where ϵ is a small positive tolerance.

- $H(\hat{x})$: Defined for any IFS $\hat{x} \in K^0$, $H(\hat{x}) = \{x : cx = c\hat{x}\}$ is the objective plane through $\hat{x}$

- $\bar{\hat{x}}$: Defined for any IFS $\hat{x} \in K^0$, it is $= \hat{x} - \delta(\hat{x})c^T/||c||$ = the bottom point of $B(\hat{x})$ in the direction $-c^T$, the point where the objective plane touches $B(\hat{x})$ when it is moved down from its present position $H(\hat{x})$, in the direction $-c^T$ until it becomes a tangent plane to $B(\hat{x})$

- $\bar{\hat{x}}^i$: Defined for any IFS $\hat{x} \in K^0$ and $i \in T(\hat{x})$, it is $= \hat{x}^i - c^T[(c\hat{x}^i - c\bar{\hat{x}})/cc^T]$ = the orthogonal projection of $\hat{x}^i$ on $H(\bar{\hat{x}})$.

- c^i : For $i = 1$ to m, $c^i = c^T - A_{i.}^T[(A_{i.}c^T)/(A_{i.}A_{i.}^T)]$, the orthogonal projection of c^T on $\{x : A_{i.}x = 0\}$.

- **The set** Γ **:** This set is used to store the output points along with the value for the objective function cx at them, from various descent steps in each iteration; with $\Gamma = \emptyset$ at the beginning of the iteration. At the end of the descent cycle, the best point in Γ by objective value is taken as the output point in the iteration.

- α, γ: The parameter α with some subscripts, or superscripts, or both, is used to denote various points along selected straight lines in the algorithm. The parameter γ with some subscripts is used to denote the step lengths in descent steps in the algorithm.

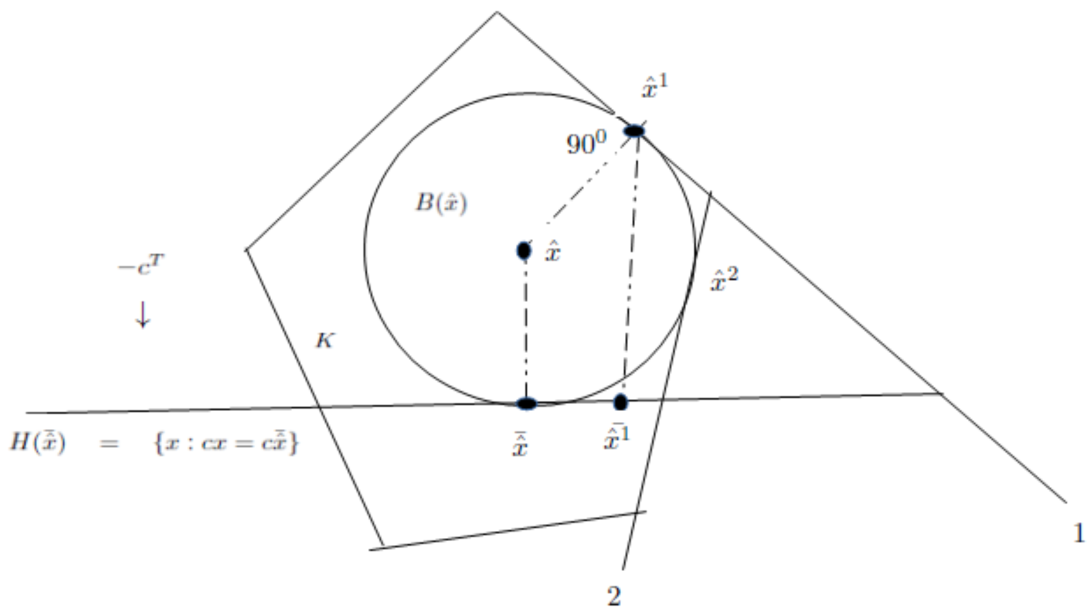


Figure 1: $\hat{x}$ is an IFS of K, $B(\hat{x})$ is the largest sphere with center $\hat{x}$ as center inside K. $\bar{\hat{x}}$ is the point in $B(\hat{x})$ with the smallest value for cx, and $H(\bar{\hat{x}})$ is the objective plane through $\bar{\hat{x}}$, it is the tangent plane to $B(\hat{x})$ at $\bar{\hat{x}}$. Facets 1, 2 of K are tangent planes to $B(\hat{x})$ with touching points $\hat{x}^1, \hat{x}^2$ respectively, so $T(\hat{x}) = \{1, 2\}$. $\bar{\hat{x}}^1$ is the orthogonal projection of $\hat{x}^1$ on $H(\bar{\hat{x}})$. Thanks to Madhusri Katta, Vijaya Katta for Figures 1, 2, 4.

SM-7-3 is based on feasible descent steps (starting with a feasible solution, maintaining feasibility throughout, with objective value improving monotonically), but not using any pivot steps at all. The 1st iteration begins with the given IFS $\hat{x}$, all subsequent iterations begin with the best solution (by objective value) obtained in the descent steps in the previous iteration. Each iteration of this method consists of 2 steps, a Centering step, followed by a descent cycle consisting of several descent steps.

2 A General Iteration in the Sphere Method, SM-7-3, for the LP (1)

In every iteration of SM- 7-3, we face a problem of finding the interval of values of a real parameter ν say, satisfying a given system of linear inequalities in the parameter. Now we give the procedure, we will call it **Subroutine 1** for computing this interval.

Subroutine 1: Let the system of inequalities in ν be

$$a_t + g_t\nu \geq 0, \qquad t = 1, ..., \ell \tag{2}$$

In systems like this that we encounter in SM- 7; for any t if $g_t = 0$, a_t will be ≥ 0, and hence that constraint is a redundant constraint in the system. Let

$$\begin{aligned} \nu^1 &= \text{maximum}\{(-a_t/g_t) : \text{over all } t \text{ satisfying } g_t > 0\} \\ \nu^2 &= \text{minimum}\{(-a_t/g_t) : \text{over all } t \text{ satisfying } g_t < 0\} \end{aligned}$$

Here define the maximum [minimum] in the empty set to be $-\infty[+\infty]$ respectively. If $\nu^1 > \nu^2$ system (2) has no solution. Otherwise the required interval for ν feasible to this system is $\nu^1 \leq \nu \leq \nu^2$.

Now we will describe the general iteration in this method beginning with an initial IFS $\hat{x}$.

2.1 Centering step

The centering step beginning with the initial IFS $\hat{x}$ consists of 2, 3 repitions of the following substep:

Substep: Find $\delta(\hat{x})$, $T(\hat{x})$. If $\delta(\hat{x})$ is too small, in implementing this algorithm go to **1. Implementation detail** below. Otherwise go to **2** below.

