ALGEBRAIC TOPOLOGY – MICHAELMAS 2015

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Date: January 22, 2016.

1. Lecture 1 – Overview and Applications

Algebraic Topology is concerned with the connectivity properties of topological spaces.

Definition 1.1. A space X is *connected* if when we write $X = U \cup V$ for $U, V \subseteq X$, U, V open, $U \cap V = \emptyset$, one of U, V is empty.

Example 1.1. $\mathbf{R}$ is connected, but $\mathbf{R} - \{0\}$ is not.

Corollary 1.1. If $f : \mathbf{R} \to \mathbf{R}$ is continuous and there exist x, y such that $f(x), f(y)$ have opposite signs, then there exists some z such that $f(z) = 0$.

Proof. If not, then $f^{-1}(-\infty, 0)$ and $f^{-1}(0, \infty)$ would disconnect $\mathbf{R}$. □

Remark 1.2. "Nice" spaces are connected if and only if they are path connected. First examples of such spaces are open subsets of $\mathbf{R}^n$ or $\mathbf{C}^n$.

Definition 1.2. X is *path-connected* if for all $x, y \in X$, there exists some $\gamma : [0,1] \to X$ such that $\gamma(0) = x, \gamma(1) = y$.

Another way of stating the above definition is to say that any two maps $\{\text{pt}\} \to X$ can be deformed into each other. This is an example of studying the zero homotopy group of X. The following definition concerns the first homotopy group of X.

Definition 1.3. A connected space X is simply connected if any two continuous maps $S^1 \to X$ can be continuously deformed into one another.

Example 1.2. $\mathbf{R}^2$ is simply connected, $\mathbf{R}^2 - \{0\}$ is not.

However, the fact that $\mathbf{R}^2 - \{0\}$ is not simply connected allows us to define the *winding number* of a curve $\gamma : S^1 \to \mathbf{R}^2 - \{0\}$. Intuitively, this is the number of times that curve wraps around the origin.

Definition 1.4. if $\gamma : S^1 \to \mathbf{R}^2 - \{0\}$ is continuous, then we may define

$$\deg(\gamma) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z} \tag{1.3}$$

which we know from complex analysis is invariant under homotoping γ.

Example 1.4. The loop $t \xrightarrow{\gamma_n} (\cos(2\pi n t), \sin(2\pi n t))$ has degree $n \in \mathbf{Z}$

This theory allows us to prove the following impressive corollary:

Corollary 1.3 (Fundamental Theorem of Algebra)**.** A non-constant polynomial $f \in \mathbf{C}[x]$ has a root (over $\mathbf{C}$).

Proof. Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ and further suppose that f is non-vanishing, that is $\mathrm{Im}(f) \subset \mathbf{C} - \{0\}$.

We can examine f along concentric circles in $\mathbf{C}$ in the following way:

$$\gamma_R(t) := f(Re^{2\pi i t}) : S^1 \to \mathbf{C} - \{0\} \tag{1.5}$$

and note that when $R = 0$, $\gamma_0(t)$ is constant for all t and therefore $\deg(\gamma_0) = 0$.

Now, take $R > \sum |a_i| + 10$.

Let $\gamma_{R,s}(t) = z^n + s(a_1 z^{n-1} + \cdots + a_n)|_{Re^{2\pi i t}}$ for $0 \leq s \leq 1$. This defines a homotopy of our original map f, since $\gamma_{R,1}(t) = \gamma_R(t)$.

Since we chose R to be very large, the z^n term dominates and so $\gamma_{R,1}(t)$ does in fact wrap around the origin n times. However, we may now homotope in two directions: First, we shrink s to zero, then we shrink R to zero. This gives the following:

$$0 = \deg(\gamma_0) = \deg(\gamma_R) = \deg(\gamma_{R,1}) = \deg(\gamma_{R,0}) = \deg(\gamma_n) = n \tag{1.6}$$

which is a contradiction when f is non-constant, i.e. when $n \neq 0$. □

We have an analogous notion of degree of maps of m-dimensional spheres:

Example 1.7. Any two maps from S^m to $\mathbf{R}^{m+1}$ can be continuously deformed into each other. But, a map

$$S^m \to \mathbf{R}^{m+1} - \{0\}$$

has an integer-valued degree, invariant under deformation. The degree of the constant map is zero and the degree of the inclusion map is one.

Definition 1.5.

$$B^n = \{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x}\| = 1\} \tag{1.8}$$

Corollary 1.4 (Brouwer Fixed Point Theorem)**.** Any map $f : B^n \to B^n$ has a fixed point.

Remark 1.5. When $n = 2$, this has a particularly elegant proof. A map without a fixed point would define a homotopy retract from D^2 to S^1, which cannot be allowed since it would imply an isomorphism of fundamental groups. We can extend this proof using homologies of spheres, but we will not calculate $H_*(S^n)$ until the next lecture.

Proof. Suppose f has no fixed point. Then, the map

$$\begin{aligned} \gamma : S^{n-1} &\to \mathbf{R}^n - \{0\} \\ v &\to v - f(v) \end{aligned} \tag{1.9}$$

indeed has image in $\mathbf{R}^n - \{0\}$. As in the previous proof, we will fit the map (1.9) into a two-parameter family of maps and use degrees to arrive at a contradiction.

This family is given by

$$\begin{aligned} \gamma_{R,s} : S^{n-1} &\to \mathbf{R}^n \\ v &\to Rv - sf(Rv). \end{aligned} \tag{1.10}$$

for $0 \leq R, s \leq 1$.

Here, we first note that $\gamma_{R,1}$ never vanishes since f has no fixed points on the *entire* ball, not just the shell, or S^{n-1}. Moreover, $\gamma_{1,s}$ never vanishes because v lies on S^{n-1} and so $\|v\| > \|sf(v)\|$ when $s < 1$ and when $s = 1$, of course $\gamma_{1,1}$ is just our original (non-vanishing) γ.

However, $\deg(\gamma_{0,1}) = 1$, since it is the inclusion map and $\deg(\gamma_{0,1}) = 0$, since it is a constant map. □

Warning 1.11. I'm not entirely confident in this proof. In particular, I'm not sure that for $R, s < 1$, $\gamma_{R,s}$ is always non-vanishing.

Now, we formally introduce homotopy:

Definition 1.6. Let X, Y be topological spaces and $f, g : X \to Y$ be continuous maps. We say f, g are *homotopic* if there exists a map

$$F : X \times [0, 1] \xrightarrow{\text{cts}} Y \tag{1.12}$$

such that $F(x, 0) = f(x), F(x, 1) = g(x)$. We write $f \simeq g$ or $f \simeq_F g$, the latter when he homotopy, F is ambiguous.

We say spaces X, Y are *homotopy equivalent* if there exist

$$\begin{aligned} f &: X \to Y \\ g &: Y \to X \end{aligned} \tag{1.13}$$

such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$

Remark 1.6. Homotopy equivalence does in fact constitute an equivalence relation.

Example 1.14. $S^{n-1} \simeq R^n - \{0\}$.

To show this, let $f : S^{n-1} \to R^n - \{0\}$ be given by inclusion and $g : R^n - \{0\} \to S^{n-1}$ be given by $v \mapsto v/\|v\|$.

We note that $g \circ f$ is in fact the identity on S^{n-1}.

In the other direction, $f \circ g : v \to v/\|v\|$. We want to show this map is homotopic to the identity on $\mathbf{R}^n - 0$. To do this, let

$$\begin{aligned} F : (\mathbf{R}^n - \{0\}) \times [0,1] &\to \mathbf{R}^n - \{0\} \\ (v,t) &\to (1-t)v + t\frac{v}{\|v\|}. \end{aligned} \tag{1.15}$$

Another broad statement: Algebraic topology is the study of functors

$$\{\text{spaces}\} / \simeq \dashrightarrow \{\text{Groups}\} / \text{isomorphism}. \tag{1.16}$$

We care especially about manifolds.

Definition 1.7. A manifold of dimension n is a topological space locally homeomorphic to $\mathbf{R}^n$.

Example 1.17. $\mathbf{RP}^n = S^n / \pm I$

A first example of our inquisition in the vein of (1.16) is the fact that loops in a space X, which start at a fixed base point b, can be concatenated. In this way, they forma group, the fundamental group.

Fact (1).

$$\left\{\text{cts maps } (S^k, *) \to (X, x)\right\} / \{\text{homotopies preserving base points}\} \tag{1.18}$$

form a group, $\pi_k(X,x)$, the k^{th} homotopy group of X.

Fact (2). If X is a connected, simply connected compact manifold and we know $\pi_k(X,x)$ for all k, then X is a point. In other words, computing homotopy groups is VERY hard.

An alternative is to study **cohomology**, which is less intuitive to define but has a richer algebraic structure, making it more computable.

2. Lecture 2 – Chain Complexes and Homology

We will define invariants of a topological space, X in two stages. First, we associate to X a chain complex/co-chain complex. Then, we take the homology of that complex.

Definition 2.1. A *chain complex* is a sequence of abelian groups and homomoprhisms

$$\cdots \to C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} C_{i-1} \to \cdots \tag{2.1}$$

with $\{C_i\}$ indexed by $\mathbf{Z}$ or $\mathbf{N}$ and $d_{i-1} \circ d_i = 0$ for all i.

A *co-chain complex* is a sequence $\{C^i\}_{i \in \mathbf{Z} \text{ or } \mathbf{N}}$

$$\cdots \to \cdots C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} C^{i+2} \to \cdots \tag{2.2}$$

such that $d^{i+1} \circ d^i = 0$ for all i.

As an abuse of notation, we will often write

$$\cdots \xrightarrow{d} C_i \xrightarrow{d} C_{i-1} \xrightarrow{d} C_{i-2} \xrightarrow{d} \cdots \tag{2.3}$$

and analogously for co-chain complexes.

Remark 2.1. The condition that $d_{i-1} \circ d_i = 0$ for chain complexes (and analogously for cochain complexes) amounts to saying $\mathrm{im}(d_i) \subset \ker(d_{i-1})$. However, the aformentioned subsets are in fact subgroups of C_{i-1} and since C_{i-1} is abelian, we have $\mathrm{im}(d_i) \trianglelefteq \ker(d_{i-1})$. When the two groups are equal, we say that the sequence is *exact* at C_{i-1}. Otherwise, the quotient is a group that measures how far our complex is from being exact This is the basic idea of homology. To lift this idea to very high levels of abstraction, homology measures relations among relations among ... among relations of say, answers to a question.

Definition 2.2. Given a chain complex $\{C_\bullet, d\}_{\bullet \in \mathbf{Z}}$, the *homology* groups $H_i(C_\bullet) := \ker(d_i)/\mathrm{im}(d_{i+1})$. The *cohomology* of a cochain complex $\{C^\bullet, d\}_{\bullet \in \mathbf{Z}}$ is given by $H^i(C_\bullet) = \ker(d^i)/\mathrm{im}(d^{i-1})$.

Elements of $\ker(d)$ are often called *(co)-cycles*. Elements of $\mathrm{im}(d)$ are often called *(co)-boundaries*.

Now, we introduce the objects that build our (mostly singular) chain complexes.

Definition 2.3. An *n-simplex* is the convex hull of $n+1$ ordered points $\langle v_0, \cdots, v_n \rangle \subset \mathbf{R}^{n+1}$ such that $\{v_i - v_0\}$ are linearly independent. Any such is canonically hte image of the standard simplex

$$\Delta^n = \left\{ (t_0, \cdots, t_n) \in \mathbf{R}^{n+1} \mid t_i \geq 0, \sum t_i = 1 \right\} \tag{2.4}$$

via the map

$$\begin{aligned} \Delta^n &\to \sigma = [v_0, \cdots, v_n] \\ e_i &\to v_i \end{aligned} \tag{2.5}$$

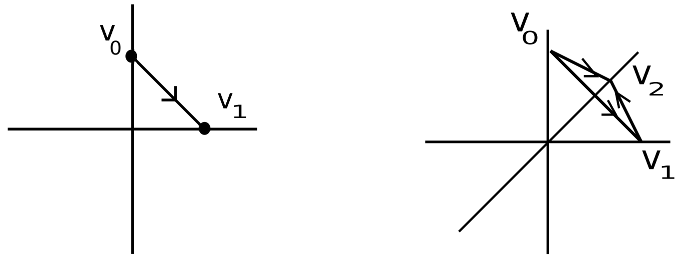
extended linearly.

The *faces* of a simplex σ are the images of $\Delta_i^{n-1} = \{t_i = 0\} \subset \Delta^n$ for $0 \leq i \leq n$.

Remark 2.2. We often use hat notation to mean an index has been omitted. In this context, when $\sigma = [v_0 \cdots v_n]$, $[v_0 \cdots \hat{v}_i \cdots v_n]$ is meant to mean the i^{th} face of σ.

Remark 2.3. All edges of a simplex are canonically oriented.

Example 2.6.



(A) Δ^1 (B) Δ^2

FIGURE 1. Examples of Oriented Simplices

Definition 2.4. Let X be a topological space. Define

$$C_n(X) \;=\; C_n(X, \mathbf{Z}) \;=\; \left\{ \sum_{\text{finite}} h_i \sigma_i \mid h_i \in \mathbf{Z},\; \sigma_i : \Delta^n \to X \right\} \tag{2.7}$$

We say that σ_i is an *n-simplex in X*.

We define the bounary map by

$$d\sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0 \ldots \hat{v}_i \ldots v_n]} \tag{2.8}$$

extending linearly to all of $C_n(X)$.

Of course, we must check that $d_i \circ d_{i+1} = 0$.

Lemma 2.4. $d^2 \;=\; 0$.

Proof.

$$\begin{aligned}
(d \circ d)(\sigma) \;&=\; d\left(\sum_{i=0}^{n} (-1)^n \sigma|_{[v_0 \ldots \hat{v}_i \ldots v_n]} \right) \\
&=\; \sum_{j<i} (-1)^{i+j} \sigma|_{[v_0 \ldots \hat{v}_j \ldots \hat{v}_i \ldots v_n]} \\
&\quad + \sum_{j>i} (-1)^{i+j-1} \sigma|_{[v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n]} \\
&=\; 0 \qquad \text{(simply swap } i, j\text{)}
\end{aligned}$$

□

We define the co-boundary map as the adjoint of the boundary map and the cocahins as duals to the chains.

Definition 2.5. Let X be a topological space and $C_n(X)$ a chain complex on X. Define the co-chain groups

$$C^i(X) \;=\; C^i(X, \mathbf{Z}) \;=\; \mathrm{Hom}_{\mathbf{Z}}(C_i(X), \mathbf{Z}) \tag{2.9}$$

and the boundary

$$\begin{aligned}
d : C^i &\to C^{i+1} \\
\psi &\to \psi \circ d_{i+1} \\
(d^i \psi)(\sigma) \;&:=\; \psi(d_{i+1}(\sigma)).
\end{aligned} \tag{2.10}$$

Remark 2.5. Since $d^2(\psi) = \psi \circ d^2$, we immediately have that $d^{i+1} \circ d^i = 0$. So, we have a legitimate co-chain complex.

Definition 2.6. The associated (co)-homology groups $H_*(X, \mathbf{Z})$ and $H^*(X, \mathbf{Z})$ are called *singular (co)-homology*. They are graded by $* \in \mathbf{N} = \{0, 1, 2, \cdots\}$.

Remark 2.6. There are other forms of homology. In particular, simplicial, or cellular homology is a far more intuitive theory. However, singular (co)-homology more easily exemplifies abstract properties of homology.

Remark 2.7.
1. The invariants are <u>functorial</u>. If $f : X \xrightarrow{\text{cts}} Y$ and $\sigma : \Delta^n \to X$ is an n-simplex, then $f \circ \sigma$ is an n-simplex in Y. By inspection, the association
$$C_n(X) \ni \sigma \to f \circ \sigma \in C_n(Y) \tag{2.11}$$
extended linearly to a map $f_\sharp : C_n(X) \to C_n(Y)$ satisfies $df_\sharp = f_\sharp d$.
2. Singular homology and cohomology are homeomorphism invariants.

Of 2.7(1). It is clear that $f \circ \sigma$ is in fact a simplex in Y, since composition of maps preserves continuity. It remains to check that $f_\sharp$ commutes with the boundary map. Intuitively, the boundary of the image is the image of the boundary. However, we can calculate

$$\begin{aligned}
(f_\sharp \circ d)(\sigma) &= f_\sharp \left(\sum_{i=0}^{n} (-1)^i \sigma|_{[v_0 \ldots \hat{v_i} \ldots v_n]} \right) \\
&= \sum_{i=0}^{n} (-1)^i f \circ (\sigma|_{[v_0 \ldots \hat{v_i} \ldots v_n]}) \\
&= \sum_{i=0}^{n} (-1)^i (f \circ \sigma)|_{[v_0 \ldots \hat{v_i} \ldots v_n]} && \text{(by continuity of } f\text{. EXPLAIN BETTER)} \\
&= d \circ f_\sharp(\sigma).
\end{aligned}$$

□

Remark 2.7(2) will become clear after the following lemma:

Lemma 2.8. A continuous $f : X \to Y$ induces a map
$$f_* : H_*(X) \to H_*(Y)$$
and similarly a map
$$f^* : H^*(X) \to H^*(Y)$$
.

Moreover,
$$\begin{aligned} \mathrm{id}_* &= \mathrm{id} \\ (f \circ g)_* &= f_* \circ g_* \end{aligned} \tag{2.12}$$
and analogously
$$\begin{aligned} \mathrm{id}^* &= \mathrm{id} \\ (f \circ g)^* &= f^* \circ g^* \end{aligned} \tag{2.13}$$

Proof. First, we must show there is an induced map on (co)-homology. Let γ be an element of $C_{n+1}(X)$ and σ a cycle in $C_{n+1}(X)$. Then, by linearity, we have $f_\sharp(\gamma + \sigma) = f_\sharp(\gamma) + f_\sharp(\sigma)$. Note that since σ is a cycle, $d\sigma = 0$. So, $df_\sharp(\sigma) = f_\sharp(d\sigma) = 0$. We have a similar story for (co)-homology. So, these maps respect quotients by cycles and therefore descend to well-defined maps on (co)-homology.

Now, we must show the functoriality properties. We will only show them for homology. First, let $[\sigma]$ be a class in $H_n(X)$ for some n. Then, following definitions, we see that $\mathrm{id}_*([\sigma]) = [\sigma]$ since composition with the identity preserves any chain σ representing $[\sigma]$.

Now, choose σ to be a representative of $[\sigma]$.

$$\begin{aligned}(f \circ g)_*(\sigma) &= (f \circ g) \circ \sigma \\ &= f \circ (g \circ \sigma) \\ &= (f_* \circ g_*)(\sigma)\end{aligned}$$

□

After the last proof, it should be clear why Remark 2.7(2) holds; if $f_* \circ f_*^{-1} = \mathrm{id}$ and $f_*^{-1} \circ f_* = \mathrm{id}$, as homology maps, then both f_* and f_*^{-1} must be identity maps.

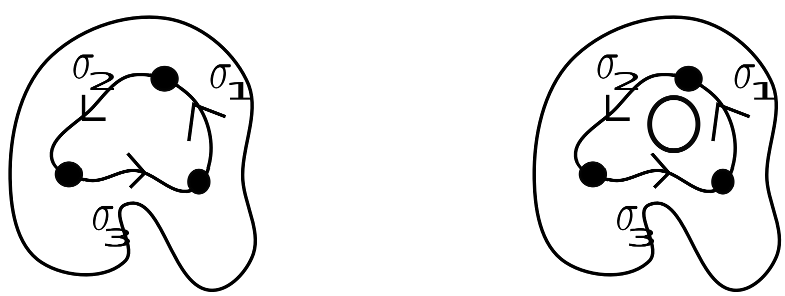
We continue with a few remarks:

Remark 2.9. (3) In general, $H^i(X) \neq \mathrm{Hom}(H_i(X), \mathbf{Z})$

(4) We can replace $\mathbf{Z}$ by any coefficient (abelian) group G, wihch gives us the chain complex

$$C_i(X, G) := \left\{ \sum_{\text{finite}} h_g \sigma \mid h_g \in G \right\} \tag{2.14}$$

Example 2.15. Here is an example that illustrates the intuition behind cycles. Both figures contain chains with cancelling boundary. However, the first chain is clearly itself the boundary of a disk. The second is not the boundary of a disk, intuitively, because the hole gets in the way.



(A) A cycle that is also a boundary

(B) A cycle that is not a boundary

Figure 2. The intuition behind cycles and boundaries

Example 2.16. Here, the idea is that boundaries correspond to physical boundaries of simplices. Cycles are similarly simplices that go in a cycle. Imagine a closed loop on a surface, say the torus. Place finitely many vertices on that loop so that it is the image of n simplices, with agreeing orientations. Then, applying the boundary map kills that cycle. However, that cycle is the boundary of a two-simplex, or a disc on the donut unless it is one of the two core curves, meridional or longitudinal. So, geometrically, homology is capturing the number of holes in the torus.

Finally, we will compute the homology of a point. This turns out to be more than a trivial example. On one hand, it allows us to compute homologies of spheres (after we learn Mayer-Vietoris). On a more general note, once a homology system is shown to have certain properties (such as Mayer-Vietoris, Excision, LES for pairs), a homology system is completely determined by what it evaluates as the homology of a point.

Example 2.17.

$$H_*(\mathrm{pt}) = \begin{cases} \mathbf{Z} & * = 0 \\ 0 & \mathrm{o/w} \end{cases} \tag{2.18}$$

and

$$H^*(\mathrm{pt}) \;=\; \begin{cases} \mathbf{Z} & * = 0 \\ 0 & \text{o/w} \end{cases} \tag{2.19}$$

Proof. Note that for each i, there is only one map from Δ^i into $\{\mathrm{pt}\}$, call this map σ^i. Hence each cochain $C_i(\{\mathrm{pt}\}) = \mathbf{Z}$– it is a free group on one generator. Examining the definition of the boundary map, we see that when i is odd $d(\sigma^i) = 0$, since the map is the same on all faces. However, when i is even, there are an odd number of faces and $d(\sigma^i) = \sigma^{i-1}$, so the induced map on chain complexes is $\mathbf{Z} \xrightarrow{+1} \mathbf{Z}$. This gives the following diagram:

$$\begin{array}{ccccc} \cdots \xrightarrow{+1} & \mathbf{Z} \xrightarrow{0} & \mathbf{Z} \xrightarrow{+1} & \mathbf{Z} \xrightarrow{0} & \mathbf{Z} \\ & C_3 & C_2 & C_1 & C_0 \end{array}$$

which, after computing yields the result in (3.2).

For cohomology, the story is similar. Here, each of the chain complexes are $\mathbf{Z}$ and the cochain maps are given by composition with either the zero map or the identity. So, we have

$$\begin{array}{ccccc} \cdots \xleftarrow{+1} & \mathbf{Z} \xleftarrow{0} & \mathbf{Z} \xleftarrow{+1} & \mathbf{Z} \xleftarrow{0} & \mathbf{Z} \\ & C^3 & C^2 & C^1 & C^0 \end{array}$$

and a quick computation yields (2.19) □

3. Lecture 3 – Homology of Spheres

Recall from last time, we defined

$$C_n(X) \;=\; C_n(X,\mathbf{Z}) \;=\; \left\{ \sum_{\text{finite}} h_i\sigma_i \mid h_i \in \mathbf{Z},\ \sigma_i : \Delta^n \to X \right\} \tag{3.1}$$

and showed

$$H_*(\mathrm{pt}) \;=\; \begin{cases} \mathbf{Z} & * = 0 \\ 0 & \text{o/w.} \end{cases} \tag{3.2}$$

We will now give some intuition for $H_0(X)$ for a general space X.

Lemma 3.1. Let X be a topological space with a set of path components $\pi_0(X)$. Then, $H_0(X) = \bigoplus_{\pi_0(X)} \mathbf{Z}$.

Proof. We begin by noting that the continuous image of a simplex Δ^n, for any n, lies in exactly one path component of X. This is because the simplices are path-connected and continuity preserves path-connectedness.

It then follows that

$$H_*(X) \;=\; H_*(\bigsqcup_{\pi_0(X)} \pi_0(X)) \;=\; \bigoplus_{\pi_0(X)} H_*(\pi_0(X)) \tag{3.3}$$

and so it suffices to show that $H_0(X) \cong \mathbf{Z}$ when X is path-connected.

First, we give an intuitive explanation. Note that 0-simplices are just formal sums of points. Again, from first definitions, $H_0(X) = C_0(X)/\mathrm{im}(d)$. So, we allow differences by boundaries of paths $\gamma : [0,1] \to X$, or $\gamma(1) - \gamma(0)$. Since the space is path connected, we can have $\gamma(1), \gamma(0)$ be whichever two points we choose – all points in X are interchangeable for the sake of $H_0(X)$. The only difference between two chains $\sum h_i p_i$ and $\sum n_j q_j$ is the totals $\sum h_i$ and $\sum n_j$.

Formally, to set ourselves on the track laid out above, we consider the morphism

$$\begin{aligned} \varphi : C_0(X) &\to \mathbf{Z} \\ \sum h_i p_i &\to \sum h_i \end{aligned} \tag{3.4}$$

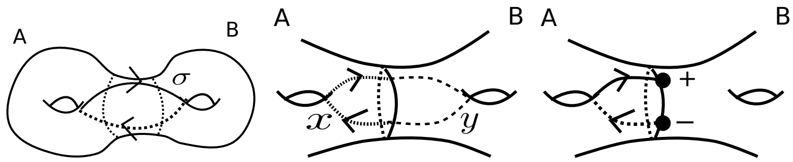
and note that when X is nonempty this map is onto. Also, note that for any 1-simplex $\sigma \subset X$, we have $\varphi(d\sigma) = \varphi(\sigma(1) - \sigma(0)) = 1 - 1 = 0$, so this map descends to a map on the quotient $C_0(X)/\mathrm{im}(d)$, or $H_0(X)$. Let's hope it's an isomorphism.

Suppose $\varphi\left(\sum h_i p_i\right) = 0$. Then, choose some p_i with a negative coefficient and some p_j with a positive one. Let γ_0 be a path from p_i to p_j. Continue this process until we have "used up" all of the coefficients, or formally until $d\left(\sum \gamma_i\right) - \sum h_i p_i = 0$. Note that this process we lay out will in fact terminate, since our sum is finite. This exhibits our chain $\sum h_i p_i$ as a boundary of a 1-simplex. In other words, $\overline{\varphi} : H_0(X) \to \mathbf{Z}$ is injective and thus an isomorphism. □

From the definition of C_*, this is all we can compute. However, homology is rendered useful and computational by 2 properties which we introduce after a brief aside.

Now, recall our principle of functoriality; if $f : X \to Y$ is continuous, it induces a map

$$\begin{tikzcd} \cdots \arrow[r, "d"] & C_i(X) \arrow[r, "d"] \arrow[d, "f_*"] & C_{i-1}(X) \arrow[r, "d"] \arrow[d, "f_*"] & C_{i-2}(X) \arrow[r, "d"] \arrow[d, "f_*"] & \cdots \\ \cdots \arrow[r, "d"] & C_i(Y) \arrow[r, "d"] & C_{i-1}(Y) \arrow[r, "d"] & C_{i-2}(Y) \arrow[r, "d"] & \cdots \end{tikzcd} \tag{3.5}$$



(A) A chain in Σ_2 (B) Subchains x, y (C) The Boundary Map

FIGURE 3. Explicit Mayer-Vietoris Map

such that all squares commute (i.e. f_* is a *chain map*) and so induces a map on homology $f_* : H_*(X) \to H_*(Y)$. We now introduce the definition hinted at in Remark 2.1.

Definition 3.1. An *exact sequence* is a (co-)chain complex with vanishing (co-)homology, i.e. a sequence of abelian groups and homomorphisms

$$\cdots \xrightarrow{d} D_i \xrightarrow{d} D_{i-1} \xrightarrow{d} D_{i-2} \xrightarrow{d} \cdots \tag{3.6}$$

such that $\ker(d_{i-1}) = \operatorname{im}(d_i)$.

Theorem 3.2 (Homotopy Invariance)**.** If $f, g : X \to Y$ are homotopic, then $f_* = g_* : H_*(X) \to H_*(Y)$. In particular, if $X \simeq Y$ are homotopy equivalent, then they have isomorphic homology. The same holds for cohomology.

Theorem 3.3 (Mayer-Vietoris)**.** If $X = A \cup B$ is a union of open subsets, there is an exact sequence

$$\cdots \to H_{i+1}(X) \xrightarrow{\partial_{\mathrm{MV}}} H_i(A \cap B) \xrightarrow{(i_A, i_B)} H_i(A) \oplus H_i(B) \xrightarrow{j_A - j_b} H_i(X) \xrightarrow{\partial_{\mathrm{MV}}} \cdots \tag{3.7}$$

where the maps are induced by inclusions

$$\begin{tikzcd} A \cap B \arrow[r, "i_A"] \arrow[d, "i_B"] & A \arrow[d, "j_A"] \\ B \arrow[r, "j_B"] & X \end{tikzcd} \tag{3.8}$$

and where ∂_{MV} (the Mayer-Vietoris boundary) is defined as follows:

$$\partial_{\mathrm{MV}} : Z_{i+1}(X) \to C_i(A \cap B). \tag{3.9}$$

Take an $i+1$-cycle in X, here, σ, and write it as a union of $(i+1)$-chains, lying in A and B, say $z = x + y$. It may not be true that x and y are cycles. But, $dz = dx + dy = 0$. So, x and y must have "cancelling boundary". Then, we set $dz = dx (= -dy)$.

In cohomology, the Mayer-Vietoris sequence looks like

$$\cdots \to H^{i-1}(A \cap B) \xrightarrow{\partial^*_{\mathrm{MV}}} H^i(X) \xrightarrow{(j_A - j_B)^*} H^i(A) \oplus H^i(B) \xrightarrow{(i_A, i_B)^*} H^i(A \cap B) \xrightarrow{\partial^*_{\mathrm{MV}}} H^{i+1}(X) \to \cdots \tag{3.10}$$

where again the maps are induced by the inclusions in (3.8).

Example 3.11. The following is an example of when neither x nor y is a cycle.

We will now compute the singular homology of a circle:

Example 3.12.

$$H_*(S^1) = \begin{cases} \mathbf{Z} & * = 0, 1 \\ 0 & \text{o/w} \end{cases} \tag{3.13}$$

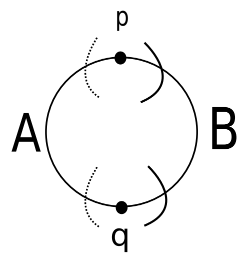


FIGURE 4. Decomposition Of a Circle

Proof. Here, we take the following decomposition of the circle:
and we note that $A \simeq \{\mathrm{pt}\} \simeq B$. Similarly, $A \cap B \simeq \{\mathrm{pt}\} \sqcup \{\mathrm{pt}\} = p \sqcup q$.

We now examne the Mayer-Vietoris sequence. At the lowest level, we have

$$\cdots \to H_1(A \cap B) \xrightarrow{\alpha'} H_1(A) \oplus H_1(B) \xrightarrow{\beta'} H_1(S^1) \xrightarrow{\partial} H_0(A \cap B) \xrightarrow{\alpha} H_0(A) \oplus H_0(B) \xrightarrow{\beta} H_0(S^1). \tag{3.14}$$

We can already deduce that $H_0(S^1) = \mathbf{Z}$, from Lemma 3.1. For $H_1(S^1)$, we examine the surrounding maps.

By homotopy equivalence, $H_1(A \cap B), H_1(A), H_1(B)$ are all zero. The same is true of higher homologies. This traps all of the higher homologies of S^1 between chains of zeros, showing $H_*(S^1) = 0$ for $* > 1$.

We know $\beta : (a, b) \to (a - b)$ by the diagram (3.8) and so $\ker(\beta) \cong \mathrm{im}(\beta) \cong \mathbf{Z}$. Moreover, $\alpha : (c, d) \to (c + d, c + d)$ which has kernel and image isomorphic to $\mathbf{Z}$. In particular, $\ker(\alpha) = \mathrm{im}(\partial)$. But, ∂ is injective since β' is the zero map. So, $H_1(S^1) \cong \mathbf{Z}$. □

We might now be lead to conjecture the following. If we did, we'd be right!

Proposition 3.4.

$$H_*(S^n) = \begin{cases} \mathbf{Z} & * = 0, n \\ 0 & \text{o/w} \end{cases} \tag{3.15}$$

Proof. Here, we use the analogous decomposition

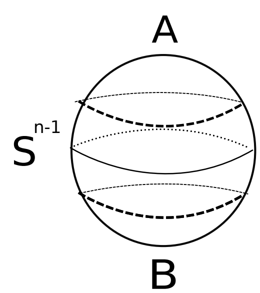


FIGURE 5. Decomposition of S^n

where homotopy equivalence shows $A \cap B \simeq S^{n-1}$, while each of A and B are homotopy equivalent to a point.

Path-connectedness immediately gives us $H_0(S^n)$. For other homologies, we examine the Mayer-Vietoris sequence:

$$\cdots \to H_i(A) \cap H_i(B) \xrightarrow{\beta} H_i(S^n) \xrightarrow{\partial} H_{i-1}(S^{n-1}) \xrightarrow{\alpha} H_{i-1}(A) \oplus H_{i-1}(B) \to \cdots \tag{3.16}$$

and we note that when $i \geq 2$, the groups on the outside of (3.16) are both zero, which establishes $H_i(S^n) \cong H_{i-1}(S^{n-1})$. By induction, this gives us $H_i(S^n)$ for $i \geq n \geq 2$.

All that remains is to consider $* = 1$. Here, the key is that when $n \geq 2$, our pieces A, B intersect in *one* copy of S^{n-1}, not two, as they did when $n = 1$. So, at the lowest level we have

$$0 \to H_1(S^n) \xrightarrow{\partial} H_0(S^{n-1}) \xrightarrow{\alpha} H_0(A) \oplus H_0(B) \xrightarrow{\beta} H_0(S^n) \tag{3.17}$$

where, from first definitions, we know $\alpha : a \mapsto (a, a)$, with zero kernel and image isomorphic to $\mathbf{Z}$. Since ∂ is injective (by exactness), we have that $H_1(S^n) \cong 0 = \ker(\alpha)$. Induction completes the proof, since $H_k(S^n) \cong H_1(S^{n-k+1})$ so that when $k < n, H_k(S^n) = 0$. $\square$

We now arrive at a result which seems trivial, but is very hard to establish directly.

Corollary 3.5. If $R^n \cong \mathbf{R}^m$, then $m = n$.

Proof. Any such homeomorphism induces a homeomorphism from $R^n - \{0\}$ to $R^m - \{0\}$, which are isomorphic to S^{n-1}, S^{m-1}, respectively. Remark 2.7(2) then implies that S^{n-1}, S^{m-1} have the same homology, which we just showed implies $n = m$. $\square$

Remark 3.6. Also note that a map $f : S^n \to S^n$ has a defining degree, $\deg(f) : H_n(S^n) \to H_n(S^n)$, since $H_n(S^n) \cong \mathbf{Z}$. This is one way to complete our proof of the Brouwer fixed point theorem.

4. Lecture 4 – Degree Maps of Spheres

Recall from last time that we showed

$$H_*(S^n) = \begin{cases} \mathbf{Z} & * = 0, n \\ 0 & \text{o/w} \end{cases} \tag{4.1}$$

and that a map $f : S^n \to S^n$ induces $\deg(f) : H_*(S^n) \to H_*(S^n)$.

