Parallel Computing
Advanced Computing

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Linear Equations

Linear Equations

$$\left\{\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N & = & b_2 \\ \cdots & & \cdots \\ a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N & = & b_M \end{array}\right.$$

N=M: the problem has a solution **if** the equations are linearly independent

Linear Equations

- When solving a linear system of equations numerically, we may be confronted with two effects caused by the same numerical problem:
  - Equations can "become" linearly dependent due to rounding errors.
  - Accumulation of rounding errors can lead to a "wrong" solution.

Linear Equations

$$\left\{\begin{array}{lcl}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 & = & \omega_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 & = & \omega_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 & = & \omega_3 \\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 & = & \omega_4
\end{array}\right.$$

$$\Leftrightarrow \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix}$$

Linear Equations

- The basic idea to solve such a system is a straight-forward substitution:

$$\left\{\begin{array}{rcl} 3x+2y & = & 5 \\ x+y & = & 3 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} x & = & \frac{5-2y}{3} \\ x+y & = & 3 \end{array}\right. \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{rcl} x & = & \frac{5-2y}{3} \\ \frac{5-2y}{3}+y & = & 3 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} x & = & \frac{5-2y}{3} \\ y & = & 4 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} x & = & -1 \\ y & = & 4 \end{array}\right.$$

Linear Equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}-\frac{a_{21}a_{12}}{a_{11}} & a_{23}-\frac{a_{21}a_{13}}{a_{11}} & a_{24}-\frac{a_{21}a_{14}}{a_{11}} \\ 0 & a_{32}-\frac{a_{31}a_{12}}{a_{11}} & a_{33}-\frac{a_{31}a_{13}}{a_{11}} & a_{34}-\frac{a_{31}a_{14}}{a_{11}} \\ 0 & a_{42}-\frac{a_{41}a_{12}}{a_{11}} & a_{43}-\frac{a_{41}a_{13}}{a_{11}} & a_{44}-\frac{a_{41}a_{14}}{a_{11}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 - \frac{a_{21}\omega_1}{a_{11}} \\ \omega_3 - \frac{a_{31}\omega_1}{a_{11}} \\ \omega_4 - \frac{a_{41}\omega_1}{a_{11}} \end{pmatrix}$$

...

Upper triangular matrix:

$$\begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & a_{34}^{(3)} \\ 0 & 0 & 0 & a_{44}^{(4)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \omega_1^{(1)} \\ \omega_2^{(2)} \\ \omega_3^{(3)} \\ \omega_4^{(4)} \end{pmatrix}$$

At this point, the problem is solved!

Linear Equations

- The general formula for the construction of the upper triangular matrix associated with the **Gauss elimination method** for a system of n equations is the following:

$$a_{jk}^{(m+1)} = a_{jk}^{(m)} - \frac{a_{jm}^{(m)} a_{mk}^{(m)}}{a_{mm}^{(m)}} \qquad \begin{aligned} a_{jk}^{(1)} &= a_{jk} \\ j, k &= m+1, \ldots, n \end{aligned}$$

$$\omega_{j}^{(m+1)} = \omega_{j}^{(m)} - \frac{a_{jm}^{(m)} \omega_{m}^{(m)}}{a_{mm}^{(m)}} \qquad \begin{aligned} \omega_{j}^{(1)} &= \omega_{j} \\ j \quad &= m+1, \ldots, n \end{aligned}$$

Linear Equations

- The solution is then obtained by **Backward Substitution**:

$$x_m = \frac{1}{a_{mm}^{(m)}}\left(\omega_m^{(m)} - \sum_{k=m+1}^{n} a_{mk}^{(m)} x_k\right)$$

$$m = n, n-1, n-2, \ldots, 1$$

This method requires ***n***

(of algebraic calculations)

Linear Equations

- However, this elimination method has a "weak point":

$$a_{jk}^{(m+1)} = a_{jk}^{(m)} - \frac{a_{jm}^{(m)} a_{mk}^{(m)}}{a_{mm}^{(m)}}$$

If this number is small, the division result can be a large number that will be added to a potentially small number...

One way to avoid this type of problems is to "move" the small numbers away from the diagonal, exchanging lines and/or columns of the matrices/vectors of the problem.

Linear Equations

Partial pivoting (per column):
exchange the lines of A and the vector/matrix on the right side in order to place the largest value in the "problematic" column, i.e. in the diagonal.

$$\begin{pmatrix} 1 & 3 & 4 & 6 \\ 0 & 10^{-8} & 198 & 19 \\ 0 & -91 & 51 & 9 \\ 0 & 7 & 76 & 541 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 4 & 6 \\ 0 & -91 & 51 & 9 \\ 0 & 10^{-8} & 198 & 19 \\ 0 & 7 & 76 & 541 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_3 \\ \omega_2 \\ \omega_4 \end{pmatrix}$$

Linear Equations

- In order to minimise rounding errors, the pivoting operation should be performed at each step of the triangularisation process, systematically placing the highest value element of the column under analysis on the diagonal.
- But this can bring a lot of overhead to the process, so it's sometimes more efficient to only pivot when a problematic small number appears in the diagonal.
- The choice depends on the balance between required precision and available computing resources.

Linear Equations

Complete pivoting (both by row and column): exchange the rows and columns of A and the vectors X and ω in order to place the largest diagonal value in the column and row of the "problematic" element.

$$\begin{pmatrix} 1 & 3 & 4 & 6 \\ 0 & 10^{-8} & 198 & 19 \\ 0 & -91 & 51 & 9 \\ 0 & 7 & 76 & 541 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 3 & 6 \\ 0 & 198 & 10^{-8} & 19 \\ 0 & 51 & -91 & 9 \\ 0 & 76 & 7 & 541 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix}$$

Linear Equations

- Ideally, from the point of view of numerical accuracy, we should do a complete pivoting in order to ensure that the largest element of the sub-matrix still to be treated is always in the diagonal...
- But we can choose to pivot only when there is a very small number on the diagonal...
- Be sure to keep a record of all line and column exchange operations!