1. **Implementation detail:** Let $H(\hat{x}) = \{x : cx = c\hat{x}\}$, the objective plane through $\hat{x}$. Find

$d^0 = (\sum_{i \in T(\hat{x})} A_{i.}^T)/||T(\hat{x}||$ = direction given by average of directions orthogonal to the facetal hyperplane $\{x : A_{i.}x = b_i\}$ towards the feasible region, for $i \in T(\hat{x})$.

Let d^{00} be the orthogonl projection of d^0 on the plane $\{x : cx = 0\}$. So $d^{00} = d^0 - c^T(cd^0)/||c||^2$, therefore $cd^{00} = 0$. In $\hat{x} + \alpha d^{00}$, increasing the value of α from 0 typically helps increase $\delta(\hat{x} + \alpha d^{00})$ while keeping $c(\hat{x} + \alpha d^{00}) = c\hat{x}$. Now find the value of α maximizing $\delta(\hat{x} + \alpha d^{00})$ subject to $\alpha \geq 0$, and satisfying feasibility of $\hat{x} + \alpha d^{00}$ to (1). This is the 2-variable LP:

$$
\begin{aligned}
&\max \quad \delta \\
&\text{subject to} \quad \delta||A_{p.}|| - \alpha[A_{p.}d^{00}] \leq A_{p.}\hat{x} - b_p \qquad p = 1, ..., m \\
&\text{and } \delta \geq 0, \quad \alpha \geq 0
\end{aligned}
$$

For (δ, α) feasible to this system, the constraint $\delta \geq 0$ in the system implies that $\hat{x} + \alpha d^{00}$ will be feasible to (1). If the optimum solution of this 2-variable LP is (δ_0, α_0) then with $(\hat{x} + \alpha_0 d^{00})$ as the initial IFS continue. For ease of notation, let us discuss the rest of this Substep with $\hat{x}$ denoting this initial IFS.

2. Substep Continued: Find $\delta(\hat{x})$, $T(\hat{x})$, $\bar{\hat{x}}$, and $\bar{\hat{x}}^i$ for each $i \in T(\hat{x})$.

If $\bar{\hat{x}}$ is a boundary point of K, i.e. satifies $A_{i.}\bar{\hat{x}} = b_i$ for some $i = 1$ to m, then $H(\bar{\hat{x}})$ must be the same as $\{x : A_{i.}x = b_i\}$, so $\bar{\hat{x}}$ is an optimum solution of the origional LP, terminate the algorithm with this conclusion. Otherwise $\bar{\hat{x}}$ is an IFS of K, continue.

For each $i \in T(\hat{x})$ find using Subroutine 1, the interval of values of the parameter α satisfying

$$A_{p.}(\alpha\bar{\hat{x}} + (1 - \alpha)\hat{x}^i) \geq b_p \qquad \text{for } p = 1 \text{ to } m.$$

This interval includes $0 \leq \alpha \leq 1$ since for these values of α the point $\alpha\bar{\hat{x}} + (1-\alpha)\hat{x}^i$ is on the line segment joining $\hat{x}^i$ to $\bar{\hat{x}}$. Suppose this interval is $0 \leq \alpha \leq \alpha^{i2}$.

If $\alpha^{i2} = \infty$ for any $i \in T(\hat{x})$, then the objective function in the original LP (1) divergs to $-\infty$ along the half-line $\{(\alpha\bar{\hat{x}} + (1-\alpha)\hat{x}^i) : \quad \alpha \geq 0\}$, terminate the algorithm with this conclusion. Otherwise continue.

For each $i \in T(\hat{x})$, $\hat{x}^{i2} = \alpha^{i2}\bar{\hat{x}} + (1-\alpha^{i2})\hat{x}^i$ is the other boundary point of K on the straight line joining $\hat{x}^i$ to $\bar{\hat{x}}$.

If $c\bar{\hat{x}}$ - minimum$\{c\hat{x}^{i2}$: $\quad i \in T(\hat{x})\}$ is $>$ some tolerance for decrease in objective value in an iteration, then in this case let r be a value of i attaining the minimum above; terminate this iteration and with $\tilde{x} = \bar{\hat{x}} + (1-\epsilon^0)(\hat{x}^{r2} - \bar{\hat{x}})$ as the new initial IFS, where ϵ^0 is a small positive tolerance, go to the next iteration.

Otherwise define $\tilde{x}, r$ as in the previous line and continue.