For today's results, we need the following addendum to the Mayer-Vietoris Theorem:

Theorem 4.1 (Mayer-Vietoris Addendum)**.** The Mayer-Vietoris sequence is *natural* in the sense that if $X = A \cup B$ and $Y = A' \cup B'$ and A, A', B, B' are all open and $f : X \to Y$ respects $f(A) \subset A', f(B) \subset B'$, then we have

$$\begin{tikzcd} \cdots \arrow[r] & H_i(A \cap B) \arrow[r] \arrow[d, "f_*"] & H_i(A) \oplus H_i(B) \arrow[r] \arrow[d, "f_*"] & H_i(X) \arrow[r] \arrow[d, "f_*"] & H_{i-1}(A \cap B) \arrow[r] \arrow[d, "f_*"] & \cdots \\ \cdots \arrow[r] & H_i(A' \cap B') \arrow[r] & H_i(A') \oplus H_i(B') \arrow[r] & H_i(Y) \arrow[r] & H_{i-1}(A' \cap B') \arrow[r] & \cdots \end{tikzcd} \tag{4.2}$$

such that each square commutes.

Proof. This will become clear from the eventual proof of the Mayer-Vietoris Theorem. □

With this fact, we may prove the following result:

Proposition 4.2. Any element $g \in O(n+1)$ acts on S^n with $\deg(g) = \det(g) = \pm 1$

Proof. We know that each element $g \in O(n+1)$ is invertible. So, $\deg(gg^{-1}) = \deg(g)\deg(g^{-1}) = 1$ and so $\deg(g) \subset \{\pm 1\}$. However, we need to show that 1) We can attain a degree of -1 and 2) Degrees coincide with determinants. We will find that reflections have degree -1. Here, we will use the reflection and decomposition given by

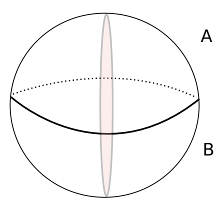


Figure 6. A plane of Reflection and a MV-decomposition

We recall that for $n \geq 1$, the Mayer-Vietoris boundary gives us an isomorphism between $H_n(S^n)$ and $H_{n-1}(S^{n-1})$ which by the Addendum 4.1 fits into the commutative diagram

$$\begin{tikzcd} H_n(S^n) \arrow[r, "\partial_{\mathrm{MV}}", "\cong"'] \arrow[d, "r_*"] & H_{n-1}(S^{n-1}) \arrow[d, "r_*"] \\ H_n(S^n) \arrow[r, "\partial_{\mathrm{MV}}", "\cong"'] & H_{n-1}(S^{n-1}) \end{tikzcd} \tag{4.3}$$

this reduces us to the case when $n = 1$, which we will examine explicitly.

We have the diagram in Figure 7. where reflection across the dashed line stabilizes A and B. However, it swaps p and q. Examining the Mayer-Vietoris sequence, we have

$$0 \to H_1(S^1) \xrightarrow{\partial} H_0(A \cap B) \xrightarrow{\alpha} H_0(A) \oplus H_0(B) \xrightarrow{\beta} H_0(S^1). \tag{4.4}$$

and since ∂ is injective and the sequence is exact, we can read off a generator of $H_1(S^1)$ as the kernel of α, which as we recall, sends $(a+b)$ to $(a+b, a+b)$, where a counts copies of p and b counts copies of q. So,

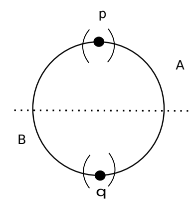


FIGURE 7. The case $n = 1$

the kernel in this case is generated by $(1, -1)$, or by $p - q$. Since r swaps p and q, it sends $p - q$ to $q - p$ and thus acts by -1 on $H_1(S^1)$. □

We can immediately obtain the following very useful corollary:

Corollary 4.3. The antipodal map $a : S^n \to S^n$, sending x to $-x$ has degree $(-1)^{n+1}$.

Proof. The antipodal map is a composition of $n + 1$ reflections, one on each coordinate. □

The following result is interesting, though it will not be used extensively in this course. To fully explain it, we need one definition.

Definition 4.1. Let G be a group acting on a space X. We say G acts *freely* on X if $g.x = x$ implies $g = e$.

This next proposition is one of several that shows even dimensional spheres are far more rigid than odd dimensional ones. The sphere S^{2k-1} naturally lives in $\mathbf{C}^k$ and so admits the structure of a Lie group. This has fairly far reaching implications, from the vector fields even vs. odd dimensional spheres admit, to the group actions the admit.

Corollary 4.4. If G acts freely on S^{2k}, then $G \leq \mathbf{Z}/2\mathbf{Z}$. Contrast this to $S^{2k+1} \subset \mathbf{C}^{(2k+2)}$. In this case, the sphere is a Lie group and admits actions by cyclic subgroups of S^1 of arbitrarily large order.

Proof. Suppose that g is a non-identity element in G. Then, the following is a homotopy between $g : S^{2k} \to S^{2k}$ and $a : S^{2k} \to S^{2k}$;

$$\begin{aligned} F : S^{2k} \times [0,1] &\to S^{2k} \\ (x,t) &\mapsto \frac{t(g.x) - (1-t)x}{\|t(g.x) - (1-t)x\|}. \end{aligned} \tag{4.5}$$

Similarly to the proof of Corollary 1.4, the Brouwer fixed point theorem, the denominator never vanishes because for $t \neq 1/2$, the norm of one of the terms is larger than the norm of the other. When $t = 1/2$, non-vanishing is guaranteed by our assumption that there are no fixed points. Since homotopic maps induce the same map on homology (Theorem 3.2), any non-identity map has degree -1.

The previous statement gives us a way to inject G into $\mathbf{Z}/2\mathbf{Z}$. Namely, we use

$$\begin{aligned} \varphi : G &\to \mathbf{Z}/2\mathbf{Z} \\ g &\mapsto \deg(g) \end{aligned} \tag{4.6}$$

which is a homomorphism from the definition of degree. So, all fibres have the same size. $\phi^{-1}(1) = \{e\}$, so G lives inside $\mathbf{Z}/2\mathbf{Z}$. □

Remark 4.5. The previous result does NOT imply that any such group is equal to $\{e, a\}$. There are exotic maps of the sphere.

The next corollary is a particularly famous result:

Corollary 4.6 (Hairy Ball Theorem)**.** S^n has a nowhere vanishing vector field if and only if n is odd.

Remark 4.7. Any (smooth) manifold M has a tangent bundle, TM and a vector field is a section of TM. For S^n, it is easy to explicitly describe the tangent bundle:

$$T_pS^n = \left\{v \in \mathbf{R}^{n+1} \mid \langle v, p\rangle = 0 \ \langle\cdot,\cdot\rangle, \text{ is the usual inner product}\right\} \tag{4.7}$$

which give the description of the total space

$$TS^n = \left\{(p, v) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \mid \|p\| = 1, \ \langle v, p\rangle = 0\right\}. \tag{4.8}$$

which naturally projects onto S^n via

$$\begin{aligned} \pi : TS^n &\to S^n \\ (p, v) &\mapsto p \end{aligned} \tag{4.9}$$

Note that the fibres of π are all linear spaces.

which allows us to define sections, an important concept in algebraic topology and differential geometry.

Definition 4.2. A *section* (or vector field), is a map $S^n \xrightarrow{\sigma} TS^n$ such that $\pi \circ \sigma = \mathrm{id}_{S^n}$.

Proof Of Corollary 4.6. When n is odd, S^n is a Lie group. So, pick a non-zero tangent vector at $v \in T_pS^n$. Then, the map $g : T_pS^n \to T_{g.p}S^n$ given by (complex) multiplication is a (smooth) isomorphism of tangent spaces. So, this gives a nowhere vanishing vector field on S^n.

When n is even, a non-zero tangent vector in T_pS^n specifies a path from p to $-p$. Therefore, a smooth, nowhere vanishing vector field specifies a homotopy fromthe identity map to the antipodal map, which cannot exist since these maps have opposite degrees. □

When discussing degree maps of spheres, we did not use anything special about the sphere, except that its top homology equals $\mathbf{Z}$. We will see on example sheet one that

$$H_*(T^2) = \begin{cases} \mathbf{Z} & * = 0, 2 \\ \mathbf{Z}^2 & * = 1 \\ 0 & \text{o/w} \end{cases} \tag{4.10}$$

and so given $f : T^2 \to S^2$, we have an induced map on second homology, to whcih we can associate a degree. This is an example of such a map.

Example 4.11. Let $K = K \sqcup K_2$ be two-component link in $\mathbf{R}^3$ (two disjoint embedded images of circles). We can obtain a map $T^2 = S^1 \times S^1$ as follows.

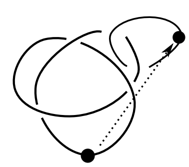


Figure 8. The Linking Number Map

Using the diagram, we define a map by

$$\begin{aligned} &g : T^2 \to S^2 \\ &(\theta, \phi) \mapsto \text{unit vector in direction} K_1(\theta) \text{ to } K_2(\phi) \end{aligned} \tag{4.12}$$

This map in fact detects linking.

Lemma 4.8. If K_1, K_2 are unlinked in the sense that there is an isotopy such that K_1, K_2 lie in disjoint $3-$balls, then $|\deg(g)| = \mathrm{link}(K_1, K_2) = 0$

Proof. If the two knots are not linked, pull them very far apart. Then, the image of this map lies in a small circle on the sphere, which can be contracted to a point. So, the degree of the map must be zero. □

Another nice application to knot theory is given by the following example.

Definition 4.3. The *Hopf map* $S^3 \xrightarrow{h} S^2$ is the map induced by

$$\begin{aligned} S^3 \subset \mathbf{C}^2 - \{0\} \to \mathbf{CP}^1 = S^2 \\ (z_1, z_2) \to [z_1, z_2] \end{aligned} \tag{4.13}$$

that is, we identify points on S^3 if they differ by multiplication by $(e^{i\theta}, e^{i\theta})$ for some $\theta \in S^1$.

Definition 4.4. A *fibre bundle* is a map $p : E \to B$ of topological spaces such that for all $b \in B$ theres exists an open neighborhood $U_b \ni b$ of b and a fibre-preserving homeomorphism

$$\begin{array}{ccc} p^{-1}(U_b) & \xrightarrow{\cong} & U_b \times F \\ \Big\downarrow p & & \Big\downarrow \pi_1 \\ U_b & \xrightarrow{=} & U_b \end{array} \tag{4.14}$$

where $F = p^{-1}(b)$ is the *fibre* of p.

Remark 4.9. Fibre bundles are locally, though not necessarily glocally trivial.

Example 4.15. (1) The Hopf map is a fibre bundle.
(2) Given $p, q \in S^2$ the circles $h^{-1}(p), h^{-1}(q)$ are linked in $\mathbf{R}^3$.
(3) The Hopf map has no section, i.e. there is no $\sigma : S^2 \to S^3$ such that $h \circ \sigma = \mathrm{id}_{S^3}$. It is not globally trivial.

5. Lecture 5 – Homotopy Invariance

recall that if C_* and D_* are chain complexes, then a chain map is a map $\varphi : C_* \to D_*$ (i.e. $\varphi_n : C_n \to D_n$) such that $d\varphi_n = \varphi_n d$.

Definition 5.1. We say that two chain maps, φ and ψ are *chain homotopic* if there exist maps

$$P : C_{n-1} \to D_n$$

for all n such that

$$dP \pm Pd = \varphi - \psi \tag{5.1}$$

The picture is the following:

$$\begin{tikzcd} \cdots \arrow[r, "d"] & C_{n+1} \arrow[r, "d"] \arrow[d, "\varphi,\psi"] \arrow[dl, "P"] & C_n \arrow[r, "d"] \arrow[d, "\varphi,\psi"] \arrow[dl, "P"] & C_{n-1} \arrow[r, "d"] \arrow[d, "\varphi,\psi"] \arrow[dl, "P"] & \cdots \\ \cdots \arrow[r, "d"] & D_{n+1} \arrow[r, "d"] & D_n \arrow[r, "d"] & D_{n-1} \arrow[r, "d"] & \cdots \end{tikzcd} \tag{5.2}$$

This notion is useful because of the following lemma.

Lemma 5.1. Chain homotopic maps induce the same map on homology. Namely,

$$\varphi_* = \psi_* : H(C_*, d) \to H(D_*, d) \tag{5.3}$$

Proof. First, we will show that this map is well-defined on homology. Let $d\beta$ be a boundary in C_n. Then,

$$\begin{aligned} (dP \pm Pd)(d\beta) &= (dP)(d\beta) \pm P(d^2\beta) \\ &= dP(d\beta) && (d^2 = 0) \end{aligned}$$

which is a boundary in D_n. So, the map is well-defined on homology.

Now, let σ be a cycle in C_n. Then,

$$\begin{aligned} (dP \pm Pd)(\sigma) &= dP(\sigma) \pm P(d\sigma) \\ &= dP(\sigma). && (\sigma \text{ is a cycle}) \end{aligned}$$

which is a boundary in D_n and thus a zero element of homology. □

But, what we really want to prove is

Theorem 5.2 (Homotopy Invariance)**.** If $f, g : X \to Y$ are homotopic, then $f_* = g_* : H_*(X) \to H_*(Y)$. In particular, if $X \simeq Y$ are homotopy equivalent, then they have isomorphic homology. The same holds for cohomology.

Proof. We will develop a "prism operator" from dividing a product $\Delta^n \times I$ into $n+1$-simplices. We eventually want a map that turns n-simplices in X to $(n+1)$-simplices in Y. Indeed, a lot of work as been done for us here. The homotopy $F : X \times [0,1] \to Y$ gives us the increase in dimension we want and also, via composition, there is a natural map

$$F \circ (\sigma \times \mathbf{1}) : \Delta^n \times [0,1] \to X \times [0,1] \to Y \tag{5.4}$$

where here σ is any simplex in X. However, the image of this map will not necessarily be a simplex, since $\Delta^n \times [0,1]$ is not a simplex for $n > 0$. This is why we need to break up $\Delta^n \times [0,1]$ into $(n+1)$-simplices. The general idea is contained in this picture. Here, $[v_0 v_1]$ is the copy of Δ^1 living in $\Delta^1 \times \{0\}$ and $[w_0 w_1]$ is the copy living in $\Delta^1 \times \{1\}$.

$$\begin{tikzcd} w_0 \arrow[r, dash] \arrow[d, dash] & w_1 \arrow[d, dash] \\ v_0 \arrow[r, dash] \arrow[ur, dash] & v_1 \end{tikzcd} \tag{5.5}$$

which is a subdivision of $\Delta^1 \times [0,1]$ into two 2-simplices. We want to make sure that in general, we can always have such a subdivision.

Claim. $\Delta^n \times [0,1]$ has a partition into $n+1$ $(n+1)$-simplices, namely $[v_0 \cdots v_i w_i \cdots w_n]$, where $[v_0 \cdots v_n] = \Delta^n \times \{0\}$ and $[w_0 \cdots w_n] = \Delta^n \times \{1\}$

Proof of Claim. We will consider the maps

$$\begin{aligned} \varphi_i : \Delta^n &\to [0,1] \\ (t_0, \cdots, t_n) &\mapsto \sum_{j=i+1}^{n} t_j \end{aligned} \tag{5.6}$$

for $-1 \leq i \leq n$. Note that $\mathrm{gr}(\varphi_{-1}) = [w_0 \cdots w_n]$ and $\mathrm{gr}(\varphi_n) = [v_0 \cdots v_n]$, where $\mathrm{gr}(\varphi_i)$ denotes the graph of φ_i. We note that $\varphi_{i+1} \leq \varphi_i$. Moreover, we see $v_0, \cdots, v_i$ and $w_{i+1}, \cdots, w_n$ are in the graph of φ_i, but no other vertices are. Since our functions are convex, we conclude that in fact $\mathrm{gr}(\varphi_i) = [v_0 \cdots v_i w_{i+1} \cdots w_n]$ so that the region between $\mathrm{gr}(\varphi_i)$ and $\mathrm{gr}(\varphi_{i-1})$ is the $(n+1)$-simplex $[v_0 \cdots v_i w_i \cdots w_n]$. As we have

$$0 = \varphi_n \leq \varphi_{n-1} \leq \cdots \leq \varphi_0 \leq \varphi_{-1} = 1 \tag{5.7}$$

and each graph of the corresponding $n+1$-simplices intersects the next in a graph of a φ_i, we find that the collection of $n+1$ $(n+1)$-simplices specified in the claim do in fact fill up $\Delta^n \times [0,1]$. ☐

with this established, we can define our prism map, relative to the homotopy $F : X \times [0,1] \to Y$.

Definition 5.2.

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathbf{1})|_{[v_0 \ldots v_i w_i \ldots w_n]} \tag{5.8}$$

for $\sigma : \Delta^n \to X$.

We have thus used our subdivision to map n-simplices in X into $n+1$-simplices in Y. Let's pause for a minute to describe our goal geometrically.

$$(g_\sharp - f_\sharp)(\sigma) \;=\; F \circ (\sigma \times 1)\,|[w_0 \cdots w_n] - F \circ (\sigma \times 1)\,|[v_0 \cdots v_n] \;=\; \text{Prism Top} - \text{Prism Bottom} \tag{5.9}$$

note that in equation (5.9), we specify a restriction to an n-simplex, whereas in Definition 5.2, we take formal sums of $(n+1)$-simplices.

The prism gets us fairly close. Since our $(n+1)$-simplices end up stacked on top of each other, sharing boundary in the graphs of the φ_i adjacent simplices share a boundary. If we take the boundary of this, we will be left with the Prism Top minus the Prism bottom, but also the outer walls of the prism, since those boundaries are not shared. We cancel these by taking the prism of the boundary. So, this theorem amounts to showing that

$$\text{Prism Top - Prism Bottom} \;=\; \text{Bdry of Prism} + \text{Prism of Bdry}$$

which is shown using some light symbol-pushing. Afterwards, we will include an example that will hopefully clarify this process.

$$\begin{aligned} dP(\sigma) \;&=\; d\left(\sum_{i=0}^{n+1} (-1)^i F \circ (\sigma \times \mathbf{1})|_{[v_0 \ldots v_i w_i \ldots w_n]}\right) \\ &=\; \sum_{j \leq i} (-1)^{i+j} F \circ (\sigma \times \mathbf{1})|[v_0 \cdots \hat{v}_j \cdots v_i w_i \cdots w_n] \\ &\quad + \sum_{j \geq i} (-1)^{i+j+1} F \circ (\sigma \times \mathbf{1})|[v_0 \cdots v_i w_i \cdots \hat{w}_j \cdots w_n]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} Pd(\sigma) &= \sum_{i\leq j}(-1)^{i+j}F\circ(\sigma\times\mathbf{1})|[v_0\cdots v_i w_i\cdots\hat{w}_j\cdots w_n] \\ &\quad+\sum_{j\leq i}(-1)^{i+j-1}F\circ(\sigma\times\mathbf{1})|[v_0\cdots\hat{v}_j\cdots v_i w_i\cdots w_n] \end{aligned}$$

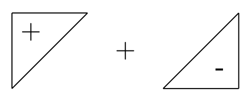
and so in $dP+Pd$, everything cancels except the terms in (5.9), which is precisely what we wanted to show. □

Example 5.10. Here is an example that should clarify how the whole process works. We will represent $\Delta^1 = X$ by $I := [0,1]$ and $Y = I^2$. Here, we have

$$\begin{aligned} f &: X \to I\times\{0\} \\ g &: X \to I\times\{1\} \end{aligned} \tag{5.11}$$

where the maps are standard inclusions. They are clearly homotpic via the identity $\mathrm{Id} : I^2 \to I^2$. In this case, let σ be $I = X$. Then, we have

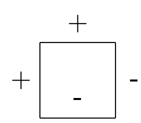
$P(\sigma)$ =



(5.12)

and so

$dP(\sigma)$ =



(5.13)

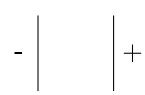
while

$d(\sigma)$ =

(5.14)

which makes

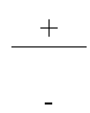
$Pd(\sigma)$ =



(5.15)

so that

$(dP+Pd)(\sigma)$ =



$= (g_\sharp - f_\sharp)(\sigma)$ (5.16)

and notice that $dP(\sigma)$ takes care of the "inside" of our simplex, while $Pd(\sigma)$ takes care of the outer edges.

Next, we will develop more algebra before proving Mayer-Vietoris.

Recall an exact sequence is a chain complex with vanihsing homology.

Definition 5.3. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

Remark 5.3. In this figure, α is injective and β is surjective.

Definition 5.4. A *short exact sequence of chain complexes* is given by a complex of the form

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & B_{n+1} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{5.17}$$

such that the rows are exact and all squares commute.

Note that in the above definition, while the rows are exact, the columns may have interesting homology.

Finally, we define a notion of relative homology. It is a way to capture the homology of quotient spaces.

Definition 5.5. If $A \subset X$ is a subspace, then we have $C_*(A), C_*(X)$ and a natural inclusion $C_i(A) \hookrightarrow C_i(X)$ and this is a chain map (commutes with the boundary).

Define $C_i(X, A) := C_i(X)/C_i(A)$ to be the quotient. Then, by definition, there is a short exact sequence of chain complexes

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0 \tag{5.18}$$

where the bounaries $d : C_i(X, A) \to C_{i-1}(X, A)$ are inherited from $C_*(X)$.

The *homology of X relative to A* is the homology of this chain complex.

We will not see until later that this "essentially" measures the homology of quotient spaces. But, for now it will be convenient to introduce a special example of relative homology.

Definition 5.6. In the definition above, take A to be a point. We define the *reduced homology* of X, denoted $\widetilde{H}_n(X)$ to be $H_n(X, \mathrm{pt})$, or $\ker(H_n(X) \to H_n(X, \mathrm{pt}))$.

Remark 5.4. Note that for $n > 0$, $\widetilde{H}_n(X) = H_n(X)$.

6. Lecture 6 – Long Exact Sequences and Excision

Lemma 6.1 (Snake Lemma)**.** Given a short exact sequence of chain complexes

$$0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0 \tag{6.1}$$

there is a boundary map $\delta : H_k(C_*) \to H_{k-1}(A_*)$ such that the resulting sequence

$$\cdots \to H_k(A) \to H_k(B) \to H_k(C) \xrightarrow{\delta} H_{k-1}(A) \to H_{k-1}(B) \to H_{k-1}(C) \to \cdots \tag{6.2}$$

is exact

.

Proof. The first step is to construct δ.

Construction of δ**:** For this construction, it is helpful to have the picture (5.17) from Definition 5.4 in mind. Let $\sigma \in C_n$ be a cycle. Since $\beta : B_* \to C_*$ is surjective, $\sigma = \beta(b)$ for some $b \in B_n$. Moreover, since our diagram commutes and $d\sigma = d\beta(b) = 0$, we also know that $\beta(db) = 0$ and so $db \in \ker(\beta)$. But, since our rows are exact, $db \in \operatorname{Im}(\alpha : A_{n-1} \to B_{n-1})$, say $b = \alpha(a)$. Since α is injective, this choice of a is unique. We set $\delta(\sigma) = a$.

In the above construction, we made many choices. So, we need to show that δ is in fact well-defined. First, suppose $\sigma = d\sigma'$ for some $\sigma' \in C_{n+1}$. We want to show that this implies a is a boundary as well. As β is surjective, there is some $b' \in B_{n+1}$ such that $\beta(b') = \sigma'$. We will let $d(b') = b$ and note that $db = d \circ \beta b' = \sigma$. However, $db = d^2 b' = 0$. Since α is injective, there is only one pre-image of 0, namely, $0 \in A_{n-1}$. So, in fact our map δ kills boundaries.

Now, we need to show that our choice of a (as a homology class) is independent of our choice of b. Suppose $\beta(b_1) = \beta(b_2) = \sigma$. We want do show $d(b_1) = d(b_2)$, or equivalently that $d(b_1 - b_2) = 0$. If $\beta(b_1 - b_2) = 0$, then $b_1 - b_2 \in \ker(\beta) = \operatorname{im}(\alpha)$. So, there exists some $a' \in A_n$ with $\alpha(a') = b_1 - b_2$. By commutativity, $\alpha(d(a')) = d(\alpha(a')) = d(b_1 - b_2)$. However, $d(a')$ is definitionally a boundary, so for the sake of homology, the choice of b does not matter.

We also need to check that δ is a homomorphism. However, this is inherited from the other maps in the diagram being homomorphisms – i.e. for $\sigma_1 + \sigma_2$ a sum of cycles, $\alpha(a_1 + a_2) = d(b_1 + b_2)$ for the a_i, b_i playing their roles as before.

Finally, we must check that the sequence is exact. That $\operatorname{im}(\alpha) = \ker(\beta)$ is given to us for free. However, we must check

- $\operatorname{im}(\beta_*) = \ker(\delta)$
- $\operatorname{im}(\delta) = \ker(\alpha_*)$.
- $\operatorname{im}(\alpha_*) = \ker(\beta_*)$.

Just as a reminder, we can no longer necessarily assume that α_*, β_* are injective and surjective, respectively – we passed to quotients. We begin with the first item.

im$(\beta_*) \subset$ **ker**(δ). Let $[\sigma]$ be a class in $H_n(C)$ and suppose $[\sigma] = \beta_*([b])$ for some $b \in B_n$. Since $[b]$ is a homology class, $d[b] = 0$, and therefore $[a] = [0]$ by definition.

im$(\beta_*) \supset$ **ker**(δ). Let σ be a representative of $[\sigma] \in H_n(C)$. If $\sigma \in \ker(\delta)$, then there exists some $a' \in C_n(A)$ such that $da' = \delta\sigma = a$. As a map on chain complexes, β is surjective. So, there is some $b \in C_n(B)$ such that $\beta(b) = \sigma$. However, we do not know that b is a cycle. Thankfully, $b - \alpha(a')$ is. This is because $d(b - \alpha(a')) = d(b) - \alpha(d(a')) = 0$, the latter equality holds by definition of a. So, $\beta_*([b] - [\alpha(a')]) = [\beta(b)] = [\sigma]$, which is what we wanted to show.

$\mathbf{im}(\delta) \subset \mathbf{ker}(\alpha_*)$. Let $a \in A_{n-1}$ with pre-image σ, a cycle in C_n. From surjectivity of β, we have some $b \in B_n$ with $\beta(b) = \sigma$. We defined a so that $\alpha(a) = d(b)$, so the image of a is clearly a boundary.

$\mathbf{im}(\delta) \supset \mathbf{ker}(\alpha_*)$. Suppose $a \in A_{n-1}$ such that $b' = \alpha(a) = d(b)$ for some $b \in B_n$. Then, since $\beta \circ \alpha = 0$ (i.e. $\beta(b') = \beta(d(b)) = 0$) and our diagram commutes, we know that $d\beta(b) = 0$. However, by the definition of a, $\delta([\beta(b)]) = [a]$. So, we have proven the claim.

$\mathbf{im}(\alpha_*) \subset \mathbf{ker}(\beta_*)$. By functoriality, $(\beta_* \circ \alpha_*)(a) = (\beta \circ \alpha)_*(a) = 0_*(a) = 0$.

$\mathbf{im}(\alpha_*) \supset \mathbf{ker}(\beta_*)$. Let $b \in B_n$ and suppose $b \in \ker(\beta_*)$. Then, there exists some $\sigma' \in C_{n+1}$ such that $d\sigma' = \sigma = \beta(b)$. As β is surjective, we may find some $b' \in B_{n+1}$ such that $\beta(b') = \sigma'$. Note that this b' satisfies $\sigma = d(\beta(b')) = \beta(d(b'))$. So, $\beta(b) = \beta(db') = \sigma$ and hence $\beta(d - db') = 0$. By exactness of the maps on chain complexes, this implies there exists some $a \in A_n$ such that $\alpha(a) = b - db'$. Now, we need to show that a is a cycle. But, b is a cycle, so noting that $d(\alpha(a)) = db = 0 = \alpha(d(a))$ and that α is injective, we learn that a is a cycle and thus $\alpha_*[a] = [b - db'] = [b]$. □

The next result is another consequence of diagram chasing, but is essential for several Theorems.

Lemma 6.2 (5-Lemma)**.** Given a diagram

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "\alpha"] & B \arrow[r, "f_2"] \arrow[d, "\beta"] & C \arrow[r, "f_3"] \arrow[d, "\gamma"] & D \arrow[r, "f_4"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\ A' \arrow[r, "g_1"] & B' \arrow[r, "g_2"] & C' \arrow[r, "g_3"] & D' \arrow[r, "g_4"] & E' \end{tikzcd} \tag{6.3}$$

such that the rows are exact and all squares commute, then if $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, γ is an isomorphism as well.

Proof. First, we will show injectivity. For this part, exactness of the rows is key; we can show an element is zero by showing it lies in the image of $f_{i+1} \circ f_i$ for $i = 1, 2, 3$.

Suppose $\gamma(c) = 0$ for some $c \in C$. Then, $g_3(\gamma(c)) = g_3(0) = 0$. However, by commutativity, we also have $\delta(f_3(c)) = 0$. But, δ is an isomorphism. So, $f_3(c) = 0$, which, by exactness implies there is some $b \in B$ such that $f_2(b) = c$. Again, by commutativity, $g_2(\beta(b)) = 0$ and so there is some $a' \in A'$ such that $g_1(a') = \beta(b)$. Since α, β are isomorphisms and all diagrams commute, we know that $f_1(\alpha^{-1}(a')) = b$. So, $c \in \operatorname{im}(f_2 \circ f_1) = 0$.

Now, we will show that γ is surjective. Let $c' \in C$. Let $g_3(c') = d' \in D'$. By exactness and the fact that δ and ε are isomorphisms, we have that $d := \delta^{-1}(d')$ lies in $\ker(f_4)$. Again, by exactness, there exists a $c_1 \in C$ such that $f_3(c_1) = d$. Consider $c' - \gamma(c_1)$. By commutativity, $g_3(c') = g_3(\gamma(c_1)) = d'$ and so $g_3(c' - c_1) = 0$. By exactness, we may find some $b' \in B'$ such that $g_2(b') = c' - c_1$. Let $b = \beta^{-1}(b')$. Let $c_2 = f_2(b)$. By commutativity, $\gamma(c_2) = c' - \gamma(c_1)$ and so $c' = \gamma(c_1 + c2)$. □

We need one final ingredient to complete our proof of Mayer-Vietoris. This is a condition called *locality*. Essentially, locality says that for the purpose of computing homology, our simplices can be as small as we want them to be. The right picture is taking a triangulation of some surface and making it finer and finer until the maximum diameter on any face is less than some ε. In that picture, we have brushed under the rug that our surface is embedded in some metric space. In a metric space, smallness makes sense in terms of our metric. For a general topological space, we define smallness in terms of covers.

Definition 6.1. Let X be a space and $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of X by open sets, or a cover by sets whose interiors cover X. We define $C_*(X, \mathcal{U})$ to be the *complex of restricted singular chains*, to have as chain groups

$$C_k(X, \mathcal{U}) := \left\{ \sum_{\text{finite}} h_i \sigma_i \mid \sigma_i : \Delta^k \xrightarrow{\text{cts}} X,\ \text{there exists } \alpha(i) \ \text{ such that } \operatorname{im}(\sigma_i) \subset \mathring{U}_{\alpha_i} \right\} \tag{6.4}$$

Remark 6.3. This locality condition is preserved by the boundary operators, so using the naturally inherited d, we in fact have a chain complex. Moreover, this chain compex produces isomorphic homology on X.

Theorem 6.4 (Locality)**.** The inclusion $C_*(X,\mathcal{U}) \hookrightarrow C_*(X)$ induces an isomorphism on homology.

We prove this theorem in the next lecture. For now, we will highlight some applications of the Theorem. Our first Corollary is Mayer-Vietoris (for homology).

Proof of Theorem 3.3. If $\mathcal{U} = \{A, B\}$, one usually writes $C_*(A+B)$ to mean $C_*(X,\mathcal{U})$. Then, we have a short exact sequence of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*(A \cap B) & \longrightarrow & C_*(A) \oplus C_*(B) & \longrightarrow & C_*(A+B) & \longrightarrow & 0 \\ & & \sigma & \longrightarrow & (\sigma, \sigma) & & & & \\ & & & & (u, v) & \longrightarrow & (u - v) & & \end{array} \tag{6.5}$$

note that we have defined $C_*(A+B)$ to make the righthand map onto.

The long exact sequence resulting from this short exact sequence of chain complexes is the Mayer-Vietoris sequence. Here, we use Theorem 6.4 to show that $H_*(A+B) = H_*(X,\mathcal{U}) = H_*(X)$. □

The second application is another fundamental result in homology, excision.

Theorem 6.5 (Excision)**.** Suppose $Z, A \subset X$ and $\mathrm{cl}(Z) \subset \mathrm{int}(A)$. Then, the inclusion of pairs $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism on homology $H_*(X \setminus Z, A \setminus Z) \overset{\cong}{\hookrightarrow} H_*(X, A)$.

Equivalently, for subspace $A, B \subset X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$ for all n.

Proof. We will prove the second formulation of Excision. So, our first task is to show that the two formulations are equivalent. We do this by setting $B = X \setminus Z$, wihch makes $A \setminus Z = A \cap B$.

Now, we examine the following diagram:

$$\begin{tikzcd} H_*(A) \arrow[r] \arrow[d, "="] & H_*(A+B) \arrow[r] \arrow[d, "\cong", "\text{Thm 6.4}"'] & H_*(A+B, A) \arrow[r] \arrow[d, "\varphi"] & H_{*-1}(A) \arrow[r] \arrow[d, "="] & H_{*-1}(A+B) \arrow[d, "\cong", "\text{Thm 6.4}"'] \\ H_*(A) \arrow[r] & H_*(X) \arrow[r] & H_*(X, A) \arrow[r] & H_{*-1}(A) \arrow[r] & H_{*-1}(X). \end{tikzcd} \tag{6.6}$$

We can see that all squares commute by examining the maps involved – it does not matter where the inclusion of the small simplices in the top row occurs. The five lemma then gives us that φ is an isomorphism.

However, we can now identify $H_n(X, A)$ with $H_n(A+B, A)$, the latter of which is clearly the same as $H_n(B, A \cap B)$, since both are generated by cycles that lie in B, but not in A. □

We now introduce some particularly nice subspaces, the interpretation of excision as homology of quotient spaces, and examples of excision.

Definition 6.2. We say a subspace A of a topological space X is *good* if A is closed and there exists neighborhoods V of A in X such that $A \hookrightarrow X$ is a deformation retract.

Example 6.7. If X is a smooth manifold and $A \subset X$ is a smooth, closed submanifold, then the Tubular Neighborhood Theorem implies there exists a good neighborhood of A.

Proposition 6.6 (Interpretation of Relative Homology)**.** Let $A \subset X$ be a good subspace. Then,

$$H_*(X, A) \cong \widetilde{H}_*(X/A) \tag{6.8}$$

Remark 6.7. One can easily show that $\widetilde{H}_*(X) = H_*(X)$ for $* > 0$ and that for $* = 0$, we have

$$\widetilde{H}_0(X) \oplus \mathbf{Z} \cong H_0(X). \tag{6.9}$$

However, the isomorphism in (6.9) is non-canonical.

Proof of Proposition 6.6. . The proof is technically contained in a single commutative diagram. We have

$$\begin{tikzcd}
H_*(X,A) \arrow[r, "\cong", "(\dagger)"'] \arrow[d, "q_*"] & H_*(X,V) \arrow[d, "q_*"] & & H_*(X \setminus A, V \setminus A) \arrow[ll, "\text{Exc.}"', "\cong"] \arrow[d, "q_*", "\cong"'] \\
H_*(X/A, \mathrm{pt}) \arrow[r, "\cong", "(\dagger)"'] & H_*(X/A, V/A) & H_*((X \setminus A)/A, (V \setminus A)/A) \arrow[l, "\text{Exc.}"', "\cong"] \arrow[r, phantom, "="] & H_*(X/A - A/A, V/A - A/A)
\end{tikzcd} \tag{6.10}$$

where here Exc. is short for excision, and ($\dagger$) is shown using homotopy equivalence of A, V and the five-lemma. This is where we use that A is a good subspace. q_* is the map on homology induced by the quotient map.

