Compliant Reference

Maxime Tournier

October 17, 2013

Abstract

Quick notes about the mathematical framework used in Compliant. We review the time-stepping scheme, kinematic constraint handling, potential forces as compliant constraints, and KKT systems.

TODO

- finish sections :-)

- references

- wikipedia links

- introduction ?

1   Lagrangian Dynamics

Given an n-dimensional state space Q, we consider a Lagrangian $\mathcal{L}$ defined as usual:

$$\mathcal{L}(q, v) = \frac{1}{2} v^T M v - V(q) \tag{1}$$

where $V(q)$ is the potential energy of the form:

$$V(q) = \frac{1}{2} \left\| g(q) \right\|_N^2 \tag{2}$$

for a matrix norm N in the suitable deformation space $Im(g)$. To be fully general, we include a Rayleigh dissipation function $\frac{1}{2} v^T D v$, where D is PSD. The Euler-Lagrange equations of motions are then:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial v} = \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial D}{\partial v} \tag{3}$$

For simplicity, we restrict to the case where M is constant, which yields:

$$M\dot{v} = -\nabla_q V - Dv \tag{4}$$

1

2 Time Stepping

We now discretize time using a fixed time step h. We use forward differences for $\dot{v}$ as follows:

$$h\,\dot{v}_{k+1} \approx v_{k+1} - v_k \tag{5}$$

Equation (4) becomes:

$$Mv_{k+1} = Mv_k - h\left(\nabla_q V_{k+1} + Dv_{k+1}\right) \tag{6}$$

Or, calling $p_k = Mv_k$ the momentum of the system and $f_k = -\nabla_q V_k$ the potential forces:

$$(M + h\,D)\,v_{k+1} = p_k + hf_{k+1} \tag{7}$$

In order to control how implicit our integrator is on forces, we introduced a blending parameter $\alpha \in [0,1]$ as follows:

$$(M + h\alpha D)\,v_{k+1} = p_k - h(1-\alpha)Dv_k + h\left((1-\alpha)f_k + \alpha f_{k+1}\right) \tag{8}$$

We linearize the above using the Hessian of the potential energy (*i.e.* stiffness matrix):

$$\nabla_q V_{k+1} \approx \nabla_q V_k + \nabla_q^2 V_k\,(q_{k+1} - q_k) \tag{9}$$

Finally, we use the following position time-stepping scheme:

$$q_{k+1} = q_k + h\left((1-\beta)v_k + \beta v_{k+1}\right) \tag{10}$$

where $\beta \in [0,1]$ is again a blending parameter. Letting the stiffness matrix be $K = \nabla_q^2 V_k$ (watch out) and $d_k = -Dv_k$ be the damping forces, we put everything together as:

$$\left(M + h\alpha D + h^2\alpha\beta K\right)v_{k+1} = p_k + h\left(f_k + (1-\alpha)d_k\right) - \alpha(1-\beta)h^2 Kv_k \tag{11}$$

(and yes, this is very ugly). Fortunately, in a sane world everyone would use the much friendlier, fully implicit scheme $\alpha = \beta = 1$ as follows:

$$\left(M + hD + h^2 K\right)v_{k+1} = p_k + hf_k \tag{12}$$

3 Response Matrix, Net Momentum

From now on, we will refer to the *response matrix* defined as follows:

$$W = \left(M + h\alpha D + h^2\alpha\beta K\right) \tag{13}$$

We will also refer to the *net momentum* of the system at time step k:

$$c_k = p_k + h\left(f_k + (1-\alpha)d_k\right) - \alpha(1-\beta)h^2 Kv_k \tag{14}$$

The time-stepping scheme (11) thus involves solving the following linear system:

$$Wv_{k+1} = c_k \tag{15}$$

The numerical solvers for time-stepping will be described in section 12.

4 Constraints

We now introduce holonomic constraints of the form:

$$g(q) = 0 \tag{16}$$

where g again maps kinematic DOFs to a suitable deformation space $Im(g)$. Such constraints are satisfied by introducing constraint forces $J^T \lambda$, where $J = dg$ is the constraint Jacobian matrix, and λ are the *Lagrange multipliers*:

$$M\dot{v} = -\nabla_q V - Dv + J^T \lambda \tag{17}$$

$$g(q) = 0 \tag{18}$$

Again, we discretize time and $\dot{v}$ as follows:

$$Mv_{k+1} = p_k + h\left(f_{k+1} + d_{k+1} + J^T_{k+1}\lambda_{k+1}\right) \tag{19}$$

