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An Exact Algorithm for the Capacitated Vehicle Routing Problem Based on a Two-Commodity Network Flow Formulation

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The capacitated vehicle routing problem (CVRP) is the problem in which a set of identical vehicles located at a central depot is to be optimally routed to supply customers with known demands subject to vehicle capacity constraints. In this paper, we describe a new integer programming formulation for the CVRP based on a two-commodity network flow approach. We present a lower bound derived from the linear programming (LP) relaxation of the new formulation which is improved by adding valid inequalities in a cutting-plane fashion. Moreover, we present a comparison between the new lower bound and lower bounds derived from the LP relaxations of different CVRP formulations proposed in the literature. A new branch-and-cut algorithm for the optimal solution of the CVRP is described. Computational results are reported for a set of test problems derived from the literature and for new randomly generated problems.

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Introduction

The capacitated vehicle routing problem (CVRP) is the problem of designing optimal delivery routes for a fleet of vehicles to supply a set of customers with given demands. The objective is to supply all customers minimizing the total cost of all the routes.

In real-world CVRPs the cost of a vehicle route includes many elements, such as the cost of fuel, tires, maintenance costs, driver wages, the cost of distance travelled, and time spent to visit all customers. In addition to vehicle capacity restrictions, real-world CVRPs (see Bodin et al. 1983, 1999; Christofides and Mingozzi 1990; Ball et al. 1995; Toth and Vigo 2000b) involve complicated constraints such as time windows to visit customers, customer-vehicle incompatibilities, mixed deliveries or collections on the same route, multiple interacting depots, and so on.

The practical importance of the CVRP provides the motivation for the effort involved in the development of heuristic algorithms (see the surveys of Golden et al. 1998, Laporte and Semet 2000, Gendreau et al. 2000). In particular, Laporte et al. (2000) report a computational comparison of some of the most important families of heuristics.

The CVRP has been shown to be NP-hard. The fact that the largest CVRP instances that can be solved consistently by the most effective exact algorithms proposed so far contain about 50 customers reflects the difficulty of this problem. In view of the large number of practical constraints that appear in real-world CVRPs, most of the exact methods investigate a basic problem, called the basic CVRP, which is the core of all vehicle routing problems.

The basic CVRP considered in this paper involves a fleet of identical vehicles located at a central depot and a set of customers, each with a given demand of goods to be supplied from the depot. Every route performed by a vehicle must start and end at the depot and the load carried must be less than or equal to the vehicle capacity. It is assumed that the cost matrix of the least-cost paths between each pair of customers is known. The cost of a route is computed as the sum of the costs of the arcs forming the route. The objective is to design vehicle routes (one route for each vehicle), so that all customers are visited exactly once and the sum of the route costs is minimized. In this paper, the cost matrix is assumed to be symmetric.

Annotated bibliographies on the CVRP are given by Laporte and Osman (1995) and by Laporte (1997). The bibliography of Laporte and Osman (1995) contains over 500 articles on vehicle routing.

Laporte and Nobert (1987) present an extensive survey that is entirely devoted to exact methods for the CVRP, and give a complete and detailed analysis of the state of the art to the late 1980s. Toth and Vigo (1998) provide an update of that survey, describing exact algorithms recently proposed for both symmetric and asymmetric CVRPs. Other surveys covering exact algorithms are given by Magnanti (1981), Laporte (1992), and Fisher (1995). The chapters of Toth and Vigo (2000c), Naddef and Rinaldi (2000), and Bramel and Simchi-Levi (2000) in the book edited by Toth and Vigo (2000a) survey the most effective exact methods proposed in the literature so far.

Exact methods can be classified into the following categories: branch-and-cut, branch-and-bound, dynamic programming, and set-partitioning methods. The best known of these algorithms are briefly surveyed below.

(i) Branch-and-cut methods extend to the symmetric CVRP the successful results of polyhedral combinatorics developed for the traveling salesman problem (TSP) by Chvátal (1973) and by Grötschel and Padberg (1979, 1985). These methods are mainly based on cutting-plane techniques used to strengthen the linear relaxation of the integer CVRP formulation by adding families of valid inequalities. Laporte et al. (1985) present the first exact method for the CVRP with distance restrictions based on a cutting-plane approach. They are able to solve optimally randomly generated problems with 50 to 60 customers and average vehicle utilization approximately equal to 0.74. Laporte and Nobert (1984) describe valid inequalities for the CVRP based on the comb inequalities developed by Chvátal (1973) and Grötschel and Padberg (1979) for the TSP. Using comb inequalities within a branch-and-cut method Cornuéjols and Harche (1993) solve the 50-customer test problem described in Christofides and Eilon (1969).

A more sophisticated branch-and-cut algorithm for the symmetric CVRP is proposed by Augerat et al. (1995). In addition to capacity constraints, these authors use new classes of valid inequalities, such as comb and extended comb inequalities, generalized capacity constraints, and hypotour inequalities. These new inequalities lead to significant improvements in the quality of the bound. The resulting branch-and-cut algorithm is able to solve a 135-customer problem, which is the largest CVRP problem solved to date.

Araque et al. (1994) describe a branch-and-cut procedure based on multistar inequalities for the case where customers have unit demands, which can consistently solve to optimality instances with up to 60 vertices. Further valid inequalities are introduced by Hill (1995) who is able to optimality solve three instances involving 44, 71, and 100 customers (in the last case, return routes between the depot and a single customer were disallowed). For a complete survey of branch-and-cut methods for the CVRP see Naddef and Rinaldi (2000).

(ii) The effectiveness of branch-and-bound algorithms largely depends on the quality of the lower bounds used to limit the search tree; therefore, the derivation of such bounds will be the focus of this discussion. Christofides et al. (1981a) describe a lower bound based on Lagrangean relaxation and on the computation of the degree center tree (K-DCT), which is a tree where the depot vertex has a fixed degree $k = 2M$, where M is equal to the number of routes. The resulting lower bound is embedded into a branch-and-bound algorithm that has successfully solved to optimality CVRP problems involving up to 25 customers. The lower bound produced by Fisher (1994) is computed using a generalization of spanning trees, called M-trees, and is incorporated within an exact branch-and-bound algorithm that has optimally solved a well-known problem with 100 customers and several real-world problems including 25 to 71 customers.

Miller (1995) dualizes vehicle capacity constraints in a Lagrangian fashion and works on the resulting b-matching relaxation that can be solved relatively efficiently. Using this approach, he solves several problems containing up to 51 vertices.

(iii) Dynamic programming has been applied to solve several types of CVRP or to obtain tight lower bounds. Christofides et al. (1981b) present three formulations of the CVRP and introduce the state space relaxation method for relaxing the dynamic programming recursions to obtain valid lower bounds on the value of the optimal solutions. The computational results show that the ratio "lower bound/optimum" varies between 93.1% and 99.6% when these state space relaxations are used. Christofides (1985) reports that a CVRP involving 50 customers has been solved exactly by this approach. Problems involving up to 125 customers are solved within 2% of the optimum in less than 15 minutes on a CYBER 855.

(iv) The set-partitioning formulation of the basic CVRP initially was introduced by Balinski and Quandt (1964). Christofides et al. (1981a) present a lower bound to the CVRP based on the set-partitioning formulation of the problem. Hadjiconstantinou et al. (1995) describe a method for computing a feasible dual solution of the LP relaxation of the set-partitioning formulation that is based on the computation of k-shortest paths and q-paths. The resulting CVRP lower bound is superior to the lower bound obtained by Christofides et al. (1981a) and the branch-and-bound algorithm is able to optimally solve problems involving up to 50 customers. Mingozzi et al. (1994) present a new method for solving the set-partitioning formulation of the CVRP. They describe a procedure for computing a valid lower bound to the cost of the optimal CVRP solution that finds a feasible solution of the dual of the LP relaxation of the set-partitioning formulation without generating the entire set-partitioning matrix. The dual solution obtained is then used to limit the set of the feasible routes containing the optimal CVRP solutions. The resulting set-partitioning problem is solved by using a branch-and-bound algorithm. Mingozzi et al. (1994) report optimal solutions of problems for up to 50 customers.

Exact algorithms for the basic CVRP with asymmetric costs are proposed by Laporte et al. (1986) and by

Fischetti et al. (1994). The latter method can exactly solve problems with 300 customers. However, neither of these two papers reports any computational results on CVRP instances with symmetric costs. Several commodity flow formulations of both symmetric and asymmetric CVRPs are developed by Petersen (2000).

This paper describes an exact algorithm for the basic CVRP with symmetric costs based on a new integer programming formulation that has the form of a two-commodity network flow problem. This new formulation is interesting in many ways. It can be shown that its LP relaxation satisfies a weak form of the capacity constraints. The formulation can also be modified to accommodate different constraints and, therefore, is capable of being extended to different routing problems. A lower bound based on the linear relaxation of the new formulation strengthened by a set of valid inequalities is derived. The resulting lower bound is then embedded in a branch-and-cut procedure to optimally solve the problem. The computational results for a set of CVRP instances known in the literature and for a set of newly generated test problems show that the lower bound obtained is tight and that the corresponding branch-and-cut algorithm is capable of solving to optimality instances containing up to 80 customers. Moreover, the branch-and-cut algorithm was able to solve to optimality an instance involving 135 customers, which is the largest instance ever solved in the literature.

This paper is organized as follows. Section 1 gives a formal description of the basic CVRP addressed in this paper and presents three of the more interesting formulations already presented in the literature. In §2, the new two-commodity flow formulation of the CVRP is given. In §3, a lower bound derived from the LP relaxation of this formulation and improved by valid inequalities is discussed. A comparison between the proposed new lower bound and the lower bounds obtained from LP relaxation of other CVRP formulations presented in this paper is also provided in §3. The branch-and-cut algorithm for the exact solution of the CVRP is described in §4, and computational results are reported in §5. Section 6 presents conclusions, and proofs are found in the Appendix.

1. The Basic CVRP and Its Formulations

In this section, a description of the problem and the notation used are given, followed by an analysis of some of the more interesting formulations of the CVRP presented in the literature. These formulations will be compared with the new two-commodity formulation given in §2.

Problem Description. The basic CVRP considered in this paper is described as follows: An undirected graph $G=(V,E)$ is given where $V=\{0,1,\dots,n\}$ is the set of $n+1$ nodes and E is the set of edges. Node 0 represents the depot and the remaining node set $V'=V\backslash\{0\}$ corresponds to n customers. Every edge $\{i,j\}\in E$ is assigned a nonnegative cost c_{ij}. Henceforth, $i\in V'$ will be used to refer both to a customer and to its node location. Each customer $i\in V'$ requires a supply of q_i units from depot 0 (we assume $q_0=0$). A set of M identical vehicles of capacity Q is stationed at depot 0 and must be used to supply the customers. For a subset F of E, $G(F)$ denotes the subgraph $(V(F),F)$ induced by F, where $V(F)$ is the set of nodes incident to at least one edge of F.

