Arc-length Method

弧长法公式推导

系统方程组:

$$
\begin{aligned}
\boldsymbol{R}(\boldsymbol{d},\lambda) = \lambda \boldsymbol{F}^{ext} - \boldsymbol{F}^{int}(\boldsymbol{d}) = \mathbf{0} \\
f(\boldsymbol{d},\lambda) = (\lambda-\lambda_0)^2 + (\boldsymbol{d}-\boldsymbol{d}_0)^T(\boldsymbol{d}-\boldsymbol{d}_0) - \delta a^2 = 0
\end{aligned} \tag{1}
$$

规定: $(\cdot)_n^i$

n : 表示第n个荷载步

i: 表示第n个荷载步的第i次修正

给定第n个荷载步的点 $(\boldsymbol{d}_n, \lambda_n)$

现需要通过牛顿迭代求出在弧长约束面下的下一个节点 $(\boldsymbol{d}_{n+1}, \lambda_{n+1})$

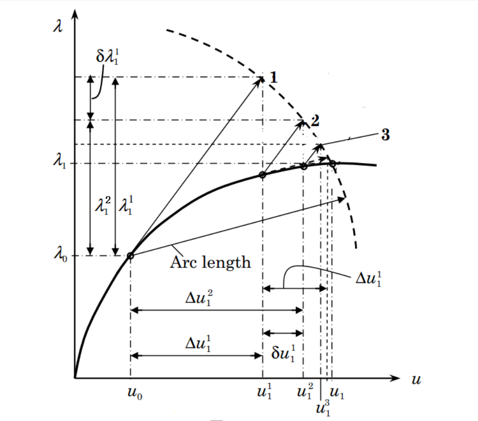

图 1

在迭代过程中求出的增量表示为 $(\Delta\boldsymbol{d}^{(i)}, \Delta\lambda^{(i)})$

在迭代过程中求出的新的节点表示为 $(\boldsymbol{d}_{n+1}^{(i)}, \lambda_{n+1}^{(i)})$

迭代过程如下:

$$
\begin{aligned}
\boldsymbol{d}_{n+1} \leftarrow \boldsymbol{d}_{n+1}^{(i)} + \Delta\boldsymbol{d}^{(i)} \\
\lambda_{n+1} \leftarrow \lambda_{n+1}^{(i)} + \Delta\lambda^{(i)}
\end{aligned} \tag{2}
$$

我们需要将系统方程组分别在 $\boldsymbol{d}_{n+1}^{(i)}$ 和 $\lambda_{n+1}^{(i)}$ 进行线性化（一阶泰勒展开）：

$$
\begin{aligned}
\mathbf{0} \equiv \boldsymbol{R}(\boldsymbol{d},\lambda) &\approx \boldsymbol{R}(\boldsymbol{d}_{n+1}^{(i)},\lambda_{n+1}^{(i)}) + \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{d}}\Big|_{(\boldsymbol{d}_{n+1}^{(i)},\lambda_{n+1}^{(i)})}\Delta \boldsymbol{d}^{(i)} + \frac{\partial \boldsymbol{R}}{\partial \lambda}\Big|_{(\boldsymbol{d}_{n+1}^{(i)},\lambda_{n+1}^{(i)})}\Delta\lambda^{(i)} \\
&= \lambda_{n+1}^{(i)}\boldsymbol{F}^{ext} - \boldsymbol{F}^{int}(\boldsymbol{d}_{n+1}^{(i)}) - \boldsymbol{K}(\boldsymbol{d}_{n+1}^{(i)})\Delta\boldsymbol{d}^{(i)} + \boldsymbol{F}^{ext}\Delta\lambda^{(i)} \\
&= \left(\lambda_{n+1}^{(i)} + \Delta\lambda^{(i)}\right)\boldsymbol{F}^{ext} - \left(\boldsymbol{F}^{int}(\boldsymbol{d}_{n+1}^{(i)}) + \boldsymbol{K}(\boldsymbol{d}_{n+1}^{(i)})\Delta\boldsymbol{d}^{(i)}\right)
\end{aligned}
$$

其中:

$$
\boldsymbol{K} = \frac{\partial \boldsymbol{F}^{int}}{\partial \boldsymbol{d}} \tag{3}
$$

