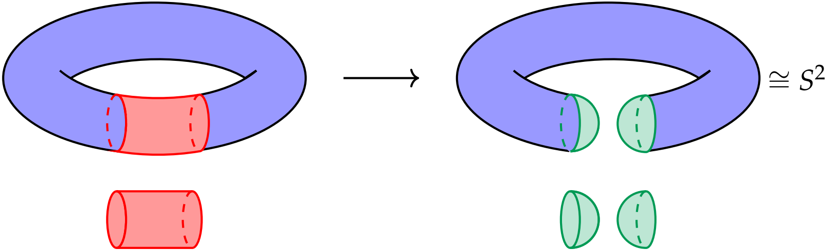
Combinatorial Tools in Equivariant Algebraic Topology

slides available at www.davidmehrle.com/csu.pdf

Part I
The Conjectures

Kervaire Problem (1960)

Can every smooth n-manifold become a sphere via surgery?



Surgery

Replace a submanifold by another with the same boundary

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(1963–1984) counterexamples with $n = 2, 6, 14, 30, 62$

(1969) Browder: counterexamples must have $n = 2^k - 2$

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Theorem (Lin–Wang–Xu, December 2024)

There exist counterexample 126-manifolds (nonconstructive)

Telescope Conjecture (Ravenel, 1984)

Two ways of computing $\pi_k S^n$ are the same

Theorem (Burklund–Hahn–Levy–Schlank, 2023)

The Telescope Conjecture is false!

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- Group actions are rigid; homotopy theory is floppy
- Computations are tough, even for small finite groups
- Needs new algebraic tools: Mackey/Tambara functors

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The better we understand these tools, the more we can do!

Part II
Equivariant Algebraic Topology

Let G be a finite group.

DEFINITION

A G-space is a topological space X with a G-action $G \times X \to X$:

$$1 \cdot x = x$$
$$g \cdot (h \cdot x) = (gh) \cdot x$$

such that $x \mapsto g \cdot x$ is continuous for all $g \in G$.

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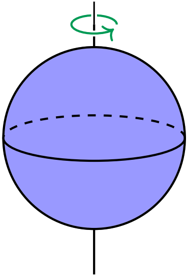
DEFINITION

A G-equivariant map $f\colon X \to Y$ is a continuous map such that

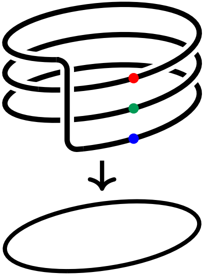
$$f(g \cdot x) = g \cdot f(x)$$

for all $g \in G$ and all $x \in X$.

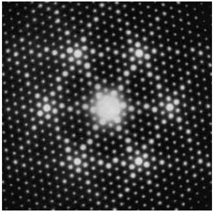
Examples



Rotation
$\mathbb{Z}/n$



Covering Spaces
$\mathbb{Z}/3$



$PbCr_3S_4$ crystals
Dihedral Group D_6

In algebraic topology, we study spaces using **invariants**

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$$X \simeq Y \implies F(X) \cong F(Y)$$

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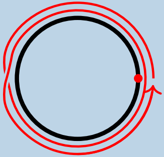
EXAMPLE

The fundamental group $\pi_1(X)$ is the group of homotopy classes of based loops in X.



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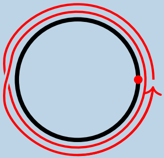

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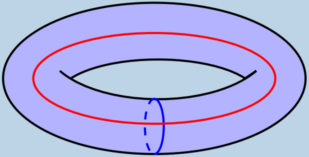

$\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$

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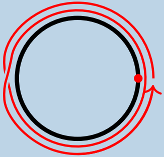

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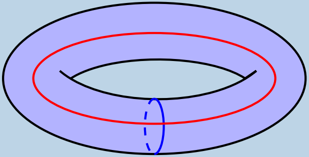

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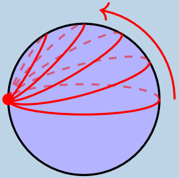
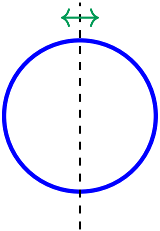


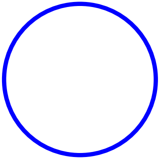
$\pi_1(S^2) = 0$

QUESTION

How do you see group actions on spaces using invariants?