A *route* is defined as a nonempty subset $R\subset E$ of edges for which the induced subgraph $G(R)$ is a simple cycle containing depot 0, and such that the total demand of the nodes in $V(R)\setminus\{0\}$ does not exceed the vehicle capacity Q. Such a route represents the trip of one vehicle leaving the depot, delivering the demands of the nodes in $V(R)\setminus\{0\}$, and returning to the depot. The cost of a route corresponds to the sum of the costs of the edges forming the route. The problem objective is to design M routes, one for each vehicle, so that all customers are visited exactly once and the sum of the route costs is minimized.

1.1. A Two-Index Vehicle Flow Formulation

All branch-and-cut methods presented in the literature are based on the following formulation of the CVRP, originally proposed by Laporte et al. (1985), which allows single-customer routes:

Let $\mathcal{S}=\{S\colon S\subseteq V', |S|\geqslant 2\}$. For a given $S\in\mathcal{S}$ we denote by $\bar{S}$ the complementary set of nodes $V\setminus S$. Let $r(S)$ be the minimum number of vehicles of capacity Q needed to satisfy the demand of customers in $S\in\mathcal{S}$. Also, let $q(S)$ indicate the total demand of the node subset $S\subseteq V'$, that is $q(S)=\sum_{i\in S}q_i$. Let ξ_{ij} be an integer variable that might take value $\{0,1\}$ $\forall\{i,j\}\in E\setminus\{\{0,j\}\colon j\in V'\}$ and value $\{0,1,2\}$ $\forall\{0,j\}\in E, j\in V'$. Note that $\xi_{0j}=2$ when a route including the single customer j is selected in the solution.

The basic CVRP can be formulated as the following integer program:

$$(F1)\quad z(F1)=\text{minimize}\sum_{\{i,j\}\in E}c_{ij}\xi_{ij} \tag{1}$$

subject to

$$\sum_{\substack{j\in V\\ i<j}}\xi_{ij}+\sum_{\substack{j\in V\\ i>j}}\xi_{ji}=2\quad\forall i\in V', \tag{2}$$

$$\sum_{i\in S}\sum_{\substack{j\in\bar{S}\\ i<j}}\xi_{ij}+\sum_{i\in\bar{S}}\sum_{\substack{j\in S\\ i<j}}\xi_{ij}\geqslant 2r(S)\quad\forall S\in\mathcal{S}, \tag{3}$$

$$\sum_{j\in V'}\xi_{0j}=2M, \tag{4}$$

$$\xi_{ij}\in\{0,1\}\quad\forall\{i,j\}\in E\setminus\{\{0,j\}\colon j\in V'\}, \tag{5}$$

$$\xi_{0j}\in\{0,1,2\},\quad\forall\{0,j\}\in E, j\in V'. \tag{6}$$

Constraint (2) is the degree constraint for each customer. Constraint (3) is the capacity constraint (also called generalized subtour elimination constraint) which, for any subset S of customers that does not include the depot, impose that $r(S)$ vehicles enter and leave S. It is NP-hard to compute $r(S)$ because it corresponds to solve a bin-packing problem where $r(S)$ is the minimum number of bins of capacity Q that are needed for packing the quantities q_i $\forall i \in S$. However, formulation $F1$ remains valid if $r(S)$ is replaced by a lower bound on its value, such as $\lceil q(S)/Q \rceil$, where $\lceil x \rceil$ denotes the smallest integer not less than x. Constraint (4) states that M vehicles must leave and return to the depot and constraints (5) and (6) are the integrality constraints.

Laporte et al. (1985) give a two-index vehicle flow formulation derived from $F1$ by replacing constraint (3) with the following subtour elimination constraint (7), which is obtained by combining constraints (2) and (3):

$$\sum_{i\in S}\sum_{\substack{j\in S\\ i<j}} \xi_{ij} \leqslant |S| - r(S) \quad \forall S \in \mathcal{S}. \tag{7}$$

1.2. A Multicommodity Flow Formulation

Commodity flow formulations of the CVRP combine assignment constraints for modelling vehicle routes and multicommodity flow constraints for modelling movements of goods. A formulation of this type was first proposed by Garvin et al. (1957) in an oil delivery problem, and was later extended by Gavish and Graves (1979, 1982).

The multicommodity flow formulation explicitly introduces arc orientation. Let ξ_{ij} be a binary variable equal to 1 if and only if arc (i, j) is in the optimal solution, $i, j \in V$, $i \neq j$. Let y_{ij}^l be the flow variables specifying the amount of demand destined to customer $l \in V'$ that is transported on arc (i, j), $i, j \in V$, $i \neq j$.

$$(F2) \quad z(F2) = \text{minimize} \sum_{\substack{i,j\in V\\ i\neq j}} c_{ij}\xi_{ij} \tag{8}$$

subject to

$$\sum_{i\in V} \xi_{ij} = 1 \quad \forall j \in V', \tag{9}$$

$$\sum_{j\in V} \xi_{ij} = 1 \quad \forall i \in V', \tag{10}$$

$$\sum_{j\in V'} \xi_{0j} = M, \tag{11}$$

$$\sum_{j\in V'} \xi_{j0} = M, \tag{12}$$

$$\sum_{i\in V} y_{ij}^l - \sum_{i\in V} y_{ji}^l = \begin{cases} q_l, \ j = l & \forall l \in V', \\ 0, \ j \neq l & \forall j, l \in V', \\ -q_l, \ j = 0 & \forall l \in V', \end{cases} \tag{13}$$

$$y_{ij}^l \leqslant q_l \xi_{ij} \quad \forall i, \ j \in V, \ i \neq j, \ \forall l \in V', \tag{14}$$

$$\sum_{j\in V'}\sum_{l\in V'} y_{ij}^l \leqslant Q - q_i \quad \forall i \in V, \tag{15}$$

$$y_{ij}^l \geqslant 0 \quad \forall i, \ j \in V, i \neq j, \ \forall l \in V', \tag{16}$$

$$\xi_{ij} \in \{0, 1\} \quad \forall i, \ j \in V, i \neq j. \tag{17}$$

Constraints (9) and (10) ensure that a customer is visited exactly once. Constraints (11) and (12) impose that a vehicle must be used exactly once. Constraint (13) is the commodity flow constraint that guarantees that the demand of each node is satisfied. Finally, constraint (15) ensures that the vehicle capacity is never exceeded.

Let $LF1$ and $LF2$ denote the LP relaxations of formulations $F1$ and $F2$, respectively, and $z(LF1)$ and $z(LF2)$ the corresponding optimal solution costs. Laporte and Nobert (1987) show that $z(LF1) \geqslant z(LF2)$ but $z(LF1) \leqslant z(LF2)$ if the value of $r(S)$ in $F1$ is replaced by the lower bound $r(S) = q(S)/Q$. The latter result derives from the observation that every fractional $LF2$ solution satisfies the subtour elimination constraint

$$\sum_{\substack{i\in S\\ j\in \bar{S}}} \xi_{ij} + \sum_{\substack{i\in \bar{S}\\ j\in S}} \xi_{ij} \geqslant 2 \quad \forall S \in \mathcal{S}, \tag{18}$$

which may be violated by an $LF1$ solution for the subsets $S \in \mathcal{S}$ having $q(S) < Q$.

1.3. A Set-Partitioning Formulation

A set-partitioning formulation associates a binary variable with each feasible route. Let $\mathcal{R}$ be the index set of all feasible routes and a_{ij} be a binary coefficient that is equal to 1 if customer i belongs to route $j \in \mathcal{R}$, and that takes the value 0 otherwise. Each route $j \in \mathcal{R}$ has an associated cost $\hat{c}_j$. Let ζ_j be a (0-1) variable, which is equal to 1 if and only if route $j \in \mathcal{R}$ is in the optimal solution. The set-partitioning formulation is

$$(F3) \quad z(F3) = \text{minimize} \sum_{j\in\mathcal{R}} \hat{c}_j\zeta_j \tag{19}$$

subject to

$$\sum_{j\in\mathcal{R}} a_{ij}\zeta_j = 1 \quad \forall i \in V', \tag{20}$$

$$\sum_{j\in\mathcal{R}} \zeta_j = M, \tag{21}$$

$$\zeta_j \in \{0, 1\} \quad \forall j \in \mathcal{R}. \tag{22}$$

Problem $F3$ has a possible exponential number of variables and cannot be used directly to solve instances of large size. However, model $F3$ is very general and can take into account several route constraints (e.g., time windows) because the route feasibility is implicity considered in the definition of set $\mathcal{R}$. Moreover, the LP relaxation of $F3$ (called $LF3$) is typically very tight (see Table 2, §5).

Let $z(LF3)$ be the optimal solution value of relaxation $LF3$. In the following we show that no dominance relation exists between lower bounds $z(LF3)$ and $z(LF1)$, but $z(LF3) \geqslant z(LF2)$.

To compare relaxations $LF1$ and $LF3$, let us consider the CVRP relaxation, called $\overline{LF1}$, which is obtained from $LF1$ by replacing the capacity constraint (3) with the multistar inequalities and adding the subtour elimination constraints. Problem $\overline{LF1}$ is as follows:

$$(\overline{LF1}) \quad z(\overline{LF1}) = \text{minimize} \sum_{\{i,j\}\in E} c_{ij}\xi_{ij} \tag{23}$$

subject to (2), (4), (5), (6) and

$$\sum_{\substack{i\in S\\ j\in \bar{S}\\ i<j}} \xi_{ij} + \sum_{\substack{i\in \bar{S}\\ j\in S\\ i<j}} \xi_{ij} \geqslant 2\frac{q(S)}{Q} + \frac{2}{Q}\left(\sum_{\substack{i\in S\\ j\in \bar{S}\\ i<j}} q_j\xi_{ij} + \sum_{\substack{i\in \bar{S}\\ j\in S\\ i<j}} q_i\xi_{ij}\right), \quad \forall S\in \mathcal{S}, \tag{24}$$

$$\sum_{\substack{i\in S\\ j\in \bar{S}\\ i<j}} \xi_{ij} + \sum_{\substack{i\in \bar{S}\\ j\in S\\ i<j}} \xi_{ij} \geqslant 2 \quad \forall S\in \mathcal{S}. \tag{25}$$

Multistar inequalities have been introduced by Araque et al. (1990) for the CVRP case where customers have unit demands (CVRPUD), and have been extended to the CVRP by several authors independently (Achuthan et al. 1998, Fisher 1995, Gouveia 1995). It can be shown (see Araque et al. 1990) that multistar inequalities are facet defining for the polytope of the solutions of $F1$ when customers have unit demands and that a fractional $LF1$ solution might violate a multistar inequality (24). Recently, Letchford et al. (2002) have defined new classes of multistar inequalities that generalize all previously known multistar inequalities and describe a polynomial time separation algorithm to detect multistar constraints violated by an $LF1$ solution. Moreover, they present computational results that indicate that the lower bound obtained by adding, in a cutting-plane fashion, multistar inequalities to $LF1$ dominates $z(LF1)$.