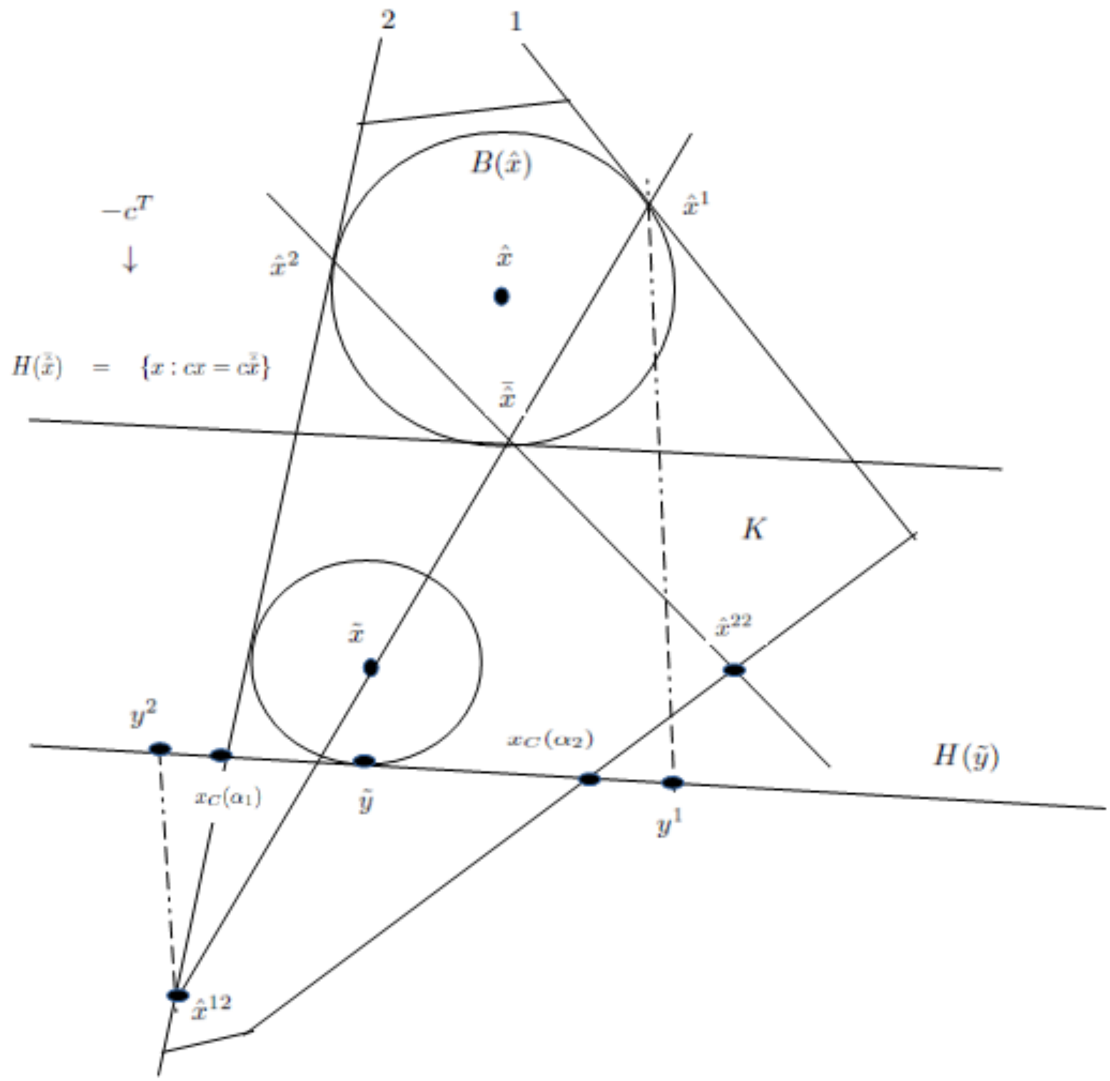


Figure 2: $\hat{x}$ is an IFS of K, $B(\hat{x})$ is the largest sphere with center $\hat{x}$ as center inside K. $\bar{\hat{x}}$ is the point in $B(\hat{x})$ with the smallest value for cx, and $H(\bar{\hat{x}})$ is the objective plane through $\bar{\hat{x}}$, it is the tangent plane to $B(\hat{x})$ at $\bar{\hat{x}}$. Facets 1, 2 of K are tangent planes to $B(\hat{x})$ with touching points $\hat{x}^1, \hat{x}^2$ respectively, so $T(\hat{x}) = \{1, 2\}$. The lines joining $\hat{x}^1, \hat{x}^2$ to $\bar{\hat{x}}$ intersect the boundary of K again at $\hat{x}^{12}, \hat{x}^{22}$ respectively. $c\hat{x}^{12} = \text{minimum}\{c\hat{x}^{12}, c\hat{x}^{22}\}$. So the index r defined above is

1 in this example. The point $\tilde{x}$ on the line joining $\hat{x}^{12}, \hat{x}^1$ is shown in the figure, and $B(\tilde{x})$ is the largest sphere with center $\tilde{x}$ inside K. $\tilde{y}$ is the boudary point of $B(\tilde{x})$ with the smallest value for the objective function cx inside $B(\tilde{x})$. y^1, y^2 are orthogonal projections of $\hat{x}^1, \hat{x}^{12}$ respectively on $H(\tilde{y})$, the objective plane through $\tilde{y}$. In this example both y^1, y^2 are outside K. L is the straight line joining y^1, y^2, in this 2-dimensional example L is the same as $H(\tilde{y})$. $L \cap K$ is the line segment joining $x_C(\alpha_1), x_C(\alpha_2)$.

Find $\delta(\tilde{x})$, $\quad \tilde{y} = \tilde{x} - \delta(\tilde{x})c^T/||c||$ = the point where the objective plane touches $B(\tilde{x})$ when it is moved down in the direction $-c^T$ to become a tangent plane to $B(\tilde{x})$. Let $y^1, \quad y^2$ be the orthogonal projections of $\hat{x}^r, \quad \hat{x}^{r2}$ on the objective plane $H(\tilde{y})$. Let

$$x_C(\alpha) = y^1 + \alpha(y^2 - y^1) \text{ and } L = \{x_C(\alpha)\text{: } \alpha \text{ real parameter}\}$$

L is the straight line joining y^1, y^2 on the objective plane $H(\tilde{y})$. Find $L \cap K = \{x_C(\alpha)$: $\alpha_1 \leq \alpha \leq \alpha_2\}$, where α_1, α_2 are the minimum and maximum values of the parameter α feasible to the system of linear inequalities

$$A_{p.}x_C(\alpha) \geq b_p, \qquad p = 1 \text{ to } m$$

which can be calculated using Subroutine 1. Clearly $\alpha_2 - \alpha_1 > 0$. Now we solve the line search problem: maximize $\quad \delta(x_C(\alpha))$ over the interval $\alpha_1 \leq \alpha \leq \alpha_2$.

There are two different approaches for solving it, which we will discuss below.

Approach 1: By solving a 2-variable LP: We verify that $A_{p.}x_C(\alpha) = A_{p.}y^1 + \alpha A_{p.}(y^2 - y^1)$, and from the definition of $\delta(x)$ for $x \in K$, we can see that this problem is the same as the 2-variable LP in the variables δ, α:

$$\begin{aligned} &\max \quad \delta \\ &\text{subject to} \quad \delta||A_{p.}|| - \alpha[A_{p.}(y^2 - y^1)] \leq A_{p.}y^1 - b_p \qquad p = 1, ..., m \\ &\text{and } \delta \geq 0, \quad \alpha_1 \leq \alpha \leq \alpha_2 \end{aligned} \tag{3}$$

The maximum value of the variable δ in this 2-variable LP is the maximum value of $\delta(x_C(\alpha))$ over the interval $\alpha_1 \leq \alpha \leq \alpha_2$.

Approach 2: Using a Line Search algorithm: Using the formula $\delta(x_C(\alpha))$ = Minimum $\{(A_{p.}x_C(\alpha) - b_p)/||A_{p.}|| : p = 1 \text{ to } m\}$ to compute $\delta(x_C(\alpha))$ for any value of α in the interval $\alpha_1 \leq \alpha \leq \alpha_2$, we can use a popular line search algorithm like "Quadratic interpolation" (also known as"Quadratic fit line search method") in Nonlinear Programming for solving this problem of maximizing $\delta(x_C(\alpha))$ over the interval $\alpha_1 \leq \alpha \leq \alpha_2$ (for example see Pages 558-560 in [1]).

Let $\bar{\alpha}$ be the optimum value for α obtained for this problem. Then the output point of this substep is $\bar{\hat{x}}^C = x_C(\bar{\alpha})$.

If the maximum value of δ in this problem is $+\infty$, then clearly cx is unbounded below on K, terminate the algorithm with this conclusion. Otherwise the maximum value of δ is finite, continue.

Now repeat this substep with $\bar{\hat{x}}^C$ as the initial IFS instead of $\hat{x}$. It can be repeated a few more times like this, as long as the δ-value at the output point keeps increasing or if the objective value decreases at a good rate. The output point at the end of the final repetition of this substep, denoted by $\bar{\bar{\hat{x}}}^C$, is called the **Center** in this iteration in SM-7-3. With it go to the Descent Steps.