The surprising part of this Theorem is the introduction of quotients. We have not seen quotients in any other part of the course. One should look at this diagram as beginning at the top left, going to the top right, then going to the bottom right and finally to the bottom left. Going from the top right to the bottom right is a trivial introduction of quotients. Contracting A to a point is a homeomorphism on $X \setminus A$. So, using the five lemma and the LES for pairs, we see that we can take a "trivial" quotient by A with respect to these homology groups. However, using excision and the fact that quotients commute with set subtraction, we arrive at an equivalent, but somehow non-trivial quotient by A in the middle part of the bottom row. □

Example 6.11. The folowing is an example of a simplex that is not a cycle in $C_*(X)$, but is a cycle in $C_*(X, A)$:

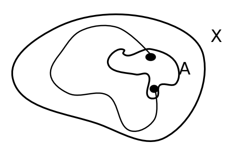


FIGURE 9. A relative cycle

Another classic example is with wedge sums of spaces.

Definition 6.3. Let X, Y be topological spaces with distinguished points x_0, y_0. Then, the *wedge* of X and Y, denoted $X \vee Y$, is given by

$$X \vee Y \ := \ X \sqcup Y / x_0 \sim y_0 \tag{6.12}$$

Example 6.13. Here is a picture of a wedge sum of spheres.

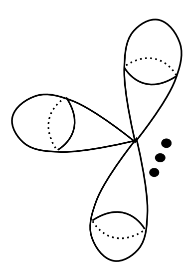


FIGURE 10. A Wedge Sum of Spheres

Corollary 6.8 (To Proposition 6.6)**.** For a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism $\oplus_\alpha i_{\alpha *} : \oplus_\alpha \widetilde{H}_n(X_\alpha) \to \widetilde{H}_n(\bigvee_\alpha X_\alpha)$, provided that the wedge sum is formed ad basepoints $x_\alpha \in X_\alpha$ and the pairs (X_α, x_α) are good.

Proof. Recall $\bigvee_\alpha X_\alpha = (\bigsqcup_\alpha X_\alpha) / \bigsqcup_\alpha \{x_\alpha\}$. However, the pairs (X_α, x_α) are good. So, Proposition 6.6 tells us we have

$$H_* \left(\bigsqcup_\alpha X_\alpha, \bigsqcup_\alpha \{x_\alpha\} \right) \cong \widetilde{H}_n \left(\bigvee_\alpha X_\alpha \right). \tag{6.14}$$

where the lefthand side is clearly isomorphic to $\oplus_\alpha \widetilde{H}_*(X_\alpha)$, and the isomorphism in (6.14) is induced by inclusion (c.f. proof of Theorem 6.5). □

This final application of Excision may be the most important. We will state and prove the proposition first, then explain its significance.

Proposition 6.9. Let X be an n-dimensional manifold for $n > 0$. Then, for $x \in X$,

$$H_n(X, X - \{x\}) \cong H_n(U, U - \{x\}) \cong \mathbf{Z} \tag{6.15}$$

where U is any open neighborhood of X.

Proof. First, we will demonstrate the righthand isomorphism. Note that $U - \{x\} \simeq S^{n-1}$. By the LES for pairs, we have

$$\cdots \longrightarrow H_n(S^{n-1}) \longrightarrow H_n(U) \longrightarrow H_n(U, U - \{x\}) \longrightarrow H_n(S^n) \longrightarrow 0 = H_{n+1}(U). \tag{6.16}$$

For $n > 0$, $H_n(U) = 0$ by homotopy invariance. So, $H_n(U, U - \{x\})$ are isomorphic by exactness.

Using $Z = X - U$ in excision shows that the inclusion of pairs $(U, U - \{x\}) \hookrightarrow (X, X - \{x\})$ induces an isomorphism on homology. □

The interpretation we are given in Proposition 6.6 tells us that if we replace x with a small neighborhood V of x, the result above should be obvious. However, it turns out to also be true when we just use $\{x\}$.

The homology group $H_n(X, X - \{x\})$ is a measure of *local homology.* Whether or not we can patch together local homology generators globally will give us a notion of the orientation of a manifold, although much of this work will be done with analogous statements for cohomology.

7. Lecture 7 – Locality

Locality is a condition that says that for the sake of generating homolgy groups, our simplices can be "as small" as we like. We make our simplices small by subdividing them in an algorithmic way. Then, we use the notion of the Lebesgue number of a cover to complete the proof. The *Lebesgue number* of an open cover of a compact metric space X is a number $\delta > 0$ such that any subset of X having diameter less than δ is contained in some member of the cover.

Throughout, we will let X be a topological space and $\{U_\alpha, \alpha \in I\}$ be a collection of subsets whose interiors cover X. The aim is to show that $C_*(X, U) \hookrightarrow C_*(X)$ is a homotopy equivalence. We will construct $\varphi : C_*(X) \to C_*(X, U)$, a chain map so that $\iota_*\varphi_*$ is chain homotopic to the identity via some map $D : C_n(X) \to C_{n+1}(X)$. We will have

(a) $dD + Dd = \mathbf{1} - \iota\varphi$ so that $\iota_*\varphi_*$ is the identity on homology.
(b) $\varphi\iota = \mathrm{id}$, which induces the identity map on homology in the other direction.

Given a simplex $\sigma \subset \mathbf{R}^n$, its barycenter of mass can be found by taking its "center of mass". The barycenter of σ is the point $\frac{1}{n+1}(1, \cdots, 1)$ under the natural linear homeomorphism $\sigma \xrightarrow{\cong} \Delta^n$.

The subdivision $s(\sigma)$ is the union of the simplices $[b, w_0, \cdots, w_{n-1}]$, where $b = \mathrm{bary}(\sigma)$ and $[w_0, \cdots, w_{n-1}]$ lies in one of the already subdivided faces of σ.

Figure 11 gives examples of barycentric subdivision.

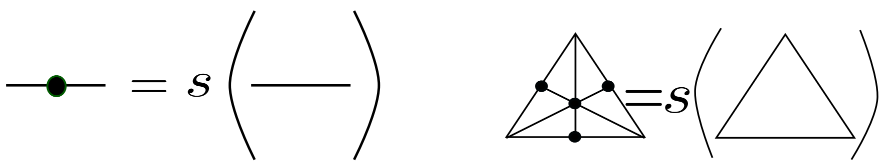


Figure 11. Examples of barycentric subdivision

Lemma 7.1. (1) There exists an operator $T : C_n(X) \to C_{n+1}(X)$ such that $dT + Td = \mathbf{1} - s$.

(2) If $\widetilde{\sigma} \in s(\sigma)$ is one of the simplices in the subdivision of σ, then with respect to the Euclidian metric on σ (and hence on $\widetilde{\sigma}$), we have

$$\mathrm{diam}(\widetilde{\sigma}) \leq \frac{n}{n+1}\mathrm{diam}(\sigma) \tag{7.1}$$

Remark 7.2. The second part of the above lemma says that subidividing makes things smaller. The first part will be proved in an analogous way to our construction of the Prism operator for homotopy equivalence. We will universall subdivide $\Delta^n \times [0,1]$ into $n+1$-simplices so that on the base, we see Δ^n, but on the top, we see $s(\Delta^n)$.

Proof given the Lemma. Given $\sigma : \Delta^n \to X$, the image of σ is compact and the induced cover of $\mathrm{int}(U_\alpha)$ has a Lebesgue number (with respect to the Euclidian metric on σ). So, there exists an $m(\sigma)$, perhaps large such that each simplex in $s^m(\sigma)$ lies inside $\mathrm{int}(U_\alpha)$ for some $\alpha \in I$. This uses part 2 from the previous Lemma.

Define $D_0 = 0$ and define $D_m = \sum_{i=0}^{m-1} Ts^i$ and from here we define

$$\begin{aligned} D : C_n(X) &\to C_{n+1}(X) \\ \sigma &\to D_{m(\sigma)}(\sigma) \end{aligned} \tag{7.2}$$

Then,

$$\begin{aligned} dD_m - D_m d &= \sum_{i=0}^{m-1} dTs^i + Ts^i d \\ &= \sum_{i=0}^{m-1} (dT + Td)s^i \\ &= \sum_{i=0}^{m-1} (\mathbf{1} - s)s^i \\ &= \mathbf{1} - s^m \end{aligned}$$

which gives us that

$$(dD - Dd)(\sigma) \;=\; \sigma - \underbrace{s^{m(\sigma)} + D_{m(\sigma)}(d\sigma) - D(d\sigma)}_{\varphi(\sigma)} \tag{7.3}$$

where the key is that $\varphi(\sigma) \in C_n(X, U)$.

Let σ_j denote the j^{th} face of σ. Then, $m(\sigma_j) \leq m(\sigma)$. Therefore, the terms $Ts^i(\sigma_j)$ in $D(d\sigma)$ are all also terms in $D_{m(\sigma)}(d\sigma)$, so the difference $D_{m(\sigma)}(d\sigma) - D(d\sigma)$ is a sum of terms $Ts^i(\sigma_j)$ with $i \geq m(\sigma_j)$, so $\varphi(\sigma)$ indeed lies in $V_n(X, U)$. Therefore,

$$(dD + Dd)(\sigma) = \sigma - i\varphi(\sigma) \tag{7.4}$$

To see that φ is a chain map, note that $d\varphi(\sigma) = d\sigma - dD(d\sigma)$ and using (7.4) and $d^2 = 0$, and also that $dD(d\sigma) + D(d^2\sigma) = d\sigma - \varphi(d\sigma)$, we see $d\varphi = \varphi d$, as required.

finally, if $\sigma \in C_n(X, U)$, then $m(\sigma) = 0$ and $D_0 = 0$ and so $\varphi \circ i(\sigma) = \sigma$. This establishes our condition and the Theorem. $\square$

Remark 7.3. We obtained $C_*(X, U) \hookrightarrow C_*(X)$ as an isomorphism on homology. If $A \subset X$, all of our constructions (s, T, φ, D) preserve the property of lying in A. Therefore, the equations $dD + Dd = \mathbf{1} - i\varphi$ descend to the quotient

$$\frac{C_*(X, U)}{C_*(A)} \to \frac{C_*(X)}{C_*(A)} \tag{7.5}$$

where $A \subset U_\alpha$ for some α. We get that the map on quotients defines an isomorphism on H_* and used this in deducing excision.

One of the main applications of excision we will use is **local orientation**. Let M be a manifold, i.e. a topological space locally homeomorphic to $\mathbf{R}^n$ (Hausdorff, second countable, etc ..). If $x \in M$, and $U \ni x$ is a neighborhood of x homeomorphic to $\mathbf{R}^n$, then by excising $X \setminus U$, we see

$$\begin{aligned} H_*(M, M \setminus x) &\cong H_*(U, U \setminus x) \\ &\cong H_*(\mathbf{R}^n, \mathbf{R}^n \setminus 0) \\ &\cong \mathbf{Z} \end{aligned} \tag{7.6}$$

where the final equivalence holds by the long exact sequence for pairs.

A *local orientation* for M at x is a choice of generators for this group $H_n(M, M \setminus x)$. We say M is orientable if it admits an open cover by sets $U_\alpha \cong \mathbf{R}^n$ such that local orientation for pionts in set U_α agree with points in set U_β under transition maps.

8. Lecture 8 – Local Degree, Cell Complexes

For this chapter, it will be especially important to recall

$$\begin{aligned} X \setminus Y &= X - Y \\ X/Y &= \text{quotient of } X \text{ that sends } Y \text{ to a point.} \end{aligned}$$

and the statement of Proposition 6.9, which says $H_n(M, M - \{x\}) \cong H_n(U, U - \{x\}) \cong \mathbf{Z}$ for any n-dimensional manifold M, $x \in M$ and U and neighborhood of x.

Remark 8.1. The following analysis applies to any $f : M \to N$, a map of closed n-manifolds, provided we know $H_n(M), H_n(N)$ are both isomorphic to $\mathbf{Z}$.

Consider $f : S^n \to S^n$ and there exists $y \in S^n$ such that $f^{-1}(y) = \{x_1, \cdots, x_n\}$. We can then choose disjoint disk neighborhoods $\mathbf{R}^n \cong U_i \ni x_i$ such that $f|_{U_i} \hookrightarrow V$ (shrink V if necessary). These maps induce maps of pairs $(U_i, U_i - \{x_i\}) \to (V, V - \{y\})$ and consequently induce maps on local homology. Since we showed local homology groups are isomorphic to $\mathbf{Z}$, we can define a local degree.

Definition 8.1. The *local degree* of f at x_i, denoted $\deg_{x_i}(f)$, is defined by

$$\deg_{x_i}(f) : H_n(U_i, U_i - \{x_i\}) \xrightarrow{f_*} H_n(V, V - \{y\}) \tag{8.1}$$

A priori, $\deg_{x_i}(f) \in \mathbf{Z}$ is well-defined, up to sign, but may not be well-defined overall, since we have to choose generators for $\mathbf{Z}$.

Being careful about signs, we have the following lemma:

Lemma 8.2.

$$\deg(f) \;=\; \sum_{i=1}^{k} \deg_{x_i}(f) \tag{8.2}$$

Remark 8.3. One of the more striking features of this lemma is the appearance of the sum. Hopefully, this explains why such a thing "should" appear.

When we choose the U small enough to be disjoint, excision (choosing Z as the complement of the U_i) gives us isomorphisms

$$H_n(S^n, S^n - \{x_1, \cdots, x_k\}) \cong H_n(\amalg_i U_i, \amalg_i U_i - \{x_i\}) \cong \bigoplus_{i=1}^{k} H_n(U_i, U_i - \{x_i\}). \tag{8.3}$$

and of course, there is a well-defined map of pairs $(S^n, S^n - \{x_1, \cdots, x_k\}) \to (S^n, S^n - \{y\})$ just given by f.

While this is all contained in the proof, I feel it is this point that makes the lemma believable at face value.

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
 & H_n(S^n) \arrow[r, "\deg(f)"] \arrow[dl, "\text{Pair LES}"', "\cong"] \arrow[d, hook, "\phi"] & H_n(S^n) \arrow[d, "\text{Pair LES}"', "\cong"] \\
H_n(S^n, S^n - \{x_i\}) \arrow[r] \arrow[dr, "\cong", "\text{Exc.}"'] & H_n(S^n, S^n - \{x_1, \cdots, x_k\}) \arrow[r, "f_*"] & H_n(S^n, S^n - \{y\}) \\
 & H_n(U_i, U_i - \{x_i\}) \arrow[u, "\iota_*"] \arrow[r, "\deg_{x_i}(f)"] & H_n(V, V - \{y\}) \arrow[u, "\cong", "\text{Exc.}"']
\end{tikzcd} \tag{8.4}$$

And that's it! Just kidding, this diagram needs a fair amount of explanation. First, the map ϕ is constructed by going around the lefthand side to $H_n(S^n, S^n - \{x_i\})$, then via excision to the bottom,

$H_n(U_i, U_i - \{x_i\})$, and finally into $H_n(S^n, S^n - \{x_1, \cdots, x_k\})$ via the map induced by inclusion, ι_*. A very important note here is that the excision isomorphism is induced by inclusion and the isomorphism from the LES for pairs is essentially induced by a quotient, (as we saw in the Proof of Proposition 6.6) both of which preserve the signs of generators. So, $\phi(1) = (1, 1, \cdots, 1)$.

Going along the bottom row, we can read off f_*, which sends $(1, 1, \cdots, 1)$ to $\sum_{i=1}^k \deg_{x_i}(f)$. Again, since the LES of pairs preserves the sign of generators, we have shown $\deg(f) = \sum_{i=1}^k \deg_{x_i}(f)$. □

An nice corollary of this fact is

Lemma 8.4. If $p(z)$ is a complex polynomial, then $\deg(p(z) : S^2 \to S^2) = \deg(p)$.

Proof. This proof will use a key fact, known as the Whitney-Graustein Theorem. This says that two curves $S^1 \to \mathbf{C} - \{0\}$ are homotopic if and only if they have the same winding number.

Let $p(z) = \prod_{i=1}^n (z - \lambda_i)^{k_i}$. We will consider the local degree by constructing small disjoint balls U_i around λ_i, so that they do not intersect. Taking a contour $\gamma_i = p(B(\lambda_i, \varepsilon))$ in U_i, we see that

$$\frac{1}{2\pi i} \int_{\gamma_i} \frac{dz}{z} \;=\; \text{degree of pole at } \lambda_i \;=\; k_i. \tag{8.5}$$

By Whitney-Graustein, for all small ε along $B(\lambda_i, \varepsilon)$, our map is homotopic to the map $f_i : z \to z^{k_i}$, which we know has degree (in the map of spheres sense) equal to k_i. So, the local degree of p near λ_i is k_i, which concludes the proof. □

An important, but related aside is Sard's Theorem. It is a nice local descriptionof smooth maps between smooth manifolds.

Theorem 8.5 (Sard's Theorem)**.** If M, N are smooth n-manifolds and $f : M \to N$ is smooth, then for a dense set of $y \in N$, the set $f^{-1}(y)$ is finite. Indeed we can find neighborhood U_i, V suh that $f|_{U_i} : U_i \to V$ is a diffeomorphism with local degree ± 1. Locally, f looks like the linear map df_{x_i}.

Now, we move on to cell complexes.

Definition 8.2. A *cell complex* (or *CW Complex*) is a space

$$X = \bigcup_{k \geq 0} X_k$$

defined inductively as follows:

- X_0 is a finite set.
- X_{i+1} is obtained from X_i via attaching finitely many i-cells, i.e. discs D^{i+1} attached via the maps
$$\partial D \xrightarrow{e_i} X_i. \tag{8.6}$$
That is, $X_{i+1} = \left(X_i \amalg_{j=1}^{n(i)} D_j^{i+1}\right) / e_j : \partial D_j^{i+1} \to X_i$
- We equip X with the weak topology, that is, $U \subset X$ is open if and only if $U \cap D^i$ is open for every cell D^i.

If X is a cell complex, the *i-skeleton* of X, denoted X_i, is the disjoint union of the interiors of the i-cells of X.

Example 8.7.
(1) S^n has a cell complex structure with one 0-cell and one n-cell, where the attaching map is the only thing possible.
(2) T^2 has a cell complex structure consisting of one 0-cell, two 1-cells, and one 2-cell. The 1-skeleton is $S^1 \vee S^1$ and the attaching map for the two-cell is given by $aba^{-1}b^{-1}$. Analogously, Σ_g has a structure as a cell complex with $2g$ 2-cells.
(3) $S^n \times S^m$ has a product cell structure with one 0-cell, one n-cell, one m-cell, and one $(n+m)$-cell.

(4) Since

$$\begin{aligned} \mathbf{CP}^{n+1} &= \mathbf{C}^{n+1} \cup \mathbf{CP}^{n} \\ \mathbf{RP}^{n} &= \mathbf{R}^{n} \cup \mathbf{RP}^{n} \end{aligned}$$

inductively, one gets a cell structure on $\mathbf{CP}^n$ with one cell in every even dimension at most $2n$ and a cell structure on $\mathbf{RP}^n$ with one cell in every dimension at most n.

Cell complexes are important in part because of the following Theorem, to be proved in Morse Theory.

Theorem 8.6. A smooth closed manifold admits the struture of a cell complex.

Remark 8.7. If X is a cell complex of finite dimension, or has only finitely many cells, X is compact.

Another nice fact (proven on exercise sheet 2) is

Lemma 8.8. If X is a cell complex and $A \subset X$ is a closed subcomplex (i.e. a union of cells of X), then (X, A) is a good pair in the sense that A is a deformation retract of open neighborhoods in X.

Proof. See exercise sheet 2. □

Lemma 8.9. Let $X = \bigcup_{k\geq 0} X_k$ be a cell complex. Then,

(1) $H_i(X_k, X_{k-1}) = \begin{cases} \mathbf{Z}^{n_k} & i = k \\ 0 & \text{o/w} \end{cases}$

(2) The inclusion $X_k \hookrightarrow X$ induces an isomorphism $H_i(X_k) \to H_i(X)$ for all $i < k$ and $H_j(X_k) = 0$ for all $j > k$.

Proof. Since X_{k-1} is a subcomplex of X_k, (X_k, X_{k-1}) is a good pair. So, by Proposition 6.6, $H_i(X_k, X_{k-1}) = \widetilde{H}_*(X_k/X_{k-1})$, but that quotient space is just a wedge sum of n_k n-spheres. So, the result follows from Corollary 6.8 and the facts we know about homologies of spheres/relative homology.

For part (2), consider the LES of the pair (X_k, X_{k-1}). Part of that long exact sequence is

$$\cdots \longrightarrow H_{i+1}(X_k, X_{k-1}) \longrightarrow H_i(X_{k-1}) \longrightarrow H_i(X_k) \longrightarrow H_i(X_k, X_{k-1}) \longrightarrow \cdots \tag{8.8}$$

so that when $i > k$, the outermost homologies are zero, establishing an isomorphism $H_i(X_{k-1}) \to H_i(X_k)$. Since $H_i(X_0) = 0$ for $i > k \geq 0$, induction finishes the proof of the final remark.

Similarly, we can bump up the indices on the skeletons in (8.8) to get

$$\cdots \longrightarrow H_{i+1}(X_{k+1}, X_k) \longrightarrow H_i(X_k) \longrightarrow H_i(X_{k+1}) \longrightarrow H_i(X_{k+1}, X_k) \longrightarrow \cdots \tag{8.9}$$

so that when $i < k$, we have an isomorphism $H_i(X_k) \to H_i(X_{k+1})$. If X is finite dimensional, $X = X_N$ for some large N, which establishes the final claim.

If, on the other hand, X has ininitely many cells, any simplex (or chain – the union of finitely many simplices representing a homology class) has compact image in X and, by a result on exercise sheet 2, meets only finitely many cells of X.

So, a nice fact is that every class in $H_*(X)$ already lies in $H_*(X_n)$ for some finite n. □

9. Lecture 9 – Cellular Homology

Recall we're studying cell complexes $X = \bigcup_{k \geq 0} X_k$ with finitely many (say n_k) k-cells for $k \geq 0$.

Last time, we showed $H_k(X_k, X_{k-1}) \cong \mathbf{Z}^{n_k}$ naturally generated by the k-cells in X.

Definition 9.1. The *cellular chain complex*, $C_*^{\text{cell}}(X)$, which depends on a choice of cell structure on X, has chain groups

$$C_k^{\text{cell}}(X) \;=\; H_k(X_k, X_{k-1}) \qquad (\cong \mathbf{Z}^{n_k}). \tag{9.1}$$

and the differential d_k^{cell} is defined as follows:

$$\begin{tikzcd} H_k(X_k, X_{k-1}) \arrow[rr, "d_k^{\text{cell}}"] \arrow[dr] & & H_{k-1}(X_{k-1}, X_{k-2}) \\ & H_{k-1}(X_{k-1}) \arrow[ur] & \end{tikzcd} \tag{9.2}$$

where the map on the left is the boundary map from the long exact sequence of pairs and the map on the right is the other natural map coming from a second LES of pairs.

First, we show that this does in fact make a chain complex.

Lemma 9.1. $(d^{\text{cell}})^2 = 0$, i.e. $d_{k-1}^{\text{cell}} \circ d_k^{\text{cell}} = 0$

Proof. This diagram represents the composition.

$$\begin{tikzcd} & H_k(X_n) \arrow[dr, "\varphi"] & & & \\ H_{k+1}(X_{k+1}, X_k) \arrow[ur] \arrow[rr, "d_{k+1}^{\text{cell}}"] & & H_k(X_k, X_{k-1})_{d_k^{\text{cell}}} \arrow[rr] \arrow[dr, "\psi"] & & H_{k-1}(X_{k-1}, X_{k-2}) \\ & & & H_{k-1}(X_{k-1}) \arrow[ur] & \end{tikzcd} \tag{9.3}$$

and we see that φ and ψ are consecutive terms in a long exact sequence for pairs. $\square$

Of course, we want to show that cellular homology and singular homology agree on cell complexes. This is the content of our next Theorem.

Theorem 9.2. Let X be a cell complex. Then, $H_*^{\text{cell}}(X) \cong H_*(X)$.

Proof. The reason this argument works is part (2) of Lemma 8.9 and the fact that the cellular homology maps are obtained from long exact sequences in our ordinary homology. Apart from that, the proof is largely a diagram chase.

Consider this augmented diagram.

$$\begin{tikzcd}
 & & & 0 = H_k(X_{k+1}, X_k) & \\
0 = H_k(X_{k-1}) \arrow[dr] & & H_k(X_{k+1}) \cong H_k(X) \arrow[ur] & & \\
 & H_k(X_k) \arrow[ur] \arrow[dr, "\varphi"] & & & \\
H_{k+1}(X_{k+1}, X_k) \arrow[ur, "\partial_{k+1}"] \arrow[rr, "d_{k+1}^{\mathrm{cell}}"] & & H_k(X_k, X_{k-1}) \arrow[rr, "d_k^{\mathrm{cell}}"] \arrow[dr, "\psi"] & & H_{k-1}(X_{k-1}, X_{k-2}) \\
 & & & H_{k-1}(X_{k-1}) \arrow[ur, "i"] & \\
 & & H_{k-1}(X_{k-2}) = 0 \arrow[ur] & &
\end{tikzcd} \tag{9.4}$$

Then, reading of the diagram, we have

$$\begin{aligned}
H_k(X) \;=\; H_k(X_{k+1}) &= H_k(X_k)/\mathrm{im}(\partial_{k+1}) \\
&= \varphi(H_k(X_k))/\mathrm{im}(d_{k+1}^{\mathrm{cell}}) && (\varphi \text{ is injective, } \varphi \circ \partial_{k+1} = d_{k+1}^{\mathrm{cell}}) \\
&= \ker(\psi)/\mathrm{im}(d_{k+1}^{\mathrm{cell}}) && (\text{exactness}) \\
&= \ker(d_k^{\mathrm{cell}})/\mathrm{im}(d_{k+1}^{\mathrm{cell}}) && (i \text{ in injective}) \\
&= H_k^{\mathrm{cell}}(X)
\end{aligned}$$

□

Corollary 9.3. If X is a cell complex,

(1) $H_k(X)$ is a finitely generated abelian group of rank at most n_k, the number of k-cells in X.
(2) If $H_k \neq 0$, then any cell structure on X must have k-cells.
(3) If X is compact $H_*(X, \mathbf{Q})$ is a finite-dimensional graded vector space.
(4) If X admits a cell structure with only even dimensional cells, then for this structure, $H_*^{\mathrm{cell}}(X) = H_*(X) = C_*^{\mathrm{cell}}(X)$

Remark 9.4. Although 4 from the above Corollary may seem derived, various spaces in algebraic geometry over $\mathbf{C}$ satisfy it. Examples include $\mathbf{CP}^k$ and $\mathrm{Gr}_k(\mathbf{C}^n)$.

Equipped with the language of cell complexes, we can generalize one of the most natural descriptions of a space – Euler characteristic. One typically begins by defining Euler characteristic for surfaces, through triangulations. However, from this description, it is not obvious that Euler characteristic is invariant under homotopy or homeomorphism of our surface and it is hard to show that Euler Characteristic does not depend on our choice of triangulation.

Definition 9.2. The j^{th} *Betti Number* of a cell complex, denoted $b_i(X)$ is $\mathrm{rk}(H_j(X))$ or $\dim_{\mathbf{Q}}(X, \mathbf{Q})$. The *Euler Characteristic of* X is

$$\chi(X) = \sum_i (-1)^i b_i(X) \tag{9.5}$$

We note that $\chi(X)$ is well-defined iff X is compact. The following Lemma shows us that the naïve computation of Euler characteristic as "vertices - edges + faces" is in fact well-defined and is the general way to compute Euler characteristic for higher-dimensional cell complexes (including all smooth manifolds).

Lemma 9.5. For a compact cell complex, $\chi(X) = \sum_{i\geq 0}(-1)^i n_i$, where n_i is the number of i cells in X.

Proof. First, we will assume that $\mathrm{rk}(\ker(d_i)) + \mathrm{rk}(\mathrm{im}(d_i)) = \mathrm{rk}(C_i)$, where C_i is the i^{th} chain group. Under this assumption, we have

$$\begin{aligned}\sum_i (-1)^i \mathrm{rk}(H_i(C_*, d)) &= \sum_i (-1)^i \ker(d_i) - (-1)^i d_{i+1} \\ &= \sum_i (-1)^i \left(\ker(d_i) + \mathrm{im}(d_i)\right) \\ &= \sum_i (-1)^i \mathrm{rk}(C_i) \qquad \text{(from our assumption)}\end{aligned}$$

which conditionally establishes our claim.

Now, we will prove the claim. Consider the short exact sequence

$$0 \longrightarrow Z_k \longrightarrow C_k \xrightarrow{d_k} B_{k-1} \longrightarrow 0. \tag{9.6}$$

We claim that in this case, the boundary group B_{k-1} is free, finitely generated Abelian group. So, the sequence splits as $C_k = Z_k \oplus B_{k-1}$, which gives the result. □

Remark 9.6. The Euler characteristic $\chi(X)$ is the simplest (with respect to computability) algebro-topological invariant. It distinguishes $S^4, S^1 \times S^3, \mathbf{CP}^2$.

Remark 9.7. For a product space $X \times Y$ satisfies $\chi(X \times Y) = \chi(X)\chi(Y)$ since if X, Y are cell complexes, $X \times Y$ admits a natural produt structure where open cells are the products of cells in X with cells in Y.

However, $b_j(X \times Y) \neq b_j(X)b_j(Y)$ in general.

Finally, we will provide a lemma that helps us in computing cellular homology. First, let's explain why this lemma is sensible. d_k^{cell} maps the boundary of a bouquet of spheres to a bouquet of spheres of lower dimension. The lemma below allows us to compute the cell boundary map.

Lemma 9.8. For a k-cell D_α^k in a cell complex, X, we have

$$d_k^{\text{cell}}\left[D_\alpha^k\right] = \sum_\beta d_{\alpha\beta}\left[D_\beta^{k-1}\right] \tag{9.7}$$

where the coefficients $d_{\alpha\beta}$ are given as follows:

$$S_\alpha^{k-1} = \partial D_\alpha^k \xrightarrow{\varphi_\alpha} X_{k-1} \to (X_{k-1}/X_{k-2}) = \bigvee_\beta S_\beta^{k-1} \xrightarrow{q} S_\beta^{k-1} \tag{9.8}$$

where final map q collapses all but one sphere in the bouquet and φ_α is the gluing map. The overall map then has a degree $n \in \mathbf{Z}$.

Proof. We have

$$\begin{tikzcd} \mathbf{Z} = H_k(D_\alpha^k, \partial D_\alpha^k) \arrow[r, "\partial", "\cong"'] \arrow[d, "\varphi"] & H_{k-1}(\partial D_\alpha^k) \arrow[r, dashed] \arrow[d, "\varphi|\partial D_\alpha^k"] & H_{k-1}(S_\beta^{k-1}) \\ H_k(X_k, X_{k-1}) \arrow[r] \arrow[dr, "d_k^{\text{cell}}"] & H_{k-1}(X_{k-1}) \arrow[r, "\alpha"] \arrow[d, "\psi"] & H_{k-1}(X_{k-1}/X_{k-2}) \arrow[u, "q_\beta"'] \arrow[d, "\cong"', "k\geq 2"] \\ & H_{k-1}(X_{k-1}, X_{k-2}) \arrow[r, "\cong"] & H_{k-1}(X_{k-1}/X_{k-2}, \mathrm{pt}) \end{tikzcd}$$

.

The key here is that φ, as a map on homology, is induced by inclusion. So, it sends a generator $\left[D_\alpha^k\right]$ of $H_k(D_\alpha^k, \partial D_\alpha^k)$ to a generator of the summand in $H_k(X_k, X_{k-1})$ corresponding to the cell α. However, we can also read off where $\left[D_\alpha^k\right]$ goes via the map $\psi \circ \varphi|\partial D_\alpha^k \circ \partial$. However, q_β is induced by projection. So, the effect can also be read off all of the image spheres S_β^{k-1}, essentially "going the wrong way", along q_β. Commutativity then gives the formula. $\square$

10. Lecture 10 – Real Projective Space and Fibre Bundles

Recall $\mathbf{RP}^n = S^n/\pm I = \mathbf{R}^n \cup \mathbf{RP}^{n-1}$. We will see inductively that $\mathbf{RP}^n$ as a cell structure with one cell of each dimension i, $0 \le i \le n$ so that C_*^{cell} is given by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow \cdots \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\ & & n & & n-1 & & 0. & & \end{array}$$

and recall $d^{\text{cell}} : \partial D^k = S^{k-1} \longrightarrow \mathbf{RP}^{k-1}/\mathbf{RP}^{k-2} = (X_{k-1}/X_{k-2})$ and $\mathbf{RP}^{k-1}/\mathbf{RP}^{k-2} = S^{k-1}$. So, by Lemma 9.8, the degree of d^{cell} is given by the degree of this map.

By our geometric construction, the map is 2:1 for a general point in S^{k-1}. Any point in the image has two pre-images that differ by composition with the antipodal map. So, the local degrees of the pre-images differ by a factor of $(-1)^k$. So,

$$\begin{array}{ccc} d_k^{\text{cell}} : & \mathbf{Z} \rightarrow & \mathbf{Z} \\ & k & k-1 \end{array} \tag{10.1}$$

has degree $1 + (-1)^k$. The chain complexes then look like

$$0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{0} \mathbf{0} \tag{10.2}$$

when n is even and

$$0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{0} \mathbf{0} \tag{10.3}$$

which establishes the following corollary.

Corollary 10.1. The homology of $\mathbf{RP}^n$ is given by

$$H_*(\mathbf{RP}^n; \mathbf{Z}) = \begin{cases} \mathbf{Z} & * = 0, * = n \text{ if } n \text{ odd.} \\ \mathbf{Z}/2 & 0 < * < n, \ * \text{ odd.} \end{cases} \tag{10.4}$$

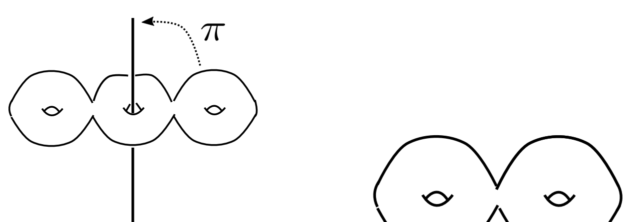
.

Note that in $\mathbf{Z}/2$, multiplication by 2 is the zero map and so $H_*(\mathbf{RP}^n, \mathbf{Z}/2) = \mathbf{Z}/2$ for $0 \le * \le n$. Now, recall from earlier lectures, Definition 4.4.

Definition 10.1. A fibre bundle such that the fibre, F, is a discrete topological space is called a *covering map*. Then, the projection ρ is a local homeomorphism.

Example 10.5. (1) $\mathbf{R} \to S^1$, $t \to e^{2\pi i t}$.

(2) The second example is



(A) Total Space plus ρ, rotation by π

(B) Base Space

Figure 12. Pictures of animals

(3) $S^n \xrightarrow{\rho} \mathbf{RP}^n$ is a covering space.

The following fact from point set topology will be useful.