It can be shown that a fractional $\overline{LF1}$ solution might violate capacity constraint (3). For example, consider any CVRP instance where $q(V') = 1.5Q$. The value of the right-hand side of constraint (24) is smaller than

$$2\frac{q(S)}{Q} + \frac{2}{Q}(q(V') - q(S)) = 3 \quad \forall S\in \mathcal{S},$$

and the value of the right-hand side of constraint (3) is equal to 4 for any $S\in \mathcal{S}$ such that $q(S) > Q$. Because a $\overline{LF1}$ solution might not be a feasible $LF1$ solution, we have the following result.

Corollary 1. *No dominance relation exists between the two relaxations $LF1$ and $\overline{LF1}$.*

Lemma 1. *Every $LF3$ solution $\boldsymbol{\zeta}$ of cost $z(LF3)$ corresponds to a solution $\boldsymbol{\xi}$ of $\overline{LF1}$ of cost $z(\overline{LF1}) = z(LF3)$.*

Proof. See the appendix. □

Corollary 2. *$z(LF3) \geqslant z(\overline{LF1})$.*

Proof. See the appendix. □

Corollary 3. *No dominance relation exists between relaxations $LF3$ and $LF1$.*

Proof. See the appendix. □

Lemma 2. *$z(LF3) \geqslant z(LF2)$.*

Proof. See the appendix. □

We can observe that the formulation $LF2$ can be tightened to satisfy multistar inequalities by adding the following constraint:

$$q_j\xi_{ij} \leqslant \sum_{l\in V'} y_{ij}^l \leqslant (Q-q_i)\xi_{ij} \quad \forall i,\ j\in V',\ i\neq j. \tag{26}$$

1.4. Other CVRP Formulations

Many other formulations of the CVRP have been presented in the literature (see Laporte and Nobert 1987, Toth and Vigo 2000a). Several of these formulations have been introduced to overcome some of the drawbacks associated with formulations $F1$ and $F2$ in representing additional constraints such as time-window constraints, nonhomogenous vehicle fleet, and so on. Golden et al. (1977) formulate the CVRP using three-index formulation variables ξ_{ij}^k indicating whether vehicle k travels directly from customer i to customer j. The resulting formulation involves $O(n^2M)$ variables and requires an exponential number of subtour elimination constraints. The two-index formulation $F1$ can be derived from the three-index formulation by aggregating all ξ_{ij}^k variables into a single variable ξ_{ij} indicating whether or not a vehicle travels directly from customer i to customer j.

2. A New Two-Commodity Flow Formulation of the CVRP

In this section, we present a new integer programming formulation of the symmetric CVRP that is based on the two-commodity flow formulation of the TSP introduced by Finke et al. (1984). In their formulation, one unit of the first commodity, say A, must be delivered at each node and one unit of the second commodity, say B, must be collected at each node. The salesman starts the tour at node 0 with n units of commodity A and 0 units of commodity B. At the next node of the tour, he leaves one unit of A and picks up one unit of B. This exchange of one unit of each commodity continues until the salesman arrives back at node 0 with 0 units of commodity A and n units of B. At any arc of the tour, the salesman carries a combined total load of n units. The amount of commodity A indicates the number of nodes left to be visited by the salesman and the amount of commodity B denotes the number of nodes already visited.

The above TSP formulation has not received much attention in the literature. Langevin et al. (1993) extend this approach to formulate the TSP with time windows (TSPTW) and to develop an exact algorithm that can solve TSPTW instances of up to 40 nodes. Lucena (1986) uses the two-commodity approach for the TSP to derive a new formulation of the CVRP and a lower bound based on the LP relaxation of the resulting formulation.

The quality of the lower bound was evaluated on a set of 10 symmetric CVRP instances. Because the computational results of Lucena (1986) show his lower bounds to be weaker than those obtained by Christofides et al. (1981a), this approach is not further investigated. Baldacci et al. (2003) use the two-commodity approach to design a branch-and-cut algorithm for the TSP with deliveries and collections capable of solving to optimality problems involving up to 200 customers.

In this section, we describe a new integer formulation of the symmetric CVRP based on a two-commodity network flow approach. To model single-customer routes, this formulation requires the extended graph $\overline{G}=(\overline{V},\overline{E})$ obtained from G by adding node $n+1$, which is a copy of depot node 0. We have $\overline{V}=V\cup\{n+1\}$, $V'=\overline{V}\setminus\{0,n+1\}$, $\overline{E}=E\cup\{\{i,n+1\}, i\in V'\}$, and $c_{in+1}=c_{0i}\ \forall\, i\in V'$. Note that graph $\overline{G}$ has the same family of subsets $\mathcal{S}$ as graph G (see §1). Also in the case of graph $\overline{G}$, for a given subset $S\in\mathcal{S}$ we denote by $\overline{S}$ the complementary set of nodes $\overline{V}\setminus S$.

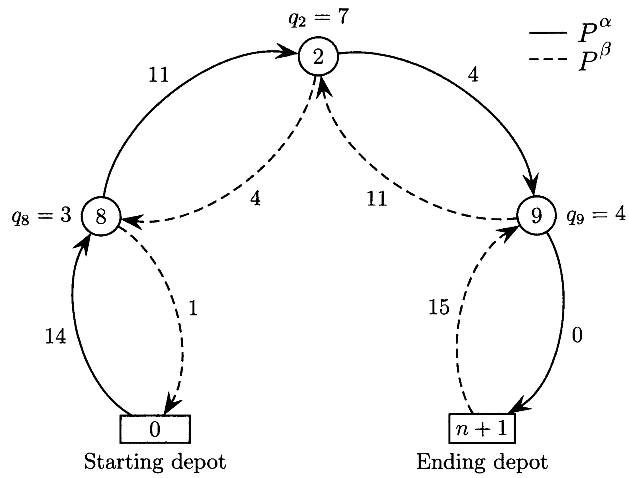
In graph $\overline{G}$, a route $R=(i_1,i_2,\dots,i_{|R|})$ is a simple path from node $i_1=0$ to node $i_{|R|}=n+1$. For a given route R, $\overline{V}(R)$ and $\overline{E}(R)$ represent the set of nodes and the set of edges of $\overline{G}$ traversed, respectively. $V'(R)=\overline{V}(R)\setminus\{0,n+1\}$ will be used to denote the subset of nodes of $\overline{G}$ corresponding to customers.

This formulation uses two flow variables, x_{ij} and x_{ji}, to represent an edge $\{i,j\}\in\overline{E}$ of a feasible CVRP solution along which the vehicle carries a combined load of Q units. If a vehicle travels from i to j, then the flow x_{ij} represents the load of the vehicle and the flow x_{ji} represents the empty space on the vehicle (i.e., $x_{ji}=Q-x_{ij}$). The flow variables x_{ij}, $i,j\in\overline{V}, i\neq j$, define two flow paths for any route of a feasible solution: One path from node 0 to node $n+1$ is given by the flow variables representing the vehicle load, and the second path from node $n+1$ to node 0 is given by the flow variables representing the empty space in the vehicle.

Figure 1 shows an example of a three-customer route for a vehicle of capacity $Q=15$ and the two paths P^α and P^β represented by the flow variables $\{x_{ij}\}$ defining the route. Path P^α is given by the variables representing the vehicle load: $(x_{08},x_{82},x_{29},x_{9n+1})$. For example, the flow $x_{08}=14$ indicates that the vehicle leaves the depot with a load equal to the total demand of the three customers. Path P^β is defined by the variables representing the empty space in the vehicle: $(x_{n+19},x_{92},x_{28},x_{80})$. For example, $x_{n+19}=15$ indicates that the vehicle arrives empty at the depot. Note that for every edge $\{i,j\}$ of the route we have $x_{ij}+x_{ji}=Q$.

Let ξ_{ij} be a 0-1 binary variable equal to 1 if edge $\{i,j\}\in\overline{E}$ is in the solution, 0 otherwise. Let x_{ij} and x_{ji} be the two flow variables associated with edge $\{i,j\}\in\overline{E}$.

Figure 1. Flow paths for a route of three customers.



The new mathematical formulation for the CVRP is as follows:

$$(F4)\quad z(F4)=\text{minimize}\sum_{\{i,j\}\in\overline{E}} c_{ij}\xi_{ij} \tag{27}$$

subject to

$$\sum_{j\in\overline{V}}(x_{ji}-x_{ij})=2q_i\quad \forall\, i\in V', \tag{28}$$

$$\sum_{j\in V'} x_{0j}=q(V'), \tag{29}$$

$$\sum_{j\in V'} x_{j0}=MQ-q(V'), \tag{30}$$

$$\sum_{j\in V'} x_{n+1j}=MQ, \tag{31}$$

$$x_{ij}+x_{ji}=Q\xi_{ij}\quad \forall\{i,j\}\in\overline{E}, \tag{32}$$

$$\sum_{\substack{j\in\overline{V}\\ i<j}}\xi_{ij}+\sum_{\substack{j\in\overline{V}\\ i>j}}\xi_{ji}=2\quad \forall\, i\in V', \tag{33}$$

$$x_{ij}\geqslant 0, x_{ji}\geqslant 0\quad \forall\{i,j\}\in\overline{E}, \tag{34}$$

$$\xi_{ij}\in\{0,1\}\quad \forall\{i,j\}\in\overline{E}. \tag{35}$$

Equations (28)–(31) and the nonnegative constraint (34) define a feasible flow pattern from the source nodes 0 and $n+1$ to sink nodes in $V'\cup\{0\}$. The outflow at source node 0 (Equation (29)) is equal to the total customer demand, and the inflow at source $n+1$ (Equation (31)) corresponds to the total capacity of the vehicle fleet. Equation (28) states that the inflow minus the outflow at each customer $i\in V'$ is equal to $2q_i$, and the inflow at node 0 (Equation (30)) corresponds to the residual capacity of the vehicle fleet. Constraint (32) defines the edges of a feasible solution and constraint (33) forces any feasible solution to contain two edges incident to each customer.