2.2 Descent Steps:

For simplicity, in this step we will denote the center obtained in this iteration by the symbol $\bar{x}$. In keeping with the notation developed in Section 1, $\bar{x}^i$ denotes the orthogonal projection of $\bar{x}$ on FH_i for $i \in T(\bar{x})$, $\bar{\bar{x}}$ the bottom point of $B(\bar{x})$ in the direction $-c^T$, $\bar{\bar{x}}^i$ the orthogonal projection of $\bar{x}^i$ on $H(\bar{\bar{x}})$. If $\bar{\bar{x}}$ is a boundary point of K, i.e. satifies $A_{i.}\bar{\bar{x}} = b_i$ for some $i =$ 1 to m, then $H(\bar{\bar{x}})$ must be the same as $\{x : A_{i.}x = b_i\}$, so $\bar{\bar{x}}$ is an optimum solution of the origional LP, terminate the algorithm with this conclusion. Otherwise $\bar{\bar{x}}$ is an IFS of K, continue.

A General Descent Step

Consider a descent step from a point $\underline{x}$ in a descent direction d, i.e., a direction satisfying $cd < 0$.

If $\underline{x}$ is an feasible solution in K, the maximum possible step length γ^2 from $\underline{x}$ in the direction d inside K is the maximum value of γ satisfying $A_{i.}(\underline{x} + \gamma d) \geq b_i$ for all $i = 1$ to m; which is minimum$\{(b_i - A_{i.}\underline{x})/(A_{i.}d)$: over all $i = 1$ to m satisfying $A_{i.}d < 0\}$.

If $\{i : 1 \leq i \leq m : \quad A_{i.}d < 0\} = \emptyset$, then this step length $\gamma^2 = +\infty$; we terminate the algorithm with the conclusion that the objective function cx in (1) is unbounded below on K, with $\{\underline{x} + \gamma d : \gamma \geq 0\}$ providing a feasible half-line along which cx diverges to $-\infty$. If $\{i : 1 \leq i \leq m : \quad A_{i.}d < 0\} \neq \emptyset$, Then the maximum step length is γ^2 defined above. We take the actual step length to be $\gamma^2 - \epsilon$, where ϵ is a small positive tolerance, to make sure that the output point of this step is an IFS; leading to the output point $\underline{x} + (\gamma^2 - \epsilon)d$, with its objective value of $c(\underline{x} + (\gamma^2 - \epsilon)d)$.

Descent Steps in SM-7

From the center $\bar{x}$ obtained in this iteration, several descent steps are carried out, and the output point in each of these descent steps is stored along with the objective value at it in a set Γ (which is initially $\emptyset$) set up to collect all the output points of descent steps carried out in this iteration. Here are some of the possible descent steps, the most productive among them need to be determined through computational tests.

(a). From the center $\bar{x}$ take descent steps in the directions $-c^T$, average of $-c^i$ for $i \in T(\bar{x})$, avearge of vectors in the set $\{A_{i.}^T : i \in T(\bar{x})$ and satisfying $cA_{i.}^T < 0\} \cup \{-A_{i.}^T : i \in T(\bar{x})$ and satisfying $cA_{i.}^T > 0\}$, and (current center $\bar{x}$) $-$(center obtained in the previous iteration) [this direction called the direction of the path of centers being generated, is only used from iteration 2 onwards].

(b). Descent Steps D5.1: For each $i \in T(\bar{x})$ the point $(1-\epsilon)\bar{x}^i + \epsilon\bar{x}$, ϵ distance away from the touching point $\bar{x}^i$ on the line segment joining it to $\bar{x}$, is known as the NTP (Near Touching Point) corresponding to that index $i \in T(\bar{x})$. For each $i \in T(\bar{x})$, take a descent step from the NTP corresponding to it in the direction $-c^i$.

(c). Descent steps D5.7: For this step in this iteration define for each $i \in T(\bar{x})$, and for parameter $\alpha \in R^1$

$$x^i(\alpha) = \bar{\bar{x}}^i + \alpha(\bar{\bar{x}} - \bar{\bar{x}}^i) \tag{4}$$

For each $i \in T(\bar{x})$, let us, as in the Substep discussed in Section 2.1, denote the line segment joining $\bar{\bar{x}}$ and $\bar{\bar{x}}^i$ contained on the objective plane $H(\bar{\bar{x}})$ by $L_i = \{x^i(\alpha) : \alpha \in R^1\}$, in parametric representation with parameter α.

Since $x^i(\alpha = 1) = \bar{\bar{x}}$, an interior point of K, L_i passes through the interior of K, it must intersect K at two boundary points of K if $L_i \cap K$ is a line segment. To find those boundary points of $L_i \cap K$ we need to solve the following system of linear inequalities in the parameter α:

$$A_{p.}x^i(\alpha) - b_p = A_{p.}\bar{\bar{x}}^i - b_p + \alpha A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) \geq 0 \text{ for } p = 1 \text{ to } m.$$

This system can be solved using Subroutine 1. Let $\alpha_{i1} \leq \alpha \leq \alpha_{i2}$ be the values of the parameter α feasible to this system. There are two cases to consider here.

Case 1: Either $\alpha_{i1} = -\infty$, or $\alpha_{i2} = +\infty$: α_{i1} will be $-\infty$ [α_{i2} will be $+\infty$] if $A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) \leq 0$ [$A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) \geq 0$] for all $p = 1$ to m. In both instances consider the case in which all the inequalities hold as strict inequalities.

If $A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) < 0$ for all $p = 1$ to m, then from the definition of $\delta(x^i(\alpha))$, it can be verified that it diverges to $+\infty$ as $\alpha \to -\infty$. And then the point $x^i(\alpha) - \delta(x^i(\alpha))c^T/||c||$ is feasible for all $\alpha \to -\infty$ and the objective value at it diverges to $-\infty$. In fact if β is a small positive number $< \min\{-A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) : p = 1 \text{ to } m\}$, then the growth rate in $\delta(x^i(\alpha))$ as α decreases is going to be larger than β; and the half-line $\{\bar{\bar{x}}^i + [(\bar{\bar{x}} - \bar{\bar{x}}^i) + \beta(-c^T)]\alpha : \alpha \leq \alpha_{i2}\}$ is feasible and the objective value diverges to $-\infty$ along it.

If $\alpha_{i2} = +\infty$ and $A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) > 0$ for all $p = 1$ to m; using the same argument with the change of "α negative" by "α positive", it can be verified that the same conclusion holds. So, we terminate the algorithm with this conclusion in this case.

If there is a p between 1 to m satisfying $A_{p.}(\bar{\bar{x}} - \bar{\bar{x}}^i) = 0$, when $\alpha_{i1} = -\infty$ [$\alpha_{i2} = +\infty$], for some $\alpha < 0$ [$\alpha > 1$] carry out the descent steps in **(a), (b)** from $x^i(\alpha)$, store the output points in the set Γ. Terminate this iteration, and with the best point in Γ by objective value at this stage, as the initial IFS, go to the next iteration.