Theorem 10.2 (Path-Lifting)**.** if $\rho : E \to B$ is a covering map and $\sigma : \Delta^i \to B$ is an i-simplex, then σ lifts to E, i.e. there exists a map $\widetilde{\sigma}$ such that the following diagram commutes

$$\begin{tikzcd} & E \arrow[d, "\rho"] \\ \Delta^i \arrow[ur, "\widetilde{\sigma}"] \arrow[r, "\sigma"'] & B. \end{tikzcd} \tag{10.6}$$

Indeed, two lifts of any path connected space X mapping to B which agree at a point agree everywhere. If $\rho : E \to B$ is a finite covering, i.e. $\text{Fibre}(F) \cong \{n \text{ point set}\}$, then σ has exactly n lifts to E.

The next lemma introduces homology mod 2. The geometric interpretation of homology mod 2 is "homology without orientation". That is, two simplices cancel if and only if they overlap. However, the main application of this notion will be to detect whether a map of spheres has even or odd degree.

Lemma 10.3. Let $\rho : X \to B$ be a double covering, so $\rho^{-1}(b) = \{x, y\}$ has two points. Then, there exists a long exact sequence.

$$\cdots \to H_r(B, \mathbf{Z}/2) \to H_r(X, \mathbf{Z}/2) \to H_r(B, \mathbf{Z}/2) \to H_{r-1}(B, \mathbf{Z}/2) \to H_{r-1}(X, \mathbf{Z}/2) \to \cdots \tag{10.7}$$

Proof. Note that in this case a simplex σ has precisely two lifts, say $\widetilde{\sigma}_1, \widetilde{\sigma}_2$. So, we can consider the following short exact sequence of chain complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*(B, \mathbf{Z}/2) & \xrightarrow{\tau} & C_*(X, \mathbf{Z}/2) & \xrightarrow{\rho} & C_*(B, \mathbf{Z}/2) & \longrightarrow & 0 \\ & & \sigma & \longrightarrow & \widetilde{\sigma}_1 + \widetilde{\sigma}_2 & & & & \\ & & & & \gamma & \longrightarrow & \rho \circ \gamma & & \end{array} \tag{10.8}$$

The key is that since coefficients are mod 2, $\rho \circ \tau = 0$ and the sequence is exact. We then use the snake lemma to extract our long exact sequence in homology. $\square$

Now, for our main application of homology mod 2.

Proposition 10.4 (Odd Maps Have Odd Degree)**.** Let $f : S^n \to S^n$ satisfy $f(x) = -f(-x)$. Then, f has odd degree.

Proof. Note that showing f has odd degree is equivalent to showing that f is an isomorphism in $\mathbf{Z}/2$ homology. This is how we proceed.

Our first observation is that the symmetry $f(x) = -f(-x)$ shows that f descends to a well-defined map, $\overline{f}$ from $\mathbf{RP}^n$ to itself.

We begin with the short exact sequence of chain complexes in (10.8) and show that such an f induces a (chain) map from this sequence to itself. Examining how we build a long exact sequence from a short exact sequence via the Snake Lemma, we see it is in fact sufficient to show that f commutes with the short exact sequence (10.8). First, $\overline{f}_\sharp \tau = \tau f_\sharp$, since f takes antipodal points to antipodal points and the lifts of σ are related by the antipodal map. Second, we have $\overline{f}_\sharp \rho = \rho f_\sharp$ by definition of ρ.

We can now examine the long exact sequence in homology given by the following diagram. All homologies are in $\mathbf{Z}/2$.

$$H_{n+1}(\mathbf{RP}^n) \xrightarrow{\partial_*} H_n(\mathbf{RP}^n) \xrightarrow{\tau_*} H_n(S^n) \xrightarrow{\rho_*} H_n(\mathbf{RP}^n) \to \cdots \to H_0(\mathbf{RP}^n) \xrightarrow{\tau_*} H_0(S^n) \xrightarrow{\rho_*} H_0(\mathbf{RP}^n) \tag{10.9}$$

Note that because of the cell complex structure on $\mathbf{RP}^n$ and Proposition 9.3, $H_{n+1}(\mathbf{RP}^n) = 0$. So, the first τ_* map is injective and thus an isomorphism (the homologies in question are finite dimensional vector spaces of the same rank). Following the diagram, we see that the final $\delta_* : H_1(\mathbf{RP}^n) \to H_0(\mathbf{RP}^n)$ is an isomorphism, the final τ_* is the zero map, and the final ρ_* is an isomorphism.

Now, we examine the effect of f_* on our final square, namely

$$\begin{tikzcd} H_0(S^n) \arrow[r, "\rho_*", "\cong"'] \arrow[d, "f_*"] & H_0(\mathbf{RP}^n) \arrow[d, "\overline{f}_*"] \\ H_0(S^n) \arrow[r, "\rho_*", "\cong"'] & H_0(\mathbf{RP}^n). \end{tikzcd} \tag{10.10}$$

As $\mathbf{RP}^n$ is path-connected, in $\mathbf{Z}_2$ coefficients the generator of $H_0(\mathbf{RP}^n)$ is any sum of k points, where k is odd. Clearly, $\overline{f}_*$ sends a generator to a generator since it sends one point to one point. So, three of the four maps in (10.10) are isomorphisms, forcing the final one to be an isomorphism as well.

Since in each degree, either ρ_* or τ_* is an isomorphism, we can apply this same argument, moving up the chain in (10.9) to show that $f_* : H_n(S^n) \to H_n(S^n)$ is an isomorphism – i.e. it is of odd degree. ☐

That proposition is reassuring, since it tells us that one of our notions of odd implies the other. It is also nice because it gives us the following two corollaries:

Corollary 10.5. Let $g : S^n \to \mathbf{R}^n$ be continuous. Then, there exists some $x \in S^n$ such that $g(x) = g(-x)$. Colloquially, there are always antipodal points on the earth with the same height and temperature.

Proof. As suggested by the previous proposition, we will assume our conclusion is false and demonstrate an odd map of spheres that has even degree.

Note that $g : S^n \to \mathbf{R}^n$ defines a map

$$\begin{aligned} g' : S^n &\to \mathbf{R}^n - \{0\} \\ x &\mapsto g(x) - g(-x) \end{aligned} \tag{10.11}$$

which (by dividing by magnitude) can be extended to a map $\widetilde{g} : S^n \to S^{n-1}$, which, from basic facts of homologies of spheres has degree zero.

However, we can also examine the induced map given by inluding S^{n-1} as the equator of S^n, namely

$$S^{n-1} \xrightarrow{\iota} S^n \xrightarrow{\widetilde{g}} S^{n-1} \tag{10.12}$$

which is also odd. However, $\deg(\iota \circ \widetilde{g}) = \deg(\iota)\deg(\widetilde{g}) = 0$, that is the composite map factors through a trivial map on homology. So, it has degree zero and is not odd, a contradiction. ☐

Finally, we have the Proposition that easily wins the "best name" award – the Cheese and Pickle Sandwich Theorem. Despite the silly name, this Theorem was one of the great early successes of Algebraic Topology. Geometers wrestled with this problem and could not solve it without removing certain convexity assumptions until they incorporated techniques from Algebraic Topology.

Corollary 10.6 (Cheese and Pickle Sandwich Theorem)**.** Let $A_1, \cdots, A_n$ be bounded open sets in $\mathbf{R}^n$. Then, there exists some hyperplane $H \in \mathbf{R}^n$ simultaneoulsy dividing all A_i into equal volume.

Proof. First, note that any hyperplane $H \in \mathbf{R}^n$ has a normal vector (and vice-versa) and can thus be associated to a point on the sphere, p. However, this correspondence is not one-to-one. Antipodal points on the sphere produce the same hyperplane (though crucially the positive and negative sides get switched). Moreover, each pair of antipodal points on S^{n-1} are associate to a family of hyperplanes, which differ by translations.

Pick $x \in S^{n-1}$. Then, since A_n is bounded, we can choose some hyperplane H_x^1 so that A_n lies entirely on the negative side of H_x^1 and similarly an H_x^2 so that A_n lies entirely on its positive side. By the intermediate value theorem, there is some hyperplane, H_x which cuts A_n into regions of equal volume.

For any $x \in S^n$ $i = 1, \cdots, n-1$, let $z_i = \lambda(H_x^+ \cap A_i)/\lambda(A_i) - 1/2$ so that z_i is zero precisely when H_x cuts A_i into regions of equal volume. We now examine the map

$$\begin{aligned} \varphi : S^{n-1} &\to \mathbf{R}^{n-1} \\ x &\mapsto (z_1, \cdots, z_{n-1}). \end{aligned} \tag{10.13}$$

By an earlier observation, $H_{-x} = H_x$ as a set, but has different positive and negative sides. As such, φ is an odd map. However, by Corollary 10.5, there is some $y \in S^{n-1}$ such that $\varphi(y) = \varphi(-y) = -\varphi(-y)$ and so $\varphi(y) = 0$ and H_y divides all the A_i into regions of equal volume. $\square$

11. Lecture 11 – Generalized Homology Theories

Axioms

Definition 11.1. An assignment $(X, A) \to h_*(X, A)$ of graded abelian groups to a pair (X, A) of a topological space X and a subspace $A \subset X$ is called a *generalized homology theory* if

(i) It's functorial: a map $f : (X, A) \to (Y, B)$ induces $f_* : h_*(X, A) \to h_*(Y, B)$ such that $\mathrm{id}_* = \mathrm{id}$ and $(f \circ g)_* = f_* \circ g_*$
(ii) Setting $h_*(X) := h_*(X, \emptyset)$ ther exists a long exact sequence

$$\cdots \longrightarrow h_i(A) \longrightarrow h_i(X) \longrightarrow h_i(X, A) \longrightarrow h_{i-1}(A) \longrightarrow \cdots \tag{11.1}$$

(iii) It satisfies homotopy invariance. If $f \simeq g$ (through maps of pairs), then $f_* = g_* : h_*(X, A) \to h_*(Y, B)$.
(iv) It satisfies excision. If $\mathrm{cl}(Z) \subset \mathrm{int}(A)$, then inclusion gives $h_*(X \setminus Z, A \setminus Z) \xrightarrow{\cong} h_*(X, A)$.
(v) It respects disjoint unions in the sense that

$$h_*\left(\bigsqcup_\alpha X_\alpha\right) = \bigoplus_\alpha h_*(X_\alpha) \tag{11.2}$$

Example 11.3. Singular homology is such a theory.

One nice feature of generalized homology theories is that they are determined by the values they take on a point. These values are called *coefficients*.

Proposition 11.1. Suppose h_* and k_* are generalized homology theories and that $\Phi : h_* \to k_*$ is a natural transformation of theories in the sense that for all pairs (X, A) we have maps $h_*(X, A) \to k_*(X, A)$ compatible with maps of spaces, LES for pairs, excision isomorphisms, and so on. If Φ is an isomorphism $h_*(\mathrm{pt}) \to k_*(\mathrm{pt})$, then Φ is an isomorphism for all good pairs (such as when X is a cell complex and A is a sub-complex).

Proof. We will show that h_* and k_* agree on cell complex pairs by induction on dimension. For zero dimensional complexes, the hypotheses and the union axiom imply Φ is an isomorphism.

Now, suppose Φ_* is an isomorphism for pairs (X, A) such that X is a cell complex of dimension at most $n-1$ and A is a subcomplex. Let X be an n-dimensional subcomplex and X_{n-1} be its $(n-1)$-skeleton. We get

$$\begin{tikzcd}
\cdots \arrow[r] & h_{*+1}(X, X_{n-1}) \arrow[r] \arrow[d, "\Phi"] & h_*(X_{n-1}) \arrow[r] \arrow[d, "\Phi", "\cong"'] & h_*(X) \arrow[r] \arrow[d, "\Phi"] & h_*(X, X_{n-1}) \arrow[r] \arrow[d, "\Phi"] & h_{*-1}(X_{n-1}) \arrow[r] \arrow[d, "\Phi", "\cong"'] & \cdots \\
\cdots \arrow[r] & k_{*+1}(X, X_{n-1}) \arrow[r] & k_*(X_{n-1}) \arrow[r] & k_*(X) \arrow[r] & k_*(X, X_{n-1}) \arrow[r] & k_{*-1}(X_{n-1}) \arrow[r] & \cdots
\end{tikzcd} \tag{11.4}$$

where the isomorphisms above come from our inductive hypothesis.

Note that by the five-lemma, it suffices to show that Φ is an isomorphism for the pair (X, X_{n-1}). Here, we appeal to excision and the fact the Φ is compatible with the excision isomorphisms. Excision, the five-lemma, and the long exact sequence for pairs say

$$h_*(X, X_{n-1}) \cong h_*(\sqcup D^n, \sqcup \partial D^n) \tag{11.5}$$

and the same for k_*.

So, it suffices to prove Φ is an isomorphism for the pair $(D^n, \partial D^n) = (D^n, S^{n-1})$.

This holds because D^n is homotopy equivalent to a point and S^{n-1} is an $(n-1)$-dimensional cell complex. More specifically, we have

$$\begin{array}{ccccccccc} h_*(\partial D^n) & \longrightarrow & h_*(D^n) & \longrightarrow & h_*(D^n, \partial D^n) & \longrightarrow & h_{*-1}(\partial D^n) & \longrightarrow & h_{*-1}(D^n) \\ \cong\big\downarrow\Phi & & \cong\big\downarrow\Phi & & \big\downarrow\Phi & & \cong\big\downarrow\Phi & & \cong\big\downarrow\Phi \\ k_*(\partial D^n) & \longrightarrow & k_*(D^n) & \longrightarrow & k_*(D^n, \partial D^n) & \longrightarrow & k_{*-1}(\partial D^n) & \longrightarrow & k_{*-1}(D^n) \end{array} \tag{11.6}$$

as part of the induced map on the long exact sequences of pairs. □

Remark 11.2. Two remarks:

(1) This doesn't mean that $h_*(\mathrm{pt})$ "algorithmically" determines $h_*(X)$ for each cell complex X.
(2) There's a generalized homology theory called stable homotopy theory, for which

$$h_i(X) \;=\; \lim_{n\to\infty} \left(\pi_{i+n}\left(\Sigma^n X\right)\right) \tag{11.7}$$

where here Σ denotes suspension and π_i is the i^{th} homotopy group. Here, $h_*(\mathrm{pt})$ is unknown. This is called the "stable stem" problem and is a major open question in algebraic topology.

There is an analogous definition of a generalized cohomology theory. It must be contravariantly functorial, have a long exact sequence for pairs, satisfy homotopy invariance, satisfy excision (though the arrow goes the other way), and respect unions in the previous sense.

Lemma 11.3. Singular cohomology defines a generalized cohomology theory in this sense.

Proof (sketch). In example sheet one, we understood $H^*\left(\sqcup_\alpha X_\alpha\right)$ and set up a long exact sequence for a pair. Functoriality here is clear. In proving homotopy invariance or excision ofr H_*, we constructed a prism operator P such that $\partial P + P\partial = f_* - g_*$ and a subdivision operator D such that $\partial D + D\partial = \mathbf{1} - \iota\rho$ where ρ is a map that subdivides complexes.

Dualizing everywhere, we get

$$\partial^* P^* + P^*\partial^* = f^* - g^*$$
$$\delta * D^* + D^*\partial^* = \mathbf{1} - \rho^*\iota^*$$

where ∂^* is the adjoint of ∂. □

A version of the Universal Coefficient Theorem says that $H^*(X, G)$ is determined by $H_*(X, Z)$ and the coefficient group, G.

There is an exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X, B) \longrightarrow \mathrm{Hom}(H_n(X), G) \longrightarrow 0 \tag{11.8}$$

where H_* is the group with $\mathbf{Z}$ coefficients and for abelian groups, G, H, one has

$$\mathrm{Ext}(H, G) \;=\; \{\text{SES's} \quad 0 \to G \to J \to H \to 0\} \,/ \cong \tag{11.9}$$

where $\cong$ is the natural notion of isomorphism.

Proposition 11.4. Let X be a finite dimensional cell complex. Then, $H^i(X, \mathbf{Z}) \cong \mathrm{Hom}(H_i(X, \mathbf{Z}), \mathbf{Z}) \oplus \mathrm{Torsion}\left(H_{i-1}(X, \mathbf{Z})\right)$

Proof. This is really a result about (co)-homology of chain complex such that the chain groups are finitely generated.

Recall $H_*(X) = H_*(C_*^{\text{cell}}, d^{\text{cell}})$. Exactly analogously, we let $C^*_{\text{cell}} = \mathrm{Hom}(C_*^{\text{cell}}, d^{\text{cell}})$, we set d^*_{cell} to be the adjoint of d_*^{cell}. This gives $H^*(X)$.

Suppose the cellular chain complex is

$$0 \longrightarrow \mathbf{Z}^{n_k} \longrightarrow \mathbf{Z}^{n_{k-1}} \longrightarrow \cdots \longrightarrow \mathbf{Z}^{n_1} \longrightarrow \mathbf{Z}^{n_0} \longrightarrow 0 \tag{11.10}$$

where n_i is the number of i-cells and $k = \dim(X)$.

As with our discussion of χ, the Euler characteristic, we break this into short exact sequences

$$0 \longrightarrow Z_n \xrightarrow{\iota} C_n \xrightarrow{d} B_{n-1} \longrightarrow 0 \tag{11.11}$$

and

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H_n \longrightarrow 0 \tag{11.12}$$

and all groups are finitely generated and free, so we can choose a splitting, i.e. $C_n \cong Z_n \oplus B_{n-1}$. Then, C_* looks like

$$\begin{tikzcd} Z_{n+1} & Z_n & Z_{n-1} \\ \oplus & \oplus & \oplus \\ B_n \arrow[uur] & B_{n-1} \arrow[uur] & B_{n-2} \end{tikzcd} \tag{11.13}$$

with boundary map as shown.

So, C_* can be described by maps $A_n : B_n \to Z_n$. By Smith normal form, there exists a $\mathbf{Z}$-linear change of basis such that the matrix A_n has the form of a torus $(d_1, d_2, \cdots, d_k, 0, \cdots, 0)$ where $d_1 \mid d_2, d_2 \mid d_3$ and so on.

So now, C_* breaks into sums of

$$0 \longrightarrow \mathbf{Z} \xrightarrow{0} \mathbf{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbf{Z} \xrightarrow{p} \mathbf{Z} \longrightarrow 0$$

for $p = d_i$ for some i. We then claim that for these complexes, homology and cohomology is trivial. This calculation itself is worthwhile. However, it comes down to the arrows in (11.13) changing direction and the fact that the adjoint of our maps in Smith normal form are just the transposes of these matrices. So, the d_i present in the k^{th} boundary map that gave us torsion in rank k homology will now give us the exact same torsion in the $(k+1)^{\text{st}}$ homology once the arrows have been reversed. □

Remarks on Cohomology

At this stage in the course, we will begin to focus more heavily on cohomology than homology. From simply looking at the definitions, they are analogous theories. Indeed, the homology groups of a space determine its cohomology groups and when chain groups are finitely generated, the converse holds as well (see the end of the previous lecture). These remarks follow closely Hatcher's exposition in "The Idea of Cohomology".

However, contravariant functors, as opposed to covariant functors, give cohomology extra structure. For example, the diagonal map $\Delta : X \to X \times X$ induces a map on cohomology which induces the *cup product*, a map $\smile: C^k(X) \times C^l(X) \to C^{k+l}(X)$ that makes cohomology a graded, (graded) commutative unital ring. There is no analogous ring structure for homology, since there is no privileged map from $X \times X$ to X.

The only immediate disadvantage of cohomology is that it is less obviously geometric.One way to interpret cohomology is through sheaves – cohomology measures the obstructions in extending certain local forms to global ones.

First, we will start by giving some meaning to the simplest cohomology group of X, the zero cohomology group. This group is generated by maps $\phi : X \to G$ such that $d\phi = 0$, or such that for any path in X, $\phi(v_1) - \phi(0) = 0$. So, the maps ϕ that generate this group are the maps constant on each path-component of X. Hence, $H^0(X) = \bigoplus_{\pi_0(X)} G$.

Now, let's examine the case when $H^1(X)$ is trivial. A cochain $\psi \subset C^1(X)$ assigns to every one-simplex in X an integer. This cochain is a coboundary if there is some (necessarily global) 0-cochain ϕ such that $\psi(\sigma) = \phi(\sigma(1)) - \phi(\sigma(0))$. If such an element exists, it will be unique up to an element of $\ker(d^0)$, or a constant function. This makes sense since in this case ψ can be defined by *changes in ϕ*. The problem is thus reminiscent of the calculus problem of finding an anti-derivative of a function, the solution of which is also unique up to a constant factor.

We can always find such a ϕ when X is a tree, since we can start at one vertex and assign it an arbitrary integer value. This determines the values ϕ takes If X is a tree, there are no cycles and we do not have to worry about any compatibility issues between our choices. In other words, patching is trivial.

Below is an example of a non-trivial co-cycle. In this figure, all edge values take into account orientation, so explicit orientation is omitted for the sake of clarity. Notice that it is not possible to assign values

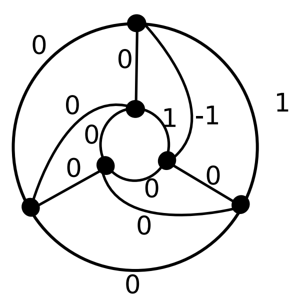


Figure 13. A Non-Trivial Co-Boundary

to the six vertices whose oriented differences give the values on the edges.

First cohomology provides a nice analogy to calculus. When $\psi \in C^1(X)$ satisfies $d\psi = 0$, we can think of ψ as a vector field whose curl is (locally) zero. In this case, ψ is a boundary when every line integral along a closed curve through that vector field vanishes. Note that for $\mathbf{R}^2$, these notions are equivalent. This is another way of saying $\mathbf{R}^2$ has trivial first (De Rham) cohomology.

More generally, let $\phi \in C^k(X)$. The condition that $d\phi = 0$ can be seen as a local condition, since it asks whether given a $(k+1)$-simplex in X, ϕ will cancel along its boundary. Asking whether ϕ itself is a co-boundary is a global question, since it asks us to create a global assignment of values to $(k-1)$-chains in X that patch together so that when we take appropriate signed differences of boundaries of k-simplices, we have ϕ.

In the case of manifolds, local patching is always possible. In differential geometry, this is known as the Poincaré Lemma. Cohomology then measures the obstruction in patching together local forms to an appropriate global form.

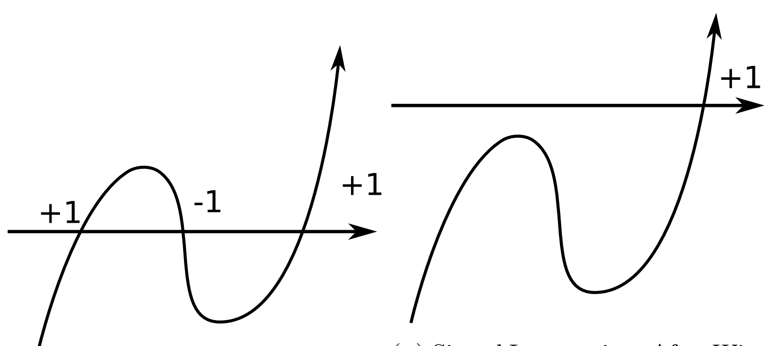
There is also a more geometric way to think of cohomology classes. The following exposition is due to Burt Totaro's excellent notes on Algebraic Topology in the Princeton Companion to Mathematics. Here, we are explicitly focused on closed, connected, oriented manifolds, since we will use the notion of Poincaré duality and its associated isomorphisms $H_{n-1}(X) \cong H^i(X)$ for an n-dimensional manifold X.

The idea of intersection number first appears in calculus, via the intermediate value Theorem. Consider

$$p(x) \;=\; x^3 + 3x - 4. \tag{11.14}$$

We know the end behavior of this polynomial. As x tends toward positive infinity, so does $p(x)$ and as x tends toward negative infinity, $p(x)$ tends toward negative infinity as well. Using basic properties of connectedness, we can deduce that p must have a zero somewhere on the real line, without necessarily knowing where that zero lives.

We can examine the graph of $p(x)$, or $\mathcal{G}(p) \subset \mathbf{R}^2$ and also place the positively oriented real line in the same copy of $\mathbf{R}^2$. and we note that the intersection number of these submanifolds does not change.



(A) Signed Intersections

(B) Signed Intersections After Wiggling

FIGURE 14. Intersection number is invariant under continuous deformations

To be technical, this required a few things. First, the manifolds had to intersect only in points, not line segments – they had to intersect *transversely.* Second, we needed orientations on both submanifolds to assign signs to each intersection.

Poincare duality tells us that when X is closed and oriented, we can write down an element of $H^i(X)$ by choosing a closed, oriented codimension i submanifold. Another way to find elements of $H^i(X)$ is to consider a “nice” map $f : X \to M^i$, where M^i is an i-dimensional manifold. Then, $f^{-1}(m)$ will be an element of $H^i(X)$.

As we will see, there is a natural ring structure on $H^*(X)$. Moreover, the idea of intersection number allows us to give a geometric description of multiplication in that ring. Let S, T be closed oriented submanifolds of X of codimension i, j respectively. Then, $[S] \in H^i(X), [T] \in H^j(X)$. An operation called the *cup product* allows us to define $[S] \smile [T] \in H^{i+j}(X)$, which should correspond to a codimension $i+j$ submanifold. By wiggling S and T, we may assume they intersect transversely (in a manifold of codimension $i+j$) and in this case the intersection manfiold $S \cap T$ inherits an orientation from S and T. It turns out these two pairings $H^i(X) \times H^j(X) \to H^{i+j}(X)$ are the same – cup products measure intersections of submanifolds.

12. Lecture 12 – The Cup Product

The key feature of cohomology as opposed to homology is that it is naturally a ring, with multiplication given by the *cup product*.

Definition 12.1. If $\phi \in C^k(X), \psi \in C^l(X)$, we define the *cup product* $\phi \smile \psi$ to be the cochain given by

$$(\phi \smile \psi)\,[v_0 \cdots v_{k+l}] \;:=\; (\phi[v_0 \cdots v_k]).(\psi[v_k \cdots v_l]) \tag{12.1}$$

where multiplication on the righthand side of (12.1) is multiplication of integers.

Remark 12.1. Sometimes the cup product of $\phi \in C^k(X)$ and $\psi \in C^l(X)$ is denoted by $\phi.\psi$. We will try to avoid this convention as often as possible.

This defines a product $C^k(X,R) \times C^l(X,R) \to C^{k+l}(X,R)$ for any coefficient ring R. We will now show that this descends to a well-defined map on cohomology.

Lemma 12.2. The cup product satisfies

$$d(\phi \smile \psi) \;=\; (d\phi) \smile \psi + (-1)^k(\phi) \smile (d\psi) \tag{12.2}$$

where $\phi \in C^k(X), \psi \in C^l(X)$

Proof. Recall that the cohomological boundary map is just the adjoint of the boundary map. So, we have

$$\begin{aligned}
d(\phi \smile \psi)[v_0 \cdots v_{k+l+1}] \;&=\; (\phi \smile \psi)\left(\sum_{i=0}^{k+l+1} (-1)^i [v_0 \cdots \hat{v}_i \cdots v_{k+l+1}]\right) \\
&=\; \sum_{i \le k+1} (-1)^i (\phi[v_0 \cdots \hat{v}_i \cdots v_{k+1}]).(\psi[v_{k+1} \cdots v_{k+l+1}]) \\
&\;+\; \sum_{i \ge k} (-1)^i (\phi[v_0 \cdots v_k]).(\psi[v_k \cdots \hat{v}_i \cdots v_{k+l+1}]) \\
&=\; (d\phi) \smile \psi + (-1)^k(\phi) \smile (d\psi)
\end{aligned}$$

□

Corollary 12.3. The cup product descends to cohomology, giving a well-defined product $H^i(X) \times H^j(X) \to H^{i+j}(X)$.

Proof. The formula shows that the cup product of two cocycles is again a cocycle.

Now, suppose ψ is a cycle (or a cohomology class). Cupping ψ with a boundary again gives a boundary since

$$d\phi \smile \psi \;=\; d(\phi \smile \psi) \pm \phi \smile d\psi \tag{12.3}$$

where the last term vanishes since ψ is a cycle. □

We now make a series of observations about the cup product:

Remark 12.4. Observe

1. Let $1 \in C^0(X)$ be the cocahin such that $1(\rho) = 1 \in \mathbf{Z}$ for each 0-simplex ρ in X. Then,
   - (i) 1 is a cocyle.
   - (ii) 1 is a chain-level unit for the cup product.

   So, $H^*(X)$ is a graded unital ring.
2. The cup productive is associative on $C^*(X)$ and so is associative on $H^*(X)$.
3. The cup product is functorial. If $f : X \to Y$ is continuous, then $f^\sharp : C^*(Y) \to C^*(X)$ satisfies
   $$f^\sharp(\phi \smile \psi) \;=\; f^\sharp(\phi) \smile f^\sharp(\psi) \tag{12.4}$$
   Moreover, $f^*(1) = 1$, so f^* is a unital ring homolomorphism $H^*(Y) \to H^*(X)$.

(4) If X, Y are spaces, there's a *cross-product*, or *external cup product*

$$H^k(X) \times H^l(Y) \xrightarrow{\times} H^{k+l}(X \times Y) \tag{12.5}$$

which is induced from the chain level products

$$(\phi, \psi) \mapsto (\pi_x^* \phi) \smile (\pi_y^* \psi) \tag{12.6}$$

where π_X is projection onto X and π_y is projection onto Y.

When $X = Y$, we use the natural map $\Delta : X \to X \times X$ sending x to (x, x) to obtain

$$H^*(X) \times H^*(X) \xrightarrow{\times} H^*(X \times X) \xrightarrow{\Delta^*} H^*(X) \tag{12.7}$$

which is the cup product. As noted in the introduction, the lack of a privileged map from $X \times X$ back to X is what fails when we try to make homology into a ring.

(5) The cup and cross products are bilinear and descend to $H^*(X) \otimes H^*(X)$ and $H^*(X) \otimes H^*(Y)$.

Proof of Remark 12.4. The only sub-remark that isn't immediate is (3). We prove it by noting that for $\phi \in C^k(X), \psi \in C^l(X)$ we have

$$\begin{aligned} f^\sharp(\phi \smile \psi)\sigma[v_0 \cdots v_{k+l}] &= (\phi \smile \psi)(f\sigma[v_0 \cdots v_{k+l}]) \\ &= \phi(f\sigma|[v_0 \cdots v_k]).\psi(f\sigma|[v_k \cdots v_{k+l}]) \\ &= f^\sharp(\phi) \smile f^\sharp(\psi) \end{aligned}$$

□

Finally, we show the promised (graded) commutativity. The proof here is somewhat technical and unilluminating. It is reminiscent of the proof of Theorem 3.2. In that proof, we were given $f, g : X \to Y$ and a homotopy $f \simeq_F g$. We used this to construct $P : C_n(X) \to C_{n+1}(Y)$ via

$$\Delta^{n+1} \xrightarrow{j} \Delta^n \times [0,1] \xrightarrow{(\sigma, \mathrm{Id})} X \times [0,1] \xrightarrow{F} Y \tag{12.8}$$

where the map j relies on the decomposition of $\Delta^n \times [0,1]$ into $n+1$ $(n+1)$-simplices $[v_0 \cdots v_i w_i \cdots w_n]$ and sends Δ^{n+1} to $\sum(-1)^i [v_0 \cdots v_i w_i \cdots w_n]$. Here, the Prism gets modified a bit.

Proposition 12.5. The cup product is (graded) commutative, namely

$$[\phi] \smile [\psi] = (-1)^{kl} [\psi] \smile [\phi] \tag{12.9}$$

where $\phi \in C^k(X), \psi \in C^l(X)$ are cocyles and the identiy also holds on cohomology.

Proof. For this proof, let $\varepsilon_n = (-1)^{n(n+1)/2}$. We will also define

$$\begin{aligned} \rho : C_n(X) &\to C_n(X) \\ [v_0 \cdots v_n] &\mapsto \varepsilon_n [v_n \cdots v_0] \end{aligned} \tag{12.10}$$

Claim. We claim ρ is chain map, chain homotopic to the identity.

Given the claim, we have

$$\begin{aligned} (\rho^*\phi) \smile (\rho^*\psi)[v_0 \cdots v_{k+l}] &= \phi(\varepsilon_k [v_k \cdots v_0]) \smile \psi(\varepsilon_l [v_{k+l} \cdots v_k]) \\ &= \varepsilon_k \varepsilon_l (\psi[v_{k+l} \cdots v_k])(\phi[v_k \cdots v_0]) \\ &= (-1)^{kl} \rho^*(\psi \smile \phi). \end{aligned}$$

where the last line usese that $\varepsilon_{k+l} = (-1)^{kl} \varepsilon_k \varepsilon_l$.

Using homotopy equivalence and functoriality, the previous calculation is sufficient for our result.

Proof of Claim. First, we will show that ρ is a chain map. This is true because

$$\begin{aligned} d\rho(\sigma) &= d\left(\varepsilon_n \sigma([v_n \cdots v_0])\right) \\ &= \varepsilon_n \sum_{i=0}^{n} (-1)^i \sigma|[v_n \cdots \hat{v}_{n-i} \cdots v_0] \end{aligned}$$

and

$$\begin{aligned} \rho(d\sigma) &= \rho\left(\sum_{i=0}^{n} (-1)^i \sigma|[v_0 \cdots \hat{v}_i \cdots v_n]\right) \\ &= \varepsilon_{n-1} \sum_{i=0}^{n} (-1)^{n-i} \sigma|[v_n \cdots \hat{v} n - i \cdots v_0] \end{aligned}$$

and these two terms are equal since $\varepsilon_n = (-1)^n \varepsilon_{n-1}$.

Now, we want to show ρ is chain homotopic to the identity by finding $P : C_n(X) \to C_{n+1}(X)$ such that $dP + Pd = \rho - \mathbf{1}$. Here, P will be a modified Prism operator. As our functors are now contravariant, we can use the map $\pi : \Delta^n \times [0,1] \to \Delta^n$ given by projection onto the first factor. As before, we use the division of $\Delta^n \times [0,1]$ into the $n+1$ simplices $[v_0 \cdots v_i w_i \cdots w_n]$ where the w_i are the vertices of Δ^n on $\Delta^n \times \{1\}$ and the v_i live in $\Delta^n \times \{0\}$.

We define

$$P(\sigma[v_0 \cdots v_n]) = \sum_i (-1)^i \varepsilon_{n-i} (\sigma \circ \pi)|[v_0 \cdots v_i w_n \cdots w_i] \tag{12.11}$$

□

and now one computes

$$\begin{aligned} dP &= \sum_{j \le i} (-1)^i (-1)^j \varepsilon_{n-i} [v_0 \cdots \hat{v}_j \cdots v_i w_n \cdots w_i] \\ &\quad + \sum_{j \ge i} (-1)^i (-1)^{i+1+n-j} \varepsilon_{n-i} [v_0 \cdots v_i w_n \cdots \hat{w}_j \cdots w_i] \end{aligned} \tag{12.12}$$

and

$$\begin{aligned} Pd &= \sum_{i<j} (-1)^i (-1)^j \varepsilon_{n-i-1} [v_0 \cdots v_i w_n \cdots \hat{w}_j \cdots w_i] \\ &\quad + \sum_{i>j} (-1)^{i-1} (-1)^j \varepsilon_{n-i} [v_0 \cdots \hat{v}_j \cdots v_i w_n \cdots w_i] \end{aligned} \tag{12.13}$$

so that if we group together the $i = j$ terms in (12.12), we get

$$\varepsilon_n [w_n \cdots w_0] - [v_0 \cdots v_n] + \sum_{i>0} \varepsilon_{n-i} [v_0 \cdots v_{i-1} w_n \cdots w_i] + \sum_{i<n} (-1)^{n+i+1} \varepsilon_{n-i} [v_0 \cdots v_i w_n \cdots w_{i+1}] \tag{12.14}$$

where the final two terms cancel under an index shift. The remainig terms then show our desired chain homotopy, since $\varepsilon_{n-i} = (-1)^{n-i} \varepsilon_{n-i-1}$.