The validity of formulation $F4$ is shown by the following lemma:

LEMMA 3. *The set of solutions to* (28)–(35) *corresponds to the set of solutions to CVRP.*

PROOF. See the appendix. □

3. A Lower Bound Based on the New Formulation

A lower bound of the CVRP is obtained by the optimal solution cost $z(LF4)$ of the LP relaxation of formulation $F4$, called $LF4$. Let $\overline{LF4}$ denote the problem resulting from $LF4$ by removing variables $\{\xi_{ij}\}$, using Equation (32), and adding the following $O(n^2)$ inequalities, called *flow inequalities*.

$$\left.\begin{aligned}(Q-q_j)x_{ij}-q_jx_{ji}\geqslant 0\\ (Q-q_i)x_{ji}-q_ix_{ij}\geqslant 0\end{aligned}\right\}\quad \forall\{i,j\}\in\bar{E}, \tag{36}$$

which are satisfied by any $F4$ solution (see Lemma 4 below).

Problem $\overline{LF4}$ is defined as follows:

$$(\overline{LF4})\quad z(\overline{LF4})=\text{minimize } \frac{1}{Q}\sum_{\substack{i,j\in\bar{V}\\ i<j}} c_{ij}(x_{ij}+x_{ji}) \tag{37}$$

subject to (28), (29), (30), (31), (34), (36), and

$$\sum_{j\in\bar{V}}(x_{ij}+x_{ji})=2Q\quad \forall i\in V', \tag{38}$$

$$x_{ij}+x_{ji}\leqslant Q\quad \forall\{i,j\}\in\bar{E}. \tag{39}$$

LEMMA 4. *Any feasible solution to* $F4$ *satisfies flow inequalities* (36).

PROOF. See the appendix. □

LEMMA 5. *Every feasible solution* $\mathbf{x}$ *of* $\overline{LF4}$ *satisfies the multistar inequalities* (24) *and Equation* (2) *when the values of variables* ξ_{ij} $\forall\{i,j\}\in\bar{E}$, *are defined according to Equation* (32).

PROOF. See the appendix. □

$\overline{LF4}$ involves $O(n^2)$ variables and $3|\bar{E}|+2n+3$ constraints. In our computational experience it was found that only a few constraints (39), called *trivial inequalities*, and a few flow constraints (36), are saturated by an optimal $\overline{LF4}$ solution. Thus, it was computationally worthwhile solving $\overline{LF4}$ using a cutting-plane procedure in which constraints (39) and (36) are initially ignored and only those that are violated in the current solution are added to $\overline{LF4}$.

3.1. Comparison of Various CVRP Relaxations

As a result of Lemma 5 we can conclude that no dominance relation exists between $LF1$ and $\overline{LF4}$, because an $LF1$ solution might violate the multistar constraints, but an $\overline{LF4}$ solution might not satisfy the capacity constraint (3). However, $z(\overline{LF4})$ dominates the lower bound given by relaxation $LF1$ where the value of $r(S)$ in constraint (3) is replaced by the lower bound $r(S)=q(S)/Q$. It is easy to observe that $z(\overline{LF1})\geqslant z(\overline{LF4})$ because any $\overline{LF1}$ solution satisfies both subtour and multistar constraints.

In comparing relaxations $LF2$ and $\overline{LF4}$ we note that neither of the formulations dominates the other because an $LF2$ solution might violate multistar inequalities, but an $\overline{LF4}$ solution might not satisfy subtour elimination constraints.

Finally, $LF3$ is a stronger relaxation than $\overline{LF4}$ because any $LF3$ solution can be transformed into an $\overline{LF4}$ solution (details are omitted for the sake of brevity), which satisfies both subtour and multistar inequalities.

In comparing $\overline{LF4}$ with the other CVRP relaxations, the former has several advantages:

(1) The number of variables and constraints in $\overline{LF4}$ increase polynomially with the size of the problem, whereas tighter relaxations $\overline{LF1}$ and $LF3$ are exponential in size. Note that the relaxation $LF1$ requires the addition of an exponential number of multistar inequalities to become stronger than $\overline{LF4}$ and relaxation $LF2$ (involving $O(n^3)$ variables and constraints) requires $O(n^2)$ constraint (26) to become stronger than $\overline{LF4}$.

(2) Relaxation $\overline{LF4}$ becomes stronger than $LF1$ and $LF2$ by adding capacity constraint (3) and is not dominated by $LF3$ whose solution may violate capacity constraint (3).

3.2. Improving the Lower Bound $z(\overline{LF4})$

The lower bound $z(\overline{LF4})$ can be strengthened by adding to $\overline{LF4}$, in a cutting-plane fashion, capacity constraint (3) and all other inequalities known for the CVRP that are violated in the current solution such as comb, extended comb, generalized capacity, and hypotour inequalities, and so on (see Naddef and Rinaldi 2000). These latter are expressed in terms of variables $\{\xi_{ij}\}$ defined in $F1$, but can be added to $\overline{LF4}$ once variables $\{\xi_{ij}\}$ are replaced with variables $\{x_{ij}\}$ using Equation (32). However, implementing these inequalities for the CVRP not only requires a large amount of computational effort (see Augerat et al. 1995), but also makes little contribution to the relevant theory. Therefore, the main focus of this paper is to investigate an exact branch-and-cut algorithm for the CVRP based on the new relaxation $\overline{LF4}$ strengthened with only capacity constraint (3).

Capacity Constraints. Let $\mathbf{x}$ be a feasible solution to $\overline{LF4}$. The solution $\boldsymbol{\xi}$ obtained from $\mathbf{x}$ using Equation (32) might violate the capacity constraint (3) that is implicitly satisfied by any feasible solution to $F4$. In that case, the constraint

$$\frac{1}{Q}\sum_{i\in S}\sum_{j\in\bar{S}}(x_{ij}+x_{ji})\geqslant 2\lceil q(S)/Q\rceil\quad \forall S\in\mathcal{S}, \tag{40}$$

that is obtained from inequality (3) by substituting ξ_{ij} with $(x_{ij} + x_{ji})/Q$ is a valid inequality for the CVRP and is added to $\overline{LF4}$ in a cutting-plane fashion.

3.3. Separation of Capacity Constraints

The separation problem for capacity constraint (3) (and, in a similar way, for (40)) is NP-complete (see Augerat et al. 1995). Augerat et al. (1995, 1998) and Ralphs et al. (2003) design several separation heuristics for these constraints. In our bounding procedure, we used the heuristic, called the *greedy randomized algorithm*, proposed by Augerat et al. (1995).

The greedy randomized algorithm is an iterative procedure that is applied to a number of subsets of customers $\overline{\mathcal{S}} \subset \mathcal{S}$ generated a priori. At each iteration, the following procedure is repeated for each $S \in \overline{\mathcal{S}}$. Let $i^* \in V'\backslash S$ be the customer such that

$$\sum_{j\in S}(x_{i^*j} + x_{ji^*}) = \max_{i\in V'\backslash S}\left[\sum_{j\in S}(x_{ij} + x_{ji})\right].$$

If the current solution $\mathbf{x}$ violates the capacity constraint (40) corresponding to the subset $S' = S \cup \{i^*\}$, then this inequality is added to $\overline{LF4}$, S is updated as $S = S'$, and the procedure is repeated until S contains all customers V'. In our implementation the initial family $\overline{\mathcal{S}}$ was built by randomly generating $10n$ subsets of customers. Augerat et al. (1995) show that the overall time complexity of this algorithm is $O(n^3)$.

4. A Branch-and-Cut Algorithm

In this section, we describe a branch-and-cut algorithm for the exact solution of the CVRP based on relaxation $\overline{LF4}$.

At each node of the branch and cut the lower bound is computed by solving problem $\overline{LF4}$ containing all trivial, flow, and capacity inequalities identified so far; and the constraints derived from branching. A node of the branch and cut is removed if the bound solution is a feasible CVRP solution or the lower bound obtained is not less than the current upper bound. If a node is not removed, the incomplete solution is further divided into two subproblems by branching on a given inequality in the way described below. The subproblem chosen for expansion has the minimum lower bound of those not expanded. The algorithm is built around the CPLEX code (CPLEX 2001). Below we discuss some important implementation issues.

4.1. Computation of the Lower Bound

The lower bound to the CVRP is computed using a cutting-plane based LP procedure. Each iteration of the bounding procedure consists of two sets of inner iterations. First, a maximum number of iterations *ITERTF* is performed to identify violated trivial and flow inequalities and, second, a maximum number of iterations *ITERC* is performed to identify violated capacity constraints. A maximum number of 200 distinct constraints are generated during the identification of violated capacity constraints. Each of these inner iterations solves the linear program resulting from $\overline{LF4}$ and those inequalities identified during the previous iterations. The overall bounding procedure terminates when at least one of the following conditions is satisfied:

1. No further violated inequalities can be generated.
2. The lower bound did not increase during a certain number of iterations *ITERLB*.
3. A maximum number of iterations *ITERMAX* has been achieved.

The following setting of parameters was used to obtain the computational results reported in §5: $ITERTF = 5$, $ITERC = 5$, $ITERLB = 5$, and $ITERMAX = 10$.

All the violated capacity constraints found are permanently stored in a global data structure, called the *pool*. Whenever a capacity constraint introduced in the LP problem is slack for four consecutive LP solutions it is removed from the LP problem. The pool is checked before applying the cutting-plane procedure for violated capacity constraints.

4.2. Variable Reduction

At the root node of the branch-and-cut algorithm the number of $\overline{LF4}$ variables is reduced based on the following observations: Denote by $\bar{z}(i,j)$ the optimal solution cost of $\overline{LF4}$ obtained by imposing $x_{ij} + x_{ji} = Q$ (i.e., forcing edge $\{i,j\} \in \bar{E}$ to be in the optimal CVRP solution). The two flow variables x_{ij} and x_{ji} can be removed from problem $\overline{LF4}$ (hence edge $\{i,j\}$ can be removed from $\bar{E}$) if $\bar{z}(i,j) \geqslant z(UB)$, where $z(UB)$ is a known upper bound to the CVRP. A possible way to compute a lower bound to $\bar{z}(i,j)$ is as follows: Let $\{c'_{ij}\}$ be the reduced costs of variables $\{x_{ij}\}$ corresponding to the optimal solution of $\overline{LF4}$. A lower bound for $\bar{z}(i,j)$ can be obtained by fixing edge $\{i,j\}$ in the $\overline{LF4}$ solution. This can be computed by solving the following LP:

$$\bar{z}(i,j) = z(\overline{LF4}) + \frac{1}{Q}\min_{\Delta_{ij}^{\min}\leqslant\Delta\leqslant\Delta_{ij}^{\max}}\left[c'_{ij}\Delta + c'_{ji}(Q-\Delta)\right], \quad (41)$$

where Δ is the value of flow variable x_{ij} and, consequently, $Q - \Delta$ is the value of x_{ji} satisfying the additional constraint $x_{ij} + x_{ji} = Q$.