Case 2: Suppose $\alpha_{i1} < \alpha_{i2}$ are both finite. For simplicity let us denote $x^i(\alpha_{i1}), x^i(\alpha_{i2})$ by x^{i1}, x^{i2} respectively. So in this case $L_i \cap K$ is the line segment $[x^{i1}, x^{i2}]$ joining x^{i1}, x^{i2}. Define

$$x^{0i}(\alpha) = x^{i1} + \alpha(x^{i2} - x^{i1})$$

Therefore $L_i = \{x^{0i}(\alpha) : \alpha \in R^1\}$ and $L_i \cap K = \{x^{0i}(\alpha) : 0 \leq \alpha \leq 1\}$.

Let $I_2 = \{p : A_{p.}(-c^T) < 0\} = \emptyset$. So, $A_{p.}(-c^T) \geq 0$ for $1 \leq p \leq m$. For any $0 \leq \alpha \leq 1$ the descent step from $x^{0i}(\alpha) \in K$ in the direction $-c^T$ has step length $= +\infty$ in this case, since $A_{p.}(x^{0i}(\alpha) - b_p + \lambda A_{p.}(-c^T) \geq 0$ for all $\lambda \geq 0, \quad 1 \leq p \leq m$ in this case.

So for any $0 \leq \alpha \leq 1$, $\{x^{0i}(\alpha) + \lambda(-c^T) : \lambda \geq 0\}$ is a feasible half-line along which the objective function cx in (1) diverges to $-\infty$. So, we terminate the algorithm with this conclusion in this subcase.

Let I_2 be nonempty. In this case the line L_i and the direction $-c^T$ together determine a 2-dimensional half-space, a general point in which is

$$x^i(\alpha, \lambda) = x^{i1} + \alpha(x^{i2} - x^{i1}) + \lambda(-c^T)$$

where α is real and $\lambda \geq 0$. The intersection of this 2-dimensional half-space with K determines a 2-dimensional polyhedron. Te problem of minimizing cx in this 2-dimensional polyhedron is the 2-variable LP obtained by substituting $x = x^i(\alpha, \lambda)$ in (1). Now we will select one of these 2-variable LPs and solve it.

Since the symbol r was already used in Section 2.1 to denote something else, we will use the symbol $\tilde{r}$ here to denote the index $i \in T(\bar{x})$ corresponding to the maximum value for the number of inequalities in (1) satisfied as equations by either x^{i1} or x^{i2}, and in case of a tie the one with maximum value for $||x^{i1} - x^{i2}||$ among those tied (i.e., those which correspond to the maximum numer of inequalities in (1) satisfied as equations by either x^{i1} or x^{i2}).

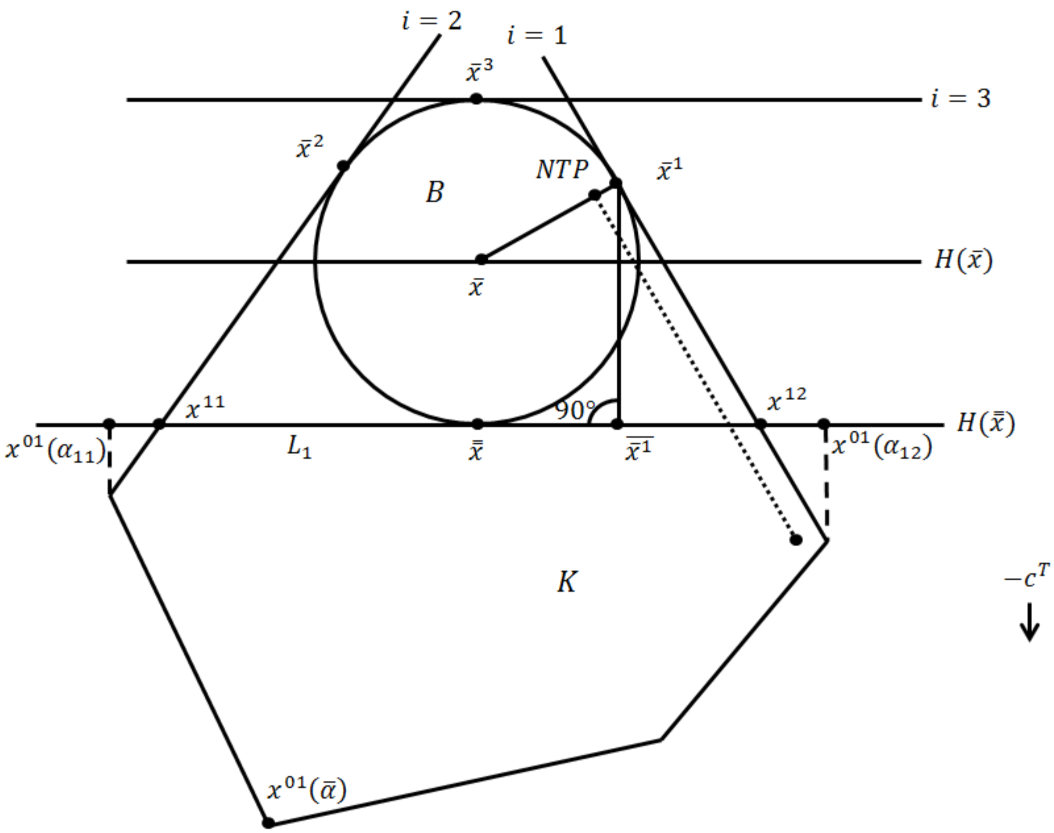
The 2-variable LP to solve is

$$\begin{array}{rll} & \min & c[x^{\tilde{r}1} + \lambda(-c^T)] \\ & \text{subject to} & A_{i.}[x^{\tilde{r}1} + \alpha(x^{\tilde{r}2} - x^{\tilde{r}1}) + \lambda(-c^T)] \geq b_i \qquad i = 1, ..., m \quad (6) \\ \text{and } \lambda \geq 0, \quad \alpha \quad real & & \end{array}$$

Let $(\bar{\alpha}^F, \bar{\lambda}^F)$ be the optimum solution of this 2-variable LP. The output point corresponding to it is $x^{\tilde{r}}(\bar{\alpha}^F, \bar{\lambda}^F)$, this will be a boundary point of K. So, we take the output point of this descent step as $x^{\tilde{r}}(\bar{\alpha}^F, \bar{\lambda}^F - \epsilon)$ if it is an IFS of K, otherwise we take the output point of this step as $x^{\tilde{r}}(\bar{\alpha}^F$ + or - ϵ, $\bar{\lambda}^F - \epsilon)$ whichever is an IFS, and store it in the set Γ along with its objective value.

FIGURE 3 NEEDS TO BE UPDATED. I BELIEVE THIS CONSISTS OF ELIMINATING POINTS MARKED ON OBJ. PLANE OUTSIDE FEASIBLE REGION.