以及:

$$
\begin{aligned}
\mathbf{0} \equiv f(\boldsymbol{d},\lambda) &\approx f(\boldsymbol{d}_{n+1}^{(i)},\lambda_{n+1}^{(i)}) + \frac{\partial f}{\partial \boldsymbol{d}}\Big|_{(\boldsymbol{d}_{n+1}^{(i)},\lambda_{n+1}^{(i)})}\Delta \boldsymbol{d}^{(i)} + \frac{\partial f}{\partial \lambda}\Big|_{(\boldsymbol{d}_{n+1}^{(i)},\lambda_{n+1}^{(i)})}\Delta\lambda^{(i)} \\
&= \left[(\lambda_{n+1}^{(i)} - \lambda_n)^2 + (\boldsymbol{d}_{n+1}^{(i)} - \boldsymbol{d}_n)^T(\boldsymbol{d}_{n+1}^{(i)} - \boldsymbol{d}_n) - \delta a^2\right] + 2(\boldsymbol{d}_{n+1}^{(i)} - \boldsymbol{d}_n)^T\Delta\boldsymbol{d}^{(i)} \\
&+ 2(\lambda_{n+1}^{(i)} - \lambda_n)\Delta\lambda^{(i)} \\
&= f(\boldsymbol{d}_{n+1}^{(i)},\lambda_{n+1}^{(i)}) + 2(\boldsymbol{d}_{n+1}^{(i)} - \boldsymbol{d}_n)^T\Delta\boldsymbol{d}^{(i)} + 2(\lambda_{n+1}^{(i)} - \lambda_n)\Delta\lambda^{(i)}
\end{aligned}
$$

其中:

$$
f(\boldsymbol{d}_{n+1}^{(i)},\lambda_{n+1}^{(i)}) = (\lambda_{n+1}^{(i)} - \lambda_n)^2 + (\boldsymbol{d}_{n+1}^{(i)} - \boldsymbol{d}_n)^T(\boldsymbol{d}_{n+1}^{(i)} - \boldsymbol{d}_n) - \delta a^2 \tag{4}
$$

整理可得:

$$
\begin{aligned}
\boldsymbol{K}(\boldsymbol{d}_{n+1}^{(i)})\Delta\boldsymbol{d}^{(i)} - \Delta\lambda^{(i)}\boldsymbol{F}^{ext} &= \lambda_{n+1}^{(i)}\boldsymbol{F}^{ext} - \boldsymbol{F}^{int}(\boldsymbol{d}_{n+1}^{(i)}) \\
2(\boldsymbol{d}_{n+1}^{(i)} - \boldsymbol{d}_n)^T\Delta\boldsymbol{d}^{(i)} + 2(\lambda_{n+1}^{(i)} - \lambda_n)\Delta\lambda^{(i)} &= -f(\boldsymbol{d}_{n+1}^{(i)},\lambda_{n+1}^{(i)})
\end{aligned}
$$

矩阵形式

$$
\begin{bmatrix} \boldsymbol{K}(\boldsymbol{d}_{n+1}^{(i)}) & -\boldsymbol{F}^{ext} \\ 2(\boldsymbol{d}_{n+1}^{(i)} - \boldsymbol{d}_n)^T & 2(\lambda_{n+1}^{(i)} - \lambda_n) \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{d}^{(i)} \\ \Delta\lambda^{(i)} \end{bmatrix} = \begin{bmatrix} \lambda_{n+1}^{(i)}\boldsymbol{F}^{ext} - \boldsymbol{F}^{int}(\boldsymbol{d}_{n+1}^{(i)}) \\ -f(\boldsymbol{d}_{n+1}^{(i)},\lambda_{n+1}^{(i)}) \end{bmatrix} \tag{5}
$$

求出迭代公式:

$$
\begin{aligned}
\boldsymbol{K}(\boldsymbol{d}_{n+1}^{(i)})\Delta\boldsymbol{d}^{(i)} &= \Delta\lambda^{(i)}\boldsymbol{F}^{ext} + \lambda_{n+1}^{(i)}\boldsymbol{F}^{ext} - \boldsymbol{F}^{int}(\boldsymbol{d}_{n+1}^{(i)}) \\
&= \Delta\lambda^{(i)}\boldsymbol{F}^{ext} + \boldsymbol{R}(\boldsymbol{d}_{n+1}^{(i)},\lambda_{n+1}^{(i)})
\end{aligned}
$$