$\Delta_{ij}^{\min}$ and $\Delta_{ij}^{\max}$ represent the minimum and maximum values of variable x_{ij} in any feasible CVRP solution containing edge $\{i,j\}$, respectively. The following cases are considered:

1. $i, j \in V'$. Then, $x_{ij} \geqslant q_j$ and $x_{ji} \geqslant q_i$, so that $\Delta_{ij}^{\min} = q_j$ and $\Delta_{ij}^{\max} = Q - q_i$.
2. $i = 0$ and $j \in V'$. Then, $x_{0j} \geqslant \max[0, q(V') - (M-1)Q]$ and $x_{j0} \geqslant 0$, so that $\Delta_{0j}^{\min} = \max[0, q(V') - (M-1)Q]$ and $\Delta_{0j}^{\max} = Q$.
3. $i \in V'$ and $j = 0$. Similar to case 2, $\Delta_{i0}^{\min} = 0$ and $\Delta_{i0}^{\max} = Q - \max[0, q(V') - (M-1)Q]$.
4. $i = n+1$, $j \in V'$. Then, $x_{ij} = Q$, so that $\Delta_{n+1j}^{\min} = \Delta_{n+1j}^{\max} = Q$.

4.3. Branching

Two branching strategies have been implemented: branching on sets and branching on variables. Branching on sets has been used in the branch-and-cut algorithm for the CVRP by Augerat et al. (1995). This strategy consists of choosing a subset S, $S \subseteq V'$, $S \neq \varnothing$, for which $0 < p(S) < 2$, where

$$p(S) = \frac{1}{Q}\sum_{i\in S}\sum_{j\in \bar{S}}(x_{ij}+x_{ji}) - 2\lceil q(S)/Q\rceil,$$

and creating two subproblems: one by adding the constraint

$$\frac{1}{Q}\sum_{i\in S}\sum_{j\in \bar{S}}(x_{ij}+x_{ji}) = 2\lceil q(S)/Q\rceil,$$

and the other by adding the constraint

$$\frac{1}{Q}\sum_{i\in S}\sum_{j\in \bar{S}}(x_{ij}+x_{ji}) \geqslant 2\lceil q(S)/Q\rceil + 2.$$

The selection of subset S is carried out in two steps: First, a candidate list of subsets is built heuristically and, second, one subset is selected from this list according to some criterion. The list of candidate subsets is constructed by the same heuristic algorithm used for identifying the capacity constraints. An initial subset of customer subsets is randomly generated and the greedy randomized algorithm is used to expand this subset to generate a new list where each subset S satisfies $0 < p(S) < 2$. Four subsets are selected according to the following criterion (proposed by Augerat et al. 1995):

- Select set S_1 with maximum demand.
- Select set S_2 which is farthest from the depot.
- Select set S_3 such that $(1/Q)\sum_{i\in S_3}\sum_{j\in \bar{S}_3}(x_{ij}+x_{ji})$ is as close as possible to 3.
- Select set S_4 such that $(1/Q)\sum_{i\in S_4}\sum_{j\in \bar{S}_4}(x_{ij}+x_{ji})$ is as close as possible to 2.75.

One out of the above four subsets is subsequently chosen for branching using the method described by Applegate et al. (1994) for the TSP. For each subset $S \in \{S_1, S_2, S_3, S_4\}$ we solve the two corresponding subproblems and compute the minimum of the increases in the lower bounds with respect to the lower bound of the current node. The subset that leads to the maximum of these minimum increases is finally selected.

When a suitable set S cannot be found using the greedy randomized algorithm, the branching on variables strategy is adopted. This involves the selection of an edge $\{i, j\}$ having a fractional value of $(x_{ij}+x_{ji})/Q$, and the generation of two subproblems: one by fixing $x_{ij}+x_{ji} = Q$ and the other by fixing $x_{ij}+x_{ji} = 0$. The edge $\{i, j\}$ is selected in such a way that is as close as possible to 0.5 and ties are broken by choosing the edge $\{i, j\}$ having maximum cost c_{ij}.

5. Computational Results

The new branch-and-cut algorithm described in this paper has been coded in Fortran 77 and run on an IBM PC equipped with a Pentium III 933 MHz processor. CPLEX 6.5 (CPLEX 2001) was used as the LP solver.

We have considered two classes of test problems.

The first class consists of test problems taken from the literature. Table 1 provides details of these problem instances: source reference, number of customers, number of vehicles, vehicle capacity, and vehicle capacity utilization $\%UT = 100q(V')/MQ$. The data of all instances can be found in the TSPLIB (Reinelt 1991) and are also available at www.branchandcut.org/VRP/. Following Naddef and Rinaldi (2000), for all problems reported in Table 1, the edge cost c_{ij} is an integer value computed as $c_{ij} = \lfloor e_{ij} + 0.5\rfloor$, where e_{ij} is the Euclidean distance between nodes i and j. Tables 2–4 present computational results for this class of test problems. Furthermore, these tables give results for three more problem instances whose names are preceded by an "f." Each one of these corresponds to an instance whose identification code start with "F" but whose edge cost c_{ij} is a real value computed by taking the first three decimal digits of the corresponding real Euclidean distance.

The second class consists of randomly generated Euclidean instances in which the coordinates of the customers were uniformly generated in the interval [0, 100] and the depot position was fixed at coordinate [50, 50]. The cost c_{ij} is a real value computed by taking the first two decimal digits of the Euclidean distance between nodes i and j. Customer demands were uniformly generated in the interval [5, 35], whereas the vehicle capacity was specified in such a way to allow control over the vehicle

Table 1. Problem data.

Problem	n	M	Q	$\%UT$	Source
C-n16-k5	15	5	55	93.8	CMT81
C-n16-k3	15	3	90	95.6	CMT81
C-n21-k6	20	6	58	94.5	CMT81
C-n21-k4	20	4	85	96.8	CMT81
E-n22-k4	21	4	60	93.8	CMT81
E-n22-k6	21	6	40	93.8	CMT81
E-n23-k3	22	3	4,500	75.5	CE69
E-n30-k3	29	3	4,500	94.4	CE69
E-n33-k4	32	4	8,000	91.8	CE69
F-n45-k4	44	4	2,010	89.8	FIS94
E-n51-k5	50	5	160	97.1	CE69
F-n72-k4	71	4	30,000	95.7	FIS94
E-n76-k10	75	10	140	97.4	CE69
E-n76-k7	75	7	220	88.6	CE69
E-n76-k8	75	8	180	94.7	CE69
E-n76-k14	75	14	100	97.4	CE69
E-n101-k8	100	8	200	91.1	CE69
M-n101-k10	100	10	200	90.5	CMT79
F-n135-k7	134	7	2,210	94.5	FIS94

Note. CE69: Christofides and Eilon (1969); CMT79: Christofides et al. (1979); CMT81: Christofides et al. (1981a); FIS94: Fisher (1994).

Table 2. Test problems from the literature: lower bounds.

Problem	$z(UB)$	Two-Commodity LB_0	$\%LB_0$	LB_1	$\%LB_1$	t_{LB_1}	Augerat et al. (1995) $\%LB_0^A$	$\%LB_1^A$	$t_{LB_1^A}$	Mingozzi et al. (1994) $\%LB^M$	t_{LB^M}
C-n16-k5	333	290.509	87.24	321.445	96.53	0.9	—	—	—	97.9	1
C-n16-k3	277	243.261	87.82	265.505	95.85	0.6	—	—	—	97.4	1
C-n21-k6	430	371.778	86.46	426.560	99.20	1.0	—	—	—	100.0	1
C-n21-k4	358	309.706	86.51	346.508	96.79	0.8	—	—	—	100.0	2
E-n22-k4	375	309.975	82.66	375.000	100.00	0.7	100.00	100.00	2.0	100.0	1
E-n22-k6	495	395.555	79.91	484.011	97.78	1.3	—	—	—	97.4	2
E-n23-k3	569	485.357	85.30	569.000	100.00	0.7	100.00	100.00	1.0	—	—
E-n30-k3	534	417.535	78.19	508.475	95.22	2.4	95.22	100.00	17.0	—	—
E-n33-k4	835	737.639	88.34	832.996	99.76	2.5	99.82	100.00	8.0	—	—
F-n45-k4	724	549.661	75.92	724.000	100.00	2.4	100.00	100.00	7.0	—	—
f-n45-k4	723.541	549.529	75.95	723.541	100.00	2.3	—	100.00	19.0	—	—
E-n51-k5	521	470.671	90.34	514.540	98.76	4.7	98.76	99.34	31.0	99.3	84
F-n72-k4	237	204.294	86.20	232.497	98.10	6.6	98.10	99.16	82.0	—	—
f-n72-k4	241.974	196.071	81.03	238.658	98.63	7.5	—	99.35	29.0	—	—
E-n76-k10	830	709.318	85.46	792.152	95.44	24.4	95.11	95.61	761.0	97.8	206
E-n76-k7	683	596.874	87.39	661.007	96.78	22.7	96.82	97.27	236.0	97.4	320
E-n76-k8	735	635.996	86.53	711.921	96.86	19.3	96.76	97.09	351.0	97.8	251
E-n76-k14	1,021	854.781	83.72	960.659	94.09	33.6	—	93.42	466.0	—	—
E-n101-k8	817	730.807	89.45	795.595	97.38	43.5	97.47	97.85	494.0	97.4	404
M-n101-k10	820	734.556	89.58	820.000	100.00	47.7	99.94	100.00	472.0	99.5	382
F-n135-k7	1,162	933.899	80.37	1,157.120	99.58	408.5	99.68	99.77	1,428.0	—	—
f-n135-k7	1,162.957	934.202	80.33	1,156.094	99.41	320.9	—	99.68	1,098.0	—	—
Averages:	All instances		84.30		98.01						
	Augerat et al. (1995)		83.93		98.24			98.74			
	Mingozzi et al. (1994)		86.61		97.61					98.49	

capacity utilization. The number of vehicles M and the vehicle capacity utilization $\%UT$ were initially set to specific values and the vehicle capacity was computed as $Q = \lceil 100q(V')/\%UTM \rceil$. Each instance was checked for consistency by testing whether M was equal to the minimum number of bins of capacity Q needed to load all the demands. If this was not the case, a new instance was generated. For this computational experimentation, the following parameters were used: $n =$ 30, 40, 50, 60, 70, 80, 90, 100, $M = 3, 5, 8$, and $\%UT = 95$ (this corresponds to average vehicle utilization of many real-world CVRPs). Overall, for each pair (n, M) 10 problem instances were generated. All the random instances are available from the authors. Table 5 gives the results for this class of test problems.