Figure 3: The Ball B with center $\bar{x}$, the largest ball inside K with this center, has 3 touching facets numbered i = 1, 2, 3 with touching points $\bar{x}^1, \bar{x}^2, \bar{x}^3$ respectively. The dotted line beginning with the NTP corresponding to $i = 1$, is the descent step D5.1 from it, and the point at the other end of this line is the output point from this step. For descent steps D5.7, The objective plane $H(\bar{x})$ is moved parallel to itself in the direction $-c^T$ until it becomes a tangent



plane to B with its touching point $\bar{\bar{x}}$. Corresponding to $i = 1$, $\bar{\bar{x}}^1$ is the orthogonal projection of $\bar{x}^1$ on $H(\bar{\bar{x}})$, and the straight line joining $\bar{\bar{x}}$, $\bar{\bar{x}}^1$ is L_1. In this 2-dimensional figure L_1 is the same as $H(\bar{\bar{x}})$; but in higher dimensions $H(\bar{\bar{x}})$ will be a hyperplane and L_1 will be a straight line on it. x^{11}, x^{12} are the two boundary point of K where L_1 interesects K, they are the end points of $L_1 \cap K$. In this iteration for solving the original problem, the 2-variable LP (6) consists of minimizing cx on the portion of K just below (i.e., in the direction of $-c^T$) the line L_1, the optimum solution of it is the point $x^{01}(\bar{\alpha})$ in K. My thanks to Kayse Maass for drawing this figure.

When these line search steps are completed, this iteration is completed. The output point in this iteration is the point in the set Γ associated with the smallest objective value at this stage. Suppose it is $\hat{\hat{x}}$, with objective value $c\hat{\hat{x}}$. With $\hat{\hat{x}}$ as the initial IFS the algorithm now moves to the next iteration. The decrease in the objective value in this iteration that started with the IFS $\hat{x}$ is the difference in the objective values at the initial IFS and the final output point in this iteration, i.e., $c\hat{x}$ - $c\hat{\hat{x}}$. The algorithm is terminated in an iteration if the decrease in objective value attained in that iteration is $\leq \epsilon$, with the final output point in that iteration taken as the approximate optimum solution of the original LP.

3 Efficient Methods for Soving the 2-Variable LPs in the Centering Steps, and In Descent Steps 5.7

Here we describe methods for solving the 2-variable LPs in the centering steps (Section 2.1) and in the Descent Steps D5.7 (Section 2.2) during the algorithm for the case when K is bounded, assuming that Approach 1 is being used in both sections.

3.1 Solving the 2-Variable LPs (3) Under Approach 1 in the Centering Steps in Section 2.1

The 2-variable LP to be solved is (3), with $\alpha_1, \alpha_2, y^1, y^2$, $x_C(\alpha)$, defined in Section 2.1 and available already. Let Δ_1 denote the set of feasible solutions (α, δ) of (3) in the 2-dimensional space (α, δ) with α plotted on the horizontal axis and δ plotted on the vertical axis.

Each iteration in this method begins with an initial feasible solution (α, δ) of (3) on the boundary of Δ_1. We have the initial feasible solution for (3), $(\alpha, \delta) = (\alpha_1, 0)$ as $\delta(x_C(\alpha_1)) = 0$ since $x_C(\alpha_1)$ is a boundary point of K. Since $x_C(\alpha_2)$ is also a boundary point of K, we also have $\delta(x_C(\alpha_2)) = 0$, so $(\alpha_2, 0)$ is another boundary point of Δ_1 in the (α, δ)-space on the line $\delta = 0$.

Note that in the interval $\alpha_1 \leq \alpha \leq \alpha_2$ in the (α, δ)-space, α corresponds to the point $x_C(\alpha)$ in K. In this interval, only α_1, α_2 correspond to boundary points of K, all other values of α in this interval correspond to IFSs $x_C(\alpha)$ of K on the line segment joining $x_C(\alpha_1)$, $x_C(\alpha_2)$.

The first iteration in this method begins with the initial feasible solution $(\alpha_1, 0)$ of (3) on the boundary of Δ_1, and performs a (horizontal move + vertical move) twice and finally a diagonal move in Δ_1; and the current value of δ increases in each iteration. For any given value

α satisfying $\alpha_1 \leq \alpha \leq \alpha_2$, define $\delta[\alpha]$ = maximum value of δ such that (α, δ) is feasible to (3). $\delta[\alpha]$ = Minimum $\{[A_{i.}x_C(\alpha) - b_i]/||A_{i.}||$: $i = 1$ to $m\}$, and the point $(\alpha, \delta[\alpha])$ is contained on the boundary of Δ_1.

From well known results in LP theory we know that $\delta[\alpha]$ is piece-wise linear concave over the interval $\alpha_1 \leq \alpha \leq \alpha_2$, and we have seen already that both $\delta[\alpha_1], \delta[\alpha_2]$ are 0.

We will now discuss a general iteration, the uth, in this method beginning with the initial feasible solution $(\alpha_{u0}, \delta_{u0})$ on the boundary of Δ_1. In this Section the index u denotes the number of the iteration in this method for solving the 2-variable LP (3).

The First Horizontal Move: Fixing $\delta = \delta_{u0}$, the system of constraints in (3) becomes:

$$\delta_{u0}||A_{i.}|| - \alpha[A_{i.}(y^2 - y^1)] \leq A_{i.}y^1 - b_i, i = 1...m \tag{7}$$

(5)

Find the interval of values of α feasible to this system (7) using Subroutine 1. Since $(\alpha_{u0}, \delta_{u0})$ is a boundary point of Δ_1, α_{u0} will be one of the bounds on this interval. If α_{u0} is the unique solution of (7), then $(\alpha_{u0}, \delta_{u0})$ is an optimum solution of (3), terminate. Otherwise let α_{u1} be the other boundary point of this interval. This is the interval of values of α in the (α, δ)-space, in Δ_1, on the horizontal line $\delta = \delta_{u0}$. All values of α in this interval satisfy $\delta(x_C(\alpha)) \geq \delta_{u0}$. $\alpha^C_{u1} = (\alpha_{u0} + \alpha_{u1})/2$, is the mid-point of this interval. The point $(\alpha^C_{u1}, \delta_{u0})$ is called the **center** of Δ_1 on the horizontal line $\delta = \delta_{u0}$ in the (α, δ)-space.

The First Vertical Move: In this move, fix $\alpha = \alpha^C_{u1}$ in (3). Then the maximum value of δ subject to $(\alpha^C_{u1}, \delta) \in \Delta_1$ is

δ_{u1} = Minimum $\{[A_{i.}(y^1 + \alpha^C_{u1}(y^2 - y^1)) - b_i]/||A_{i.}|| : i = 1$ to $m\}$.

and the point in Δ_1 achieving this value is the boundary point $(\alpha^C_{u1}, \delta_{u1})$.

The Second (Horizontal + Vertical) Moves: Starting with $(\alpha^C_{u1}, \delta_{u1})$ for (3) apply the second horizontal move as described under the first horizontal move. When δ is fixed at δ_{u1}, if

α^C_{u1} is the unique value of α such that (α, δ_{u1}) is feasible to (3); then $(\alpha^C_{u1}, \delta_{u1})$ is the optimum solution for (3), terminate.