□

13. Lecture 13 – Künneth and Category

The cross product on cohomology, a bilinear map from $H^i(X) \times H^j(Y) \to H^{i+j}(X \times Y)$, induced by projections onto the various factors, gives us well-defined maps

$$\bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \xrightarrow{\times} H^n(X \times Y). \tag{13.1}$$

The Künneth Theorem tells us that this map is in fact an isomorphism if X and Y are both reasonably nice.

Theorem 13.1 (Künneth Theorem)**.** Let X be a compact cell complex and let Y be a compact cell complex such that $H^*(Y)$ is free. Then, the cross product in (13.1) is an isomorphism.

Proof. The idea of this proof is fairly straightforward. We outline it below:

- Create cohomology theories, h^*, k^* to represent the right and left sides of (13.1), respectively. Show that they are cohomology theories.
- Show that the cross product is a natural transformation between the theories.
- Prove the statement when X is a point. Use the facts about general (co)-homology theories to complete the proof for a general compact cell complex, X.

We begin with the third fact, as it is the easiest to prove. When X is a point, the only i such that $H^i(X) \neq 0$, is $i = 0$. So, we must check that the cross product gives an isomorphism between $\mathbf{Z} \otimes H^n(Y)$ and $H^n(\mathrm{pt} \times Y) \cong H_n(Y)$. To get practice with the definitions, we explicitly write out the map. It is defined by

$$a \otimes b \mapsto (\sigma \to a(\sigma|X|[v_0]).b(\sigma|Y|[v_0 \cdots v_n])) \tag{13.2}$$

which is nonzero whenever a, b are nonzero. Letting a be the map assigning 1 to each chain, we see that the map is surjective as well.

Now, we will show that $\times$ is a natural transformation between the two theories (assuming they are theories). Throughout this proof, Y is fixed, and we allow X to vary. More precisely, our cohomology theories, for a fixed CW complex Y, are

$$\begin{aligned} h^*(X) &:= \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \\ k^*(X) &:= H^n(X \times Y). \end{aligned} \tag{13.3}$$

So that the maps with which $\times$ must commute are continuous maps $f : X \to X$. However, This follows from proof of Remark 12.4, where we established the naturality of cup products. To show that it commutes with coboundary maps in long exact sequences. Here, we note that the cross product induces a map

$$\bigoplus_{i+j=n} H^i(X, A) \otimes H^j(Y) \to H^n(X \times Y, A \times Y) \tag{13.4}$$

since if $\phi \in C^i(A)$ and $\psi \in H^j(Y)$, then $\phi \smile \psi$ vanishes everywhere, since it projects down to a chain lying in A. So, we must check the commuativity of the following square:

$$\begin{tikzcd} H^i(A) \times H^j(Y) \arrow[r, "\delta \times \mathbf{1}"] \arrow[d, "\times"] & H^{i+1}(X, A) \times H^j(Y) \arrow[d, "\times"] \\ H^{i+j}(A \times Y) \arrow[r, "\delta"] & H^{i+j+1}(X \times Y, A \times Y). \end{tikzcd} \tag{13.5}$$

where, of course, if this square commutes, the tensor will inherit commutativity.

To check that this square commutes, we briefly remind ourselves of the geoemtric outcome of the boundary map in the LES for pairs in the singular homology case. Throughout this proof, we will use that

$C^*(X,A) = C^*(X/A)$, an interpretation we are allowed by Proposition 6.6. The boundary map, following the snake lemma, goes

$$C_n(X/A) \xrightarrow{\text{Lift}} C_n(X) \xrightarrow{\text{Boundary}} C_{n-1}(X) \xrightarrow{\text{Restriction}} C_{n-1}(A) \tag{13.6}$$

where the content of the snake lemma is essentially that a) our choice of lift does not matter and b) the chain living in $C_{n-1}(X)$ in fact lives in $C_{n-1}(A)$ (a consequence of exactness).

Reversing arrows, we can describe the LES boundary map fairly cleanly. Begin with a cochain $\phi \in C^n(A)$. Extend it to a cochain $\widetilde{\phi} \in C^n(X)$ via

$$\widetilde{\phi}(\sigma) \;=\; \begin{cases} \phi(\sigma) & \sigma \subset A \\ 0 & \sigma \not\subset A. \end{cases} \tag{13.7}$$

From there, we can take coboundaries to obtain the obvious element of $C^{n+1}(X)$. Finally, we introduce the map given the quotient $X \to X/A$ to arrive at $\phi' \in C^{n+1}(X,A) = C^{n+1}(X/A)$ given by

$$\delta(\phi) \;=\; \phi'(\sigma) \;:=\; \widetilde{\phi}(q_\sharp d\sigma) \;=\; \widetilde{\phi}(dq_\sharp\sigma) \tag{13.8}$$

where commutativity comes from the fact that continuous maps induce chain maps. One quickly checks that a chain in A is sent to $\phi(\text{bdry}(\text{pt})) = \phi(\emptyset) = 0$, which shows the image lives where we want it to.

Now, we check the commutativity of (13.5). Let $\phi \in C^i(A), \psi \in C^j(Y)$, both cocycles. Going around the top part of the diagram, we first send (ϕ,ψ) to (ϕ',ψ), where ϕ' is as in (13.8). Then, applying $\times$, we arrive $\pi_1^*(\phi') \smile \pi_2^*(\psi)$. Going around the bottom, we first apply $\times$ to get $\pi_1^*(\phi) \smile \pi_2^*(\psi)$ and then apply the LES boundary to obtain $\delta(\pi_1^*(\phi) \smile \pi_2^*(\psi))$. However, examining (13.8), and noting that a) q does not affect a chain living in Y b) we do not need to extend a chain living in Y, we have

$$\begin{aligned} \delta(\pi_1^*(\phi) \smile \pi_2^*(\psi)) \;&=\; q^*(d(\pi_1^*(\widetilde{\phi}) \smile \pi_2^*(\psi))) \\ &=\; \pi_1^*(\phi') \smile \pi_2^*(\psi) + (-1)^i q^*(\pi_1^*(\phi) \smile \pi_2^*(d\psi)) \\ &=\; \pi_1^*(\phi') \smile \pi_2^*(\psi) \end{aligned}$$

where the last equality holds because ψ was a cocyle.

Finally, we show that both h^*, k^* are general cohomology theories.

(1) Homotopy invariance: clear.
(2) Excision: Here we take the version of excision that states $h^*(B, A\cap B) \cong h^*(X,A)$ via inclusion (where A, B are subcomplexes of X). For h^*, this is clear. For k^*, we note that $(A \cap B) \times Y = A \cap Y \times B \cap Y$ and similarly for unions, which makes the statement trivial.
(3) Long exact sequences for pairs. In the case of k^*, we use the pairs $(A,\emptyset), (X,\emptyset)$, and (X,A) so that exactness is inherited. The construction for h^* is straightforward, but requires a little more work. We invite the reader to examine Hatcher, page 221. The key fact here is that tensoring with a free module preserves exact sequences.
(4) Disjoint Unions. For k^* this is obvious, since we can pass the disjoint unions through the product. For h^*, we need the fact that $H^*(Y)$ is free and finitely generated. Since that holds, we have a canonical isomorphism $\left(\amalg_\alpha H^i(X)\right) \otimes H^j(Y) \cong \amalg_\alpha \left(H^i(X) \otimes H^j(Y)\right)$.

□

Remark 13.2. Tnesoring with modules that have torsion does not necessarily preserve exactness. For example,

$$0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \tag{13.9}$$

is exact, but when we tensor all the elements by $\mathbf{Z}/2$, the resulting sequence is no longer exact.

The following explicit description of the cohomology ring of a sphere will be useful.

Example 13.10. $H^*(S^n) = \mathbf{Z}[x]/(x^2)$, where $|x| = n$. This isomorphism might be hard to guess, but it is not hard to verify. Consider $\varphi : \mathbf{Z}[x]/(x^2) \to H^*(S^n)$ such that $\varphi(1) = 1$, as the generator of $H^0(S^n)$ and $\varphi(x) = \psi$, the generator of $H^n(S^n)$. The map must be injective since $\psi^2 \in H^{2n} = 0$. It is also surjective, since we can write any element of $H^*(S^n)$ $(1^k \smile \psi) + 1^l$.

Example 13.11. $H^*(T^n) = H^*(S^1)^{\otimes n} = \bigwedge(x_1 \cdots x_n)$, which is the exterior algebra on n generators of degree one, i.e. freely geenrated by the x_i subject to the constraints forced by graded commutativity, that $x_i^2 = 0$ and $x_i x_j = -x_j x_i$. We can see this from Example 13.10 and the Künneth Theorem.

This fact will allow us to describe maps from the sphere to the torus very nicely.

Corollary 13.3. If $n \geq 2$, there is no map $S^n \to T^n$ of nonzero degree.

Proof. A map $f : S^n \to T^n$ induces a map $f^* : H^n(T^n) \to H^n(S^n)$. We can examine the image of a generator, $x_1 \smile \cdots \smile x_n$. By functoriality,

$$\begin{aligned} f^*(x_1 \smile \cdots \smile x_n) &= f^*(x_1) \smile \cdots \smile f^*(x_n) \\ &= 0 \end{aligned}$$

since $f^*(x_1) \in H^1(S^n) = 0$ for $n \geq 2$. □

The basic reason for why this holds is that the cohomology ring of T^n is generated by loops, but there are no non-trial loops in S^n for $n \geq 2$. An alternate proof using the following Theorem elucidates this point.

Theorem 13.4 (Existence of Lifts)**.** Let $p : (\widetilde{X}, \widetilde{b}) \to (X, b)$ be a based covering map. Let Y be path connected and locally path-connected and $f : (Y, y_0) \to (X, b)$ be some map. Then, f has a lift $\overline{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{b})$ if and only if $f_*\pi_1(Y, y_0) \subset p_*\pi_1(\widetilde{X}, \widetilde{b})$.

Of Corollary 13.3. For $n \geq 2, \pi_1(S^n, b)$, for any b is zero, so our containment is automatic. Therefore, any such map factors through $\mathbf{R}^n$, the universal cover of T^n, which has trivial homology in top dimension. □

Moreover, even if spaces have the same cohomology groups, they may have different ring structures. This gives us another tool to differentiate spaces. The first tool in this vein is cuplength.

Definition 13.1. The *cuplength* $\mathrm{cl}(X)$ of a space X is defined as

$$\mathrm{cl}(X) = \max\left\{N \mid \text{there exist } \alpha_i, 1 \leq i \leq N \in H^{>0}(X) \text{ such that } \alpha_1 \smile \cdots \smile \alpha_n \in H^*(X) \setminus \{0\}\right\} \tag{13.12}$$

The previous examples show

Example 13.13. $\mathrm{cl}(S^n) = 1$ and $\mathrm{cl}(T^n) = n$.

Definition 13.2. A subset $U \subset X$ of a space X is *contractible in* X if its inclusion is homotopic to a constant map.

Definition 13.3. The *category* (or *LS-category*) of a space X is a function

$$\nu : \{\text{subset of } X\} \to \mathbf{N}_0 \tag{13.14}$$

where $A \subset X$ has category N if A can be covered by N open sets in X, each of which is contractible, but no fewer.

Remark 13.5. A closed manifold has finite category (think of it relative to itself). By closed, we mean compact without boundary.

Proposition 13.6. Call the category of A $\nu(A)$. Then, $\nu(A)$ satisfies

(i) If $A \subset X$, then there exists open U such that $A \subset U \subset X$ and $\nu(A) = \nu(U)$
(ii) $A \subset B$ implies $\nu(A) \leq \nu(B)$
(iii) $\nu(A \cup B) \leq \nu(A) + \nu(B)$
(iv) $\nu(\emptyset) = 0,\ \nu(\mathrm{pt}) = 1$

(v) ν is a homeomorphism invariant.

Proof. Left to the reader. ☐

Proposition 13.7. $\nu(X) > \mathrm{cl}(X)$.

Proof. Exercise sheet 2 ☐

Proposition 13.8. Let M be a closed, smooth manifold. Then, any smooth function $f : M \to \mathbf{R}$ has at least $1 + \mathrm{cl}(M)$ critical points, i.e. points x such that $df_x = 0$.

Corollary 13.9. Every smooth function $f : T^n \to \mathbf{R}$ has at least $n+1$ critical points.

Proof (likely non-examinable). We show f has at least $\nu(M)$ critical points. For $c > 0$, set $M^c = f^{-1}(-\infty, c]$. Pick a Riemannian metric on M and hence obtain a gradient flow

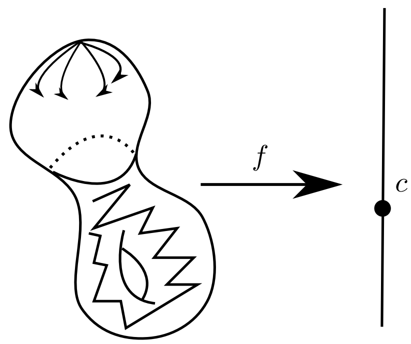


FIGURE 15. The lower portion is not in the pre-image of M^c

$$\phi^t : M \to M \tag{13.15}$$

the flow of the vector field ∇f. This is a flow by self-diffeomorphisms of M. Let $c_j = \sup\{c \mid \nu(M^c) < j\}$. Note that the image of f is bounded, so that $c_1 = \min(f), c_N = \max(f)$, where $N = \nu(M)$.

If $c \in \mathbf{R}$ is not a critical value of f, i.e. $df_x \neq 0$, then for all $x \in f^1(c)$, basic differetial topology says that we can flow a short way without hitting a critical value, i.e. there exist $t, \delta > 0$ such that $\phi^t(M^{c+\delta}) \subset M^{c-\delta}$. So, using continuity of capacity, c_j is a critical value of f for each j.

Claim. $c_j < c_{j+1}$, of $f^-1(c_j)$ contains infinitely many critical points of f.

If $f^{-1}(c_j) \cap \mathrm{crit}(f)$ is finite, it has a neighborhood, U, which is homeomorphic to an open disc and has category $\nu(U) - 1$. Now, $\nu(M^{c_j+\delta}) \leq \nu(M^{c_j+\delta} \setminus U) + 1 \leq \nu(M^{c_j-\delta}) + 1$. Since away from $\mathrm{crit}(f)$, ϕ^t does flow from $M^{c_j+\delta}$ to $M^{c_j-\delta}$, we inductively have that $\nu(M^{c_j-\delta}) + 1 < j + 1$, so $c_{j+1} \geq c_j + \delta$, which implies $c_{j+1} > c_j$. ☐

For related results, take Morse Theory.

14. Lecture 14 – Vector Bundles

Note: This discussion assumes some knowledge of vector bundles and may be better left as motivation after reading the two lectures that follow.

Before we delve into the study of vector bundles, it is natural to ask why we would want to study vector bundles (or more generally, fibre bundles) at all. One of the most natural answers is that the tangent bundle is an inherently interesting bundle on smooth manifolds. Given a smooth manifold M, of dimension n, we can embed M in $\mathbf{R}^N$ for sufficiently large N. From here, we can think of the tangent space of M at x, or TM_x as the tangent plane which best approximates M at x.

Remark 14.1. To be more formal, if M is smooth, there is a neighborhood U of x and a diffeomorphism $g_x : V \subset \mathbf{R}^n \to U \subset \mathbf{R}^N$. The tangent plane at x is the image of $\mathbf{R}^n$ under the Jacobian of g_x, evaluated at x.

The first question we might ask is whether a tangent bundle is trivial. That is, we ask if we find a smooth global basis for the (total) tangent space TM. In the cases of the circle and the 2-torus, this is fairly easy. However, the content of the Hairy Ball Theorem (Theorem 4.6) is that this cannot be done for the 2-sphere. Hopefully, our discussion of vector bundles will cast the Hairy Ball Theorem in a new light.

From our remark, one can work out, using basic functoriality properties, that the tangent space TM_x will be n-dimensional. Therefore, TM, or M with a copy of its tangent space at each point, will be a $2n$ dimensional smooth manifold. It comes with a projection map

$$\begin{aligned} \pi : TM &\to M \\ (m, v) &\mapsto m \end{aligned} \tag{14.1}$$

and we often discuss *sections* of this map, or maps $s : M \to TM$ such that $\pi \circ s = \mathrm{Id}_M$.

A trivial (but VERY important) example of a smooth section is given by the zero section $m \mapsto (m, 0)$. A first observation about this section is that we can identify it with M. We might then examine $[M] \smile [M]$, as an element of $H^{2n}(TM)$. Note that if M satisfies the above hypotheses, $H^n(M) \cong \mathbf{Z}$ and has a generator.

We claim that $[M] \smile [M]$ counts the zeros of a smooth section of TM. The following expalanation is due to Andy Putman on StackOverflow. Since M lives in $\mathbf{R}^N$, we can identify the normal bundle with the tangent bundle of M. Take a tubular neighborhood N of M in $M \times M$. This is then isomorphic to TM. A section s of the tangent bundle (that meets the zero section transversely) then gives a submanifold M' of $N \subset M \times M$ which is both isotopic to M and whose intersections with M correspond to zeros of the section. The result then follows from the Hopf index formula, which relates these zeros to the Euler characteristic.

The last step is left intentionally vague. However, through the general machinery, it should be clear that if a manifold has nonzero Euler characteristic, every smooth section ofTM must vanish somehwere.

In further lectures, will we expand on this analysis more, hopefully making vector bundles both a powerful and natural tool for distinguishing smooth manifolds.

This marks the beginning of Lecture 14. Our aim is to understand part of the ring structure on H^* for X when X is a sphere bundle.

Definition 14.1. A *vector bundle* $E \xrightarrow{\pi} X$ is a space E with continuous projection $\pi : E \to X$ such that for all $x \in X$, $\pi^{-1}(x) = E_x \cong \mathbf{R}^n$ for some fixed n, called the *rank* of E. We also require the fibration to be locally trivial in the following sense: for all $x \in X$, ther exists an open neighborhood $U \ni x$ such that $E|_U$ is trivial, i.e.

$$\begin{tikzcd} \bigcup_{x\in U} E_x := E|U \arrow[r, "\cong"] \arrow[d, "\pi"] & U \times \mathbf{R}^n \arrow[d, "\pi_U"] \\ U \arrow[r, "\text{id}"] & U \end{tikzcd} \tag{14.2}$$

such that $E_y \to y \times \mathbf{R}^n$ for each $y \in U$ is a linear homeomorphism.

Remark 14.2. We adopt the following standard notation:
(1) E is the total space, X is the base space.
(2) There exists a canonical map $\iota : X \hookrightarrow E$ defined by $x \mapsto (x, 0)$, where 0 is the vector space origin.
(3) This map is a *section*, in the sense that $\pi \circ \iota = \text{Id}_X$.
(4) Note that $\pi : E \to X$ and $\iota : X \to E$ are (inverse) homotopy equivalences, namely

$$\begin{aligned} \pi \circ \iota &= \text{Id}_X \\ \iota \circ \pi &\simeq \text{Id}_E \end{aligned} \tag{14.3}$$

where the latter homotopy is given by radial shrinking.

Remark 14.3. There are several standard operations on Vector Bundles
(1) If $E \to X$ is a vector bundle and $f : Y \to X$ is continuous, then $f^*E \to Y$, or *the pullback of E by f* has total space

$$f^*(E) \;=\; \{(e, y) \in E \times Y \mid \pi(e) = f(y)\} \tag{14.4}$$

so that the fibre $(f^*E)_y = E_{f(y)}$. It is given by the following diagram:

$$\begin{tikzcd} Y \arrow[dr, "f"] & \\ E \arrow[u, dashed, "\pi_y"] \arrow[r, "\pi"] & X \end{tikzcd} \tag{14.5}$$

(2) If $E \to X, F \to X$ are vector bundles of rank n, m, then the *Whitney Sum* $E \oplus F$ of rank $n + m$ has fibre $E_x \oplus F_x$. This has total space

$$E \oplus F \;=\; \{(e, f) \in E \times F \mid \pi(e) = p(f)\} \tag{14.6}$$

where p is the projection associated to F. Therefore, the Whitney Sum inherits a natural projection map.

Remark 14.4. Both pullbacks and Whitney sums have the following two features
(a) They take trivial bundles $(X \times \mathbf{R}^n)$ to trivial bundles.
(b) The operations commute with respect to restriction to open sets in X, namely

$$\begin{aligned} f^*(E|_U) &= (f^*E)|_{f^{-1}(U)} \\ (E \oplus F)|_U &= E|_U \oplus F|_U \end{aligned} \tag{14.7}$$

and these two properties show that local triviality of the new bundles is inherited from local triviality of the old ones.

The following example is crucial in both algebraic topology and algebraic geometry.

Example 14.8. Let $X = \text{Gr}_k(\mathbf{R}^n)$, or the set of k-planes in $\mathbf{R}^n$ (when $k = 1$, we have $\mathbf{P}^{n-1}(\mathbf{R})$). Let $E \to X$ be the tautological bundle such that $E_x = \langle \text{subspace of } \mathbf{R}^n \text{ defined by } x\rangle$. Then, globally, we have $E \subset X \times \mathbf{R}^n$, defined by $\{(x, v) \mid v \in \langle x\rangle\}$, where $\langle x\rangle$ is the subspace of $\mathbf{R}^n$ corresponding to x.

It is clear that the fibres at each point are of a fixed dimension and that there is a projection map. However, local triviality is less obvious.

Proof of Local Triviality. Here, we make an ammendment to the lecture's notes. Pick an inner product, $p(\cdot,\cdot)$ on $\mathbf{R}^n$ For $x \in \mathrm{Gr}_k(\mathbf{R}^n)$, define U_x to be

$$U_x \;:=\; \left\{ v \in \mathrm{Gr}_k(\mathbf{R}^n) \mid \pi_{\langle x\rangle}(\langle v\rangle) \cong \langle x\rangle \right\} \tag{14.9}$$

or the set of subspaces that project isomorphically onto $\langle x\rangle$. It should be clear that this set is open (look at the volume of the unit box in $\langle x\rangle$ under projection to $\langle v\rangle$) and contains x.

Now, pick your favorite linear isomorphism $f : \langle x\rangle \to \mathbf{R}^k$. For $(v,e) \in E_{U_x}$, define $\overline{f}(v,e) = (v, f(\pi_{\langle x\rangle}(e)))$, which is again a linear isomorphism. □

Analogously there is a tautlogical bundle of $\mathrm{Gr}_k(\mathbf{C}^n)$ with fibre $\mathbf{C}^k$ and hence a bundle $\mathcal{L} \to \mathbf{P}^{n-1}_{\mathbf{C}}$ when $k = 1$, called the *tautological line bundle*.

Definition 14.2. A vector bundle of rank one is called a *line bundle*

Definition 14.3. A Hausdorff space is *paracompact* if for any open over $X = \{U_i\}_{i\in I}$ there exists a partition of unity

$$\lambda_i : X \to \mathbf{R} \tag{14.10}$$

such that everywhere $\sum \lambda_i = 1$ and $\mathrm{supp}(\lambda_i) \subset U_i$ and moreover the collection is locally finite, in the sense that for each $x \in X$ we have $\#\{i \in I \mid x \in \mathrm{supp}(\lambda_i)\} < \infty$.

Example 14.11. The following are classical examples of Paracompact spaces. We do not prove these statements.

(1) Every compact space is paracompact.
(2) Every CW complex is paracompact.
(3) Every metric space is paracompact.

Lemma 14.5. If $E \to X$ is a vector bundle over a paracompact space X, then

(i) E admits an inner product $\langle\cdot,\cdot\rangle$ i.e. a family of inner products $\langle\cdot,\cdot\rangle_x$ on E_x varying continuously.
(ii) For all $x \in X$, $\xi_x \in E_x$, there exists a section $s : X \to E$ such that $s(x) = \xi_x$.
(iii) If, in addition, X is compact, then there exists a bundle $F \to X$ such that $E \oplus F \cong \mathbf{R}^N \times X$ is a trivial bundle.

Corollary 14.6. If X is compact and Hausdorff (and therefore paracompact) and $E \to X$ is a bundle, then there exists a map $f : X \to \mathrm{Gr}_k(\mathbf{R}^N)$ with $k = \mathrm{rk}(E), N > k$ such that $E = f^*E_{\mathrm{taut}}$.

Proof. Choose F as in part (iii). Then, for each $x \in X, E_x \subset \mathbf{R}^N$ defines an element of $\mathrm{Gr}_k(\mathbf{R}^n)$. Define $f(x)$ to be $[E_x]$. Then, $f^*E_{\mathrm{taut}}(x) = (x, E_x)$, as desired. □

Proof of Lemma 14.5. (i) This is an example of a classic argument of using partitions of unity to patch together local information to global information. For each $x \in X$, choose an open set U_x for which E is locally trivial. Since $E|_{U_x} \cong U \times \mathbf{R}^n$ via a linear isomorphism, we can choose an inner product $\langle\cdot,\cdot\rangle_{U_x}$ on each U_x. Choose a POU subordinate to this open cover. Now, we use that POU to smooth these inner products into a global inner product. More specifically, for $v, w \in E_x$, define

$$\langle v, w\rangle \;:=\; \sum_{\lambda_i(x)\neq 0} \lambda_i \langle v, w\rangle_{U_i} \tag{14.12}$$

which is a continuously varying global inner product. By definition, the sum is finite, so we do not have to worry about convergence.

(ii) In a neighborhood of x, we can define a constant section $s(U_x) \equiv \xi_x$. Then, patch this to a global section using a partition of unity.

(iii) To prove this lemma, it suffices to find some N and a trivial bundle $\mathbf{R}^N \times X$ such that we have the diagram

$$\begin{tikzcd} E \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & \mathbf{R}^N \times X \arrow[d, "\pi_X"] \\ X \arrow[r, "\mathrm{id}"] & X \end{tikzcd} \tag{14.13}$$

where ι denotes inclusion. Once we have this, We can fix an inner product on $\mathbf{R}^N$ and set F to be the orthogonal complement with respect to that inner product. The real content here is that we can choose some N globally. That is why we need compactness.

For any $x \in X$, part (ii) of Lemma 14.5 allows us to pick a local section $\langle s_1^x, \cdots, s_n^x \rangle$ that we can extend to a global section via our partition of unity. However, it is only guaranteed to be a frame, or a basis for the bundle, within some small U_x. Using compactness, we may take a finite subcover $U_{x_1}, \cdots, U_{x_k}$ of "framing" open sets for X. We then have kn global sections, say $\sigma_1, \cdots, \sigma_{kn}$, which span D_x at each $x \in X$. Here, it is tempting to try to fit these vectors directly into $\mathbf{R}^{kn}$, but on a minute's thought it is not immediately obvious how one would define such a map $E \to \mathbf{R}^N \times X$. So, we take the map

$$\varphi(x, \xi) \ := \ (x, \langle \xi, s_1(x) \rangle, \cdots, \langle \xi, s_{kn}(x) \rangle) \tag{14.14}$$

where $\xi \in E_x$ and $\langle \cdot, \cdot \rangle$ is a global non-degenerate inner prodcut on E, which we are guaranteed by part (i). Since there is always a local frame within the s_i, this map is injective.

To avoid confusion, we note that the immediately aforementioned inner product is one on $\mathbf{R}^n$. We need a second inner product to find the complementary bundle F.

□

Remark 14.7. Let $\mathrm{Gr}_k = \bigcup_{N \geq k+1} \mathrm{Gr}_k(\mathbf{R}^N)$ via $\mathbf{R}^N = \mathbf{R}^N \times \{0\} \subset \mathbf{R}^{N+1} \subset \mathbf{R}^{N+2} \cdots$ be a cell complex, with the weak topology.

Hopefully given the previous lemma, the following theorem is at least believable

Theorem 14.8. For a compact, Hausdorff space X,

$$\begin{aligned} \mathrm{Vect}_k(X) \ := \ \{\text{rank } k \text{ bundles on } X\} / \cong \quad &\xrightarrow{1:1} [X, \mathrm{Gr}_k] \\ f^* E_{\mathrm{taut}} &\leftarrow f \end{aligned} \tag{14.15}$$

where the quantity on the righthand side is a collection of homotopy classes of maps. Moreover, each map has a unique pullback.

15. Lecture 15 – The Thom Isomorphism

Recall from our earlier discussion of intersection numbers that, at least in the case of closed, orientable manifolds, we can think of cup products as counting intersection numbers, generalizing our notion of the picture in Figure 14a – 14b. Here, the subject of interest is vector bundles, $E \xrightarrow{\pi} B$. By definition, vector bundles are locally trivial – we can find a local frame (a local basis), say $s_1, \cdots, s_n$. However, not all vector bundles are globally trivial. By this, we mean we may not be able to extend local frames to form a global one. In particular, if we try to extend some s_i into a global section, that s_i may be forced to vanish in some places, as is the case with the sphere, which we learned through Theorem 4.6, the Hairy Ball Theorem.

We may attempt to detect this vanishing by looking at transverse intersections (which we may get by "wiggling" things a little) of the vector field with the zero section of a vector bundle. This is the notion we intend to capture by cupping with a generator of $H^n(E, E-0)$, where 0 denotes the zero section. We want to detect when a property of our manifold X forces the bundle to vanish.

Without further ado, we present an introduction to the Thom Isomorphism Theorem.

In the picture Figure 14a – 14b, we needed an orientation on the x axis and an orientation on our manifold to properly count intersection number. However, orientations on general manifolds are harder to visualize, so we use the following definition, which one easily checks agrees with our prior definition.

Definition 15.1. An *orientation* of $\mathbf{R}^n$ is a choice of generator of $H^n(\mathbf{R}^n, \mathbf{R}^n \setminus 0) \cong \mathbf{Z}$. We say a vector bundle, E is *oriented* if we have generators $\varepsilon_x \in H^n(E_x, E_x \setminus 0)$, where n is the rank of our bundle and these generators vary locally trivially in the following sense. For all $x \in X$, there exists some neighborhood $U \subset X$ that is a trivializing neighborhood for E such that

$$\begin{array}{ccc} E|_U & \xrightarrow{\cong} & U \times \mathbf{R}^n \\ \cup & & \cup \\ E_y & \longrightarrow & y \times \mathbf{R}^n. \end{array} \tag{15.1}$$

sends $\varepsilon_y \mapsto \varepsilon_n$, a fixed orientation generator for all $y \in U$.

Remark 15.1. The isomorphism $H^n(\mathbf{R}^n, \mathbf{R}^n \setminus 0) \to \mathbf{Z}$ comes from the long exact sequence of a pair. This shows it is isomorphic to $H^{n-1}(S^{n-1})$. It is this isomorphism that gives us our intuition for why a generator of $H^n(\mathbf{R}^n, \mathbf{R}^n \setminus 0)$. Namely, those transformations that reverse the orientation of $S^{n-1} \subset \mathbf{R}^n$ are those that reverse the orientation of $\mathbf{R}^n$. The key example is a reflection when n is odd.

Theorem 15.2 (Thom Isomorphism Theorem)**.** Let $\pi : E \to X$ be an oriented vector bundle of (real) rank n. Let $E^\sharp = E \setminus$ (0-section), or $E \setminus (X \times (0, \cdots, 0))$. Then,

(i) $H^k(E, E^\sharp) = 0$ for all $k < n$.
(ii) There exists a unique element $u_E \in H^n(E, E^\sharp)$ such that $u_E|E_x = \varepsilon_x$, our chosen orientation generator
(iii) The map

$$\begin{aligned} H^k(X) &\to H^{k+n}(E, E^\sharp) \\ \alpha &\mapsto \pi^*(\alpha) \smile u_E \end{aligned} \tag{15.2}$$

is an isomorphism. The element u_E is called the *Thom class* of E.

If we take the Theorem for granted, briefly, we see that, using pullbacks, we obtain an interesting cohomology class.

Definition 15.2. We obtain a class

$$\begin{aligned} H^n(E, E^\sharp) &\xrightarrow{\text{LES map}} H^n(E) \cong H^n(X) \\ u_E &\mapsto e_E \end{aligned} \tag{15.3}$$

called the *Euler class* of E.

A rule $E \mapsto C(E) \in H^*(X)$ which asociates to a vector bundle E (perhaps with structure like an orientation) a class such that $c(f^*E) = f^*c(E)$ whenever $f : X \to Y$ is continuous is called a *characteristic class* (with structure).

Remark 15.3. (i) If $E \to X$ is oriented, $f : Y \to X$, then f^*E inherits an orientation. Let $y \in Y$. Then, we have a generator $\varepsilon_{f(y)}$ given from the orientation on X. To see that it varies locally trivially, pull back a trivializing neighborhood of $f(y)$ (with the desired property) under f.

From this, we see that $f^*(U_E)|_{f^*(E_y)} = f^*(U_E|_{E(f(y))}) = f^*(\varepsilon_x) = \varepsilon_y$ and so $u_{f^*E} = f^*(u_E)$, or Thom class of the pullback is the pullback of the Thom class.

So, the Thom and Euler classes is are characteristic class of oriented vector bundles.

(ii) If E is a complex vector bundles, $E_x \cong \mathbf{C}^k$ and the isomorphisms are linear in the local trivialization, then E is canonically oriented, since complex vector spaces are canonically oriented.

It is a worthwhile exercise to see why the previous remark is true. We can look at the action of $\mathrm{GL}_n(\mathbf{R})$ on $H^{n-1}(S^{n-1}) \cong H^n(\mathbf{R}^n, \mathbf{R}^n - 0)$, and analogously for $\mathbf{C}$.

(iii) If we take coefficients in $\mathbf{Z}/2\mathbf{Z}$, $H^n(\mathbf{R}^n, \mathbf{R}^n - 0) \cong \mathbf{Z}/2\mathbf{Z}$ and since there exists a unique generator, all bundles become oriented. This meshes well with out intuition that $\mathbf{Z}/2\mathbf{Z}$ coefficients "forget orientation".

The following lemma verifies that the Euler class does what we constructed it to do – detect some cases of obstructions to global frames.

Lemma 15.4. If an oriented vector bundle $E \xrightarrow{\pi} X$ has a nowhere zero section, then $e_E = 0$.

Proof. As noted before, since $\mathbf{R}^n$ is contractible, we can view e_E as an element of $H^n(X)$. Our nowhere vanishing section gives maps

$$\begin{gathered} X \xrightarrow{s} E^\sharp E \\ H^*(E) \to H^*(E^\sharp) \xrightarrow{s^*} H^*(X). \end{gathered} \tag{15.4}$$

such that in particular, the following diagram commutes:

$$\begin{tikzcd} E^\sharp \arrow[r, "\iota"] & E \\ X \arrow[u, "s"] \arrow[ur, "s"'] & \end{tikzcd} \tag{15.5}$$

Crucially, $\pi \circ s$ is the identity on X, so the inclusion of X into E induces the same map on cohomology as s. Functoriality and commutativity of (15.5) give us commutativity of the triangle on the righthandside of (15.6)

$$\begin{tikzcd} H^n(E, E^\sharp) \arrow[r, "\mathrm{LES}"] \arrow[dr, dashed] & H^n(E) \arrow[r, "\iota^*"] \arrow[d, "\simeq s^*"] & H^n(E^\sharp) \arrow[dl, "s^*"] \\ & H^n(X) & \end{tikzcd} \tag{15.6}$$

which shows that the image of the Thom class factors through two consecutive terms in a long exact sequence and is therefore zero. □

Remark 15.5. Unfortunately, $e_E = 0$ does not imply that there exists a nowhere vanishing section. Intuitively, this is because there might exist sections with cancelling intersection numbers with the zero section. The Euler class will only pick up the sum of these numbers.