$$g(q_{k+1}) = 0 \tag{20}$$

And again:

$$g(q_{k+1}) = g_k + hJ_k\left((1-\beta)v_k + \beta v_{k+1}\right) \tag{21}$$

At this point we become lazy and approximate constraints as *affine* functions, meaning that $J_{k+1} \approx J_k$, otherwise computing the constraint Hessian d^2g would be too costly, and would result in a non-linear system to solve anyways (*i.e.* with terms involving $v^T_{k+1}d^2g_k\lambda_{k+1}$). As we will see later, such approximation is consistent with our treatment of potential forces as compliant constraints. The time-discrete system is then:

$$Mv_{k+1} = p_k + h\left(f_{k+1} + d_{k+1} + J^T_k\lambda_{k+1}\right) \tag{22}$$

$$J_k v_{k+1} = -\frac{g_k}{\beta h} - \frac{1-\beta}{\beta}J_k v_k \tag{23}$$

At this point, we *could* introduce an implicit blending between λ_k and λ_{k+1}, but this would result in unneeded complication as λ_{k+1} would need to be computed anyways. The blended force would then be:

$$J^T_k\left((1-\alpha)\lambda_k + \alpha\lambda_{k+1}\right)$$

which simply offsets λ_{k+1} with respect to λ_k. We will thus happily ignore such blending. The final linear system to solve (for v_{k+1} and λ_{k+1}) becomes:

$$Wv_{k+1} - J^T_k\lambda_{k+1} = c_k \tag{24}$$

$$J_k v_{k+1} = -b_k \tag{25}$$

where $b_k = \frac{g_k}{\beta h} + \frac{1-\beta}{\beta}J_k v_k$. Before leaving, note the sneaky accounting of h inside λ_{k+1}.

5   Compliance

We will now establish a connection between elastic forces and constraint forces through a compliance matrix. Remember that we defined potential energy as:

$$V(q) = \frac{1}{2} \left\| g(q) \right\|_N^2 \tag{26}$$

where g maps kinematic DOFs to an appropriate deformation space $Im(g)$. This means the potential forces are:

$$f = -\nabla_q V = J^T N g(q) \tag{27}$$

Now, the Hessian or stiffness matrix is :

$$K = \nabla_q^2 V = dJ^T N g(q) + J^T N J \tag{28}$$

As in section 4, we are too lazy for computing second derivatives and approximate deformation mappings as affine maps, meaning that $dJ \approx 0$. We are left with the following response matrix (13):

$$W = \left(M + h\alpha D + h^2 \alpha\beta J_k^T N J_k \right) \tag{29}$$

The net momentum (14) becomes:

$$c_k = p_k + h(1 - \alpha)d_k - hJ_k^T N \left(g_k + h\alpha(1 - \beta)J_k v_k \right) \tag{30}$$

If we write the potential force as $J_k^T \lambda_{k+1}$, we have:

$$\lambda_{k+1} = -N \left(h^2 \alpha\beta J_k v_{k+1} + h g_k + h^2 \alpha(1 - \beta) J_k v_k \right) \tag{31}$$

It turns out that this system can be rewritten as a *larger* system:

$$\begin{pmatrix} M + h\alpha D & -J_k \\ J_k & \frac{N^{-1}}{h^2\alpha\beta} \end{pmatrix} \begin{pmatrix} v_{k+1} \\ \lambda_{k+1} \end{pmatrix} = \begin{pmatrix} p_k + h(1 - \alpha)d_k \\ -\frac{g_k}{h\alpha\beta} - \frac{(1-\beta)}{\beta} J_k v_k \end{pmatrix} \tag{32}$$

(pffew !) One can notice that this system is *almost* exactly the one we obtained with kinematic constraints in (24), with $\alpha = 1$, with the exception of the N^{-1} term in the $(2, 2)$ block. We call matrix $C = N^{-1}$ the *compliance* matrix. We see that kinematic constraints simply correspond to a zero compliance matrix, *i.e.* infinite stiffness, as one would intuitively expect. It can be checked that taking α into account for constraint forces produces the same system with $C = 0$. (TODO) We now recall the main results for the unified elastic/constraint treatment:

Linear System

$$\begin{pmatrix} W & -J \\ -J & -\frac{C}{h^2\alpha\beta} \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} c_k \\ b_k \end{pmatrix} \tag{33}$$