Otherwise let α_{u2}, α_{u3} be the boundary points of the interval of values of α in Δ_1 on the horizontal line $\delta = \delta_{u1}$ in the (α, δ)-space. One of these boundary values will be will be equal to α^C_{u1}. Let $\alpha^C_{u2} = (\alpha_{u2} + \alpha_{u3})/2$. Then $(\alpha^C_{u2}, \delta_{u1})$ is called the **center** of Δ_1 on the horizontal line $\delta = \delta_{u1}$ in the (α, δ)-space.

Now carry out the second vertical move keeping α fixed at α^C_{u2} to find the maximum value δ_{u2} of δ in Δ_1 , as in the first vertical move, it is attained at $(\alpha^C_{u2}, \delta_{u2})$ on the boundary of Δ_1.

The Diagonal Move: This move involves finding the maximum value of δ in Δ_1 for points along the line joining the two centers of Δ_1 obtained in the two horizontal moves in this iteration, $(\alpha^C_{u1}, \delta_{u0}), (\alpha^C_{u2}, \delta_{u1})$, with $\delta_{u1} > \delta_{u0}$. The equation for the line joining $(\alpha^C_{u1}, \delta_{u0}), (\alpha^C_{u2}, \delta_{u1})$ in the (α, δ) space is:

$\alpha = \alpha^C_{r1} + (\delta - \delta_{r0})s$, where $s = (\alpha^C_{u2} - \alpha^C_{u1})/(\delta_{u1} - \delta_{u0})$.

Substituting this expression for α in (3) leads to the following syatem of linear inequalities in δ.

$\delta ||A_{i.}|| \leq ((\alpha^C_{u1} + (\delta - \delta_{u0})s)A_{i.}(y^2 - y^1) + A_{i.}y^1 - b_i$, $i = 1$ to m

Find the maximum value of δ, δ_{u3}, feasible to this system of inequalities in δ using Subroutine 1. So, δ_{u3} is the maximum value of δ in Δ_1 on the straight line joining $(\alpha^C_{u1}, \delta_{u0}), (\alpha^C_{u2}, \delta_{u1})$. Let $\alpha_{u3} = \alpha^C_{u1} + (\delta_{u3} - \delta_{u0})s$, $(\alpha_{u3}, \delta_{u3})$ is the point of intersection of this straight line with Δ_1.

Let δ_{u4} = maximum $\{\delta_{u2}, \delta_{u3}\}$, and let $(\alpha_{u4}, \delta_{u4})$ be the point among $(\alpha^C_{u2}, \delta_{u2}), (\alpha_{u3}, \delta_{u3})$ associated with it; i.e., $\alpha_{u4} = \alpha^C_{u2}$ if $\delta_{u4} = \delta_{u2}$, or $\alpha_{u4} = \alpha_{u3}$ if $\delta_{u4} = \delta_{u3}$. Then $(\alpha_{u4}, \delta_{u4})$ is the output of this iteration in this method, with this point go to the next iteration.

When the improvement in the value of δ becomes small in an iteration, terminate the method with the output in that iteration as an approximate optimum for (3).

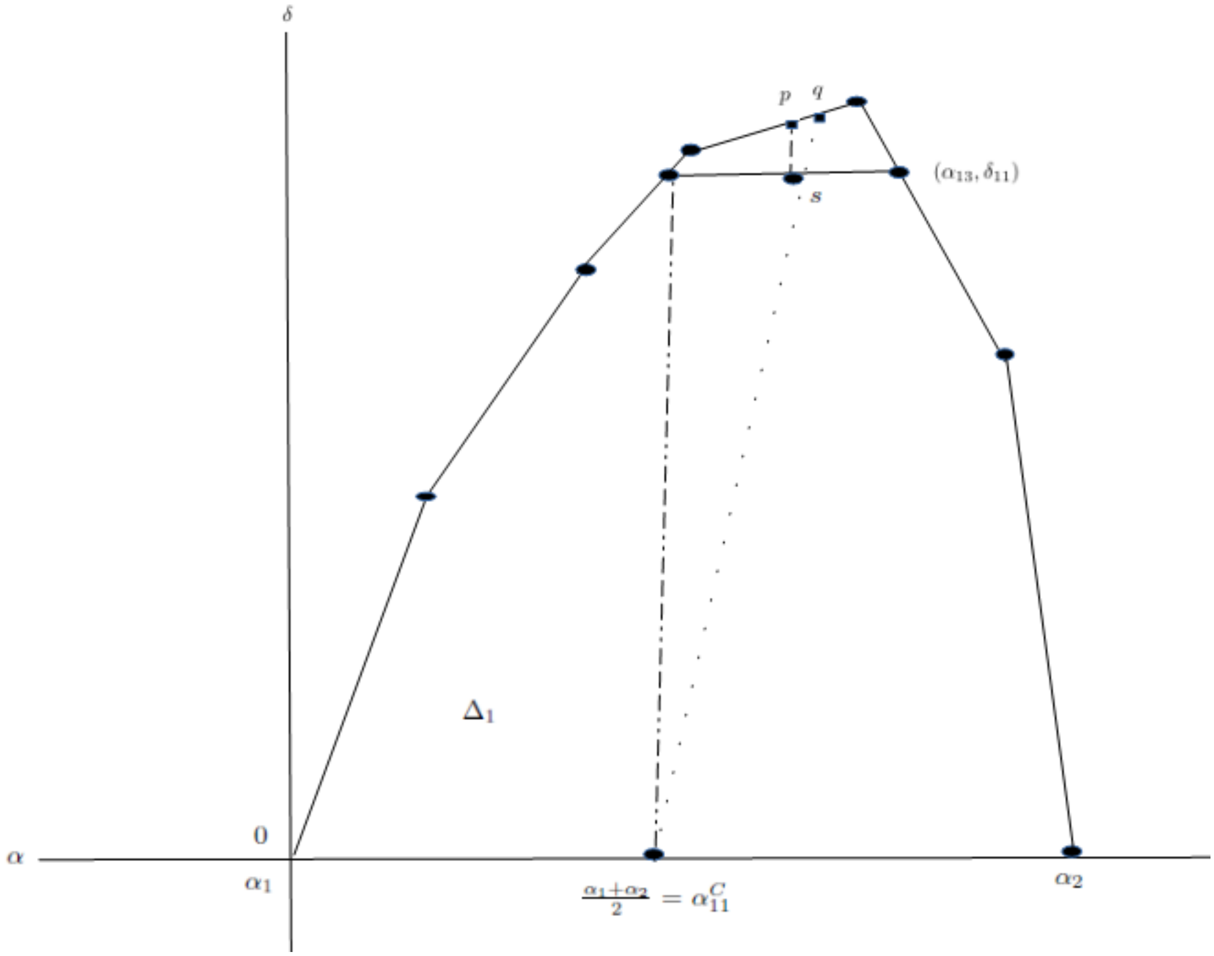


Figure 4: We illstrate iteration $u = 1$ in this algorithm. Δ_1 is the feasible region of (3) in the (α, δ)-space; we plot $\delta(x_C(\alpha))$ = maximum value of δ in (3) for given value of α. Vertical moves in this iteration are marked with dashed lines; and the diagonal move with a dotted line. $\alpha_{11}^C, \alpha_{12}^C$ are the values of α at the centers in the two horizontal moves in this iteration. In this iteration, we have $\delta_{13} > \delta_{12}$, so the value of δ goes up from 0 to $\delta_4 = \text{maximum}\{\delta_{12}, \delta_{13}\}$. The points p, q, s in the figure are $(\alpha_{12}, \delta_{12}), (\alpha_{13}, \delta_{13}), (\alpha_{12}^C, \delta_{11})$ respectively.