$\mathbb{Z}/2$ acts by reflection



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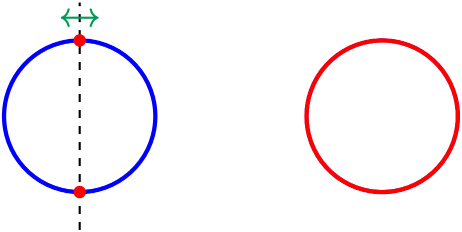

$\mathbb{Z}/2$ acts by reflection
fixed points S^0

$\mathbb{Z}/2$ acts trivially
fixed points S^1

Answer

Consider the fixed point subspaces X^G as well!

$$X^G := \left\{ x \in X \;\middle|\; g \cdot x = x \text{ for all } g \in G \right\} \subseteq X$$

The Data Pipeline

$G \circlearrowright X$

G-space

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$$\begin{tikzcd}
 & X^G \arrow[dl, hook] \arrow[dr, hook] & \\
X^H \arrow[dr, hook] & & X^K \arrow[dl, hook] \\
 & X &
\end{tikzcd}$$

G-space $\longrightarrow$ Fixed Point Subspaces of X

Theorem (Elmendorf, Piacenza, 1991)

For the purposes of homotopy theory, we may replace a G-space X by its collection of fixed point subspaces $\{X^H\}_{H \subseteq G}$

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\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \pi_1(X^G) \arrow[dl] \arrow[dr] & \\
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$$G\text{-space} \longrightarrow \boxed{\text{Fixed Point Subspaces of } X} \xrightarrow{\text{invariant}} \boxed{\text{Mackey Functors}^{*}}$$

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$$\begin{tikzcd}
 & \pi_1^{\mathrm{st}}(X^G) \arrow[dl, shift left] \arrow[dr, shift left] & \\
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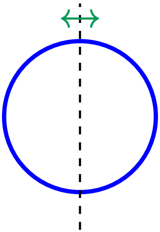
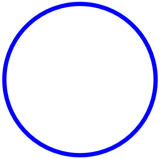
Example

reflection
action



trivial
action



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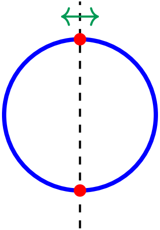

$$\longrightarrow \qquad \begin{array}{c} (S^1)^{\mathbb{Z}/2} \\ \big\downarrow \\ S^1 \end{array}$$

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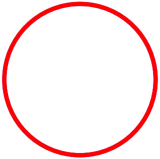

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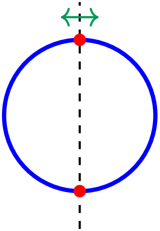
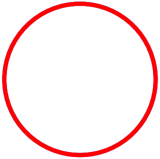


$$\begin{array}{c} \color{red}{S^0} \\ \downarrow \\ S^1 \end{array}$$

trivial action



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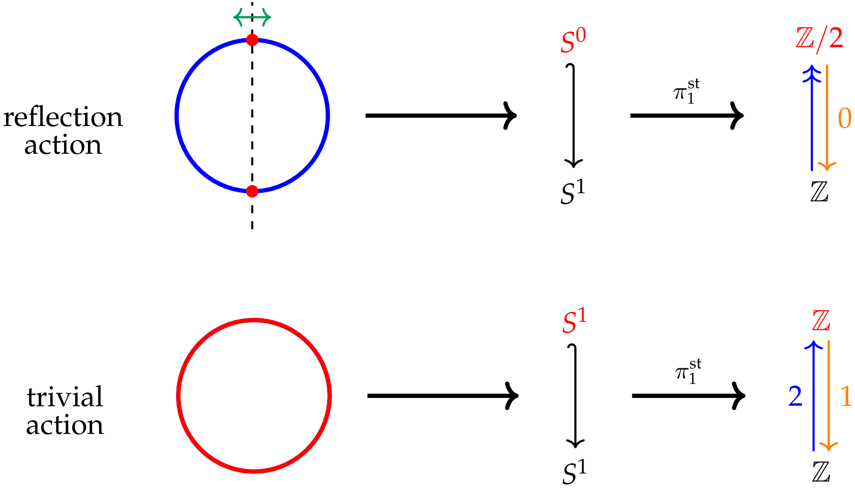

reflection action
S^0
S^1
π_1^{st}
$\mathbb{Z}/2$
$\mathbb{Z}$
0
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2
1

