MATH 1110
Spring 2017

Prelim 1 Review
Solutions

(1) Area: $A(r) = \pi r^2$

Circumference: $C(r) = 2\pi r$

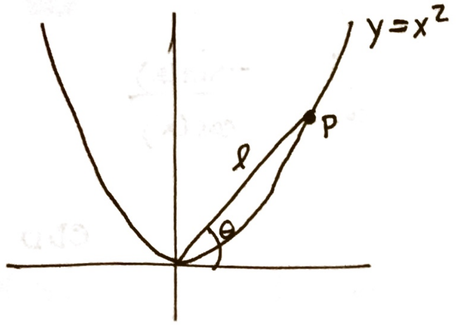
Area as function of circumference: $r = \frac{C}{2\pi}$

$$A(r) = \pi r^2$$

$$A(C) = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{\pi C^2}{4\pi^2} = \frac{C^2}{4\pi}$$

$$\boxed{A(C) = C^2/4\pi}$$

(2)



line of length ℓ forms angle θ with x-axis

P has coordinates (t, t^2) for some real number t.

$$t = \ell\cos\theta$$
$$t^2 = \ell\sin\theta$$

$$\ell^2\cos^2\theta = \ell\sin\theta$$

$$\ell = \frac{\sin\theta}{\cos^2\theta}$$

$$\boxed{\ell = \tan\theta\sec\theta.}$$

(3) Plug in $-x$ to check.

(a) $y(-x) = (-x)^2+1 = x^2+1 = y(x)$ EVEN

(b) $1-\cos(-x) = 1-\cos(x)$ EVEN

(c) $(-x)^5-(-x)^3-(-x) = (-1)^5x^5-(-1)^3x^3-(-x)$

$= -x^5+x^3+x$

$= -(x^5-x^3-x)$ ODD

(d) $\sec(-x)\tan(-x) = \dfrac{1}{\cos(-x)}\dfrac{\sin(-x)}{\cos(-x)} = \dfrac{1}{\cos(x)}\dfrac{-\sin(x)}{\cos(x)}$

$= -\sec(x)\tan(x)$ ODD

(4) Plug in $-x$ to check.

$(fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = (fg)(x)$ EVEN

$(f^3)(-x) = (f(-x))^3 = (-f(x))^3 = -(f(x))^3 = -(f^3)(x)$ ODD

$(f\cdot\sin)(-x) = f(-x)\sin(-x) = (-f(x))(-\sin(x)) = f(x)\sin(x)$ EVEN

$(g\cdot\sec)(-x) = g(-x)\sec(-x) = -g(x)\dfrac{1}{\cos(-x)} = -g(x)\dfrac{1}{\cos(x)}$

$= -g(x)\sec(x)$ ODD

(4) ctd

$$(|g|)(-x) = |g(-x)| = |-g(x)| = |g(x)| \text{ EVEN}$$

(5)

	DOMAIN	RANGE	
(a)	$(-\infty, \infty)$	$[-2, \infty)$	
(b)	$(-\infty, 1]$	$[-2, \infty)$	Must have $\sqrt{1-x}$ with ~~positive~~ nonnegative under radical, so $1-x \geq 0 \Rightarrow x \leq 1$
(c)	$[-4, 4]$	$[0, 4]$	Semi-Circle w/ radius 4 and center at the origin.
(d)	$(-\infty, \infty)$	$(1, \infty)$	Can have any number in the exponent. 3^{2-x} is always strictly greater than zero.