4

Response Matrix

$$W = M + h\alpha D \tag{34}$$

Net Momentum

$$c_k = p_k + h(1 - \alpha)d_k \tag{35}$$

Constraint Value

$$b_k = \frac{g_k}{h\alpha\beta} + \frac{(1 - \beta)}{\beta}J_k v_k \tag{36}$$

6 KKT Systems, Compliance and Relaxation

At this point, it is probably a good idea to introduce a few notions on saddle-point (or KKT) systems. Such systems are (for now, linear) systems of the form:

$$\begin{pmatrix} Q & -A^T \\ -A & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix} \tag{37}$$

KKT systems typically arise from the Karush-Kuhn-Tucker (hence the name) optimality conditions of constrained optimization problems. For instance, the KKT system (45) corresponds to the following equality-constrained Quadratic Program (QP):

$$\underset{Ax+b=0}{\mathrm{argmin}} \quad \frac{1}{2}x^T Q x - c^T x \tag{38}$$

The KKT system summarizes the optimality conditions for the following unconstrained function (also called the *Lagrangian*, this time in the context of optimization):

$$\mathcal{L}(x, \lambda) \quad = \quad \frac{1}{2}x^T Q x - c^T x \quad - \quad (Ax + b)^T \lambda \tag{39}$$

and whose critical points (in fact, saddle-points) solve the constrained problem (38). As one can see, the constrained integrator (24) is an example of such KKT systems:

$$\begin{pmatrix} W & -J \\ -J & 0 \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} c_k \\ b_k \end{pmatrix} \tag{40}$$

As symmetric *indefinite* systems, they tend to be more difficult to solve than positive definite ones: CG and Cholesky LDL^T factorizations may breakdown on such systems. However, when the $(1, 1)$ block (here, matrix Q) is invertible, one can use *pivoting* to obtain the following smaller system:

$$x = Q^{-1}\left(c - A^T \lambda\right) \tag{41}$$

$$-Ax = AQ^{-1}A^T \lambda - AQ^{-1}c = b \tag{42}$$

$$AQ^{-1}A^T \lambda = b - AQ^{-1}c \tag{43}$$

This smaller system (43) is known as the *Schur* complement of the KKT system. It is always SPD as long as A^T is full column-rank, but requires to invert Q which might be costly in practice. The Schur complement system also corresponds to an (unconstrained) optimization problem:

$$\underset{\lambda}{\text{argmin}} \quad \frac{1}{2}\lambda^T S\lambda - s^T\lambda \tag{44}$$

where $S = AQ^{-1}A^T$ is the Schur complement, and $s = b + AQ^{-1}c$. The minimized quantity has a physical interpretation: typically, holonomic constraints minimize the total kinetic energy. If we now consider systems with compliance, such as:

$$\begin{pmatrix} Q & -A^T \\ -A & -D \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix} \tag{45}$$

we see that they correspond to the following Lagrangian:

$$\mathcal{L}(x, \lambda) \quad = \quad \frac{1}{2}x^T Qx - c^T x \quad - \quad (Ax + b)^T \lambda \quad - \quad \frac{1}{2}\lambda^T D\lambda \tag{46}$$

The Schur complement becomes $S + D$ and corresponds to:

$$\underset{\lambda}{\text{argmin}} \quad \frac{1}{2}\lambda^T S\lambda - s^T\lambda \quad + \quad \frac{1}{2}\lambda^T D\lambda \tag{47}$$

We see that D acts as a form of numerical damping on the resolution of constraints, by biasing the solution of the constrained system towards zero. It is a form of *constraint relaxation*. Under their most general form, KKT systems include unilateral and complementarity constraints. We will develop these in more details in section 9.

7 KKT Systems, Optimization and Relaxation

We have seen that KKT systems correspond to Quadratic Programs:

$$\begin{pmatrix} Q & -A^T \\ -A & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix} \tag{48}$$

corresponds to the following convex QP:

$$\underset{Ax=b}{\text{argmin}} \quad \frac{1}{2}x^T Qx - c^T x \tag{49}$$

The compliant KKT system obtained in section 5 is of the form:

$$\begin{pmatrix} Q & -A^T \\ -A & -D \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix} \tag{50}$$

where both Q and C are PSD.

TODO: effect of C on the schur complement, lagrangian relaxation

8 Geometric Stiffness

TODO: what is left when we approximate g as an affine function, how to recover it

9 Unilateral Constraints

TODO: general KKT systems

10 Stabilization

TODO

11 Restitution

TODO: velocity constraints for contact with restitution (rigid), mention Generalized Reflections

12 Numerical Solvers

TODO: CG, Cholesky, Minres, GS, PGS, Sequential Impulses