SEQUENTIAL SPECTRA 2 TALK OUTLINE

Read [Mal23, Sections 2.4 and 2.5]. The main point of this talk is to discuss the ways in which $\mathcal{S}p$ is a nicer category than the category of spaces – namely, $\mathrm{ho}(\mathcal{S}p)$ has the property of *stability* but $\mathrm{ho}(\mathcal{T}op_*)$ does not.

OUTLINE

(1) Overview and Recollections

  (a) Recall the definition of spectra and maps of spectra [Mal23, Definition 2.1.1]. The category of spectra and the maps described above is denoted $\mathcal{S}p$.

  (b) Spectra are better than spaces insofar as $\mathrm{ho}(\mathcal{S}p)$ is a stable category.

  (c) There are three different ways to formulate stability:
  - Σ and Ω are inverses up to stable equivalence.
  - Cofiber sequences and fiber sequences are the same, up to stable equivalence.
  - Homotopy pushout squares are homotopy pullback squares, up to stable equivalence.

  (d) We will recall definitions of cofiber sequences, fiber sequences, homotopy pushouts, and homotopy pullbacks as we need them.

  (e) Give the warning that although homotopy pushouts are homotopy pullbacks, it is not true that the homotopy colimit of a diagram is the same as the homotopy limit of the same diagram! Finding a counterexample is a homework exercise.

(2) Stability as an equivalence $\Sigma^{-1} \simeq \Omega$.

  (a) Recall the suspension ΣX and loops ΩX of a spectrum X [Mal23, Definitions 2.3.6 and 2.3.8]. Do so without reference to the smash $K \wedge X$ or the function spectrum $F(K, X)$, as much as is possible.

  (b) Give a quick sketch argument as to why Σ is left adjoint to Ω as functors $\mathcal{S}p \to \mathcal{S}p$: these functors are adjoints on $\mathcal{T}op_*$, and we're just doing those operations levelwise on spectra.

  (c) State [Mal23, Proposition 2.4.1] and prove it.

  (d) State [Mal23, Corollary 2.4.5] and prove it. Note that this doesn't reference or require [Mal23, Corollary 2.4.3] – you can skip that one.

  (e) The consequence is that Σ and Ω are inverse functors up to stable equivalence.

(3) Stability in terms of fiber and cofiber sequences

  (a) Define fiber and cofiber sequences of spectra following [Mal23, Definition 2.4.6]. You will need to define a homotopy of maps of spectra too [Mal23, Definition 2.3.11].

  (b) Prove that any fiber sequence of spectra yields a LES in homotopy [Mal23, Lemma 2.4.9].

  (c) Prove that any cofiber sequence of spectra yields a LES in homotopy [Mal23, Lemma 2.4.10]. Make a note that this is not true for spaces!

  (d) Sketch the proof that $\Sigma\,\mathrm{hofib}(f) \simeq \mathrm{hocof}(f)$, where $\mathrm{hofib}(f)$ and $\mathrm{hocof}(f)$ are homotopy fiber and cofiber of f [Mal23, 2.4.12]. It's probably best if you don't go too much into the details, but just indicate the construction of the map and draw the diagram and sketch the proof from there.

  (e) Prove that a sequence is a fiber sequence of spectra if and only if it is a cofiber sequence of spectra [Mal23, Proposition 2.4.13]. This is the big theorem here!

(4) Stability in terms of homotopy pushouts and pullbacks

   (a) Define homotopy pushout square and homotopy pullback square following [Mal23, Definition 2.4.14].
   (b) State but do not prove [Mal23, Lemma 2.4.15]. It will be an exercise on the homework.
   (c) Use the lemma to prove [Mal23, Corollary 2.4.16]. This is perhaps the strongest of the versions of stability introduced at the beginning of the talk.
   (d) Explore some consequences of this last theorem: show that [Mal23, Corollary 2.4.16] implies both [Mal23, Proposition 2.4.13] and [Mal23, Corollary 2.4.5]. To see the latter, note that the homotopy pushout of $* \leftarrow X \rightarrow *$ is ΣX and the homotopy pullback of $* \leftarrow X \rightarrow *$ is ΩX, and consider the homotopy pushout square below (and it's dual).

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

   (e) Show that coproducts and products of spectra are weakly equivalent, following [Mal23, Proposition 2.4.19].

(5) Extra stuff to think about, if you have time or energy left after the above

   (a) Now that you know how to construct cofibers, you can define the spectrum $\mathbb{S}/p$ [Mal23, Example 2.5.29]. This is the cofiber of the map $\Sigma^{-1}\Sigma^{\infty}p\colon \mathbb{S} \to \mathbb{S}$, where $p\colon S^1 \to S^1$ is a degree p map. Explain how this is related to the universal coefficient sequence.
   (b) Define the rationalization of the sphere spectrum $\mathbb{S}_{\mathbb{Q}}$ [Mal23, Example 2.5.30]. By a homework problem, we know that $\mathbb{S}_{\mathbb{Q}}$ is equivalent to $H\mathbb{Q}$. Prove this! (it should be short)
   (c) Define p-localization of a spectrum, and p-completion [Mal23, Examples 2.5.31, 2.5.32]. This can be a quick aside after rationalization.
   (d) Explain how to rationalize or take any spectrum mod-p [Mal23, Example 2.5.33].
   (e) State the theorem that the rationalization of any spectrum is just a sum of shifted copies of $H\mathbb{Q}$, cf. [Mal23, Example 2.5.34]. Give the example for rationalized K-theory $KU_{\mathbb{Q}}$.

References

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.