(6) $(f \circ g)(x) = f(g(x))$

$$= 2 - g(x)^2$$

$$= 2 - \left(\sqrt{x+2}\right)^2$$

$$= 2 - (x+2) = x.$$

Domain: $(-\infty, \infty)$

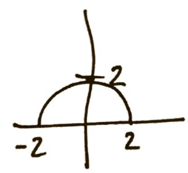
Range: $(-\infty, \infty)$

$(g \circ f)(x) = g(f(x))$

$$= g(2-x^2) = \sqrt{(2-x^2)+2} = \sqrt{4-x^2}$$

Domain: $[-2, 2]$

Range: $[0, 2]$



(7)

shifting up $1/2$ and right 3: $y = g(x-3) + 1/2$

reflect in y-axis $y = g(-x)$

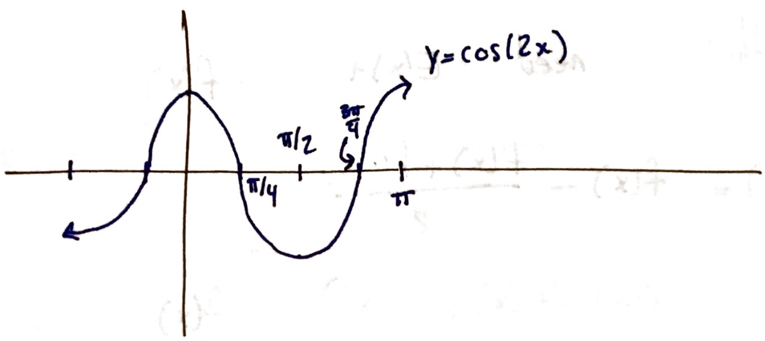
compress horizontally by factor of 5 $y = g(5x)$

(8) $y=f(x-5)$ shifted right 5

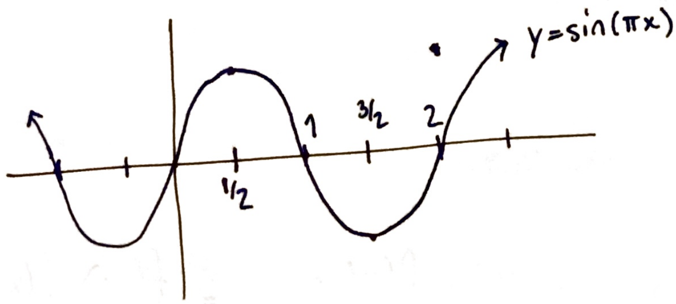
$y=f(-3x)$ reflect in y-axis and compress by factor of 3

$y=\cancel{\ }-3f(x)+1/4$ reflect in x-axis and ~~[illegible]~~ stretch by factor of 3, <u>then</u> shift up 1/4

(9)



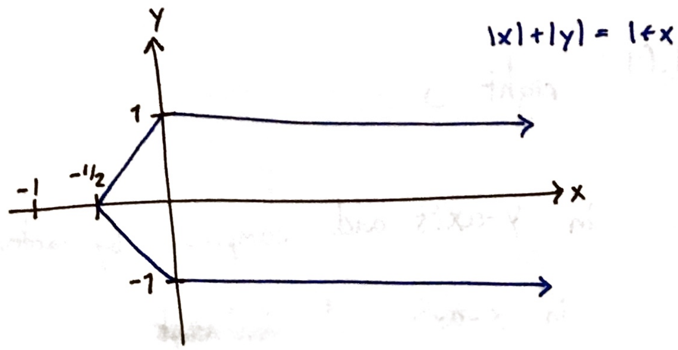
period $= \pi$



period $= 2$

(10) You do not need to be able to do this for the prelim. My apologies for including this question.

(11)



Maybe a bit too hard for a prelim question.

(12) Let $E(x) = \dfrac{f(x)+f(-x)}{2}$

To find $O(x)$, note that we need $E(x)+O(x) = f(x)$, so

$$O(x) = f(x) - E(x) = f(x) - \frac{f(x)+f(-x)}{2}$$

$$= \frac{2f(x) - (f(x)+f(-x))}{2} = \frac{f(x)-f(-x)}{2}$$

Hence $O(x) = \dfrac{f(x)-f(-x)}{2}$

Check E is even: $E(-x) = \dfrac{f(-x)+f(-(-x))}{2} = \dfrac{f(-x)+f(x)}{2} = E(x)$ ✓

Check O is odd: $O(-x) = \dfrac{f(-x)-f(-(-x))}{2} = \dfrac{f(-x)-f(x)}{2} = -O(x)$ ✓

Also, $f(x) = E(x) + O(x)$, as required.