令:

$$
\Delta\boldsymbol{d}^{(i)} = \Delta\lambda^{(i)}\bar{\boldsymbol{q}} + \Delta\bar{\boldsymbol{d}} \tag{6}
$$

其中

$$\bar{\boldsymbol{q}} = \boldsymbol{K}(\boldsymbol{d}_{n+1}^{(i)})^{-1}\boldsymbol{F}^{ext}$$
$$\Delta\bar{\boldsymbol{d}} = \boldsymbol{K}(\boldsymbol{d}_{n+1}^{(i)})^{-1}\boldsymbol{R}(\boldsymbol{d}_{n+1}^{(i)}, \lambda_{n+1}^{(i)})$$

代入约束面函数的线性化方程:

$$2(\boldsymbol{d}_{n+1}^{(i)} - \boldsymbol{d}_n)^T\left(\Delta\lambda^{(i)}\bar{\boldsymbol{q}} + \Delta\bar{\boldsymbol{d}}\right) + 2(\lambda_{n+1}^{(i)} - \lambda_n)\Delta\lambda^{(i)} = -f(\boldsymbol{d}_{n+1}^{(i)}, \lambda_{n+1}^{(i)}) \tag{7}$$

$$\Delta\lambda^{(i)} = \frac{-f(\boldsymbol{d}_{n+1}^{(i)}, \lambda_{n+1}^{(i)}) - 2(\boldsymbol{d}_{n+1}^{(i)} - \boldsymbol{d}_n)^T\Delta\bar{\boldsymbol{d}}}{2(\boldsymbol{d}_{n+1}^{(i)} - \boldsymbol{d}_n)^T\bar{\boldsymbol{q}} + 2(\lambda_{n+1}^{(i)} - \lambda_n)} \tag{8}$$

其实有:

$$\begin{aligned} \delta\boldsymbol{d}^{(i)} &= \boldsymbol{d}_{n+1}^{(i)} - \boldsymbol{d}_n \\ \delta\lambda^{(i)} &= \lambda_{n+1}^{(i)} - \lambda_n \end{aligned} \tag{9}$$

所以可以写成:

$$\Delta\lambda^{(i)} = \frac{-f(\boldsymbol{d}_{n+1}^{(i)}, \lambda_{n+1}^{(i)}) - 2\delta\boldsymbol{d}^{(i)\,T}\Delta\bar{\boldsymbol{d}}}{2\delta\boldsymbol{d}^{(i)\,T}\bar{\boldsymbol{q}} + 2\delta\lambda^{(i)}} \tag{10}$$

有不同的启动方法（预测子）：

1. 用Newton-Rapshon启动第一步

需要指定满足残差方程的初始节点 $(\boldsymbol{d}_{n+1}^{(0)}, \lambda_{n+1}^{(0)})$ 以及第一步的荷载增量，用 $\Delta\lambda^{(0)}$ 表示

执行Newton-Raphson迭代过程:

$$\begin{gathered} \lambda_{n+1}^{1} = \lambda_{n+1}^{(0)} + \Delta\lambda^{(0)} \\ \boldsymbol{R}(\boldsymbol{d}_{n+1}^{(0)}, \lambda_{n+1}^{(1)}) = \lambda_{n+1}^{(1)}\boldsymbol{F}^{ext} - \boldsymbol{F}^{int}(\boldsymbol{d}_{n+1}^{(0)}) \neq \boldsymbol{0} \\ \text{计算: } \boldsymbol{K}(\boldsymbol{d}_{n+1}^{(0)}) \\ \text{求解: } \Delta\boldsymbol{d}^{(0)} = \boldsymbol{K}(\boldsymbol{d}_{n+1}^{(0)})^{-1}\boldsymbol{R}(\boldsymbol{d}_{n+1}^{(0)}, \lambda_{n+1}^{(1)}) \\ \boldsymbol{d}_{n+1}^{(1)} = \boldsymbol{d}_{n+1}^{(0)} + \Delta\boldsymbol{d}^{(0)} \end{gathered} \tag{27}$$