3.2 Solving the 2-Variable LPs (6) Under Approach 1 in the Descent Steps D5.7 in Section 2.2

We will solve the equivalent problem:

$$\begin{aligned} &\max \quad \lambda \\ &\text{subject to} \quad \lambda(-A_{i.}c^T) + \alpha(A_{i.}(x^{\tilde{r}2} - x^{\tilde{r}1})) \geq b_i - A_{i.}x^{\tilde{r}1}, \quad i = 1, ..., m \qquad (8) \\ &\text{and } \lambda \geq 0, \quad \alpha \quad real \end{aligned}$$

Here $\tilde{r}$ is the index defined earlier. For any $\lambda_1 > 0$, if there is no feasible solution (λ_1, α) for (8), then the maximum objective value in (8) must be $< \lambda_1$.

For $\lambda \leq$ optimum objective value in (8), the interval of feasibility for α on the horizontal line corresponding to λ in the (λ, α)-space can be found using Subroutine 1. For α in this range, the maximum value of λ in (8) on the vertical line corresponding to α in this space is

$\lambda(\alpha) = \text{minimum}\{(b_p - A_{p.}(x^{\tilde{r}1} + \alpha(x^{\tilde{r}2} - x^{\tilde{r}1}))/A_{p.}(-c^T)$: over $1 \leq p \leq m$ satisfying $A_{p.}(-c^T) < 0\}$.

We will now describe the application of the method described in Section 3.1, to solve the 2-variable LP (8). The first iteration begins with the line $\lambda = 0$, on this line the interval of feasibility for α is $0 \leq \alpha \leq 1$.

We will now describe the general iteration in this method beginning on the line $\lambda = \lambda_0$.

The First Horizontal Move: Fix $\lambda = \lambda_0$ in (8), and find the interval of values of α feasible to it using Subroutine 1. If $\alpha = \alpha_1$ is the unique solution of this system, then (λ_0, α_1) is the optimum solution of (8), terminate. Otherwise, let α_1 be the midpoint of this interval. Then (λ_0, α_1) is called the **center** of the set of feasible solutions of (8) on the horizontal line $\lambda = \lambda_0$ in the (λ, α)-space.

The First Vertical Move: Compute $\lambda(\alpha_1) = \lambda_1$ say.

The Second Horizontal Move: Fix $\lambda = \lambda_1$ in (8), and find the interval of values of α feasible to it using Subroutine 1. If this solution is α_2 which is unique, then (λ_1, α_2) is an optimum solution of (8), terminate. Otherwise, let α_2 be the midpoint of this interval. Then (λ_1, α_2) is called the **center** of the set of feasible solutions of (8) on the horizontal line $\lambda = \lambda_1$ in the (λ, α)-space.

The Second Vertical Move: Compute $\lambda(\alpha_2) = \lambda_2$ say. The point corresponding to it feasible to (8) is (λ_2, α_2).

The Diagonal Move: This move finds the maximum value of λ in (8) for points along the straight line joining the two centers (λ_0, α_1), (λ_1, α_2). The equation describing this straight line is:

$$\lambda = \lambda_0 + (\alpha - \alpha_1)((\lambda_1 - \lambda_0)/(\alpha_2 - \alpha_1)).$$

Substitute this expression for λ in (8), and find the maximum value of α feasible to the resulting system, α_3 say, using Subroutine 1. Let $\lambda_3 = \lambda_0 + (\alpha_3 - \alpha_1)((\lambda_1 - \lambda_0)/(\alpha_2 - \alpha_1))$.

If $\lambda_3 > \lambda_2$, then (λ_3, α_3) is the output point in this iteration. On the other hand if $\lambda_2 > \lambda_3$, then (λ_2, α_2) is the output point in this iteration. Go to the next iteration beginning with the horizontal line through the output point.

Terminate the method with the output point in an iteration as an approximate optimum for (8) when the improvement in the value of λ in the iteration becomes small. If $(\bar{\lambda}, \bar{\alpha})$ is the output point in this method, then $x^{\tilde{r}1} + \bar{\alpha}(x^{\tilde{r}2} - x^{\tilde{r}1}) + (\bar{\lambda} - \epsilon)(-c^T)$ is the output point of this method for (6) in the space of the original variables x.

4 How to transform a general LP into the form required to apply SM-7

Consider the general LP in variables $y \in R^n$: minimize hy subject to $Ay = b$, $Dy \geq d$ with equality and inequality constraints. Any bounds on the variables are included among inequality constraints. Let A be of order $m \times n$, and D of order $p \times n$. Let e_p be the column vectors in R^p with all entries $= 1$. Let y_0 be an artificial variable.

First make $b \leq 0$, by multiplying both sides of the i-th equality constraint by -1 if the original b_i is > 0. After this, let $M_1 = \{i : 1 \leq i \leq m$, such that $b_i < 0\}$, $M_2 = \{i : 1 \leq i \leq m$, such that $b_i = 0\}$.

Consider the Phase I problem: minimize $hy+\alpha y_0+\beta_2 \sum_{i \in M_2}(A_{i.}y+y_0-b_i) + \beta_1 \sum_{i \in M_1}(A_{i.}y-b_i)$ subject to $A_{i.}y + y_0 \geq b_i$ for $i \in M_2$; $A_{i.}y \geq b_i$ for $i \in M_1$, $Dy + e_p y_0 \geq d$, $y_0 \geq 0$, where α, β_1, β_2 are large positive penalty coefficients.

For this Phase I problem, $(y = 0, y_0 = y_0^0)$ where $y_0^0 >$ maximum$\{0, \ d_j : j = 1, ..., p\}$ is an interior feasible solution. With this as the initial IFS, this Phase I problem is in the form required to apply SM-7 to solve it. Since α, β are large numbers, if the original problem has an optimum solution, the Phase I problem will output an optimum solution in which y_0 will be 0, and the y satisfies the first set of constraints as equations.

Other Phase I formulations of the original problem may also be considered.

5 Computational Results

6 References

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