(Also maybe too hard for a prelim question)

(13) $\lim_{t \to t_0} 3f(t) = 3\left(\lim_{t \to t_0} f(t)\right) = 3(-7) = -21$

$\lim_{t \to t_0} f(t)^2 = \left(\lim_{t \to t_0} f(t)\right)^2 = (-7)^2 = 49$

$\lim_{t \to t_0} f(t)\,g(t) = \left(\lim_{t \to t_0} f(t)\right)\left(\lim_{t \to t_0} g(t)\right) = (-7)(0) = 0$

$$\lim_{t \to t_0} \frac{f(t)}{g(t) - 7} = \frac{\lim_{t \to t_0} f(t)}{\lim_{t \to t_0} g(t) - 7} = \frac{-7}{0-7} = 1$$

$\lim_{t \to t_0} \cos(g(t)) = \cos\left(\lim_{t \to t_0} g(t)\right) = \cos(0) = 1.$

(14)

$f(x) = x^{1/3}$ all real numbers

$g(x) = \frac{\sin(x)}{x}$ $(-\infty, 0) \cup (0, \infty)$ (be wary of divide-by-zero)

$h(x) = x^{-2/3} = \frac{1}{\sqrt[3]{x^2}}$ $(-\infty, 0) \cup (0, \infty)$

(15) (a)

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} = \lim_{x\to 1} \frac{\cancel{(1-x)}}{\cancel{(1-x)}(1+\sqrt{x})}$$

$$= \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$$

(b) $$\lim_{x\to a} \frac{x^2-a^2}{x^4-a^4} = \lim_{x\to a} \frac{\cancel{(x-a)}(x+a)}{\cancel{(x-a)}(x^3+ax^2+a^2x+a^3)} = \lim_{x\to a} \frac{x+a}{x^3+ax^2+ax+a^3} = \frac{2a}{4a^3} = \frac{1}{2a^2}$$

(c) Sorry, this one has a typo.

(d) $$\lim_{x\to\pi^-} \csc(x) = \lim_{x\to\pi^-} \frac{1}{\sin(x)} = \infty.$$

As $x \to \pi^-$, that is, x approaches π from the left, $\sin(x)$ is positive, and approaches zero from the positive side of the number line.

(e) $$\lim_{x\to\pi} \sin\left(\frac{x}{2} + \sin(x)\right)$$

$$= \sin\left(\lim_{x\to\pi}\left(\frac{x}{2} + \sin(x)\right)\right)$$

$$= \sin\left(\frac{\pi}{2} + \sin(\pi)\right) = \sin(\pi/2 + 0) = \sin(\pi/2) = 1.$$

(f) $$\lim_{x\to 0} \frac{\cos(2x)-1}{\sin(x)} = \lim_{x\to 0} \frac{\cos^2(x)-\sin^2(x)-1}{\sin(x)}$$

$$= \lim_{x\to 0} \frac{\cancel{\cos^2(x)}-\sin^2(x)-(\cancel{\cos^2(x)}+\sin^2(x))}{\sin(x)}$$

$$= \lim_{x\to 0} \frac{-2\sin^2(x)}{\sin(x)} = \lim_{x\to 0} \frac{-2\sin(x)}{1} = 0$$

(15) ctd

(g) $$\lim_{x\to 0^+} \frac{2e^{1/z}}{e^{1/z}+1} = \lim_{x\to 0^+} \frac{\cancel{e^{1/z}}}{\cancel{e^{1/z}}} \frac{2}{1+1/e^{1/z}}$$

$$= \lim_{x\to 0^+} \frac{2}{1+1/e^{1/z}}$$

$$= \frac{2}{1+\lim_{x\to 0^+} 1/e^{1/z}} = \frac{2}{1+1/e^{\infty}}$$

$$= \frac{2}{1+1/\infty} = \frac{2}{1+0} = \boxed{2}$$

(16) Need to check that the limit as $x \to \pm 1$ of $f(x)$ exists.