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$$M(G)$$

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$$M(K) \qquad M(gKg^{-1})$$

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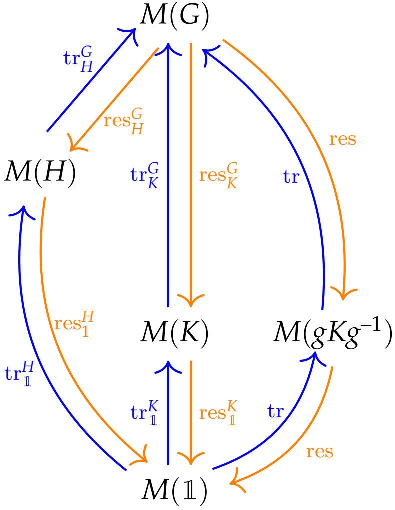

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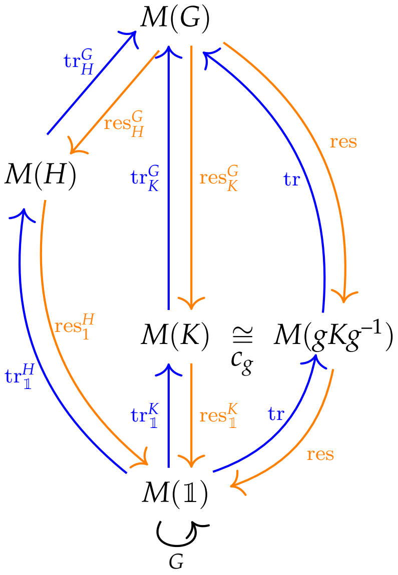
- conjugation isomorphisms

$$c_g \colon M(K) \cong M(gKg^{-1})$$

with a "double coset formula" for $\mathrm{res} \circ \mathrm{tr}$, and other conditions



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$$

$$
\mathrm{res}^{C_p}_{1} \circ \mathrm{tr}^{C_p}_{1}(m) = \sum_{g \in C_p} g \cdot m
$$

Question

Where do the transfers come from?

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Let E/F be a Galois extension with (finite) Galois group $\mathrm{Gal}(E/F)$

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Slogan

Mackey Functors $\pi_1^{\mathrm{st}}(X^H)$ encode a "Galois theory" of G-spaces

Part III
Equivariant Algebra

The category of G-Mackey functors is:

- abelian
- symmetric monoidal
- generated by finitely many projectives

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- abelian $\implies$ algebra!
- symmetric monoidal
- generated by finitely many projectives

The category of G-Mackey functors is:

- abelian $\Longrightarrow$ algebra!
- symmetric monoidal $\Longrightarrow$ commutative algebra!
- generated by finitely many projectives

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EXAMPLE

When $G = \mathbb{1}$, this is just abelian groups and commutative rings

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Transfers are sums over orbits (field-theoretic traces):

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$$a \longmapsto \sum_{\sigma \in \mathrm{Gal}(E^K/E^H)} \sigma(a)$$

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Also have field-theoretic norms:

$$a \longmapsto \prod_{\sigma \in \mathrm{Gal}(E^K/E^H)} \sigma(a)$$

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$T(G)$

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- restriction (ring) homomorphisms

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 & T(\mathbb{1}) &
\end{tikzcd}$$

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$$\mathrm{res}^H_K \colon T(H) \to T(K)$$