The results obtained by the branch-and-cut algorithm described in this paper are compared with those reported for the same test problems in Mingozzi et al. (1994)

Table 3. Test problems from the literature: optimal solutions.

Problem	z^*	Two-Commodity Nodes	Time	Augerat et al. (1995) Nodes	Time	Mingozzi et al. (1994) Time
C-n16-k5	333	18	2	—	—	1
C-n16-k3	277	68	4	—	—	1
C-n21-k6	430	4	1	—	—	1
C-n21-k4	358	52	6	—	—	2
E-n22-k4	375	0	1	0	2	1
E-n22-k6	495	80	17	—	—	2
E-n23-k3	569	0	1	0	1	—
E-n33-k4	835	32	28	0	8	—
F-n45-k4	724	0	2	0	7	—
f-n45-k4	723.541	0	2	0	19	—
E-n51-k5	521	28	52	7	54	128
F-n72-k4	237	956	3,429	51	180	—
f-n72-k4	241.973	562	1,270	31	98	—
M-n101-k10	820	2	73	0	472	—
F-n135-k7	1,162	237	6,599	423	20,570	—
f-n135-k7	1,162.955	319	8,918	633	15,774	—

Table 4. Test problems from the literature: number of cuts in the computation of lower bound LB_1.

Problem	Trivial	Flow	Capacity	Subtour	Total
C-n16-k5	17	44	89	2	150
C-n16-k3	13	29	41	2	83
C-n21-k6	19	50	170	3	239
C-n21-k4	18	44	88	0	150
E-n22-k4	16	34	142	3	192
E-n22-k6	22	58	266	0	346
E-n23-k3	17	29	36	2	82
E-n30-k3	19	55	182	14	256
E-n33-k4	21	102	300	11	423
F-n45-k4	27	89	409	22	525
f-n45-k4	26	93	540	44	659
E-n51-k5	34	101	370	19	505
F-n72-k4	40	127	507	83	674
f-n72-k4	40	114	953	148	1,107
E-n76-k10	52	226	904	17	1,182
E-n76-k7	49	148	830	13	1,027
E-n76-k8	45	177	811	6	1,033
E-n76-k14	68	315	996	8	1,379
E-n101-k8	60	219	975	37	1,254
M-n101-k10	73	278	1,877	90	2,228
F-n135-k7	39	507	4,099	310	4,645
f-n135-k7	20	359	3,313	312	3,692

and Augerat et al. (1995) reported in Naddef and Rinaldi (2000).

The following notation is used in this section:

z^*: cost of the optimal CVRP solution or cost of the best solution found by the branch-and-cut algorithm or $z(UB)$.

$z(UB)$: upper bound value; in our computational results the upper bound for a given CVRP instance is computed as $z(UB) = \min[z^1, z^2]$, where z^1 is the cost of the best solution so far known in the literature and z^2 is the cost of the solution obtained with our implementation of the Tabu search algorithm proposed by Gendreau et al. (1994). For the second class of test problems we have $z(UB) = z^2$.

LB_0: lower bound corresponding to the optimal solution cost of problem $\overline{LF4}$ without trivial inequality (39) and flow inequality (36).

LB_1: lower bound obtained at the root node after adding trivial inequalities, flow inequalities, and capacity constraints (see §4.1).

LB_0^A: lower bound produced by Augerat et al. (1995) considering only capacity constraints.

LB_1^A: lower bound produced by Augerat et al. (1995) including a complete set of valid inequalities.

LB^M: lower bound produced by Mingozzi et al. (1994).

LB_{BC}: lower bound of the branch-and-cut tree node with minimum lower bound at the end of the branch-and-cut algorithm.

Table 2 compares different lower bounds and reports the corresponding average percentage errors. The table shows the following columns:

$\%LB_0$: percentage ratio LB_0/z^*.

$\%LB_1$: percentage ratio LB_1/z^*.

t_{LB_1}: computing time of lower bound LB_1 (in seconds).

$\%LB_0^A$: percentage ratio LB_0^A/z^*.

$\%LB_1^A$: percentage ratio LB_1^A/z^*.

$t_{LB_1^A}$: computing time of lower bound LB_1^A (in seconds of a Sun Sparc 20 machine).

$\%LB^M$: percentage ratio LB^M/z^*.

t_{LB^M}: computing time of lower bound LB^M (in seconds of a Silicon Graphics Indy MIPS R4400/200 MHz processor).

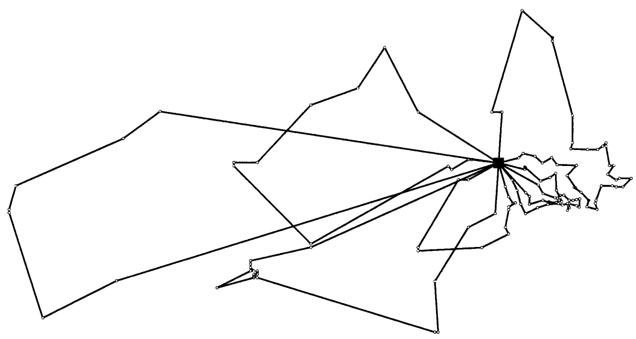
Table 2 shows that the average ratio of the lower bound LB_1 to the optimal solution value computed on all problem instances is equal to 98.01, which clearly indicates that the addition of valid inequalities substantially improves the value of lower bound LB_0. Details on the various types of cuts required to compute lower bound LB_1 are given in Table 4. (The column Subtour reports the number of capacity constraints having the bin-packing lower bound $\lceil q(S)/Q \rceil$ equal to 1.) A comparison between lower bound LB_1 and the one computed by Augerat et al. (1995) shows that LB_1 and LB_0^A are of similar quality. Moreover, the results show that the improvement on the quality of the lower bounds obtained by Augerat et al. due to other classes of valid inequalities (comb and extended comb inequalities, generalized capacity inequalities, and hypotour inequalities) is on average equal to 0.5%.

Table 3 reports the number of nodes generated (Nodes) and the total computing time (Time) of our exact method and the ones proposed by Augerat et al. (1995) and by Mingozzi et al. (1994). This table shows that our branch-and-cut algorithm was capable of solving the largest instance

Table 5. Randomly generated test problems: average results.

	$M=3$					$M=5$					$M=8$				
n	$\%z(UB)$	$\%LB_1$	$\%LB_{BC}$	n_{opt}	Time	$\%z(UB)$	$\%LB_1$	$\%LB_{BC}$	n_{opt}	Time	$\%z(UB)$	$\%LB_1$	$\%LB_{BC}$	n_{opt}	Time
30	100.01	99.52	—	10	1.63	100.00	98.98	—	10	10.93	101.59	97.66	99.43	7	39.66
40	100.00	99.48	—	10	3.20	100.00	98.73	—	10	203.19	100.43	98.29	98.14	9	609.34
50	100.03	98.76	—	10	140.76	100.02	99.04	—	10	157.56	102.46	97.55	99.29	1	18.81
60	100.02	98.65	—	10	197.16	100.05	98.99	—	10	501.86	101.78	97.76	98.92	4	1,772.69
70	100.84	98.57	99.12	9	337.69	100.51	99.06	99.83	7	349.90	102.52	97.55	98.82	0	0.00
80	101.05	98.24	99.31	7	425.16	100.72	98.55	99.25	7	834.15	102.27	97.62	98.57	2	2,247.96
90	101.31	98.26	99.37	5	921.80	101.39	98.33	98.71	5	1,251.97	102.29	97.68	98.33	1	2,323.35
100	101.41	98.14	98.77	5	615.97	100.97	98.62	98.74	6	1,304.03	102.85	97.24	97.99	0	0.00

Figure 2. Test problem F-n135-k7: Layout of the optimal solution.



ever solved in the literature (F-n135-k7 and f-n135-k7) with 135 customers. The layout and details of the optimal solution of problem F-n135-k7 are displayed in Figure 2 and Table 6, respectively.

It is worth noting that, although the branch-and-cut algorithm was able to solve optimally an instance with 135 customers, some of the most difficult instances proposed in the literature with 75 customers still could not be solved to optimality in reasonable computing time by our algorithm. These instances still remain some of the most difficult CVRPs proposed in the literature. Two of these instances (E-n76-k7 and E-n76-k8) are solved for the first time by Ralphs et al. (2003) using a parallel branch-and-cut algorithm.

Table 5 summarizes the performance of our branch-and-cut algorithm for randomly generated test problems with $M = 3, 5$, and 8 vehicles. The table gives the following average results, computed over 10 instances, for each value of M and n:

$$\%z(UB) = \begin{cases} 100z(UB)/z^* & \text{if the instance has been solved to optimality,} \\ 100z(UB)/LB_1 & \text{otherwise.} \end{cases}$$

$\%LB_1$: percentage ratio LB_1/z^*.

$\%LB_{BC}$: percentage ratio LB_{BC}/z^*.

n_{opt}: number of instances solved to optimality out of 10.

Time: average computing time in seconds for the problems solved to optimality within a time limit of 3,600 CPU seconds.

The results show the effectiveness of the branch-and-cut algorithm in solving CVRP instances with up to 80 customers within a reasonable computing time, having $M = 3$ or $M = 5$ vehicles and vehicle utilization equal to 95%. Note that only a few of the instances having $M = 8$ could be solved to optimality within the imposed time limit of one hour.

6. Conclusions

In this paper, we considered the CVRP, in which a given fleet of delivery vehicles of uniform capacity must service customers with known demands from a central depot at minimum routing cost. A new integer programming formulation of the CVRP based on a two-commodity network flow approach was investigated. We obtained a new tight lower bound derived from the LP relaxation of this formulation, which was further improved by adding valid inequalities. By comparing our lower bound to the lower bounds known in the literature, obtained from the LP relaxation of different CVRP formulations, we were able to prove the effectiveness of the new lower bound and to show that it is competitive with the best lower bound reported in the literature.

We developed a new branch-and-cut algorithm for the optimal solution of the CVRP incorporating the new lower bound. Computational experimentation with problems taken from the literature and from new randomly generated test problems has demonstrated that, overall, the algorithm has promising performance. The algorithm solved to optimality a problem involving 135 customers, which is the largest instance ever solved in the literature. Moreover, it successfully solved instances involving up to 80 customers, five vehicles, and 95% vehicle utilization.