The following lemma is a nice application of properties of the cup product.

Lemma 15.6. If E has odd rank, then E is 2-torsion, i.e. $2e_E = 0$.

Proof. Let E^{op} denote E with the reverse orientation, namely a choice of $-\varepsilon_x$ at each $x \in E$. We note that this gives us the Thom class of E^{op} as $-u_E$. This implies $e_E = -e_{E^{\text{op}}}$. However, if E is of odd rank then the map $v \to -v$ reverses orientation. So, we may identify E with E^{op} in a way that sends e_E to $e_{E^{\text{op}}}$. $\square$

Now that we have seen two nice consequences of the Thom isomorphism Theorem, we move onto a proof.

Proof. For simplicity, we assume that X admits a finite trivializing open cover for E. The general case appeals to Zorn's lemma. We'll induct on the number of sets in an open cover. We will use this strategy fairly often. An advantage is that the base case is a trivial bundle. We can then use Mayer-Vietoris for the inductive step.

This proof will appeal to a relative version of the Künneth Theorem, namely that the (relative) cross product map given by

$$\bigoplus_{i+j=n} H^i(X) \otimes H^j(Y,Z) \xrightarrow{\times} H^n(X \times Y, X \times Z) \tag{15.7}$$

is an isomorphism whenever $Z \subset Y$ are homotopy equivalent to finite dimensional cell complexes and $H^*(Z), H^*(Y)$ are finitely generated and free. We in fact proved the difficult part of the relative Künneth Theorem in our proof of the ordinary Künneth Theorem in showing that the relative version of Φ was a natural transformation. The rest can be deduced from the LES for pairs axiom.

Base Case – Trivializing Cover With One Set In this case, our bundle is globally trivial, it is $X \times \mathbf{R}^n$. So, the relative Künneth Theorem gives us an isomorphism

$$\bigoplus_{i+j=n} H^i(X) \otimes H^j(\mathbf{R}^n, \mathbf{R}^n - 0) \underset{\cong}{\xrightarrow{\times}} H^n(E, E^\sharp). \tag{15.8}$$

which in fact lets us deduce all of the Theorem fairly quickly. Using the LES for pairs, we see $H^i(\mathbf{R}^n, \mathbf{R}^n - 0) = \mathbf{Z}$ when $i = n$ and 0 otherwise. For part (i) of the Theorem, note that when $i < n$, $H^i(E, E^\sharp)$ can be expressed as a sum of tensors, one of which (the one corresponding to the pair $(\mathbf{R}^n, \mathbf{R}^n - 0)$) is always zero.

For parts (ii) and (iii), we set $u_E = 1 \otimes \varepsilon_n$, where ε_n is the (global) orientation generator of our bundle. Part (ii) is clear – this element restricts to the orientation generator and is clearly the unique cochain that agrees with our orientation generator everywhere. This element is unique because For part(iii), note that we have maps

$$\begin{tikzcd} & X \times (\mathbf{R}^n, \mathbf{R}^n - 0) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & (\mathbf{R}^n, \mathbf{R}^n - 0) \end{tikzcd} \tag{15.9}$$

and the cross product is defined by cupping pullbacks of $H^*(X), H^*(\mathbf{R}^n, \mathbf{R}^n - 0)$. In the latter case, we have only one nonzero cohomology group, whose orientation generator pulls back to the Thom class. Then, it is clear that the cup product gives our desired isomorphism $H^k(X) \to H^{k+n}(E, E^\sharp)$ for any k.

Inductive Step: For the inductive step, we write $X = U \cup V$ and assume that $E|_U, E|_U, E|_{U\cap V}$ all admit trivializing covers by a smaller number of open sets. We can assume this by taking a fintie trivializing cover of E and forming U and V from proper subcollections of sets in our finite cover. Then, a relative version of Mayer-Vietoris (to be proved in two lectures) gives

$$H^{i-1}(E|_{U\cap V}, E^\sharp|_U \cap V) \to H^i(E, E^\sharp) \to H^i(E|_U, E^\sharp|_U) \oplus H^i(E|_V, E^\sharp|_V) \to H^i(E|_{U\cap V}, E^\sharp|_{U\cap V}) \tag{15.10}$$

as part of a long exact sequence. When $i < n$, our inductive hypothesis gives

$$0 \to H^i(E, E^\sharp) \to 0 \tag{15.11}$$

as part of the sequence (15.10). This establishes part (i) in general.

When $i = n$, we have

$$0 \to H^n(E, E^\sharp) \to H^n(E|_U, E^\sharp|_U) \oplus H^n(E|_V, E^\sharp|_V) \to H^n(E|_{U\cap V}, E^\sharp|_{U\cap V}) \tag{15.12}$$

where the final map sends (a, b) to $(a - b)$. By our inductive hypothesis, we have thom classes for $E|_U, E|_U$, and $E|_{U\cap V}$. However, by the definition of the Thom class, $u_{E|U}$ and $u_{E|V}$ both restrict to a unique orientation generator on $U \cap V$. So, in our sequence $(u_{E|U}, u_{E|V})$ gets mapped to zero. By exactness, there is a unique element $u \in H^n(E, E^\sharp)$ that is sent (by inclusion) to the pair $(u_{E|U}, u_{E|V})$. This is our Thom class.

The inductive step of part (iii) is not difficult conceptually, but there is a subtle technical hurdle. We set up the diagram (which we must write in two parts)

$$\begin{tikzcd}
H^{i+n}(E|_U, E^\sharp|_U) \oplus H^{i+n}(E|_V, E^\sharp|_V) \arrow[r] & H^{i+n}(E|_{U\cap V}, E^\sharp|_{U\cap V}) \arrow[r, "\partial_{\mathrm{MV}}"] & H^{i+n+1}(E, E^\sharp) \arrow[r] & \cdots \\
H^i(U) \oplus H^i(V) \arrow[r] \arrow[u, "\mathrm{Thom}"] & H^i(U \cap V) \arrow[r, "\partial_{\mathrm{MV}}"] \arrow[u, "\mathrm{Thom}"] & H^{i+1}(X) \arrow[r] \arrow[u, "\mathrm{Thom}"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
H^{i+n+1}(E, E^\sharp) \arrow[r] & H^{i+n}(E|_U, E^\sharp|_U) \oplus H^{i+n+1}(E|_V, E^\sharp|_V) \arrow[r] & H^{i+n+1}(E|_{U\cap V}, E^\sharp|_{U\cap V}) \\
H^i(U) \oplus H^i(V) \arrow[r] \arrow[u, "\mathrm{Thom}"] & H^{i+1}(U) \oplus H^{i+1}(V) \arrow[r] \arrow[u, "\mathrm{Thom}"] & H^{i+1}(U \cap V). \arrow[u, "\mathrm{Thom}"]
\end{tikzcd} \tag{15.13}$$

If we can show the diagram commutes, the Five Lemma gives our desired result. However, it is non-trivial that this diagram commutes. The only troubling square is

$$\begin{tikzcd}
H^{i+n}(E|_{U\cap V}, E^\sharp|_{U\cap V}) \arrow[r, "\partial_{\mathrm{MV}}"] & H^{i+n+1}(E, E^\sharp) \\
H^i(U \cap V) \arrow[r, "\partial_{\mathrm{MV}}"] \arrow[u, "\mathrm{Thom}"] & H^{i+1}(X). \arrow[u, "\mathrm{Thom}"]
\end{tikzcd} \tag{15.14}$$

since the Mayer Vietoris boundary is not necessarily induced by a map of spaces. The other squares can be resolved using functoriality and properties of the Thom class. For example, in the leftmost square of (both diagrams in) (15.13), we note

$$\begin{aligned}
\iota^* \circ \mathrm{Thom}(\phi, \psi) &= \iota^*(\pi^*(\phi, \psi) \smile (u_{E|U}, u_{E|V})) \\
&= \iota^*(\pi^*(\phi, \psi)) \smile \iota^*(u_{E|U}, u_{E|V}) \\
&= \pi^*(\iota^*(\phi, \psi)) \smile (u_{E|U\cap V}, u_{E|U\cap V}) \\
&= \mathrm{Thom} \circ \iota^*(\phi, \psi)
\end{aligned} \tag{15.15}$$

where the second equality holds by functoriality, the third by the properties of the Thom class and noting that π^* and ι^* commute.

So, we are left with (15.14). We will need to analyze the Mayer-Vietoris boundary for cohomology and invite the reader to diagram 3 to gain some geometric intuition for this process. Recall that in defining ∂_{MV}, we start with an element $\alpha \in H^i(U \cap V)$ and we write α as a difference

$$\alpha = \psi_U - \psi_V \tag{15.16}$$

for $\psi_U \in C^i(U), \psi_V \in C_i(V)$ (note that as in the case of the Mayer-Vietoris for homology, these maps need not be cocycles.) Then, $\partial_{\mathrm{MV}}(\alpha) := [d^*\psi_U]$, where d^* is the cohomology boundary operator.

Going around the bottom of the square (15.14), we have

$$\alpha \mapsto \pi^*(d^*\psi_U) \smile u_E \tag{15.17}$$

and the other way, we initially have $\mathrm{Thom}(\alpha) = \pi^*(\alpha) \smile u_{E|U\cap V}$, to which we will apply the Mayer-Vietoris boundary. First, we must express it as a difference of chains in $(E|_U, E^\sharp|_U)$ and $(E|_V, E^\sharp|_V)$, which we do by espressing α as $\psi_U - \psi_V$ and using thom classes of $E|U$ and $E|V$. We write $\pi^*(\alpha) \smile u_{E|U\cap V} = \pi^*\psi_U \smile u_{E|U} - \pi^*\psi_V \smile u_{E|V}$. Then, we have

$$\begin{aligned} d^*(\pi^*\psi_U \smile u_{E|U}) &= d^*(\pi^*\psi_U) \smile u_{E|U} \pm \pi^*\psi_U \smile d^*(u_{E|U}) \\ &= d^*(\pi^*\psi_U) \smile u_{E|U} && \text{(the Thom class is a cocycle)} \\ &= \pi^*(d^*\psi_U) \smile u_{E|U} && (\pi^\sharp \text{ is a chain map, induced by a continuous map)} \\ &= \pi^*(d^*\psi_U) \smile u_E \end{aligned}$$

where the final equality holds because $\pi^*(d^*\psi_U)$ will vanish on chains outside of U, so restricting the Thom class does not change our final cochain. $\square$

16. Lecture 16 – The Gysin Sequence

One of the nice features of Thom classes is that they relate cohomology groups of X to cohomology groups of $(E, E^\sharp)$, where E is a bundle over X. The Gysin sequence will be another relationship between cohomology of a bundle and cohomology of the base space. However, this time the bundle will be a sphere bundle, not a vector bundle.

Consider the long exact sequence for $(E, E^\sharp)$ given by

$$\cdots \to H^{i+n}(E, E^\sharp) \to H^{i+n}(E) \to H^{i+n}(E^\sharp) \to H^{i+n+1}(E, E^\sharp) \to \cdots \tag{16.1}$$

Assume X is paracompact so that E admits an inner product (given by smoothing local inner products via a partition of unity). We can define $s(E) \subset E$ to be the sphere bundle over X with fibre S^{n-1}_x the unit sphere in E_x. This is a fibre bundle and is locally trivial since E was.

Moreover, this bundle is naturally related to $E^\sharp$ by the inclusion $S(E) \overset{E^\sharp}{\hookrightarrow}$, which is a homotopy equivalence. This allows us to augment the diagram in (16.1) to

$$\begin{tikzcd} H^{i+n}(E,E^\sharp) \arrow[r] & H^{i+n}(E) \arrow[r] \arrow[d, "\cong"', "\iota^*"] & H^{i+n}(E^\sharp) \arrow[r] \arrow[d, "\cong"', "\iota^*"] & H^{i+n+1}(E,E^\sharp) \\ H^i(X) \arrow[r, "\phi"] \arrow[u, "\mathrm{T}"', "\cong"] & H^{i+n}(X) \arrow[r] & H^{i+n}(S(E)) \arrow[r] & H^{i+1}(X) \arrow[u, "T"', "\cong"] \end{tikzcd} \tag{16.2}$$

where T is the Thom isomorphism and ι denotes inclusion. All maps along the bottom row of (16.2) are defined so that the diagram commutes.

Definition 16.1. The bottom row of (16.2) is called the *Gysin sequence* of the (oriented) sphere bundle.

Lemma 16.1. The map ϕ is given by $\phi(\alpha) = \alpha \smile e_E$, where e_E is the Euler class of the bundle E.

Proof. This is a quick computation and should not even be surprising since the Gysin sequence is in a sense induced by the long exact sequence for pairs maps, which give the euler class from the Thom class. Here, we will let ψ denote the LES map. Going around the top three edges of the square, we have

$$\begin{aligned} \iota^*(\psi(\pi^*\alpha \smile u_E)) &= \iota^*(\pi^*\alpha \smile e_E) \\ &= \alpha \smile e_X \end{aligned}$$

where the first equality holds because we identify X with the zero section of E. As ψ is the adjoint of the map $C^*(E) \to C^*(E)/C^*(E^\sharp)$, it has no effect on the pullback of alpha. Definitionally, it sends u_E to the Euler class. The final equality holds by functoriality and the fact that i, π are homotopy equivalences. $\square$

One of the most immediate benefits of the Gysin sequence is it may allow us to use our knowledge of (co)-homology of spheres to deduce the multiplicative structure of cohomology rings of more complicated objects. The next example illustrates this point nicely.

Example 16.3. Let $\mathcal{L} \to \mathbf{CP}^n$ be the tautologaical complex line bundle. The unite sphere bundle $S(\mathcal{L})$ is the unit sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$. The Gysin sequence here becomes

$$H^{i+1}(S^{2n+1}) \to H^i(\mathbf{CP}^n) \xrightarrow{\smile e_{\mathcal{L}}} H^{i+2}(\mathbf{CP}^n) \to H^{i+2}(S^{2n+1}) \to H^{i+1}(\mathbf{CP}^n) \tag{16.4}$$

where the 2 appears since that is the real rank of a complex line bundle.

For reference, the additive structure of $\mathbf{CP}^n$ is given by

$$H^i(\mathbf{CP}^n) = \begin{cases} \mathbf{Z} & i = 0, 2, 4, \cdots, 2n \\ 0 & \text{otherwise.} \end{cases} \tag{16.5}$$

Reading off (16.4), we see that for $i \leq 2n-2$, we have theexact segment

$$0 \to H^i(\mathbf{CP}^n) \xrightarrow{\smile e_{\mathcal{L}}} H^{i+2}(\mathbf{CP}^n) \to 0 \tag{16.6}$$

which shows cupping with the Euler class gives an isomorphism $H^i(\mathbf{CP}^n) \to H^{i+2}(\mathbf{CP}^n)$ for $i \le 2n-2$. For larger i, this map becomes the zero map for degree reasons. It follows that $H^*(\mathbf{CP}^n) \cong \mathbf{Z}[e_{\mathcal{L}}]/(e_{\mathcal{L}}^{n+1})$.

Remark 16.2. If one includes $\mathbf{CP}^k \subset \mathbf{CP}^n$ via $\mathbf{C}^{k+1} \subset \mathbf{C}^{n+1}$, then $\mathcal{L}|_{\mathbf{CP}^k}$ is the tautological bundle of $\mathbf{CP}^k$.

Stiefel manifolds are another important example of tautological bundles. They also give us a natural way to construct sphere bundles, as the next example shows. The bundle may seem complicated to describe geometrically, but cohomologically, it is essentially the simplest bundle we could ask for.

Example 16.7. Let $V_k(\mathbf{C}^n)$ be the Stiefel manifold, whose points are orthonormal k-tuples of vectors in $\mathbf{C}^n$, which is naturally topologized as a subset of $\underbrace{\mathbf{C}^n \times \cdots \mathbf{C}^n}_{n \text{ times}} = \mathbf{C}^{nk}$.

There is again a tautological bundle $E_k \to V_k(\mathbf{C}^n)$ with fibre $\langle e_1, \cdots, e_k \rangle \subset \mathbf{C}^n$ at the point corresponding to $\langle e_1, \cdots, e_k \rangle$. Note that $E_k \subset \mathbf{C}^n \times V_k(\mathbf{C}^n)$ and so naturally lives inside the trivial bundle.

We note that it is locally trivial by pulling back via $V_k(\mathbf{C}^n) \to \mathrm{Gr}_k(\mathbf{C}^n)$ and that we have a forgetful map

$$\begin{aligned} V_{k+1}(\mathbf{C}^n) &\xrightarrow{f} V_k(\mathbf{C}^n) \\ \langle e_1, \cdots, e_{k+1} \rangle &\mapsto \langle e_1, \cdots, e_k \rangle \end{aligned} \tag{16.8}$$

and that the fibre over $\langle e_1, \cdots, e_k \rangle$ is the unit sphere in $\langle e_1, \cdots, e_k \rangle^\perp \subset \mathbf{C}^n$. Thus, if $F_k \to V_k(\mathbf{C}^n)$ is orthogonal to $E_k \subset \mathbf{C}^n$, then $V_{k+1}(\mathbf{C}^n) = S(F_k)$ is the sphere bundle of a vector bundle. Here, F_k is the orthogonal complement to E_k, in the sense of the Whitney sum and a chosen inner product on $\mathbf{C}^n$.

Proposition 16.3. $H^*(V_k(\mathbf{C}^n)) \cong \bigwedge_{\mathbf{Z}}(a_{2n-2k+1}, a_{2n-2k+3}, \cdots, a_{2n-1})$ i.e. it is freely generated by elements a_i of degree i as above, subject only to relations forced by skew-commutativity.

Proof (likely non-examinable). Again, the general strategy is to use the Gysin sequence to deduce the multiplicative structure of the cohomology ring.

We induct on k. If $k = 1$, $V_k(\mathbf{C}^n) = S^{2n-1}$ and $H^*(S^{2n-1}) = \bigwedge(a_{2n-1})$. This is because the forgetful map forgets everything, so we are left with an entire S^{2n-1} of choices for our pullback.

Now, suppose the result holds for $V_k(\mathbf{C}^n)$ and consider $V_{k+1}(\mathbf{C}^n) = S(F)$, the bundle of $\mathrm{rank}_{\mathbf{C}} n - k$. So, it has real rank $2n-2k$ and $e_F = 0 \in H^{2n-2k}(V_k\mathbf{C}^n) = 0$. This is true because $V_k(\mathbf{C}^n)$ has complex rank $n-(k-1)$ as a bundle (also by induction).

We then examine the Gysin sequence

$$0 \xrightarrow{\smile e_F} H^i(V_k) \to H^i(V_{k+1}) \to H^{i-2n-2k+1} \xrightarrow{\smile e_F = 0} \to \cdots \tag{16.9}$$

and so the fact that the Euler class is zero breaks up the Gysin sequence into short exact sequences. By hypothesis, $H^{i-2n+2k+1}(V_k)$ is free. So, the sequence splits and our result follows additively. Moreover, the map given by the splitting of our sequence is in fact a ring isomorphism. $\square$

17. Lecture 17 – Cohomology with Compact Support

In proving the Thom isomorphism, we used

Lemma 17.1. Suppose $(X, Y) = (A \cup B, C \cup D), X = A \cup B, Y = C \cup D$, where $C \subset A, D \subset B$.

Then, theres a relative Mayer-Vietoris sequence

$$\cdots \to H^n(X, Y) \to H^n(A, C) \oplus H^n(B, D) \to H^n(A \cap B, C \cap D) \to H^{n+1}(X, Y) \to \cdots \tag{17.1}$$

Proof. The proof comes from the following diagram, which requires a good deal of explanation.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C^*(A+B, C+D) \arrow[r, "\varphi"] \arrow[d] & C^*(A, C) \oplus C^*(B, D) \arrow[r, "\psi"] \arrow[d] & C^*(A \cap B, C \cap D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^*(A+B) \arrow[r, "\varphi"] \arrow[d] & C^*(A) \oplus C^*(B) \arrow[r, "\psi"] \arrow[d] & C^*(A \cap B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^*(C+D) \arrow[r, "\varphi"] \arrow[d] & C^*(C) \oplus C^*(D) \arrow[r, "\psi"] \arrow[d] & C^*(C \cap D) \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd} \tag{17.2}$$

A few observations about the diagram. First, the content of locality can be summed up as saying that $C_*(A + B)$, or the chain group of chains lying entirely in A or B is naturally isomorphic to $C_*(X)$, when $X = A \cup B$. More specifically, $C_*(A+B) \hookrightarrow C_*(Y)$ is a homotopy equivalence. Its adjoint is also a homotopy equivalence. This gives us an interpretation of the first column. Locality gives us that $C^*(A + B) \cong C^*(X)$ and $C^*(C + D) \cong C^*(Y)$.

We must also give meaning to $C^*(A+B, C+D)$, so we choose to define it so that the lefthand column is exact. The other columns are exact automatically. Then, using the Five lemma and the exactness of the first column, we see $C^*(A + B, C + D) \cong C^*(X, Y)$

Finally, we note that the first row injects into the second and the second row is exact. So, in the first row $\psi \circ \varphi = 0$. This row has a long exact sequence in cohomology, which is the Mayer-Vietoris sequence (17.1). ☐

At this point, we've computed H^* of Σ_g, S^n, products, $\mathbf{CP}^n$, and $V_k(\mathbf{CP}^n)$. We have observed that for an oriented n-manifold, $H^n(M) \cong \mathbf{Z}$. Our aim is to prove this in general, inductively over the number of sets of a such a cover. But, $H^n(\mathbf{R}^n) = 0$. So, our base case fails,so we need to introduce a new variant, cohomology with compact supports, so that $H^n_{\text{ct}}(\mathbf{R}^n) \cong \mathbf{Z}$.

The idea of cohomology with compact supports for a space X is to take the cohomology of "the largest" compact set inside X. Of course, for many immediate examples of non-compact spaces, there is no largest compact set. To remedy this difficulty, we introduce the notion of a direct limit of groups. In $\mathbf{R}^n$, any compact set is contained in some compact sphere, so we should expect something like the cohomology of a sphere to surface as our answer.

Definition 17.1. Let A be a partially ordered set such that for all $a, b \in A$, there exsts $c \in A$ such that $a \leq c, b \leq c$.

A *direct system* of groups indexed by A comprises $\{G_a\}_{a \in A}$ and homs $\rho_{ab} : G_a \to G_b$ such that

(i) $\rho_{aa} = \mathrm{Id}_{G_a}$
(ii) If $a \leq b \leq c$, $\rho_{ac} = \rho_{bc} \circ \rho_{ab}$.

The *direct limit* of the system is $\xrightarrow[a]{\lim} G_a$ and is given by $(\coprod_a G_a)/\sim$, where $g \sim \rho_{ab}(g)$ for any $b \geq a$. We can also write the direct limit as

$$\bigoplus_{a \in A} G_a / \langle g - \rho_{ab}(g) \text{ for } g \in G_a, a \leq b\rangle \tag{17.3}$$

Remark 17.2. Addition under this quotient is in fact well-defined. For $g \in G_a, h \in G_b$, we find some $c \geq a, b$ and let $g' = \rho_{ac}(g), h' = \rho_{bc}(h)$. Define $[g] + [h] = [g' + h']$. Let $g \in G_a, h \in G_b$.

We will now show this addition is well-defined. Suppose $g \sim g_\alpha \in G_\alpha, h \sim h_\beta \in G_\beta$. Then, we find some $c' \geq \alpha, \beta, c$. We have

$$\begin{aligned} [g_\alpha] + [g_\beta] &= [\rho_{\alpha c'}(g_\alpha) + \rho_{\beta c'}(g_\beta)] \\ &= [\rho_{\alpha c'} \circ \rho_{a\alpha}(g) + \rho_{\beta c'} \circ \rho_{b\beta}(h)] \\ &\sim [h' + g'] \end{aligned}$$

where the last equality holds because $\rho_{\alpha c'} \circ \rho_{a\alpha} = \rho_{ac'} = \rho_{cc'} \circ \rho_{ac}$ and similarly for b, β. So, $\xrightarrow{\lim} G_a$ inherits a group structure.

Let X be a space, If $K_1 \subset K_2 \subset X$, then $X \setminus K_1 \supset X \setminus K_2$ and so nclusion of pairs $(X, X \setminus K_2) \to (X, X \setminus K_1)$ induces a map $H^*(X, X \setminus K_1) \to H^*(X, X \setminus K_2)$.

Definition 17.2. Let $A = \{\text{compact subsets of } X\}$, partially ordered by inclusion, the groups $H^*(X, X \setminus K)$ for $K \subset A$ form a direct system. Let $H^*_{\mathrm{ct}}(X)$, cohomology with compact supports as

$$\xrightarrow[K \subset X, K \text{ cpt}]{\lim} H^*(X, X \setminus K) \tag{17.4}$$

Remark 17.3. We have three main observations.

(1) In any directed system, if I have $B \subset A$ is also an indexing set for the same directed system, then for all $a \in A$, there exists $b \in B$ such that $a \leq b$. Then,

$$\xrightarrow[a \in A]{\lim} G_a = \xrightarrow[b \in B]{\lim} G_b \tag{17.5}$$

(2) We could also write

$$C^*_{\mathrm{ct}}(X) = \bigcup_{\substack{K \subset X \\ K \mathrm{ct}}} C^*(X, X \setminus K) = \{\phi \in C^*(X) \mid \text{there exists } K_\phi \subset X \text{ with compact support}, \phi | X \setminus K_\phi \equiv 0\} \tag{17.6}$$

(3) If X is compact, there exists a maximal compact set, X. So, $H^*_{\mathrm{ct}}(X) = H^*(X, X \setminus X) = H^*(X)$.

Example 17.7. If $X = \mathbf{R}^n$, every compact $K \subset X$ is contained in some $\overline{B_0(R)}$ for some $R = 1, 2, 3, \cdots$. By Remark 17.3 (1), $H^*_{\mathrm{ct}}(\mathbf{R}^n) = \xrightarrow[R]{\lim} H^*(\mathbf{R}^n, \mathbf{R}^n \setminus \overline{B_0(R)})$. But, inclusion $\mathbf{R}^n \setminus \overline{B_0(R)} \hookrightarrow \mathbf{R}^n - 0$ is a homeomorphism for all R. So, we have $H^*_{\mathrm{ct}}(\mathbf{R}^n) \cong H^*_{\mathrm{ct}}(\mathbf{R}^n, \mathbf{R}^n - 0) \cong \mathbf{Z}$.

In particular, this isomorphism passes through the limit, since all maps in the system $\xrightarrow[R]{\lim} H^*(\mathbf{R}^n, \mathbf{R}^n \setminus \overline{B_0(R)})$ are the identity.

Remark 17.4. The previous example shows that H^*_{ct} is not a homotopy invariant, since

$$\begin{aligned} H^i_{\mathrm{ct}}(\mathbf{R}^n) &= \begin{cases} \mathbf{Z} & i = n \\ 0 & \text{otherwise} \end{cases} \\ H^i_{\mathrm{ct}}(\mathrm{pt}) &= \begin{cases} \mathbf{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{17.8}$$

In fact, continuous maps need not act on H^*_{ct}. However, H^*_{ct} does have functoriality in two ways.

Definition 17.3. A *proper map* $f : X \to Y$ is a closed map such that $f^{-1}(\text{compact}) = \text{compact}$.

Remark 17.5. Note that

(a) A proper map $f : X \to Y$ induces a pullback $H^*_{\mathrm{ct}}(Y) \to H^*_{\mathrm{ct}}(X)$. In this context, most of the maps we worry about will be maps induced by inclusion. So, we often omit an explicit argument that are maps are proper and that pullbacks exist.

(b) Let $i : U \to X$ be a homeomorphism onto an open subset. If $c \in C^*_{\mathrm{ct}}(U)$, there exists a compact $K \subset U$ such that $c|_{U \setminus K} \equiv 0$. Then, in X, we can "extend c by zero", defining $H^*_{\mathrm{ct}}(U) \to H^*_{\mathrm{ct}}(X)$. To be more formal, $(\overline{c})(\sigma) = \iota(\sigma)$ when $\sigma \subset K$ and 0 otherwise. Here, $\overline{c}$ is the image of c under "extension by zero".

Indeed, excision That gives a map $H^*(X, X \setminus KK) \cong H^*(U, U \setminus K)$ by excision of $X \setminus U$. So, the $\lim_{K \subset U} H^*(U, U \setminus K)$ can be regarded as a sub-direct system of $\xrightarrow[K \subset U, K \text{ ct}]{\lim} H^*(X, X \setminus K)$.

Simply by excising $X \setminus U$, we have $H^*(X, X \setminus K) \cong H^*(U, U \setminus K)$. Perhaps this is what was intended. Even with this weaker statement, we can still regard the one system as a sub-direct system of the other.

18. Lecture 18 – Cohomology of Manifolds

Lemma 18.1. Let X be a space and $X = U \cup V$, a union of open subsets. Then, there is a Mayer-Vietoris type exact sequence

$$\cdots \to H^{i-1}_{\text{ct}}(X) \to H^i_{\text{ct}}(U \cap V) \to H^i_{\text{ct}}(U) \oplus H^i_{\text{ct}}(V) \to H^i_{\text{ct}}(X) \to H^{i+1}_{\text{ct}}(U \cap V) \cdots \tag{18.1}$$

Proof. For a fixed pair of compact sets $K \subset U, L \subset V$, inclusion induces a long exact sequence of pairs

$$\cdots \to H^i(X, X \setminus (K \cap L)) \to H^i(X, X \setminus K) \oplus H^i(X, X \setminus L) \to H^i(X, X \setminus (K \cup L)) \to \cdots . \tag{18.2}$$

From Remark 17.5 (b), in each case we can localize the cohomology groups to be $H^i(U \cap V, (U \cap V) \setminus (K \cap L))$ and so on. The basic idea here is that every compact set in $C \subset U \cap V$ can be written as the intersection of two compact sets $K_C \subset U, L_C \subset V$ (and analogously for the other localizations). So, we find that the direct limit over all such K, L, of each term in (18.2) is

$$\cdots \to H^i_{\text{ct}}(U \cap V) \to H^i_{\text{ct}}(U) \oplus H^i_{\text{ct}}(V) \to H^i_{\text{ct}}(X) \to \cdots . \tag{18.3}$$

One then shows that taking direct limits preserves exactness to complete the proof. □

The Mayer-Vietoris sequence above becomes very useful in the following context.

Definition 18.1. A manifold M^n has *finite type* if one can write $M = \bigcup_{i=1}^m U_i$ such that each U_i is open in M and each intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ is either empty or homeomorphic to $\mathbf{R}^n$.

Example 18.4. A closed smooth manifold has finite type.

Proof sketch. Given a Riemannian metric over M, find a covering by geodesically convex balls. Then, any two points in such a set are joined by a unique geodesic. The iterated intersections are convex if they are nonempty. Then, one can build a homeomorphism to $\mathbf{R}^n$. □

The following lemma illustrates that cohomology with compact support does what we would like it to.

Lemma 18.2. Let M be an n-manifold of finite type. Then,

(i) $H^i_{\text{ct}}(M) = 0$ for $i > n$.
(ii) $H^i_{\text{ct}}(M)$ is a finitely generated abelian group for all i.
(iii) If M is connected, H^n_{ct} is cyclic. If $U \hookrightarrow M$ is the inclusion of an open disc, the map $H^n_{\text{ct}}(U) \to H^n_{\text{ct}}(M)$ is surjective.

Proof. We will induct over the number of sets in a cover of the type Definition 18.1. We begin with the case of a one-set cover.

Base Case: Here, M is homeomorphic to $\mathbf{R}^n$, keeping in mind that homeomorphisms are proper maps, we see $H^i_{\text{ct}}(M)$ is $\mathbf{Z}$ in dimension n and zero elsewhere. We then get item (ii) for free. All open disks are homeomorphic, so inclusion of an open disk induces an isomorphism on H^n_{ct}.

Inductive Step: As usual, we use Mayer-Vietoris in the inductive step. Cover M by M_0, M_1 so that $M_0, M_1, M_0 \cap M_1$ all require fewer sets for a cover of the type in Definition 18.1. We examine the long exact sequence (18.1) with $U = M_0, V = M_1$. Substituting $i > n$ gives us part (i). To show (ii), note that the map $\phi i : H^i_{\text{ct}}(X) \to H^{i+1}_{\text{ct}}(U \cap V)$ presents $H^i_{\text{ct}}(X)/\ker(\phi)$ as a subgroup of a finitely generated abelian group (by induction). By exactness, we know $\ker(\phi)$ is contained in the image of a finitely generated abelian group. Using the general fact that if G/H is finitely generated and H is finitely generated, then G is, we obtain our result. One can prove this fact using that each element of G has a unique representation as a sum of elements in G/H and H and inherits addition from G.

Finally, we show part (iii). In this case, if X is connected, $M_0 \cap M_1 \neq \emptyset$. But, the real key here is to note that part of (18.1) involves

$$\to \cdots H^n_{\mathrm{ct}}(M_0 \cap M_1) \to H^n_{\mathrm{ct}}(M_0) \oplus H^n_{\mathrm{ct}}(M_1) \xrightarrow{\iota^*} H^n_{\mathrm{ct}}(X) \to 0 \tag{18.5}$$

so that ι^* is a surjective map that we can build on. Consider the inclusion of an open disk via

$$D^n \hookrightarrow M_0 \cap M_1 \hookrightarrow M_0. \tag{18.6}$$

By our inductive hypothesis, the induced maps $H^n_{\mathrm{ct}}(D^n) \to H^n_{\mathrm{ct}}(M_0)$ and $H^n_{\mathrm{ct}}(M_0 \cap M_1) \to H^n_{\mathrm{ct}}(M_0)$ are surjective. Keep in mind that these maps are induced by inclusion and so are the same as the maps in the sequence (18.5). So, by exactness, the map $H^n_{\mathrm{ct}}(M_1) \to H^n_{\mathrm{ct}}(X)$ is onto. Moreover, the map $H^n_{\mathrm{ct}}(M_0) \to H^n_{\mathrm{ct}}(X)$ has zero image. So, $H^n_{\mathrm{ct}}(X)$ is then a quotient of a cyclic group.

Finally, we show that a general inclusion of a disk $U \hookrightarrow M$ induces a surjection on cohomology. We may assume without loss of generality that this disk has image in M_1. One way to see this is to take a "finite type" cover of X and let M_0 be the interior of one covering set while M_1 is the interior of all the others. Then, if the inclusion of a disk U has image in M_0, we may homotope the inclusion map (via a proper map) to one that includes it in M_1. The last step uses that M_0 looks like $\mathbf{R}^n$ and that M is a connected manifold, so M_0, M_1 meet. □

Corollary 18.3. If M is a closed manifold of dimension N, $H^*(M) = 0$ unless $0 \leq * \leq n$ and $H^n(M)$ has rank at most the number of components of M.

Proof. Our manifold M is compact, so the first part of our statement follows from part (i) of Lemma 18.2. The second part can be seen via the unions axiom. □

Definition 18.2. A *topological manifold* M is *oriented* if for all open disks $U \subset M, U \cong \mathbf{R}^n$ there is a distinguished generator $w_n \in H^n_{\mathrm{ct}}(U)$ and if $U \hookrightarrow V \subset M$ with $U, V \cong \mathbf{R}^n$, then extension by zero sends w_u to w_v.

We invite the reader to compare this definition to Definition 15.1.

Remark 18.4. We have a few remarks on orientation.