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x(x^2-1)}{|x^2-1|}$$

Check the left and right limits separately.

$$\lim_{x\to 1^-} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x\to 1^-} \frac{x(x^2-1)}{-(x^2-1)}$$

$$= \lim_{x\to 1^-} \frac{x}{-1} = -1$$

As $x \to 1^-$, x^2 is less than 1, so $x^2-1<0$.
Therefore, $|x^2-1| = -(x^2-1)$ for $x \to 1^-$.

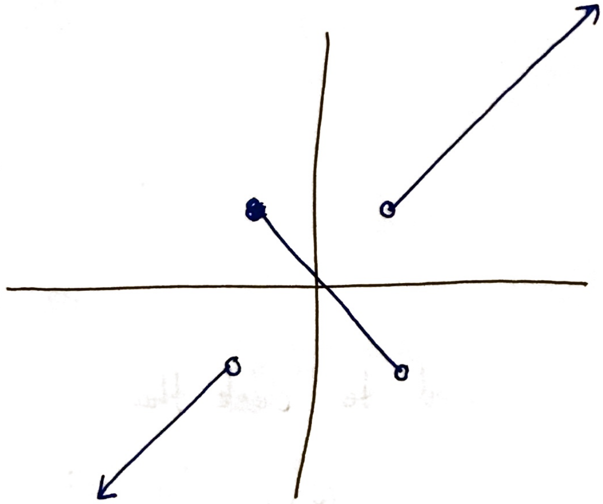
(16) ctd

$$\lim_{x\to 1^+} \frac{x(x^2-1)}{|x^2-1|} = \lim_{x\to 1^+} \frac{x(x^2-1)}{(x^2-1)} = \lim_{x\to 1^+} x = 1.$$

As $x \to 1^+$, $x^2 > 1$
so $x^2-1 > 0$.
Therefore,
$|x^2-1| = x^2-1$ for $x \to 1^+$

The left and right limits do not agree, so we cannot extend $f(x)$ to such a function.

Similarly, cannot extend when $x \to -1$



(17)

(a) $$\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to\infty} \frac{2+3/x}{5+7/x} = \frac{2}{5}$$

(b) $$\lim_{x\to-\infty} \frac{x^2-4x+8}{3x^3} = \lim_{x\to-\infty} \frac{1-4/x+8/x^2}{3x} = 0$$

(c) $-1 \le \sin(x) \le 1$, so $\frac{-1}{\lfloor x \rfloor} \le \frac{\sin(x)}{\lfloor x \rfloor} \le 1/\lfloor x \rfloor$

Squeeze Theorem (aka Sandwich Theorem) $\Longrightarrow \lim_{x\to\infty} \frac{-1}{\lfloor x \rfloor} \le \lim_{x\to\infty} \frac{\sin(x)}{\lfloor x \rfloor} \le \lim_{x\to\infty} \frac{1}{\lfloor x \rfloor}$

$\Longrightarrow 0 \le \lim_{x\to\infty} \frac{\sin(x)}{\lfloor x \rfloor} \le 0 \Longrightarrow \lim_{x\to\infty} \frac{\sin(x)}{\lfloor x \rfloor} = 0$

(17) ctd

(d) $\lim_{x \to \infty} e^{1/x} \cos(1/x) = e^0 \cdot \cos(0) = 1 \cdot 1 = 1$

(18) ~~Bq~~ ~~the~~

Use the Intermediate Value Theorem.

$f(x) = x + 2\cos(x)$

$f(0) = 0 + 2\cos(0) = 2 \quad > 0$

$f(-\pi) = -\pi + 2\cos(-\pi) = -\pi + -2 < 0$

The function $f(x)$ is continuous, $f(-\pi) < 0$, $f(0) > 0$, so there is some $x_0 \in [-\pi, 0]$ such that $f(x_0) = 0$.

This is our one solution. $\blacksquare$