- transfer homomorphisms (for $+$)

$$\mathrm{tr}^H_K \colon T(K) \to T(H)$$

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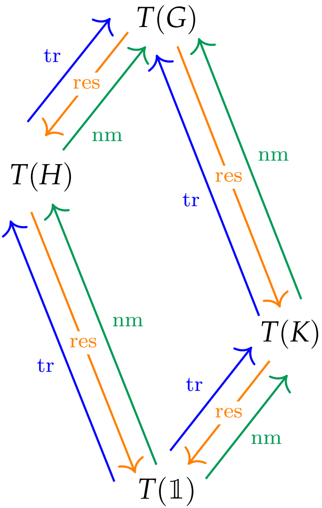
$$\mathrm{tr}^H_K : T(K) \to T(H)$$

- norm homomorphisms (for $\times$)

$$\mathrm{nm}^H_K : T(K) \to T(H)$$



DEFINITION (Tambara, 1993)

A Tambara Functor for G is the data:

- a commutative ring $T(H)$ for each subgroup $H \subseteq G$
- restriction (ring) homomorphisms

$$\mathrm{res}^H_K : T(H) \to T(K)$$

- transfer homomorphisms (for $+$)

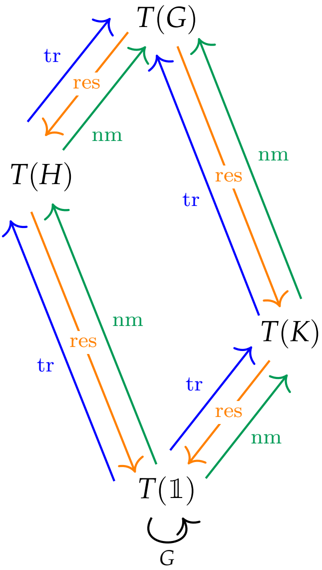
$$\mathrm{tr}^H_K : T(K) \to T(H)$$

- norm homomorphisms (for $\times$)

$$\mathrm{nm}^H_K : T(K) \to T(H)$$

- conjugation isomorphisms

with "double coset formulas" for res $\circ$ tr and res $\circ$ nm, and …



Tambara Functors for Small Groups

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A Tambara functor T for $G = C_p$ is the data:

$$\begin{tikzcd}
T(C_p) \arrow[d, "\mathrm{res}" description] \\
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$$\mathrm{nm}(0) = 0$$

$$\mathrm{res} \circ \mathrm{tr}(x) = \sum_{g \in C_p} g \cdot x \qquad \mathrm{res} \circ \mathrm{nm}(x) = \prod_{g \in C_p} g \cdot x$$

(and other conditions)

Example: Burnside Functor $\mathbb{A}$

For each G, there is a Burnside functor $\mathbb{A}$ with origin in topology. Each $\mathbb{A}(H)$ is built from finite G-sets.

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$\mathbb{A}$ for C_p:

$$\begin{tikzcd}
\mathbb{Z}[t]/(t^2 - pt) \arrow[d, "t \mapsto p"] \\
\mathbb{Z} \arrow[u, bend left, "x \mapsto tx"] \arrow[u, bend right, "x \mapsto x + \left(\frac{x^p - x}{p}\right)t"'] \arrow[loop below, "\text{trivial}"]
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$$

This Tambara functor $\mathbb{A}$ plays the role of $\mathbb{Z}$.

- Every Mackey functor is an $\mathbb{A}$-module
- $\mathbb{A}$ is the initial Tambara functor
- $\mathbb{A}$ is the unit for the tensor product

Definition

Let X be a G-space. The space X^n has actions of both G^n and S_n, which combine to an action of the wreath product $G \wr S_n$.

Representation Theory

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Work-in-Progress (Calle–Chan–Mehrle–Quigley–Spitz–Van Niel)

Every Tambara functor T has a character theory: a Tambara functor $\Gamma(T)$ with $T \hookrightarrow \Gamma(T)$, where $\Gamma(T)$ is easier to study.

Commutative Algebra

Theorem (Nakaoka, 2014)

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Theorem (Chan–Mehrle–Quigley–Spitz–Van Niel, 2024)

Use $\Gamma(\mathbb{A})$ for $G = C_p$ to describe the Tambara affine line, and relate affine Tambara algebraic geometry to invariant theory.