Appendix

Proof of Lemma 1. A feasible solution $\boldsymbol{\xi}$ of $\overline{LF1}$ can be obtained from a solution $\boldsymbol{\zeta}$ of $LF3$ as follows:

$$\xi_{ij} = \sum_{r \in \mathcal{R}_{ij}} \eta^r_{ij} \zeta_r \quad \forall \{i, j\} \in E, \tag{42}$$

where $\mathcal{R}_{ij} \subset \mathcal{R}$ is the subset of routes covering edge $\{i, j\}$ and the coefficients $\{\eta^r_{ij}\}$ are defined as follows:

• If r is a single-customer route covering customer $k \in V'$, then $\eta^r_{0k} = 2$ and $\eta^r_{ij} = 0\ \forall i, j \in V, i < j, j \neq k$.

Table 6. Test problem F-n135-k7: details of the optimal solution.

Route	Cardinality	Load	Cost	Customers
1	41	2,145	188	1 21 83 84 85 86 87 88 90 91 17 14 16 89 15 13 12 11 10 9 8 7 6 5 3 43 42 4 41 45 44 46 94 95 30 93 29 28 27 26 22 92 1
2	25	2,147	66	1 73 48 33 35 50 51 103 102 101 100 36 37 99 38 96 40 39 97 98 106 58 56 55 62 61 1
3	17	2,059	54	1 76 2 49 63 52 53 54 104 105 57 59 31 32 60 23 25 24 1
4	13	1,864	88	1 67 72 34 81 68 80 64 65 78 135 77 75 74 1
5	7	2,209	336	1 116 115 107 108 109 110 121 1
6	19	2,149	225	1 79 134 69 71 70 111 124 126 113 112 125 123 122 128 127 129 130 114 82 1
7	12	2,047	205	1 47 119 18 19 133 117 132 118 120 131 66 20 1

• If r is not a single-customer route, then $\eta_{ij}^r = 1$ for each edge $\{i, j\}$ covered by route r and is 0 otherwise.

It is easy to verify that the fractional vector $\boldsymbol{\xi}$ defined by expression (42) satisfies constraints (2) and (4) of $\overline{LF1}$; we omit the details for the sake of brevity.

Let us consider for each $S \in \mathcal{S}$ the surrogate constraint obtained by adding Equation (20) corresponding to customers S after having multiplied the equation associated with $i \in S$ by q_i:

$$\sum_{r\in\mathcal{R}(S)} b_r(S)\zeta_r = q(S), \tag{43}$$

where $b_r(S) = \sum_{i\in S} q_i a_{ir}$ $\forall\, r \in \mathcal{R}$ and $\mathcal{R}(S) = \{r\colon r \in \mathcal{R}, b_r(S) > 0\}$. Equation (43) implies that every $LF3$ solution satisfies the following inequalities:

$$\sum_{r\in\mathcal{R}(S)} \zeta_r \geqslant \mu(S), \tag{44}$$

where

$$\left.\begin{aligned} &\mu(S) = \text{minimize} \sum_{r\in\mathcal{R}(S)} \zeta_r \\ &\text{subject to (43) and} \\ &0 \leqslant \zeta_r \leqslant 1 \quad \forall\, r \in \mathcal{R}(S). \end{aligned}\right\} \tag{45}$$

Because $b_r(S) \leqslant \min[Q, q(S)]$, we have

$$\mu(S) \geqslant \max[1, q(S)/Q]. \tag{46}$$

Note that $\mu(S)$ can be smaller than $\lceil q(S)/Q \rceil$ for some $S \in \mathcal{S}$. However, we have $\mu(S) \geqslant \lceil q(S)/Q \rceil$ for any subset $S \in \mathcal{S}$ such that $q(S) \geqslant Q$ and $b_r(S) \leqslant (1/2)Q$ $\forall\, r \in \mathcal{R}(S)$.

In the following we show that constraints (43), (44), and (46) imply that the vector $\boldsymbol{\xi}$ derived from expression (42) for a given $LF3$ solution $\boldsymbol{\zeta}$ satisfies the multistar constraint (24) and subtour elimination constraint (25), thus proving that $\boldsymbol{\xi}$ is a feasible $\overline{LF1}$ solution.

(A) Inequality (43) implies multistar constraint (24). Because $b_r(S) + b_r(\bar{S}) \leqslant Q$ $\forall\, r \in \mathcal{R}$ and $\forall\, S \in \mathcal{S}$, we have

$$\sum_{r\in\mathcal{R}(S)} Q\zeta_r \geqslant \sum_{r\in\mathcal{R}(S)} b_r(S)\zeta_r + \sum_{r\in\mathcal{R}(S)} b_r(\bar{S})\zeta_r \quad \forall\, S \in \mathcal{S}. \tag{47}$$

From Equation (43) and inequality (47) we deduce that every $LF3$ solution satisfies the following relaxation of constraint (43):

$$\sum_{r\in\mathcal{R}(S)} \zeta_r \geqslant \frac{q(S)}{Q} + \frac{1}{Q} \sum_{r\in\mathcal{R}(S)} b_r(\bar{S})\zeta_r \quad \forall\, S \in \mathcal{S}. \tag{48}$$

Note that any route $r \in \mathcal{R}(S)$ contains at least two edges, one having an ending node in S and the other in $\bar{S}$. Therefore, for any subset $S \in \mathcal{S}$, we have

$$\sum_{\substack{i\in S\\ j\in\bar{S}\\ i<j}} \eta_{ij}^r\zeta_r + \sum_{\substack{i\in\bar{S}\\ j\in S\\ i<j}} \eta_{ij}^r\zeta_r \geqslant 2\zeta_r \quad \forall\, r \in \mathcal{R}(S). \tag{49}$$

For every edge $\{i, j\}$ with $i \in S$ and $j \in \bar{S}$ or with $i \in \bar{S}$ and $j \in S$ we have

$$\sum_{r\in\mathcal{R}(S)} \eta_{ij}^r\zeta_r = \sum_{r\in\mathcal{R}_{ij}} \eta_{ij}^r\zeta_r \quad \text{as } \mathcal{R}_{ij} \subseteq \mathcal{R}(S).$$

Therefore, adding inequality (49) we obtain

$$\sum_{\substack{i\in S\\ j\in\bar{S}\\ i<j}} \sum_{r\in\mathcal{R}_{ij}} \eta_{ij}^r\zeta_r + \sum_{\substack{i\in\bar{S}\\ j\in S\\ i<j}} \sum_{r\in\mathcal{R}_{ij}} \eta_{ij}^r\zeta_r \geqslant 2 \sum_{r\in\mathcal{R}(S)} \zeta_r. \tag{50}$$

Using expression (42) from inequalities (50) and (48) we have

$$\sum_{\substack{i\in S\\ j\in\bar{S}\\ i<j}} \xi_{ij} + \sum_{\substack{i\in\bar{S}\\ j\in S\\ i<j}} \xi_{ij} \geqslant 2\frac{q(S)}{Q} + \frac{2}{Q} \sum_{r\in\mathcal{R}(S)} b_r(\bar{S})\zeta_r. \tag{51}$$

From the definition of coefficients $\{\eta_{ij}^r\}$ we have

$$b_r(\bar{S}) \geqslant \sum_{\substack{i\in S\\ j\in\bar{S}\\ i<j}} q_j\eta_{ij}^r + \sum_{\substack{i\in\bar{S}\\ j\in S\\ i<j}} q_i\eta_{ij}^r, \tag{52}$$

therefore, from expression (42) and inequality (52) we obtain

$$\sum_{\substack{i\in S\\ j\in\bar{S}\\ i<j}} q_j\xi_{ij} + \sum_{\substack{i\in\bar{S}\\ j\in S\\ i<j}} q_i\xi_{ij} \leqslant \sum_{r\in\mathcal{R}(S)} b_r(\bar{S})\zeta_r. \tag{53}$$

Finally, from inequalities (51) and (53) we have the multistar inequality (24).

(B) Inequalities (44) and (45) imply subtour elimination constraint (25). From inequality (50), replacing $\sum_{r\in\mathcal{R}_{ij}} \eta_{ij}^r\zeta_r$ with ξ_{ij} and $\sum_{r\in\mathcal{R}(S)} \zeta_r$ with $\mu(S)$, according to inequality (44), we have

$$\sum_{\substack{i\in S\\ j\in\bar{S}\\ i<j}} \xi_{ij} + \sum_{\substack{i\in\bar{S}\\ j\in S\\ i<j}} \xi_{ij} \geqslant 2\mu(S). \tag{54}$$

Because $\mu(S) \geqslant 1$ the subtour elimination constraint (25) is a relaxation of inequality (54) and is satisfied by any vector $\boldsymbol{\xi}$ derived from the $LF3$ solution. It is quite simple to show that the cost $z(LF3)$ of the $LF3$ solution $\boldsymbol{\zeta}$ is equal to the cost $z(\overline{LF1})$ of the corresponding $\overline{LF1}$ solution $\boldsymbol{\xi}$ given by expression (42). □

Proof of Corollary 2. It is easy to produce CVRP examples where an optimal $\overline{LF1}$ solution $\boldsymbol{\xi}^*$ saturates the multistar constraint of a given subset S, which in the $LF3$ model implies that

$$\mu(S) > q(S)/Q + \frac{1}{Q}\left(\sum_{\substack{i\in S\\ j\in\bar{S}\\ i<j}} \xi_{ij}^* + \sum_{\substack{i\in S\\ j\in\bar{S}\\ i<j}} \xi_{ij}^*\right).$$

Therefore, the solution $\boldsymbol{\xi}^*$ cannot correspond to any $LF3$ solution, which implicity satisfies constraint (44). □

Proof of Corollary 3. From Corollaries 1 and 2 it follows that $LF1$ does not dominate $LF3$. However, according to expression (42) the $\overline{LF1}$ solution $\boldsymbol{\xi}$ corresponding to a $LF3$ solution $\boldsymbol{\zeta}$ may violate capacity constraint (3); therefore, $LF3$ does not dominate $LF1$. □

Proof of Lemma 2. Any $LF3$ solution $\boldsymbol{\zeta}$ of cost $z(LF3)$ can be transformed into a $LF2$ solution $(\mathbf{y}, \boldsymbol{\xi})$ of cost $z(LF2) = z(LF3)$ by using the following algorithm:

Step 1. Initialize $y^l_{ij} = 0\ \forall\, i, j \in V,\ i \neq j,\ l \in V'$, and $\xi_{ij} = 0\ \forall\, i, j \in V, i \neq j$.