此时，通过指定了第一步（预测子）的荷载增量 $\Delta\lambda^{(0)}$ ，求出了一个位移增量 $\Delta\boldsymbol{d}^{(0)}$

那么，可以根据这两个增量构造一个弧长 $\delta a_{n+1}^2 = \Delta \boldsymbol{d}^{(0)} \cdot \Delta \boldsymbol{d}^{(0)} + \Delta\lambda^{(0)} \cdot \Delta\lambda^{(0)}$

随后，在接下来的多步修正子过程中始终使用这个弧长

2. 用指定弧长启动第一步

需要指定满足残差方程的初始节点 $(\boldsymbol{d}_{n+1}^{(0)}, \lambda_{n+1}^{(0)}) = (\boldsymbol{d}_n, \lambda_n)$ 和弧长 δa

已知可以计算当前节点的切线刚度矩阵 $\boldsymbol{K}(\boldsymbol{d}_{n+1}^{(0)})$， 需要求解下一个节点 $(\boldsymbol{d}_{n+1}^{(1)}, \lambda_{n+1}^{(1)})$ 使其满足 弧长约束面 $f(\boldsymbol{d}_{n+1}^{(1)}, \lambda_{n+1}^{(1)}) = 0$

$$f(\boldsymbol{d}_{n+1}^{(1)}, \lambda_{n+1}^{(1)}) = (\lambda_{n+1}^{(1)} - \lambda_n)^2 + (\boldsymbol{d}_{n+1}^{(1)} - \boldsymbol{d}_n)^T (\boldsymbol{d}_{n+1}^{(1)} - \boldsymbol{d}_n) - \delta a^2 = 0 \quad (28)$$

即

$$f(\boldsymbol{d}_{n+1}^{(1)}, \lambda_{n+1}^{(1)}) = \left(\Delta\lambda^{(0)}\right)^2 + \left(\Delta \boldsymbol{d}^{(0)}\right)^T \left(\Delta \boldsymbol{d}^{(0)}\right) - \delta a^2 = 0 \quad (29)$$

有两个未知量 $(\Delta \boldsymbol{d}^{(0)}, \Delta\lambda^{(0)})$， 但是只有一个方程，需要引入关系

$$\Delta \boldsymbol{d}^{(0)} = \Delta\lambda^{(0)} \bar{\boldsymbol{q}} + \Delta \bar{\boldsymbol{d}} \quad (30)$$

其中

$$\begin{aligned} \bar{\boldsymbol{q}} &= \boldsymbol{K}(\boldsymbol{d}_{n+1}^{(0)})^{-1} \boldsymbol{F}^{ext} \\ \Delta \bar{\boldsymbol{d}} &= \boldsymbol{K}(\boldsymbol{d}_{n+1}^{(0)})^{-1} \boldsymbol{R}(\boldsymbol{d}_{n+1}^{(0)}, \lambda_{n+1}^{(0)}) = \boldsymbol{0} \end{aligned}$$

所以

$$\Delta \boldsymbol{d}^{(0)} = \Delta\lambda^{(0)} \bar{\boldsymbol{q}} \quad (31)$$

代入

$$\begin{aligned} f(\boldsymbol{d}_{n+1}^{(1)}, \lambda_{n+1}^{(1)}) &= \left(\Delta\lambda^{(0)}\right)^2 + \left(\Delta\lambda^{(0)} \bar{\boldsymbol{q}}\right)^T \left(\Delta\lambda^{(0)} \bar{\boldsymbol{q}}\right) - \delta a^2 = 0 \\ (1 + \bar{\boldsymbol{q}}^T \bar{\boldsymbol{q}}) \left(\Delta\lambda^{(0)}\right)^2 &= \delta a^2 \\ \Delta\lambda^{(0)} &= \pm \sqrt{\frac{\delta a^2}{(1 + \bar{\boldsymbol{q}}^T \bar{\boldsymbol{q}})}} \end{aligned} \quad (32)$$