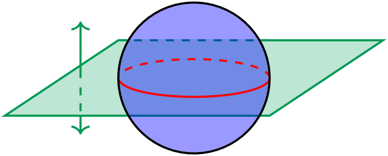
Long-Term Goal

Algebraic geometry and invariant theory of Tambara functors

Part IV
Homotopical Combinatorics

A Non-Example

$\mathbb{Z}/2$ acts on S^2 by
reflection in equator



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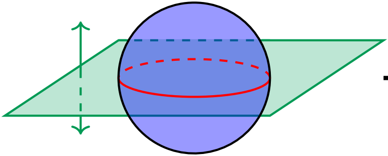

$$\begin{tikzcd} \color{red}{(S^2)^{\mathbb{Z}/2}} \arrow[d, hook] \\ S^2 \end{tikzcd}$$

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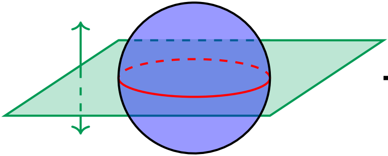

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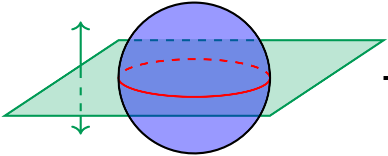

$$\begin{tikzcd} S^1 \arrow[d, hook] \\ S^2 \end{tikzcd} \xrightarrow{\pi_1^{\mathrm{st}}} \begin{tikzcd} \mathbb{Z} \arrow[d, shift left] \\ 0 \arrow[u, shift left] \end{tikzcd}$$

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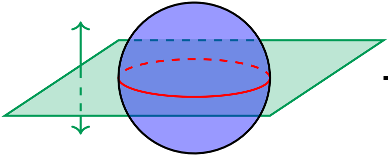

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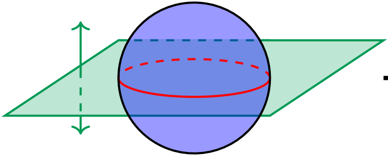

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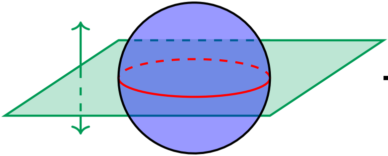

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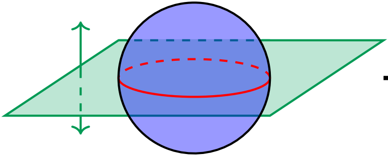

$$\begin{tikzcd} {} \arrow[r] & S^1 \arrow[d] \\ & S^2 \end{tikzcd} \xrightarrow{\pi_1^{st}} \begin{tikzcd} \mathbb{Z} \arrow[d, shift left] \\ 0 \arrow[u, shift left] \end{tikzcd}$$

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Contradiction! No map nm can exist

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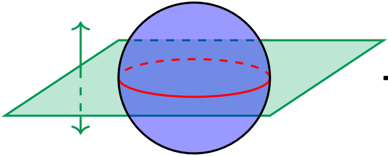

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Want to allow Tambara functors with a subset of the norms

Question

Which combinations of norms are allowable?

Definition (Rubin, 2020)

A G-transfer system is a partial order $\to$ on subgroups of G:

- (refinement) if $K \to H$, then $K \subseteq H$
- (conjugation) if $K \to H$, then $gKg^{-1} \to gHg^{-1}$ for all $g \in G$
- (restriction) if $K \to H$ and $L \subseteq H$, then $K \cap L \to L$

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Transfer systems for $G = C_p$:

$$\begin{array}{c} C_p \\ \\ \mathbb{1} \end{array} \qquad\qquad \begin{array}{c} C_p \\ \uparrow \\ \mathbb{1} \end{array}$$

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Transfer systems for $G = C_p$:

$$\begin{tikzcd} C_p & & C_p \\ & \longrightarrow & \\ \mathbb{1} & & \mathbb{1} \arrow[uu] \end{tikzcd}$$