Step 2. Let $\overline{\mathcal{R}}$ be the subset of routes that are in the $LF3$ solution $\boldsymbol{\zeta}$, that is, $\overline{\mathcal{R}} = \{r\colon r \in \mathcal{R}, \zeta_r > 0\}$. Let us indicate with $R_r = (i_0, i_1, \ldots, i_{k_r}, i_{k_r+1})$, where $i_0 = i_{k_r+1} = 0$, the sequence of customers visited by route $r \in \overline{\mathcal{R}}$. In the case of the symmetric CVRP we assume that R_r denotes one of the two possible orientations of route r. For each route $r \in \overline{\mathcal{R}}$ and for each arc (i_s, i_{s+1}), $s = 0, 1, \ldots, k_r - 1$, of route R_r, define

$$y^{i_l}_{i_s i_{s+1}} = y^{i_l}_{i_s i_{s+1}} + q_{i_l}\zeta_r, \quad l = s+1, \ldots, k_r. \tag{55}$$

Step 3. Let $\overline{\mathcal{R}}(i, j)$ be the subset of routes of $\overline{\mathcal{R}}$ covering arc (i, j). Define

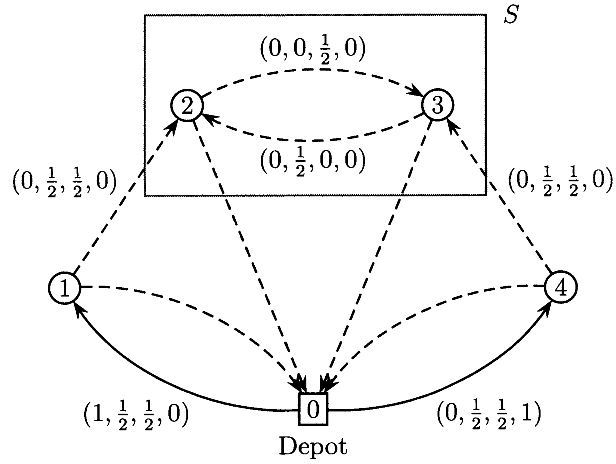
$$\xi_{ij} = \sum_{r \in \overline{\mathcal{R}}(i,j)} \zeta_r \quad \forall\, i, j \in V, i \neq j. \tag{56}$$

It is easy to verify that the vectors $\mathbf{y}$ and $\boldsymbol{\xi}$ as defined above satisfy constraints (9)–(17). Moreover, in case of the symmetric CVRP, the two vectors $\mathbf{y}$ and $\boldsymbol{\xi}$ represent a feasible $LF2$ solution independently of the orientation given in Step 2 to each route R_r, $r \in \overline{\mathcal{R}}$. It can be shown that the vector $\boldsymbol{\xi}$ defined by expression (56) satisfies the subtour elimination constraint (18) and the following version of the multistar constraint for the asymmetric CVRP:

$$\sum_{\substack{i \in S\\ j \in \bar{S}}} \xi_{ij} + \sum_{\substack{i \in \bar{S}\\ j \in S}} \xi_{ij} \geqslant 2\frac{q(S)}{Q} + \frac{2}{Q}\left(\sum_{\substack{i \in S\\ j \in \bar{S}}} q_j \xi_{ij} + \sum_{\substack{i \in \bar{S}\\ j \in S}} q_i \xi_{ij}\right) \quad \forall\, S \in \mathcal{S}. \tag{57}$$

The lemma follows from the observation that an $LF2$ solution may violate the multistar inequality (57). Figure 3 shows an $LF2$ fractional solution for a CVRP instance with $Q = 2$, $M = 2$, and $q_i = 1$, $i = 1, \ldots, 4$. The bold lines indicate variables $\xi_{ij} = 1$ and the dotted lines indicate variables $\xi_{ij} = 1/2$. The four values $(y^1_{ij}, y^2_{ij}, y^3_{ij}, y^4_{ij})$ are given for each arc (i, j) unless $y^l_{ij} = 0, l = 1, \ldots, 4$, as it is for arcs $(i, 0), i = 1, \ldots, 4$. This $LF2$ solution violates the multistar inequality (57) for $S = \{2, 3\}$ because the left-hand side of (57) is equal to 2 but the right-hand side is equal to $2 + (1/2 + 1/2) = 3$. □

Figure 3. A fractional $LF2$ solution violating a multistar inequality.



Proof of Lemma 3.

(A) Every CVRP solution provides a feasible solution $(\mathbf{x}, \boldsymbol{\xi})$ to (28)–(35) obtained as follows:

1. For each route $R = (i_1, i_2, \ldots, i_{|R|})$ of a CVRP solution, define
   - $\xi_{ij} = 1\ \forall \{i, j\} \in \bar{E}(R)$,
   - $x_{i_1 i_2} = \sum_{i \in V'(R)} q_i$ and $x_{i_k i_{k+1}} = x_{i_{k-1} i_k} - q_{i_k}$, $k = 2, \ldots, |R| - 1$,
   - $x_{i_k i_{k-1}} = Q - x_{i_{k-1} i_k}$, $k = 2, \ldots, |R| - 1$.
2. Set $\xi_{ij} = 0$ and $x_{ij} = x_{ji} = 0$, for each edge $\{i, j\}$ not belonging to any route of the CVRP solution. It is easy to verify that variables $(\mathbf{x}, \boldsymbol{\xi})$ as defined above are a feasible solution of (28)–(35).

(B) Every solution $(\mathbf{x}, \boldsymbol{\xi})$ of (28)–(35) is a CVRP solution. Let F be the subset of edges of $\bar{G}$ corresponding to all edges $\{i, j\} \in \bar{E}$ such that $\xi_{ij} = 1$; i.e., $F = \{\{i, j\}\colon \{i, j\} \in \bar{E}$ and $\xi_{ij} = 1\}$. Consider the subgraph $G(F) = (V, F)$ induced by F. The degree of each node of V' in $G(F)$ is 2 (see Equation (33)), hence the paths in $G(F)$ from node 0 to $n+1$ are simple and pairwise disjoint. By adding Equations (29) and (30) and by replacing in the resulting equation $x_{0j} + x_{j0}$ with $Q\xi_{0j}$, we show that the degree of node 0 in $G(F)$ is M. Similarly, from Equations (31) and (32) and the fact that $x_{jn+1} = 0\ \forall\, j \in V'$, the degree of node $n+1$ in $G(F)$ is M. Moreover, it can be shown that subgraph $G(F)$ does not contain subtours. Adding flow constraint (28) of customers in $S \in \mathcal{S}$, we have

$$\sum_{i \in S}\left(\sum_{j \in S}(x_{ji} - x_{ij}) + \sum_{j \in \bar{S}}(x_{ji} - x_{ij})\right) = 2q(S) \quad \forall\, S \in \mathcal{S}. \tag{58}$$

Because $\sum_{i \in S}(\sum_{j \in S}(x_{ji} - x_{ij})) = 0$ and $x_{ji} + x_{ij} \geqslant x_{ji} - x_{ij}$, from Equation (58) we obtain the following inequality:

$$\sum_{i \in S}\sum_{j \in \bar{S}}(x_{ij} + x_{ji}) \geqslant 2q(S) \quad \forall\, S \in \mathcal{S}. \tag{59}$$

Using Equation (32) and the integrality constraint (35), Inequality (59) becomes

$$\sum_{\substack{i\in S\\ j\in \bar{S}\\ i<j}} \xi_{ij} + \sum_{\substack{i\in \bar{S}\\ j\in S\\ i<j}} \xi_{ij} \geqslant 2\lceil q(S)/Q\rceil \quad \forall S\in \mathcal{S}. \tag{60}$$

Because Inequality (60) is satisfied for any subset S of customers, the subgraph $G(F)$ is connected and the total demand of the customers in any simple path from 0 to $n+1$ does not exceed the vehicle capacity. Thus, $G(F)$ corresponds to M routes defining a feasible CVRP solution. □

Proof of Lemma 4. Inequality (36) is satisfied by the two flow variables x_{ij} and x_{ji} associated with each edge $\{i, j\}$ such that $\xi_{ij} = 0$ because from Equation (32) we have $x_{ij} = x_{ji} = 0$.

Let $\{i, j\}$ and $\{j, k\}$ be the two edges incident to customer j in a given $F4$ solution (i.e., $\xi_{ij} = 1$ and $\xi_{jk} = 1$). From Equations (28) and (32) we have

$$x_{ij} - x_{ji} + x_{kj} - x_{jk} = 2q_j, \tag{61}$$

$$x_{ij} + x_{ji} = Q. \tag{62}$$

Adding Equations (61) and (62) we obtain

$$x_{ij} + x_{kj} = q_j + Q. \tag{63}$$

Because $x_{kj} \leqslant Q$ it follows from Equation (63) that $x_{ij} \geqslant q_j$. Hence, any feasible solution of $F4$ satisfies the following inequalities:

$$x_{ij} \geqslant q_j\xi_{ij} \quad \text{and} \quad x_{ji} \geqslant q_i\xi_{ij} \quad \forall \{i, j\} \in \bar{E}. \tag{64}$$

Inequality (36) follows directly from Inequality (64) and Equation (32). □

Proof of Lemma 5. Adding the flow conservation constraint (28) associated with the customers of a given subset $S \in \mathcal{S}$, we obtain

$$\sum_{\substack{i\in S\\ j\in \bar{S}}} x_{ji} = 2q(S) + \sum_{\substack{i\in S\\ j\in \bar{S}}} x_{ij} \quad \forall S\in \mathcal{S}. \tag{65}$$

Flow constraint (36) implies the following:

$$\left.\begin{aligned} x_{ji} &\leqslant (Q - q_j)(x_{ij} + x_{ji})/Q \\ x_{ij} &\geqslant q_j(x_{ij} + x_{ji})/Q \end{aligned}\right\} \quad \forall \{i, j\} \in \bar{E}. \tag{66}$$

Therefore, from Equation (65) and Inequality (66) we obtain

$$\sum_{\substack{i\in S\\ j\in \bar{S}}} (x_{ij} + x_{ji})/Q \geqslant 2q(S)/Q + \frac{2}{Q}\sum_{\substack{i\in S\\ j\in \bar{S}}} q_j(x_{ij} + x_{ji})/Q \quad \forall S\in \mathcal{S}. \tag{67}$$

Using Equation (32) to replace $\mathbf{x}$ with $\boldsymbol{\xi}$, Inequality (67) becomes the multistar Inequality (24) and Equation (38) becomes Equations (2). □

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