需要判断荷载增量的正负

考虑两种情况，$n = 0$ 和 $n > 0$ 的情况，即第一个荷载步和之后的荷载步的情况

- $n = 0$ 情况

这是第一个荷载步，没有之前的荷载步作为参考，需要手动设置荷载增量的正负，使得荷载的前进方向是往 $\boldsymbol{F}^{ext}$ 的方向推进的

- $n > 0$ 的情况

需要判断当前荷载步的初始节点切线刚度方向与上一个荷载步的切线刚度方向是否一致

即 如果 $sign(\det K(\boldsymbol{d}_{n+1}^{(0)})) == -sign(\det K(\boldsymbol{d}_{n}^{(0)}))$

说明方向发生改变，则荷载增量的方向也需要改变，即

$$\Delta\lambda^{(0)} = -sign(\delta\lambda_n^{(0)})\sqrt{\frac{\delta a^2}{(1+\bar{\boldsymbol{q}}^T\bar{\boldsymbol{q}})}} \tag{33}$$

否则

荷载增量的方向不需要改变，保持与上一荷载步荷载增量方向一致

$$\Delta\lambda^{(0)} = sign(\delta\lambda_n^{(0)})\sqrt{\frac{\delta a^2}{(1+\bar{\boldsymbol{q}}^T\bar{\boldsymbol{q}})}} \tag{34}$$

启动步求出了 荷载增量 $\Delta\lambda^{(0)}$ ，即可求出满足弧长约束面的 位移增量

$$\Delta\boldsymbol{d}^{(0)} = \Delta\lambda^{(0)}\bar{\boldsymbol{q}} \tag{35}$$

更新节点

当前位移：

$$\boldsymbol{d}_{n+1}^{(1)} = \boldsymbol{d}_{n+1}^{(0)} + \Delta\boldsymbol{d}^{(0)} \tag{36}$$

当前荷载：

$$\lambda_{n+1}^{(1)} = \lambda_{n+1}^{(0)} + \Delta\lambda^{(0)} \tag{37}$$

弧长法迭代（多步修正子）

启动迭代过程 $i = 1 : imax$

计算下一步荷载增量

$$\Delta\lambda^{(i)} = \frac{-f(\boldsymbol{d}_{n+1}^{(i)}, \lambda_{n+1}^{(i)}) - 2\delta\boldsymbol{d}^{(i)\,T}\Delta\bar{\boldsymbol{d}}}{2\delta\boldsymbol{d}^{(i)\,T}\bar{\boldsymbol{q}} + 2\delta\lambda^{(i)}} \tag{38}$$

其中

$$\begin{aligned}
\bar{\boldsymbol{q}} &= \boldsymbol{K}(\boldsymbol{d}_{n+1}^{(0)})^{-1}\boldsymbol{F}^{ext} \\
\Delta\bar{\boldsymbol{d}} &= \boldsymbol{K}(\boldsymbol{d}_{n+1}^{(0)})^{-1}\boldsymbol{R}(\boldsymbol{d}_{n+1}^{(i)}, \lambda_{n+1}^{(i)}) \\
\delta\boldsymbol{d}^{(i)} &= \boldsymbol{d}_{n+1}^{(i)} - \boldsymbol{d}_n \\
\delta\lambda^{(i)} &= \lambda_{n+1}^{(i)} - \lambda_n
\end{aligned}$$

下一步位移增量

$$\Delta\boldsymbol{d}^{(i)} = \Delta\lambda^{(i)}\bar{\boldsymbol{q}} + \Delta\bar{\boldsymbol{d}} \tag{39}$$

更新节点

当前位移:

$$\boldsymbol{d}_{n+1}^{(i+1)} = \boldsymbol{d}_{n+1}^{(i)} + \Delta\boldsymbol{d}^{(i)} \tag{40}$$

当前荷载:

$$\lambda_{n+1}^{(i+1)} = \lambda_{n+1}^{(i)} + \Delta\lambda^{(i)} \tag{41}$$

判断迭代是否收敛，即节点是否迭代到弧长约束面与残差方程相交，由于前面公式已经可以保证每个节点都在约束面上，在此，仅需判断残差是否小于指定的阈值，若小于，则停止迭代

$$\begin{aligned}
&\text{if } \|\boldsymbol{R}(\boldsymbol{d}_{n+1}^{(i+1)}, \lambda_{n+1}^{(i+1)})\| < \text{tol} \\
&\quad \text{break;} \\
&\text{end}
\end{aligned}$$