Transfer systems for $G = C_{p^2}$:

$$\begin{tikzcd} C_{p^2} \\ C_p \\ \mathbb{1} \end{tikzcd}
\qquad
\begin{tikzcd} C_{p^2} \\ C_p \\ \mathbb{1} \arrow[u] \end{tikzcd}
\qquad
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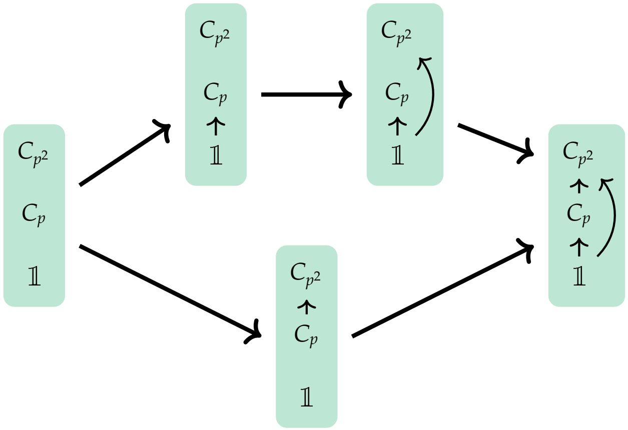

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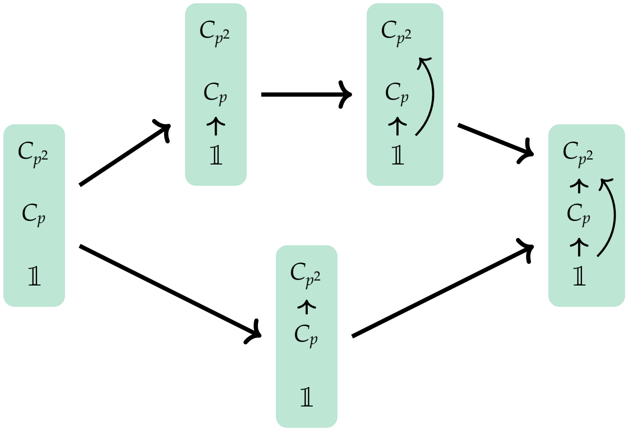

Theorem (Balchin–Barnes–Roitzheim, 2021)

There are $\mathrm{Cat}(n+1) = \frac{1}{n+2}\binom{2n+2}{n+1}$ transfer systems for C_{p^n}.

Incomplete Tambara Functors

Definition

Let τ be a transfer system for G. A τ-Tambara functor is a Tambara functor with only those norms parameterized by τ.

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$$\begin{tikzcd}
C_{p^2} \\
C_p \arrow[u] \\
\mathbb{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
T(C_{p^2}) \arrow[d, "\mathrm{res}"] \\
T(C_p) \arrow[u, bend left, "\mathrm{tr}"] \arrow[u, bend right, "\mathrm{nm}"'] \arrow[d, "\mathrm{res}"] \\
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Free Algebras

$\mathbb{Z}[x]$ is a free algebra with one generator

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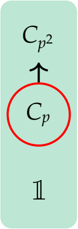

τ

$$\begin{tikzcd}
\mathbb{A}(C_{p^2}) \arrow[d, "\mathrm{res}"] \\
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Let G be solvable. Then $\mathbb{A}[\tau, H]$ is free as a Mackey functor iff:

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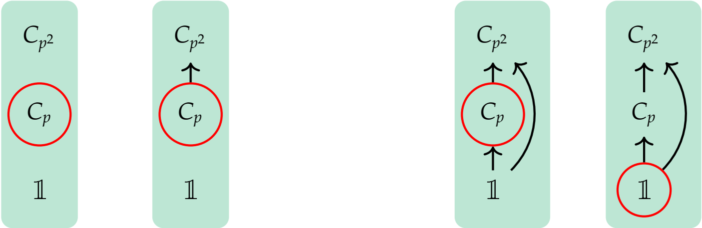

not free | free | not free | free

Combinatorial Question

How many $\mathbb{A}[\tau, H]$ are free?