(1) We say a homeomorphism $f : \mathbf{R}^n \to \mathbf{R}^n$ is *orientation preserving* if it acts by $+1$ on $H^n_{\mathrm{ct}}(\mathbf{R}^n)$. Then, M is orientable if it admits an atlas with orientation preserving maps.
(2) Since $H^n_{\mathrm{ct}}(\mathbf{R}^n) \cong H^n(\mathbf{R}^n, \mathbf{R}^n - 0)$ we can also use the group $H^n(\mathbf{R}^n, \mathbf{R}^n - 0)$.
(3) If M is a smooth manifold, M is orientable (as a manifold) if and only if the tangent bundle TM of M is oriented as a vector bundle.

The final remark in this case is the most interesting. It gives us a nice example of how the geometry of a manifold influences that of its tangent bundle and vice-versa. One can prove This remark using the Gysin sequence.

Finally, we introduce the Theorem we have been waiting for, which proves our observation about top dimension cohomology.

Theorem 18.5. Let M be a connected n-manifold of finite type.

(a) If M is orientable, $H^n_{\mathrm{ct}}(M) \cong \mathbf{Z}$. Moreover, there exists a unique isomorphism

$$\int_M : H^n_{\mathrm{ct}}(M) \to \mathbf{Z} \tag{18.7}$$

such that for all discs $M \supset U \cong \mathbf{R}^n$, the map $\int_M$ sends ε_n to 1, where ε_n is the generator coming from fixing a choice of orientation on M.

(b) If M is not orientable, $H^n_{\mathrm{ct}}(M) \cong \mathbf{Z}_2$.

Proof. Again, we use the strategy of inducting on the number of sets in a finite type cover. When there is only one such set in a cover, our manifold is homeomorphic to $\mathbf{R}^n$ and the Theorem holds trivially.

Take a cover $\{U_i\}$ of M of the kind allowed by the finite type hypothesis. Again, we use a Mayer-Vietoris type argument. For $1 \leq i \leq n+1$, let W_i be given by $U_1 \cup \cdots \cup U_i$. Suppose inductively that U_i is orientable. We know that $W_i \cap U_{i+1}$ is a union of open disks by the finite type hypothesis. Hower, these disks may meet up in interesting ways. So, in general, all we can say is that $W_i \cap U_{i+1} = V_1 \sqcup \cdots \sqcup V_p$ for open, connected, oriented V_i. The V_i inherit orientations from the disks of which they are unions.

Now, we examine the sequence (18.1) which becomes

$$H^n_{\mathrm{ct}}(V_1) \oplus \cdots \oplus H^n_{\mathrm{ct}}(V_p) \xrightarrow{\phi} H^n_{\mathrm{ct}}(W_i) \oplus H^n_{\mathrm{ct}}(U_{i+1}) \to H^n_{\mathrm{ct}}(W_{i+i}) \to 0 \tag{18.8}$$

where the final term is zero by Lemma 18.2. We have supposed that W_i is oriented. So, the map ϕ, which is induced by inclusion, can be regarded as the joint inclusion

$$D^n \hookrightarrow V_j \hookrightarrow W_i \tag{18.9}$$

which surjects onto W_i whether or not W_i is oriented. However, if W_i is oriented, we can choose orientations on the V_j to be consistent in the sense that $\phi(\varepsilon_{V_j})|\, H^n_{\mathrm{ct}}(W_i) = \phi(0, \cdots, 0, 1, 0, \cdots, 0)|\, H^n_{\mathrm{ct}}(W_i) = 1 \in H^n_{\mathrm{ct}}(W_i)$.

There is an inclusion analogous to (18.9) into U_{i+1}. However, in this case although we know the map is surjective, we cannot guarantee a consistent choice of generators. So, at best we can say $\phi(\varepsilon_{V_j})|\, H^n_{\mathrm{ct}}(U_{i+1}) = \phi(0, \cdots, 0, 1, 0, \cdots, 0)|\, H^n_{\mathrm{ct}}(U_{i+1}) = \pm 1 \in H^n_{\mathrm{ct}}(U_{i+1})$.

We know that U_{i+1}, viewed as a manifold is oriented. So, if all of the images $\phi(0, \cdots, 0, 1, 0, \cdots, 0) = (1, \pm 1)$ agree in sign, we have in fact oriented U_{i+1} in a way that is consistent with our orientation on the rest of the manifold. If not, there is no way to orient U_{i+1} in a globally consistent way, since the inclusions of the open disks as in (18.9) must respect our choice of generators.

In the case when all of the signs on the $\phi(\varepsilon_{V_j})$ agree, we can construct an explicit orientation on W_{i+1}. If all the images are $(1, 1)$, set the generator of $H^n_{\mathrm{ct}}(W_{i+1})$ to be the image of $(1, -1)$, or vice-versa. We know this is a generator by exactness of (18.8). When, W_{i+1} is not orientable, both $(1, 1)$ and $(1, -1)$ are in $\mathrm{Im}(\phi)$ and so the same exactness argument shows that $H^n_{\mathrm{ct}}(W_{i+1}) \cong \mathbf{Z}/2\mathbf{Z}$ when it is not orientable.

Now, we discuss the case when W_i is not orientable. Then, there is some least i such that W_i is not orientable. For any $j > i+1$, we have

$$H^n_{\mathrm{ct}}(V_1) \oplus \cdots \oplus H^n_{\mathrm{ct}}(V_q) \xrightarrow{\phi} H^n_{\mathrm{ct}}(W_{j-1}) \oplus H^n_{\mathrm{ct}}(U_j) \to H^n_{\mathrm{ct}}(W_j) \to 0 \tag{18.10}$$

where in this instance, we cannot choose a consistent generator for W_{j-1}, so we choose generators ε_{V_j} for the V_j (defined as before) so that the image $\phi(\varepsilon_{V_j}) = (\pm 1, 1)$ – that is, they are consistent on U_j. However, we have at least one sign disagreement in these images (by hypothesis) and so exactness shows $H^n_{\mathrm{ct}}(W_j) \cong \mathbf{Z}/2$. □

19. Lecture 19 – Cohomology of Submanifolds

This lecture, while somewhat isolated in its content, ought to be a deeply satisfying lecture. We have noted for some time the mantra that "cup product measures intersection". We will now provide some formal machinery to back that claim. In previous lectures, we have considered characteristic classes in top dimension for oriented manifolds. For dimension reasons, these have uninteresting cup products. To really see what cup product measures, we have to analyze the cohomology of submanifolds, where interesting cup products may be present. This will involve defining cohomology classes for submanifolds, in the course of which we will use two important results from differential geometry and one from differential topology.

For the rest of this lecture, let M be a connected, finite type smooth manifold, and X, Y, Z be smooth closed submanifolds of M.

Proposition 19.1 (Differential Topology)**.** Whenever, $X \subset M$ is a smooth submanifold of a smooth manifold, X has a *normal bundle* $\nu_{X/M}$ which has fibre at $x \in X$ the quotient T_xM/T_xX. If we put a metric on M, we view this as $(T_xX)^{\perp} \subset T_xM$.

Remark 19.2. While we will not prove the above fact, we should at least do a sanity check on it. If $X \subset M$ is a submanifold of lower dimension, it will have smaller tangent space, which makes for a larger normal space.

Theorem 19.3 (Tubular Neighborhood Theorem)**.** Let M be a smooth manifold and $Y \subset M$ a closed smooth submanifold.

Then, there is an open neighborhood U_Y of Y in M which is diffeomorphic to the total space of $\nu_{Y/M}$ and indeed there's a diffeomorphism φ such that ϕ takes the zero section to Y identically. Moreover, φ and the neighborhood U_Y are unique up to isotopy.

Recall remark 18.4 part 3, where we say a manifold is oriented if and only if its tangent bundle is (though we do not prove this fact). The following remark is in the same vein:

Definition 19.1. We say $Y \subset M$ is *co-oriented* if $\nu_{Y/M}$ is oriented.

Lemma 19.4. If Y, M are oriented, then Y is naturally co-oriented.

Proof. Recall that we have the identification

$$(\nu_{Y/M})_y \oplus T_yY \cong T_yM. \tag{19.1}$$

A (twice) relative version of the Künneth Theorem gives

$$\bigoplus_{i+j=n} H^i(X,A) \otimes H^j(Y,B) \xrightarrow[\cong]{\times} H^n(X \times Y, X \times B \cup Y \times A) \tag{19.2}$$

under appropriate hypotheses of homology groups being free and finitely generated and all sets being CW complexes. Our application is about as nice as possible, and we do not have to worry about these conditions.

At each point, (19.2) shows that cupping with $\pi^*(u_{T_yY})$ gives an isomorphism $H^{n-k}(\nu_{Y/M})_y \to H^n(T_yM)$. The pre-image of the Thom class under this map must be a generator of $H^{n-k}(\left(\nu_{Y/M}\right)_y, \left(\nu^{\sharp}_{Y/M}\right)_y)$. Since both Thom classes are locally consisten, this generator is locally consisten as well. $\square$

Now, suppose that M is oriented and $Y \subset M$ is closed and co-oriented. The argument above shows that in this case, Y is also oriented, since its tangent bundle is. However, we also have Thom class

$$u_{\nu_{Y/M}} \in H^{n-k}(\nu_{Y/M}, \nu^{\sharp}_{Y/M}) \;=\; H^{n-k}(U_Y, U_Y \setminus Y) \cong H^{n-k}(M, M \setminus Y) \xrightarrow{\text{Pair LES}} H^{n-k}_{\text{ct}}(M) \tag{19.3}$$

and can recover an Euler class of M from equation (19.3). The first equality is given by the Tubular Neighborhood Theorem (Theorem 19.3) and the second is given by excising $M \setminus U_Y$.

Denote the image of $u_{\nu_{Y/M}}$ via the maps above by ε_y. We say ε_y is the *cohomology class* of the submanifold Y.

Example 19.4. If $Y = \{\text{pt}\}$ and $U_Y \cong \mathbf{R}^n \subset M$ is a disc. Then, ε_y is the orientation generator of $H^n_{\text{ct}}(M)$

It might seem initially strange that the cohomology class of a submanifold belongs to the cohomology group of the entire manifold. The following lemma should make it clear why this name is sensible.

Lemma 19.5. The restriction of ε_y to Y gives $e_{\nu_{Y/M}} \in H^{n-k}(\nu_{Y/M}) \cong H^{n-k}(Y)$, where e denotes Euler class.

Proof. Note that the inclusion $Y \hookrightarrow M$ is proper. So, we in fact have a well-defined pullback by inclusion. The proof here comes from augmenting the righthand side of the diagram (19.3) to

$$\begin{array}{ccc} H^{n-k}(U_y, U_y \setminus Y) & \xrightarrow{\text{LES}} & H^{n-k}_{\text{ct}}(U_Y) \cong H^{n-k}_{\text{ct}}(Y) \\ {\scriptstyle \iota^*}\uparrow & & {\scriptstyle \iota^*}\uparrow \\ H^{n-k}(M, M \setminus Y) & \xrightarrow{\text{LES}} & H^{n-k}_{\text{ct}}(M) \end{array} \tag{19.5}$$

where ι^* denotes the adjoint of inclusion, which on the lefthand side is the excision isomorphism and so can be taken in the other direction. However, in this direction, it is clear that the diagram commutes so that the restriction (or ι^*) of ε_y is in fact our Euler class. $\square$

Lemma 19.6. For any space X, cup product defines $H^i(X) \otimes H^j_{\text{ct}}(X) \to H^{i+j}_{\text{ct}}(X)$.

Proof. The proof is an exercise in the definition and properties of the relative cup product. It is left to the reader. $\square$

Now, we have a lemma that formalizes our intuition that cup products should count intersection.

Lemma 19.7. Let M^n be a connected, oriented, smooth, finite type manifold and let $Y^k \subset M$ be oriented, smooth, and closed. Then, for all $\alpha \in H^k(M)$

$$\int_M \alpha \smile \varepsilon_y \;=\; \int_Y \alpha|_Y \tag{19.6}$$

Proof. Not to toot my own horn, but I thought this proof I came up with was pretty slick.

As an opening remark, recall that Lemma 19.4 shows us that $\nu_{Y/M}$ is oriented in this case. Before beginning a short proof, we make four observations. First, locally, by identifying $\nu_{Y/M}$ with $(T_xY)^\perp$, $Y \times \nu_{Y/M} \cong M$. Second, the LES map in (19.5) is induced by inclusion of cochains. Third, by choosing a generator of $\nu_{Y/M}$, we can ensure that the Thom isomorphism $H^k(Y) \xrightarrow{\smile u_{\nu_{Y/M}}} H^n(Y \times \nu_{Y/M})$ preserves the pre-images of 1 from the maps $\int_M, \int_Y$, respectively. Finally, the inclusion $Y \hookrightarrow M$ is a proper map.

We have

$$\begin{aligned} \int_Y \alpha|_Y &= \int_M \alpha|_Y \smile u_{\nu_{Y/M}} && \text{(Observations 1 and 3, definition of } \textstyle\int \text{ maps)} \\ &= \int_M i_*(\alpha|Y \smile u_{\nu_{Y/M}}) && \text{(inclusion preserves generators of } H^n_{\text{ct}}) \\ &= \int_M \overline{\alpha} \smile \varepsilon_y && \text{(Observation two, functoriality)} \end{aligned}$$

where $\overline{\alpha}$ denotes extension by zero, the image of $\alpha|Y$ under inclusion. $\square$

Now, we address the "transverse intersection" part of our previous discussions of the relationship between cup products and intersection.

Definition 19.2. Let $Y, Z \subset M$ be closed, oriented (and co-oriented) submanifolds. We say Y and Z intersect transversely, denoted $Y \pitchfork Z$ if for all $x \in Y \cap Z$, we have

$$T_x Y + T_x Z = T_x M \tag{19.7}$$

Remark 19.8.

(1) If $\dim(Y) + \dim(Z) < \dim(M)$, then $Y \pitchfork Z$ implies $Y \cap Z = \emptyset$.

(2) An important result from differential geometry (which is a consequence of the Implicit Function Theorem) says that if $Y \pitchfork Z$, then $Y \cap Z \subset M$ is again a closed smooth manifold, of codimension equal to $\operatorname{codim}(Y) + \operatorname{codim}(Z)$. Moreover,

$$\nu_{Y\cap Z/M} \cong \nu_{Y/M} \oplus \nu_{Z/M}. \tag{19.8}$$

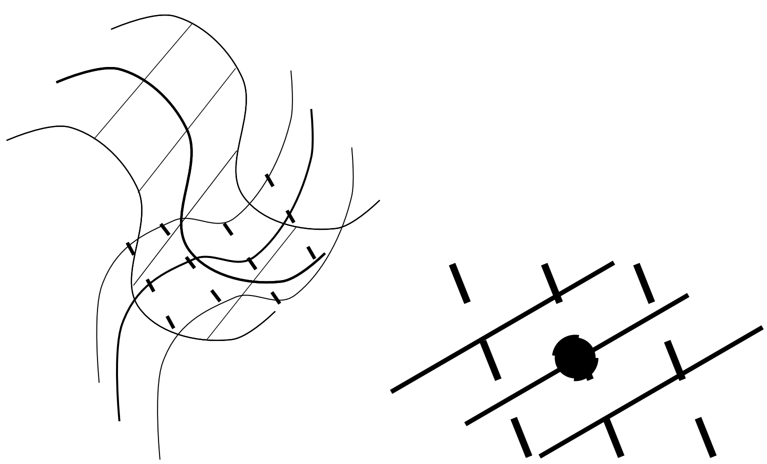
In particular, ordering $\{Y, Z\}$ gives a co-orientation of $Y \cap Z$. This works by a similar Künneth argument to the one we used before in Lemma 19.4.

Theorem 19.9 (Tubular Neighborhood Addendum)**.** Given $Y, Z \subset M$, as before with $Y \pitchfork Z$, there are neighborhood, $U_Y \supset Y, U_Z \supset Z$ such that $U_Y \cap U_Z = U_{Y\cap Z}$ is a tubular neighborhood and such that the diffeomorphism

$$\nu_{Y\cap Z/M} \to U_{Y\cap Z} \ = \ U_Y \cap U_Z \tag{19.9}$$

takes $(\nu_{Y\cap Z/M})_x \to (\nu_{Y/M})_x \times (\nu_{Z/M})_x$ for all $x \in Y \cap Z$, where $\times$ denotes the cross product., relative to the identification $U_Y \cong \nu_{Y/M}, U_Z \cong \nu_{Z/M}$

We provide an example of this "splitting of the normal space" in the following diagram:



(A) Transverse intersection of submanifolds

(B) Local Splitting of the normal space

FIGURE 16. A transverse intersection

Corollary 19.10. If $Y, Z \subset M$ are smooth, oriented, and closed, then provided $Y \pitchfork Z$, we have $\varepsilon_{Y\cap Z} = \varepsilon_Y \cup \varepsilon_Z$, where the co-orientation on $Y \cap Z$ comes from ordering $\{Y, Z\}$

Remark 19.11. The above notion is compatible with the skew-commutativity of the cup product. In fact, this is why we need to order $\{Y, Z\}$.

Proof. This is immediate from the Tubular Neighborhood Addendum and the more general fact that whenever $E \to X, F \to X$, then with respect to the product orientation of $E \oplus F$, $u_{E\oplus F} = u_E \times u_F$, where $\times$ is the cross product. $\square$

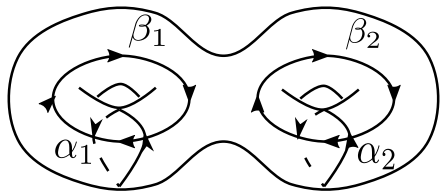


FIGURE 17

Example 19.10. Note the following example: we immediately see $\varepsilon_{\alpha_i} \smile \varepsilon_{\beta_j} = \delta_{ij}$, since $\varepsilon_{\mathrm{pt}} \in H^2(\Sigma_2)$ is a generator.

This ought to give some weight to our slogan that the cup is dual to/given by transverse intersection.

20. Lecture 20 – Poincaré Duality, Part I

We will now lay out the foundations for the strongest relationship between homology and cohomology that we will see in this course. It is based on the notion of a dual graph and known as Poincaré duality. Before we state the Theorem, we provide an important definition.

Unless otherwise stated, all homology groups are over a field, $\mathbf{F}$

Definition 20.1. Let E, F be vector spaces over a field, K. A map $(\cdot,\cdot) : E \times F \to K$ is called a *nondegenerate pairing* if it is bilinear and if $(u, v) = 0$ for all v in F implies $u = 0$ and likewise $(e, f) = 0$ for all e implies $f = 0$.

Remark 20.1. Note that if a nondegenerate pairing as described above exists between two vector spaces E, F then they must be of the same dimension. To see this, note that such a pairing gives injections $\phi_1 : E \to \mathrm{Hom}_K(F, K), \phi_2 : F \to \mathrm{Hom}_K(E, K)$.

Theorem 20.2 (Poincaré Duality, Version 1)**.** Let M be a closed, connected orientable manifold (of finite type). Working over a field $\mathbf{F}$, the pairing

$$\begin{aligned} H^i(M, \mathbf{F}) \times H^{n-i}(M, \mathbf{F}) &\to \mathbf{F} \\ (\alpha, \beta) &\to \int_M \alpha \smile \beta \end{aligned} \tag{20.1}$$

is nondegenerate

Corollary 20.3. In the setting above,

(1) $b_i(M) = b_{n-i}(M)$
(2) If M is odd dimensional, $\chi(M) = 0$
(3) If M, N are closed oriented manifolds and $f : M^n \to N^n$ has nonzero degree, then $f^*(N, \mathbf{F}) \to H^*(M, \mathbf{F})$ is injective which imples so $b_i(M) \geq b_i(N)$ for all i.

Proof of corollaries. In order,

(1) This is immediate from Remark 20.1. However, it is also worth recalling we have a surjection $H^i(X, \mathbf{Z}) \to \mathrm{Hom}(H_i(X, \mathbf{Z}), \mathbf{Z})$, with the only possible kernel coming from the torsion in $H_{i-1}(X, \mathbf{Z})$ (c.f Proposition 11.4). Over a field, we have no torsion so the aforementioned map is an isomorphism.
(2) This is immediate from the previous proposition when M is orientable. If not, take an orientable double cover $\widetilde{M}$ for M. It is not hard to show $\chi(\widetilde{M}) = 2\chi(M)$, which establishes the result in the non-orientable case.
(3) Assume neither $H^i(N), H^{n-i}(N)$ Take $\alpha \neq 0 \in H^i(N)$ and find some $\beta \in H^{n-i}(N)$ such that $\int_N \alpha \smile \beta \neq 0$. In fact, by rescaling, we may assume $\int_N \alpha \smile \beta = 1$ which implies $\alpha \smile \beta = \varepsilon_n$. Then, using the isomorphism $H^n(N) \to \mathbf{Z}$ given by the integral, we have an induced map $\overline{f} : \mathbf{Z} \to \mathbf{Z}$ of the same degree as f^*. So, $\overline{f}\left(\int_N \alpha \smile \beta\right) = \int_M f^*(\alpha) \smile f^*(\beta) \neq 0$, which implies $f^*(\alpha) \neq 0$.

□

Eventually, we will infer Poincaré Duality, version 1, from a natural relationship between chains and cochains.

Definition 20.2. On any space, X, there's a product

$$\begin{aligned} C_k(X) \times C^l(X) &\to C_{k-l}(X) \\ ([v_0 \cdots v_k], \phi) &\to \phi([v_0 \cdots v_l])[v_l \cdots v_k] \end{aligned} \tag{20.2}$$

where multiplication on the right is multiplication of a chain by an integer.

We extend this map linearly and write $\sigma \frown \phi$ for the output.

Proposition 20.4. The cap product has the following properties:

(1) **Relation to (co)-boundary**: For $\sigma \in C_k(X), \phi \in C^l(X)$, we have

$$d(\sigma \frown \phi) \ = \ (-1)^l(d\sigma \frown \phi - \sigma \frown d^*\phi) \tag{20.3}$$

where d, d^* are the boundary operators on the chain and cochain groups.

Hence $\frown$ defines a product $H_k(X) \times H^l(X) \to H_{k-l}(X)$.

(2) **Relation to Cup Product**: If $\sigma \in C_{k+l}(X), \phi \in C^k(X), \psi \in C^l(X)$, then

$$(\phi \smile \psi)(\sigma) \ = \ \psi(\sigma \frown \phi) \tag{20.4}$$

(3) **Functoriality:** If $f : X \to Y$ and $\alpha \in H_k(X), \beta \in H^l(Y)$, then

$$f_*(\alpha) \frown \beta \ = \ f_*(\alpha \frown f^*(\beta)) \tag{20.5}$$

(4) **Pairs:** If $A \subset X$, then the cap product

$$C_k(X) \times C^l(X, A) \xrightarrow{\frown} C_{k-l}(X) \tag{20.6}$$

and hence we get an induced cap product

$$C_k(X, A) \times C^l(X, A) \xrightarrow{\frown} C_{k-l}(X) \tag{20.7}$$

Proof. All of these are relatively straightforward. We will demonstrate the relation to (co)-boundary, arguably the most difficult. In two separate steps, we have

$$\begin{aligned} d(\sigma \frown \phi) \ &= \ d\left(\phi([v_0 \cdots v_l])[v_l \cdots v_k]\right) \\ &= \ \sum_{i=l}^{k} (-1)^{i-l} \phi([v_0 \cdots v_l])[v_l \cdots \hat{v}_i \cdots v_k] \end{aligned}$$

and

$$\begin{aligned} d\sigma \frown \phi \ &= \ \sum_{i \le l} (-1)^i \phi([v_0 \cdots \hat{v}_i \cdots v_{l+1}])[v_{l+1} \cdots v_k] \\ &\quad + \sum_{i > l} (-1)^i \phi([v_0 \cdots v_l])[v_l \cdots \hat{v}_l \cdots v_k] \end{aligned}$$

while

$$\sigma \frown d^*\phi \ = \ \sum_{i \le l+1} (-1)^i \phi([v_0 \cdots \hat{v}_i \cdots v_{l+1}])[v_{l+1} \cdots v_k]$$

which establishes the result. □

Finally, we will give some geometric meaning to the integral map.

Let M be a closed connected oriented manifold of finite type. We know $H^n(M, \mathbf{Z}) \cong \mathbf{Z}$ (where $n = \dim_{\mathbf{R}} M$). Let $[M]$ be the generator of $H_n(M, \mathbf{Z})$ dual to the orientation generator of H^n. This is the *fundamental class* of M. We think of the fundamental class as a sum of n-simplices, covering M once. When M is oriented, we can orient these simplices so that the boundaries cancel. When M is not oriented, this is not possible. However, using $\mathbf{Z}_2$ coefficients this will not matter – two simplices meet and each boundary point and so cancel.

Since our integral on cohomology picks up how many copies of the orientation generator ε_M live in some $\phi \in H^n(M)$, we can also count this number by taking the natural pairing with $[M]$. Going further, we can use the relation of cap product to cup product in (20.4) to obtain

$$\begin{aligned} \int_M \alpha \smile \beta &= \langle \alpha \smile \beta, [M] \rangle \\ &= (\alpha \smile \beta)([M]) \\ &= \beta([M] \frown \alpha). \end{aligned} \tag{20.8}$$

So, our integral corresponds to evaluating n-cochain eat every simplex in a triangulation of M.

Examining (20.8), we see that the fundamental class also gives a map

$$\begin{aligned} [M] \frown : H^k(M) &\xrightarrow{D} H_{n-k}(M) \\ \alpha &\mapsto [M] \frown \alpha \end{aligned} \tag{20.9}$$

so that $\int_M \alpha \smile \beta = \langle \beta, D(\alpha) \rangle$.

We can now reformulate Poincaré Duality in terms of this fundamental class map.

Theorem 20.5 (Poincaré Duality, Version 1)**.** The pairing $\int_M$ is non-degenerate over a field (if and only if D is an isomorphism over a field).

Theorem 20.6 (Poincaré Duality, Version 2)**.** For any connected oriented manifold M (of finite type), there exists a canonical isomorphism $D : H^k_{\text{ct}}(M, \mathbf{Z}) \to H_{n-k}(M, \mathbf{Z})$. If M is closed, then $D = [M] \frown$.

Remark 20.7. Clearly, as $H^n_{\text{ct}}(M)$ is torsion-free, the pairing $H^k_{\text{ct}}(M) \times H^{n-k}(M) \to \mathbf{Z}$ given by $(\alpha, \beta) \to \int_M \alpha \smile \beta$ must ill torsion classes α, β.

The first step in the proof is to define D in general. We want a notion analogous to compactly supported cohomology for homology. We start by observing that for a manifold M and $x \in M$, we have

$$\begin{aligned} H_n(M, M \setminus x) &\cong H_n(U, U \setminus x) \\ &\cong H_n(\mathbf{R}^n, \mathbf{R}^n \setminus 0) \\ &\cong \mathbf{Z} \end{aligned} \tag{20.10}$$

where the first isomorphism is given by excision of $X \setminus U$, the second comes from homotopy equivalence, and the final one comes from the LES for a pair.

We can view an orientation on M as giving generators u_X for $H_n(M, M \setminus x)$ varying locally trivially with x, which is made more precise by the following Proposition

Proposition 20.8. If M is an oriented manifold, then for every compact $K \subset M$, there is a unique $u_K \in H_n(M, M \setminus K)$ such that under $M \setminus K \hookrightarrow M \setminus x$, where $x \in K$, then $u_K \mapsto u_x$ for all $x \in K$.

First we explore the consequences of this proposition. After all, we have a suitable definition via compactly supported cohomology. We use it in cointinuing to define D. Via the proposition and the relative cap product, we get a diagram:

$$\begin{tikzcd} H_n(M, M \setminus L) \arrow[d, "\iota_*"] & \frown & H^k(M, M \setminus L) \arrow[r] & H_{n-k}(M) \arrow[d, "\iota_*"] \\ H_n(M, M \setminus K) & \frown & H^k(M, M \setminus K) \arrow[r] \arrow[u, "\iota^*"] & H_{n-k}(M). \end{tikzcd} \tag{20.11}$$

By uniqueness of the class u_K, we get that $u_K = \iota_*(u_L)$. So, we have, for $\phi \in H^k(M, M \setminus L)$,

$$\begin{aligned} u_K \frown \phi &= \iota_*(u_L) \frown \phi \\ &= \iota_*(u_L \frown \iota^*(\phi)) \end{aligned} \tag{20.12}$$

which shows $\phi \mapsto u_K \cap \phi$ is compatible with inclusions for compact sets. So, we get a map

$$D : H^k_{\mathrm{ct}}(M) \to H_{n-k}(M) \tag{20.13}$$

via $[u_K] \frown$. This is the map in Poincare Duality, version 2.

Proof of Proposition 20.8. Since this proposition so strongly echoes the Thom isomorphism, many of the arguents will be similar. The proof proceeds in the following manner. We first show that if the true for K, L, it is true for $K \cup L$ and analogously for any finite union of compact sets. Then, we argue it is true for convex sets using a homotopy argument. We use the inclusion of $\mathbf{R}^n \setminus B(R)$ into $\mathbf{R}^n \setminus K$ to establish existence for general K and then make a geometric argument to establish uniqueness.

The first step most strongly echoes the proof of the Thom isomorphism Theorem. Suppose the result is true for K, L. Then, using the relative Mayer-Vietoris sequence, we examine

$$0 = H_{n+1}(\mathbf{R}^n, \mathbf{R}^n \setminus (K \cap L)) \to H_n(\mathbf{R}^n, \mathbf{R}^n \setminus (K \cup L)) \to H_n(\mathbf{R}^n, \mathbf{R}^n \setminus K) \oplus H_n(\mathbf{R}^n, \mathbf{R}^n \setminus L) \to H_n(\mathbf{R}^n, \mathbf{R}^n \setminus (K \cap L)) \tag{20.14}$$

where the final map sends (a, b) to $(a - b)$. However, u_K and u_L must agree on $K \cap L$, since both restrict to generators at each point. By exactness, there is a unique $u_{K \cup L}$ that restricts to u_K on K and u_L on L.

Similarly, we may show that if $K = \bigcup_{i=1}^n K_i$ and the result holds for each K_i, it holds for K.

Now, we establish the result for convex K. In this case, homotopy equivalence, the long exact sequence for pairs, and the five lemma give us that

$$H_n(\mathbf{R}^n, \mathbf{R}^n \setminus K) \cong H_n(\mathbf{R}^n, \mathbf{R}^n \setminus x) \tag{20.15}$$

for any $x \in K$, which establishes the result (all maps preserve orientation). By step 1, we have established the result for finite unions of compact sets in $\mathbf{R}^n$.

Now, given a general compact set $K \subset \mathbf{R}^n$, we will first establish existence, then uniqueness of such a K. We may contain K in some ball $B(R)$. Then, inclusion gives a map

$$H_n(\mathbf{R}, \mathbf{R} \setminus B(R)) \to H_n(\mathbf{R}^n, \mathbf{R}^n \setminus K) \tag{20.16}$$

and we define u_K to be image of $u_{B(R)}$ under this map. It satisfies the desired property of restricting to generators at each point since $K \subset B(R)$ and $u_{B(R)}$ does so. However, we do not yet know that u_K is the only class with property. If it is not, subtracting two such classes gives a class $\widetilde{u} \in H_n(\mathbf{R}^n, \mathbf{R}^n \setminus B(R))$ that is nonzero and is zero on all of K. This class $\widetilde{u}$ is a finite union of simplices, each of which is individually convex. We can contain $\widetilde{u}$ in a disjoint union of balls B_i so that $\widetilde{u} \subset \bigcup \mathring{B}_i$ (cycles stay cycles). In this case, $\widetilde{u}$ is in the image of the map

$$\bigoplus H_n(\mathbf{R}^n, \mathbf{R}^n \setminus B_i) \to H_n(\mathbf{R}^n, \mathbf{R}^n \setminus K). \tag{20.17}$$

However, each of the summands on the left of (20.17) that maps to $\widetilde{u}$ is convex. So, by our convexity argument, it only vanishes on some $x \in K$ if it is the zero element in homology. So, u_K has only one pre-image. $\square$

21. Lecture 21 – Poincaré Duality, Part II

Now, we prove Poincaré Duality.

Theorem 21.1 (Poincaré Duality, Version 2)**.** The map $D : H^k_{\mathrm{ct}}(M) \to H_{n-k}(M)$ is an isomorphism for a finite type oriented M.

Proof. As usual, we will induct on type.

Suppose $M = U \cup V$ where U, V, and $U \cap V$ are of lower type. We will use the following Mayer-Vietoris diagram:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^k_{\mathrm{ct}}(U\cap V) & \longrightarrow & H^k_{\mathrm{ct}}(U) \oplus H^k_{\mathrm{ct}}(V) & \longrightarrow & H^k_{\mathrm{ct}}(M) & \longrightarrow & H^{k+1}_{\mathrm{ct}}(U\cap V) & \longrightarrow & \cdots \\ & & \cong\big\downarrow D & & \cong\big\downarrow D & & \big\downarrow D & & \cong\big\downarrow D & & \\ \cdots & \longrightarrow & H_{n-k}(U\cap V) & \longrightarrow & H_{n-k}(U) \oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \longrightarrow & H_{n-k-1}(U\cap V) & \longrightarrow & \cdots \end{array} \tag{21.1}$$

so that provided that 1) all squares commute and 2) passing to direct limits preserves exactness so the upper Mayer-Vietoris sequence is in fact exact, the Five Lemma completes our proof, provided we demonstrate a base case. We leave the second matter as an exercise to the reader, to give more familiarity with the notion of a direct limit.

First, we will demonstrate a base case, then we will show that a general square in (21.1) commutes and finally, we show that the square involving the Mayer-Vietoris boundary commutes (up to a sign depending only on k). The final task is especially difficult.

Base Case: When M has a one set cover, we know $M \cong \mathbf{R}^n$. Recall that the fundamental class, $[M]$ is defined to be dual to the orientation generator u_x

$$\begin{aligned} u_x \frown [M] &= u_x([v_0 \cdots v_n])[v_n] \\ &= [v_n] \end{aligned} \tag{21.2}$$

where $[v_n]$ is just a point in our disk and is therefore a generator of zeroth homology.

Now, we will argue that a "typical" square in (21.1) commutes. By a typical square, we mean one whose maps on the top and bottom are induced by inclusion maps. For an example, we use the square

$$\begin{array}{ccc} H^k_{\mathrm{ct}}(U\cap V) & \longrightarrow & H^k_{\mathrm{ct}}(U) \oplus H^k_{\mathrm{ct}}(V) \\ \big\downarrow D & & \big\downarrow D \\ H_{n-k}(U\cap V) & \longrightarrow & H_{n-k}(U) \oplus H_{n-k}(V). \end{array} \tag{21.3}$$

Let $\phi \in H^k_{\mathrm{ct}}(U \cap V)$ and let $\overline{\phi}$ denote its extension by zero, or $i_*(\phi)$. We note that inclusion is a proper map and one can easily verify that $\phi = i^*(\overline{\phi})$. With that in mind, going around the bottom of the diagram, we have

$$\iota^*(\overline{\phi}) \xrightarrow{D} [M] \frown \iota^*(\overline{\phi}) \xrightarrow{\iota_*} \iota_*([M] \frown \iota^*(\overline{\phi})). \tag{21.4}$$

and around the bottom, we have

$$\iota^*(\overline{\phi}) \xrightarrow{\text{Extension by zero}} \overline{\phi} \xrightarrow{D} [M] \frown \overline{\phi} = \iota_*([M]) \frown \overline{\phi} \tag{21.5}$$

where the final equality holds since $\overline{\phi}$ will vanish on chains outside a fixed compact set in $U \cap V$. The two expressions are equal by the functoriality of the cap product.