Group	# subgroups H	# transfer systems τ	# pairs (τ, H)	# free	% free
C_p	2	2	4	2	50
C_{p^2}	3	5	15	4	≈ 27
C_{p^3}	4	14	56	9	≈ 16
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C_{p^n}	$n+1$	$\mathrm{Cat}(n+1)$	$P(n)$	$F(n)$	$\frac{F(n)}{P(n)}$

$$P(n) = (n+1)\,\mathrm{Cat}(n+1) \qquad F(n) = \sum_{i=0}^{n} \mathrm{Cat}(i)$$

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$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} F(i)}{\sum_{i=0}^{n} P(i)} = 0$$

THEOREM (Hill–Mehrle–Quigley, 2023)

Fix a bijection $\mathbb{G}\colon \mathbb{N} \to \{\text{isomorphism classes of finite groups}\}$.

Let $F(G)$ be the number of pairs (τ, H) such that $\mathbb{A}[\tau, H]$ is free.

Let $P(G)$ be the total number of pairs (τ, H) for G.

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} F(\mathbb{G}(i))}{\sum_{i=1}^{n} P(\mathbb{G}(i))} = 0$$

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Let $P(G)$ be the total number of pairs (τ, H) for G.

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} F(\mathbb{G}(i))}{\sum_{i=1}^{n} P(\mathbb{G}(i))} = 0$$

SLOGAN

"Free incomplete Tambara functors are almost never free."

THEOREM (Hill–Mehrle–Quigley, 2023)

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WORK-IN-PROGRESS (Bingham-Franchere-Jones-Mehrle-Shoults-Yousef)

Computer code and recursive formulas to enumerate free transfer systems for $C_{p^n}, C_{pq^n}, C_{p^2q^n}, \ldots$

Theorem (Mehrle–Quigley–Stahlhauer, 2024)

Let G be a cyclic p-group for an odd prime p. If $\mathbb{A}[\tau, H]$ is free, we construct well-behaved Koszul resolutions.

Theorem (Mehrle–Quigley–Stahlhauer, 2024)

Let G be a cyclic p-group, any prime p. If $\mathbb{A}[\tau, H]$ is *not* free, then it is infinite dimensional: there is a module with no finite resolution.

Goal

A theory of minimal resolutions for Tambara functors

Takeaways

Will equivariant algebraic topology solve all our problems?

Not yet.

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Will equivariant algebraic topology solve all our problems?

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Work-in-Progress (Guillou–Keyes–Mehrle)

Apply what we've learned about Tambara functors to make new calculations in equivariant homotopy theory.

Thank you!

Bonus: Mackey Functors

$\mathcal{F}\text{in}^G$ = category of finite G-sets and G-equivariant functions

$\mathcal{S}\text{pan}(\mathcal{F}\text{in}^G)$ = category of finite G-sets and spans of finite G-sets

$$U \leftarrow X \rightarrow V$$

Definition

A Mackey functor is a product-preserving functor

$$M\colon\ \mathcal{S}\text{pan}(\mathcal{F}\text{in}^G) \to \mathcal{A}\text{b}$$

$$M(H) := M(G/H)$$

$$\text{res}^H_K := M\left(G/H \twoheadleftarrow G/K \xrightarrow{\text{id}} G/K\right)$$

$$\text{tr}^H_K := M\left(G/K \xleftarrow{\text{id}} G/K \twoheadrightarrow G/H\right)$$

Bonus: Tambara Functors

$\mathcal{B}\mathrm{ispan}(\mathcal{F}\mathrm{in}^G)$ = category of finite G-sets & bispans of finite G-sets

$$U \leftarrow X \rightarrow Y \rightarrow V$$

Definition

A Tambara functor is a product-preserving functor

$$T\colon\ \mathcal{B}\mathrm{ispan}(\mathcal{F}\mathrm{in}^G) \to \mathcal{S}\mathrm{et}$$

such that each $T(U)$ is a commutative ring

$$\mathrm{res}^H_K := T\left(G/H \twoheadleftarrow G/K \xrightarrow{\mathrm{id}} G/K \xrightarrow{\mathrm{id}} G/K\right)$$

$$\mathrm{tr}^H_K := T\left(G/K \xleftarrow{\mathrm{id}} G/K \xrightarrow{\mathrm{id}} G/K \twoheadrightarrow G/H\right)$$

$$\mathrm{nm}^H_K := T\left(G/K \xleftarrow{\mathrm{id}} G/K \twoheadrightarrow G/H \xrightarrow{\mathrm{id}} G/H\right)$$