The sticking point in this proof is to show that the square

$$\begin{tikzcd} H^k_{\mathrm{ct}}(M) \arrow[r] \arrow[d, "D"] & H^{k+1}_{\mathrm{ct}}(U\cap V) \arrow[d, "D", "\cong"'] \\ H_{n-k}(M) \arrow[r] & H_{n-k-1}(U\cap V) \end{tikzcd} \tag{21.6}$$

commutes, up to a global sign, depending only on k. For the duration of this proof, we will fix compact subsets $K \subset U, L \subset V$. Recall that given $\phi \in H^*(M, J \setminus (K \cup L))$, we compute the Mayer-Vietoris boundary by writing $\phi = \phi_{M\setminus K} - \phi_{M\setminus L}$, a difference of cochains in the relative groups $C^*(M, M \setminus K), C^*(M, M \setminus L)$, though these chains are not necessarily cycles. Then, we define $\partial^*_{\mathrm{MV}}(\phi) = d\phi_{M\setminus K}$, where d is the usual boundary. This gives an element of $H^{*+1}(M \setminus K)$, which gives an element of $H^{*+1}(U \setminus K)$ by the excision isomorphism.

For the proof, the following diagram will be helpful for context. As $M = U \cup V$, we note we can write

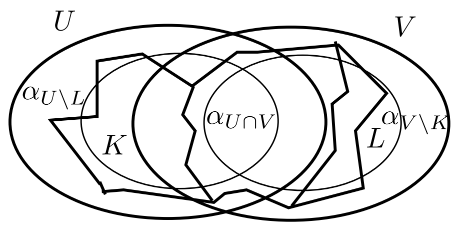


FIGURE 18. A subdivision of simplices

M as

$$M \;=\; (U \setminus L) \cup (U \cap V) \cup (V \setminus K). \tag{21.7}$$

In (21.6), the map D is supposed to be given by cupping with the fundamental class. However, we are now on a submanifold and need a new context for this idea. This is where Proposition 20.8 comes in. We view $H^k_{\mathrm{ct}}(U \cap V)$ as $H^{k+1}(M, M \setminus (K \cap L))$ and give the orientation generator corresponding to $K \cap L$, $u_{K\cap L}$, which will be naturally dual to the cohomological orientation generator. So, it locally represents $[M]$. By the uniqueness clause in Proposition 20.8, we can find this, as well as the versions for $M \setminus K$ and $M \setminus L$ that the Mayer-Vietoris boundary will require us to find.

Pick a chain representing $u_{K\cup L}$, such as the jagged chain in Figure 18. We represent this chain as a sum of chains $u_{K\cup L} \;=\; \alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}$ written (by barycentric subdivision) as sums of chains α_Y living in Y.

We notice that the chains $\alpha_{U\setminus L}, \alpha_{V\setminus L}$ live in $M\setminus(K\cap L)$. So, the vanish in $H_*(U\cap V, U\cap V\setminus(K\cap L)) \cong H_*(M, M \setminus (K \cap L))$. So, the chain $\alpha_{U\cap V}$ must represent $u_{K\cap L}$, by uniqueness of the orientation generator.

Similarly, we note that $\alpha_{V\setminus K}$ lies in $M \setminus K$ and is hence the zero element in $H_*(M, M \setminus K) \cong H_*(V, V \setminus K)$. So, u_K must be given by $\alpha_{U\cap V} + \alpha_{U\setminus L}$ and similarly, $u_L = \alpha_{U\cap V} + \alpha_{V\setminus K}$.

To summarize, we have

$$\begin{aligned} u_{K\cap L} &= \alpha_{U\cap V} \\ u_K &= \alpha_{U\cap V} + \alpha_{U\setminus L} \\ u_L &= \alpha_{U\cap V} + \alpha_{V\setminus K}. \end{aligned} \tag{21.8}$$

Now, we compare going around both sides of (21.6). Going around the top and starting with $\phi \in H^k_{\mathrm{ct}}(M)$, we obtain

$$\phi \xrightarrow{\partial_{\mathrm{MV}}} d\phi_{M\setminus K} \xrightarrow{D} \alpha_{U\cap V} \frown d\phi_{M\setminus K} \tag{21.9}$$

and around the bottom, we get

$$\begin{aligned}\phi \xrightarrow{D} (\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}) \frown \phi \xrightarrow{\partial_{\mathrm{MV}}} & d(\alpha_{U\setminus L} \frown \phi) \\ = & (-1)^k (d\alpha_{U\setminus L} \frown \phi - \alpha_{U\setminus L} \frown d\phi) \\ = & (-1)^k (d\alpha_{U\setminus L} \frown \phi) \\ = & (-1)^{k+1} (d\alpha_{U\cap V} \frown \phi)\end{aligned} \tag{21.10}$$

where the second equality holds because ϕ is a cycle and the final equality holds because $u_{K\cup L} = \alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}$ is a cycle and so $d\alpha_{U\setminus L} = -d(\alpha_{U\cap V} + \alpha_{V\setminus K})$. Finally, note that $d\alpha_{V\setminus K} \subset M \setminus K$ and so by the properties of the relative cap product $d\alpha_{V\setminus K} \frown \phi_{M\setminus K} = 0$.

At this point the final expressions in (21.9) and (21.10) look similar. Applying d again, we see both are cycles. To show they represent the same of homology (up to sign), it suffices to show they differ a boundary. But, this is apparent from the relationship between the cap product and the boundary map, given by

$$d(\sigma \frown \psi) \;=\; -1^{\deg(\psi)}(d\sigma \frown \psi - \sigma \frown d\psi) \tag{21.11}$$

which shows that indeed the square commutes up to sign. Passing to direct limits completes the proof. □

Remark 21.2. Some opt to define $C^k(X) = \mathrm{Hom}(C_k(X), \mathbf{Z})$ with $d^* = (-1)^k(\text{adjoint of } d)$ to avoid these shennanigans.

After a fairly opaque proof, we might wonder why something like Poincaré duality might be true. The case of surfaces gives us some intuition. For a given surface, Σ, take a polygonal decompsition and then use this to define a dual decomposition so that $C_i^{\mathrm{dual}} = C_{2-i}$. We illustrate this process via the dual graph in the following picture.

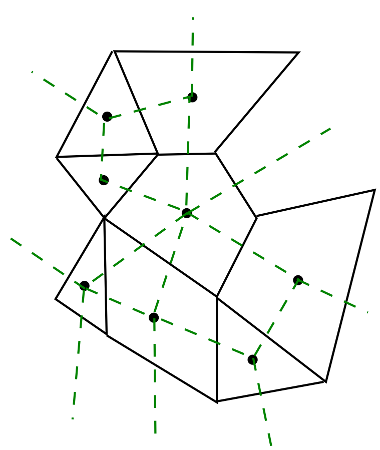


FIGURE 19. Intuition For Poincaré Duality

For any closed, smooth manifold, Morse theory lets you construct analogous dual cell decompositions such that for all i-cells in C_*^{cell} there exists a unique $(n-i)$-cell in C_*^{dual} such that these cells meet transverly in a point. Then, $H_i(M, \mathbf{Z}/2) \cong H_{n-i}(M, \mathbf{Z}/2)$ by cellular cohomology.

22. Lecture 22 – Applications

This lecture, we will explore the applications of Poincaré Duality and related ideas. Throughout this lecture, we will be using the natural notion of a smooth vector bundle over a smooth manifold and a smooth section of such.

The next lemma, on a mechanical level is not difficult. But, what it says is at once surprising and reassuring. It tells us that the Euler class of a bundle governs the cohomology of ANY section of that bundle, in the sense of its transverse intersections with the zero section.

Lemma 22.1. Let $E \to M$ be a smooth vector bundle, which is oriented, and M be a closed manifold. Let $s : M \to E$ be a smooth section which vanishes transversely, i.e. $0_M \subset E$ (the zero section) and $s(M)$ meet transversely in E. Then, $Z = s^{-1}(0) \subset M$ is naturally co-oriented and the cohomology class $\varepsilon_Z = e_E \in H^{\mathrm{rk}(E)}(M)$, where e denotes the Euler class.

Proof. We refer the reader to Remark 19.8, where we establish that if two bundles, submanifolds Y, Z of E meet transversely, then the normal bundle of the intersection is given by

$$\nu_{Y\cap Z/E} \;=\; \nu_{Y/E} \oplus \nu_{Z/E} \tag{22.1}$$

which gives it a natural co-orientation. We can be even more specific, referring to the proof of Corollary 19.10, we see that the Thom class of $\nu_{Y\cap Z/E}$ is the cross product of the Thom classes of $\nu_{Y/E}$ and $\nu_{Z/E}$. Local triviality follows from local triviality of the factors.

So, the real content of this lemma is that the orientation class of the vanishing points of the section is simply the Euler class of the bundle. The setup in this problem is

$$\begin{tikzcd}
 & T_{s(x)}E \arrow[dl, bend right, "\pi"'] \arrow[dd, "\pi"] \arrow[dr, "\pi"] & \\
TM \arrow[ur, bend right, "\overline{s}"'] \arrow[dr, "\pi"] & & E \arrow[dl, bend right, "\pi"'] \\
 & M \arrow[ur, bend right, "s"'] &
\end{tikzcd} \tag{22.2}$$

where $\overline{s}(\xi) = (\xi, s(\pi(\xi)))$ and $T_{s(x)}E \cong TM \oplus E$.

Note that we have to include a further bundle over E. This is because the transverse intersection hypothesis requires us to engage with whether or not the intersection of our sections spans the tangent space in E.

To verify the statement about the cohomology class and the Euler class, we could start with a diagram

$$\begin{tikzcd}
H^{\mathrm{rk}(E)}(\nu_{Z/M}, \nu^{\sharp}_{Z/M}) \arrow[d, "\text{LES}"] & & H^{\mathrm{rk}(E)}(E, E^{\sharp}) \arrow[d, "\text{LES}"] \\
H^{\mathrm{rk}(E)}(\nu_{Z/M}) \arrow[dr, "\pi", "\cong"'] & & H^{\mathrm{rk}(E)}(E) \arrow[dl, "\cong", "\pi"'] \\
 & H^{\mathrm{rk}(E)}(M) &
\end{tikzcd} \tag{22.3}$$

and hope that the separate Thom classes in the upper left and right corners have the same image in $H^{\mathrm{rk}(E)}(M)$.

This leads us to ask for a map between the two entries in the top row and indeed there is a natural one. Recall that we have a map $f := \pi \circ \overline{s} : TM \to E$, from diagram (22.2). We want to get to $\nu_{Z/M}$, so we use the identification as TM/TZ. This map fits that context as well, since if $(x, v) \in TM, x \in Z$, then, $s(x) = (x, 0)$ and so (x, v) is sent to zero by f. However, these are the only zeros of the map. We know it is injective, so if we can show the spaces are of the same dimension, it is an isomorphism. Note that both $s(M)$, and 0_M have codimension $\mathrm{rk}(E)$ in E. Their intersection has codimension $2\mathrm{rk}(E)$ and, since it lives in the zero section, can be read as a submanifold of dimension $n - \mathrm{rk}(E)$ in M (note when $n - \mathrm{rk}(E) \leq 0$, the transverse intersection is empty and this lemma is uninteresting). So, TM/TZ has rank $\mathrm{rk}(E)$ and our map f is an isomorphism of spaces, hence an isomorphism of homology groups, which sends Thom class to Thom class. □

For the rest of this lecture, M will be a closed smooth manifold and we will work over $\mathbf{Q}$.

Recall by Künneth that $H^*(M) \otimes H^*(M) \xrightarrow{\times} H^*(M \times M)$ is an isomorphism, where $\times$ is the cross product. Under this context, let $\{a_i\}$ be a basis of $H^*(M)$, b_i $a_i \in H^{d_i}(M)$ and $\{b_i\}$ be a dual basis, so that

$$\int_M a_i \smile b_j \;=\; \delta_{ij} \tag{22.4}$$

which is possible by non-degeneracy. Note that in this case, we must have $b_j \in H^{n-d_i}(M)$.

Proposition 22.2. If $\Delta_M = \Delta \subset M \times M$ denotes the diagonal and we orient M, then Δ is co-oriented and

$$\varepsilon_\Delta \;=\; \sum_i (-1)^{d_i}(a_i \times b_i) \tag{22.5}$$

Proof. Recall in previous lectures, we showed that the cohomology class of a submanifold essentially acts like a Dirac delta. By the non-degeneracy of the integral pairing, it should be clear that this in fact characterizes the cohomology class of a submanifold. So, it suffices to show

$$\int_{M\times M} (\xi \otimes \eta) \smile \varepsilon_\Delta \;=\; \sum_i \int_{M\times M} (-1)^{d_i} (\xi \otimes \eta) \smile (a_i \times b_i) \tag{22.6}$$

for all $\xi \in H^p(M), \eta \in H^{n-p}(M)$. Note that this representation of elements in the homology of the product space is due to the Künneth Theorem. We first expand the lefthand side. We have

$$\begin{aligned} \int_{M\times M} (\xi \otimes \eta) \smile \varepsilon_\Delta \;&=\; \int_\Delta (\xi \otimes \eta)|_\Delta \\ &=\; \int_M \xi \smile \eta \end{aligned} \tag{22.7}$$

where the final equality requires some explanation. Here, we use that $\nu_{\Delta/M\times M} \cong TM$ under the obvious identification. Then, we have that $M \hookrightarrow \Delta$ is an isomorphism. This is a general fact and is in fact why we have a co-orientation on Δ_M.

Now, we examine the righthand side of (22.6). We have

$$\begin{aligned} (-1)^{d_i} \int_{M\times M} (\xi \otimes \eta) \smile (a_i \times b_i) \;&=\; (-1)^{d_i} \int_{M\times M} (\pi^*(\overline{\xi})) \smile \pi^*(\overline{\eta}) \smile \pi^*(\overline{a_i}) \smile \pi^*(\overline{b_i}) \\ &=\; (-1)^{d_i} (-1)^{d_i(n-p)} \int_{M\times M} \pi^*(\overline{\xi}) \smile \pi^*(\overline{a_i}) \smile \pi^*(\overline{\eta}) \smile \pi^*(\overline{b_i}) \\ &=\; (-1)^{d_i} \left(\int_M \xi \smile a_i \right) (-1)^{d_i(n-p)} \left(\int_M \eta \smile b_i \right) \end{aligned} \tag{22.8}$$

where the first equality holds by definition and by examining the Künneth identification, the second holds by graded commutativity, and the third holds by realizing that the integral operator evaluates a cochain on the fundamental class of $M \times M$, which is the cross product of the fundamental classes, so the integral splits naturally. To elaborate on the final line, we showed the cross product of orientation classes is the orientation class of the product for Thom classes, so by Duality, it holds for fundamental classes in homology.

We also note that the above identity can only be nonzero if $\deg(\xi) = p = n - d_i$. So, the overall sign is $d_i + d_i^2 = +1$. Finally, since the a_i form a basis for $H^{d_i}(M)$ and our pairing is non-degenerate, it suffices to check this equality for a dual basis, where $\xi = a_i, \eta = a_i$. Thus, (22.8) reduces to

$$\int_M \xi \smile a_i \int_M \eta \smile b_i \ = \ \delta_{ii} \int_M \xi \smile \eta = \int_M \xi \smile \eta \tag{22.9}$$

□

The previous lemma tells us how to compute intersections with the diagonal Δ in $M \times M$. The first Lemma in this lecture, Lemma 22.1 gives that computation real geometric power. Using the fact that $\nu_{\Delta/M\times M} \cong TM$, along with Lemma 19.5, we see that ε_Δ restricts to e_{TM}.

Corollary 22.3 (Gauss-Bonnet Theorem)**.** If M is a smooth, oriented, and closed manifold, then

$$\int_M e_{TM} \ = \ \chi(M) \tag{22.10}$$

where $\xi(M)$ is the Euler characteristic of M. Hence, if $\xi(M) \neq 0$, M has no nowhere vanishing vector field.

Proof. From the discussion above, we see

$$\begin{aligned} \int_M e_{TM} \ &= \ \int_M \varepsilon_\Delta \\ &= \ \sum_i (-1)^{d_i} \int_M a_i \times b_i \\ &= \ \sum_i (-1)^{d_i} \int_M a_i \smile b_i \\ &= \ \sum_i (-1)^i \mathrm{rk}(H^i(M)) \end{aligned} \tag{22.11}$$

where the third equality holds since we have restricted to M. □

We now put the above Theorem in a more general context. Observe that flowing a vector field yields a diffeomorphism (invertible because we can flow backwards) whose fixed points are zeros of the vector field. So, Gauss Bonnet talks about fixed points which are isotopic to the identity. The following Theorem can be seen as a generalization of Gauss-Bonnet.

Theorem 22.4 (Lefschetz fixed point Theorem)**.** Let M be a closed oriented manifold. Let $f : M \to M$ be a smooth map with non-degenerate fixed points, meaning that Δ_M and Γ_f, the graph of f, meet transversely in $M \times M$. Then,

$$\sum_{x \in \mathrm{fix}(f)} \mathrm{sign}(x) \ = \ \sum_k (-1)^k \mathrm{Tr}(f^* : H^k(M) \to H^k(M)) \tag{22.12}$$

where the quantity on the righthand side of (22.12) is sometimes called the *supertrace* of the map f.

Remark 22.5. A few observations:

(1) When f is homotopic to the identity, the righthand side picks out the Euler characteristic of M.
(2) If Y, Z are co-oriented closed submanifolds of M (which is oriented), and they have dimensions $\dim(Y) + \dim(Z) = M$, then $Y \pitchfork Z$ implies that $Y \cap Z$ is a finite set.

    To see this, we let for $x \in Y \cap Z$, $\mathrm{sign}(x) = +1$ if $T_xM \cong (\nu_{Y/M})_x \oplus (\nu_{Z/M})_x$ is orientation preserving and -1 if it is orientation reversing. This is a remark, not necessarily a consequence of the Theorem.
(3) By approximating smooth functions with continuous ones, we see that $L(f) \neq 0$ implies that any continuous map homotopic to f has fixed points.

Proof. Recall that the cup product counts signed intersections, so we may compute the lefthand side of (22.12) via

$$\int_{M\times M} \varepsilon_\Delta \smile \varepsilon_{\Gamma_f} \;=\; \int_{\Gamma_f} \varepsilon_\Delta|\Gamma_f \tag{22.13}$$

by the characterization of the orientation class as a dirac delta.

For the other side, define $F : M \to M \times M$ as (id, f). Then, we have

$$\int_{\Gamma_f} \varepsilon_\Delta|_{\Gamma_f} \;=\; \int_M F^*(\varepsilon_\Delta) \tag{22.14}$$

where

$$\begin{aligned} F^*\varepsilon_\Delta \;&=\; (\mathrm{id}\times f)^*\left(\sum_i (-1)^{d_i}(a_i \times b_i)\right) \\ &=\; \sum_i (-1)^{d_i} a_i \times f^*(b_i). \end{aligned} \tag{22.15}$$

But, with respect to the basis $\{b_j\}$ of $H^*(M)$, if $f^*b_i \;=\; \sum_j \rho_{ij}b_j$, then this quantity is just given by the sums of the $\rho_{ii} \;=\; \int_M a_i \smile f^*(b_i)$ which just gives the supertrace. $\square$

Example 22.16. No finite group acts freely on $\mathbf{CP}^{2k}$, indeed, every map $f : \mathbf{CP}^{2k} \to \mathbf{CP}^{2k}$ has fixed points.

Remark 22.6. This is false for $\mathbf{CP}^k$ when k is odd. For $S^2 \cong \mathbf{CP}^1$, reflection has fixed points, but is homotopic to the antipodal map with none.

Proof. We have $f^* : H^*(\mathbf{CP}^{2k}) \to H^*(\mathbf{CP}^{2k})$, where $H^*(\mathbf{CP}^{2k}) = \mathbf{Z}[x]/(x^{2k+1}), |x| = 2$.

Suppose $f^*(x) = lx$ for some $l \in \mathbf{Z}$. Then, $f^*(x^j) = l^j x^j$ for each j and hence $L(f) = 1+l+\cdots+l^{2k} \neq 0$ breaks this. In the case of $\mathbf{CP}^1$, using $l = 1$ gives a root to this polynomial. $\square$

Example 22.17. Let Σ be a closed Riemann surface, $g \geq 2$. If $f : \Sigma \to \Sigma$ is a homolomrphic automorphism and factors trivially on cohomology, then $f = \mathrm{id}_\Sigma$.

Proof sketch. The key idea here is that the identity Theorem from complex analysis (extended to Riemann surfaces) says that if two complex manifolds of the same dimension intersect but not transversely, then they must coincide.

So, we consider $\Delta_\Sigma, \Delta_f \subset \Sigma \times \Sigma$. If f factors trivially on cohomology, its supertrace is $\xi(\Sigma) = 2 - 2g < 0$, by our remarks following Theorem 22.4. In complex geometry, all fixed points have positive signed intersection. So, the assumption of transverse intersection must be false and f must be the identity. $\square$

23. Lecture 23 – Complex Vector Bundles and Chern Classes

We arrived at the Euler class by looking at Sphere bundles. Spheres are simple homologically. The next best thing is $\mathbf{CP}^n$.

Let $E \to X$ be a rank n complex vector bundle, so that the fibres are $\mathbf{C}^n$. There's an associated fibre bundle $\mathbf{P}(E) \to X$ with fibre $\mathbf{CP}^{n-1}$. This itself has a tautological line bundle $\mathcal{L}_{\text{taut}} \to \mathbf{P}(E)$ with the obvious definition.

Remark 23.1. $\mathbf{P}(E) = \mathbf{P}(E \otimes V)$ for any complex line bundle $V \to X$, but $\mathcal{L}_{\text{taut}}$ is **not** naturally the same. It depends on E.

We also note that via projection, $\pi : \mathbf{P}(E) \to X$, $H^*(\mathbf{P}(E))$ becomes an $H^*(X)$ module.

Lemma 23.2. This is the free module with basis $1, t, \cdots, t^{n-1}$, with $t = e_{\mathcal{L}_{\text{taut}}} \in H^2(\mathbf{P}(E))$ and $n = \text{rk}(E)$.

Proof. We will assume, for simplicity, that E admits a finite trivializing cover.

Define

$$\begin{aligned} H^*(X) \oplus \cdots \oplus H^*(X) &\to H^*(\mathbf{P}(E)) \\ (\alpha_1, \cdots, \alpha_n) &\to \sum_i \pi^*(\alpha_i) \smile t^i \end{aligned} \tag{23.1}$$

where t is the nontrivial element of $H^2(\mathbf{CP}^{n-1})$. Note that when E is trivial this is the map given in the Künneth Theorem. When E does not admit a one set trivializing cover, we argue by induction, imitating the proof of the Thom Isomorphism Theorem, using Mayer-Vietoris and the Five Lemma.

We briefly justify why t is in fact the Euler clsas of $H^*(\mathbf{P}(E))$. Imagine a diagram such as

$$\begin{tikzcd} H^*(X) \otimes H^*(\mathbf{CP}^{n-1}, \mathbf{CP}^{n-1} - 0) \arrow[r, "\times"] \arrow[d, "(\text{Id,LES})"] & H^*(\mathbf{P}(E), \mathbf{P}(E)^\sharp) \arrow[d, "(\text{Id,LES})"] \\ H^*(X) \otimes H^*(\mathbf{CP}^{n-1}) \arrow[r, "\times"] & H^*(\mathbf{P}(E)) \end{tikzcd} \tag{23.2}$$

which commutes by naturality. It shows that our t is the image of the Thom class from the top row. $\square$

Corollary 23.3. We have $H^*(\mathbf{P}(E)) \cong H^*[t]/(p(t))$ for a unique monic polynomial $p(t)$, where

$$p(t) \;=\; t^n - c_1(E) \smile t^{n-1} + c_2(E) \smile t^{n-2} + \cdots + (-1)^n c_n(E) \tag{23.3}$$

where the coefficients $c_i(E) \in H^{2i}(X)$.

Definition 23.1. The coefficient $c_i(E)$ are called the Chern classes of E.

We will now show that these classes are characteristic, in the sense that they commute with pullback operations.

Remark 23.4.
(1) Given $E \to X$, $f : Y \to X$, with f continuous, we have pullback pundles $f^*(E) \to Y$ and hence $\mathbf{P}(f^*E) \to Y$.
Suppose I know that for $t_x = e_{\mathcal{L}_{\mathbf{P}(E)}}$ has Chern polynomial $t_n - c_1(E) \smile t^{n-1} + \cdots + (-1)^{n-1} c_{n-1}(E) \smile t + (-1)^n c_n(E) = 0 \in H^*(\mathbf{P}E)$. Then, it pulls back to zero in $H^*(\mathbf{P}(f^*E))$. However,
$$f^*t \;=\; f^* e_{\mathcal{L}_{\text{taut}}} \;=\; e_{f^*\mathcal{L}_{\text{taut}}} \;=\; t_y \tag{23.4}$$
which implies $(t_y)^n - f^*c_1(E) \smile t_y^{n-1} + \cdots + (-1)^n f^* c_n(E) = 0 \in H^*(\mathbf{P}(f^*E))$. So, the Chern classes are characteristic classes of complex vector bundles.
(2) If $E \to X$ is a rank one complex line bundle, then $\mathbf{P}(E) \equiv X$ and $c_1(E) \in H^2(X)$ is just $e_E \in H^2(X)$, viewing E as a real rank two bundle.

The key to understanding properties of Chern classes is to reduce to the case of sums of line bundles.

Lemma 23.5. If $E = L_1 \oplus \cdots \oplus L_n \to X$ is a Whitney sum of complex lines bundles, then $c_i(E) = \sigma_i(e_{L_1}, \cdots, e_{L_n})$, where e_{L_i} is the Euler class of L_i and σ_i is the i^{th} elementary symmetric function. Thus,

$$H^*(\mathbf{P}(E)) \cong H^*(X)[t] / \left(\prod_{i=1}^{n} (t - e_{L_i}) \right) \tag{23.5}$$

where $t = e_{\mathcal{L}_{\text{taut}}} \in H^2(\mathbf{P}(E))$.

Proof. Note the $\mathbf{P}(E) \to X$ has n sections

$$\begin{aligned} s_i : X &\to \mathbf{P}(E) \\ X &\to (L_i)_X \end{aligned} \tag{23.6}$$

and moreover $s_i^* \mathcal{L}_{\text{taut}} = L_i$ by construction. At any $x \in X$, the section $\{s_j(x) \mid j \neq i\}$ span a $\mathbf{P}^{n-2} \subset \mathbf{P}^{n-1} = \mathbf{P}(E_x)$. The complement, $U_{i,x}$ is then an affine space, in fact isomorphic to $\bigoplus_{j \neq i} (L_j)_x$. This is a slightly difficult claim at first, however, we can imagine a local trivialization of the bundle, or just a trivial bundle to convince ourselves of it.

If $U_i = \bigcup_{x \in X} U_{i,x}$, then $U_i \xrightarrow{\pi_i} X$ is a rank $n-1$ complex vector bundle such that s_i is the zero section. Observe that

$$t|_{U_i} = \pi_i^* e_{L_i} \tag{23.7}$$

and indeed since $\pi_i : U_i \to X$ is a homotopy equivalence via the map s_i, it suffices to check

$$\begin{aligned} s^*(t|_{U_i}) &= s_i^*(\pi_i^* e_{L_i}) \\ &= e_{L_i} \\ &= e_{s_i^* \mathcal{L}_{\text{taut}}|U_i} \end{aligned} \tag{23.8}$$

where the final equality holds by our construction and an earlier remark.

Since t and $\pi^* e_{L_i}$ (for $\pi : \mathbf{P}(E) \to X$) agree on U_i, $t - \pi^* e_{L_i}$ can be represented by a cocyle vanishing on U_i. Since the U_i cover $\mathbf{P}(E)$, the product $\prod_i (t - \pi^* e_{L_i})$, vanishes everywhere. Note this polynomial is of degree n. So, it must determine the Chern classes. □

The following general principle is very useful in analyzing vector bundles and is, in particular, a constructive result, rendering the previous method useful.

Lemma 23.6 (Splitting Principle)**.** If X is paracompact and if $E \to X$ is a complex vector bundle, then there exists a space Y and a map $f : Y \to X$ such that
(a) $f^* E \to Y$ is a direct sum of line bundles
(b) $f^* : H^*(X) \to H^*(Y)$ is injective.

Proof. This construction echoes the construction of the Stiefel manifold, but in a complex setting this time.

There is a fibre bundle $\mathrm{Fl}_k \to E$, with fibre $x \in X$ as the space of length k flags in E_x, that is of ordered, orthonormal k-tuples of lines in E_x (defined with respect to a fixed choice of inner product). Then, there's a forgetful map

$$\mathrm{Fl}_k(E) \to \mathrm{Fl}_{k-1} E \tag{23.9}$$

forgetting the last entry in a k-tuple.

Let $V^{(k)} \to \mathrm{Fl}_{k-1}(E)$ be the complex vector bundle with fibre at $(\xi_1, \cdots, \xi_{k-1})$ equal to $(\xi_1 \oplus \cdots \oplus \xi_{k-1})^\perp$. Then, $\mathrm{Fl}_k(E) \cong \mathbf{P}(V^{(k)}) \to \mathrm{Fl}_{k-1}(E)$, so that flag manifolds can be seen as projective bundles over smaller flag manifolds.

But then our last lemma shows $H^*(\mathrm{Fl}_{k-1}(E)) \to H^*(\mathrm{Fl}_k(E))$ is injective and iterating, we get an injection $H^*(X) \hookrightarrow H^*(\mathrm{Fl}_k(E))$.

Now, set $Y = \mathrm{Fl}_n(E)$, with $n = \mathrm{rk}_{\mathbf{C}}(E)$. Then, if $L_i \to \mathrm{Fl}_n(E)$ has fibre at $(\xi_1, \cdots, \xi_n)$ given by ξ_i, then if $f : \mathrm{Fl}_n(E) \to X$ is a natural map, we have $f^*E = L_1 \oplus \cdots \oplus L_n$. ☐

Corollary 23.7. If E, F are complex vector bundles over X, then $c(E \oplus F) = c(E) \smile c(F)$, i.e.

$$c_k(E \oplus F) = \bigoplus_{i+j=k} c_i(E) \smile c_j(F) \tag{23.10}$$

By naturality, $c(f^*E) = f^*c(E)$, it suffices to prove this when E, F are directed sums of line bundles. Checking that this holds amounts to using Corollary 23.3 and making an elementary computation with symmetric functions.

Corollary 23.8. If $E \to X$ is a complex bundle of complex rank n, then $c_n(E) \in H^{2n}(X)$ is just the Euler class of $E_{\mathbf{R}}$, the underlying real bundle.

Proof. Take $\eta : Y \to X$ such that $\eta^*E = L_1 \oplus \cdots \oplus L_n$ a sum of bundles and so that η^* is injective on cohomology. Then, we have

$$\begin{aligned} \eta^* c_n(E) &= c_n(\eta^* E) \\ &= c_n(L_1 \oplus \cdots \oplus L_n) \\ &= e(L_1) \smile \cdots \smile e(L_n) \\ &= e(L_1 \oplus \cdots \oplus L_n) \\ &= e(\eta^* E) \\ &= \eta^* e(E) \end{aligned} \tag{23.11}$$

☐

where the fourth equality holds by properties of the Euler (and Thom) class.

In general, Euler class is difficult to compute in the real case and easier to compute in the complex case.

24. Lecture 24 – Cobordism

Much of what we have done can be motivated by the pursuit of classifying manifolds up to homeomorphism, or more generally, homotopy equivalence. However, classifying manifolds up to diffeomorphism is intractible/algorithmically impossible. But, there is an equivalent goemetric notion that is much more tractible.

Definition 24.1. Two closed, oriented, smooth n-manifolds M, N are *(oriented)-cobordant* if there exists a smooth compact $(n+1)$-manifold with boundary ω, also oriented, such that

$$d\omega \;=\; M \coprod (-N) \tag{24.1}$$

Remark 24.1. A manifold with boundary is locally homeomorphic to either $\mathbf{R}^n$ or $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_1 \geq 0\}$. An orientation of a manifold with boundary means an orientation of its interior. The boundary becomes canonically oriented.

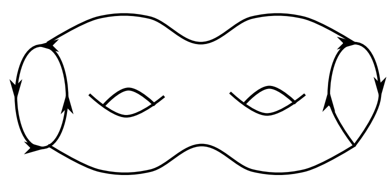


Figure 20. An oriented manifold with boundary

Oriented cobordism defines an equivalence relation on n-manifolds. Reflexivity is natural, as is symmetry (just reverse the orientation on the oriented $(n+1)$ manifold) Transitivity is more difficult. One usese a "Collar Neighborhood Theorem", which says that if ω is a manifold with boundary $\partial\omega$, compact, an open neighborhood of $\partial\omega$ is difeomorphic to $[0, \varepsilon) \times \partial\omega$ such that $\{0\} \times \partial\omega \to \partial\omega$. This allows composite cobordisms to inherit smooth structures.

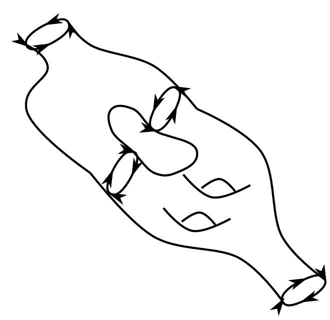


Figure 21. A composite cobordism

Let $\Omega_n = \{\text{oriented, smooth } n - \text{manifolds up to cobordism}\}$. Then, via $(M, N) \to M \coprod N$, Ω_n becomes a group, with the identity being the empty manifold. Via $(M^{n_1}, N^{n_2}) \to M^{n_1} \times N^{n_2}$, Ω_n becomes a graded ring.

Example 24.2. Ω_n is not very interesting for low values of n.

(1) $\Omega_1 = 0$, there is a boundary cobordism from the disk to S^1.
(2) $\Omega_2 = 0$. Surfaces are the boundaries of regions in $\mathbf{R}^3$.
(3) $\Omega_3 = 0$. This fact is harder, it relies on presenting 3 manifolds as "surgeries on knots".

Lemma 24.2. Let M^{2n} be a closed manifold and suppose $M = \partial\omega^{2n+1}$. Then, $\chi(M) = 2\chi(\omega)$.

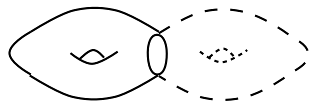


Figure 22. The "double" of a manifold.

Corollary 24.3. $\mathbf{CP}^2$ is not the boundary of any compact 5 manifold, so $\Omega_4 \neq 0$.

Proof of Lemma. Let $Z = \omega \bigcup_{\partial\omega} \omega$ be the double of ω, as illustrated in the figure below. Then, Z is a closed $(2n+1)$ manifold (the collar neighborhood theorem resolves problems near the gluing). But, Z is odd dimensional and hence has $\chi(Z) = 0$ by Poincaré Duality. But, Mayer-Vietoris gives an exact sequence

$$\to H_{i+1}(Z) \to H_i(\omega) \to H_i(M) \oplus H_i(M) \to H_i(Z) \to \cdots \tag{24.3}$$

for which the alternating sum of ranks must be zero. However, $\chi(Z)$ is already zero, which establishes that $\chi(M) = 2\chi(W)$. □

More generally, $\mathbf{CP}^{2i_1} \times \cdots \times \mathbf{CP}^{2i_k}$ is not zero in Ω_n, where $n = 4(i_1 + \cdots + i_k)$. To proceed, we take two digressions. We first establish Poincaré Duality for manifolds with boundary.

Theorem 24.4. If $Y \subset X$ and both are closed smooth, manifolds, then $H^*_{\text{ct}}(X \setminus Y) \cong H^*(X, Y)$. (see examples sheet 4).

So, Poincaré Duality says that the follwoing diagram commutes.

$$\begin{array}{ccc} H^*(M, \partial M) & \xrightarrow[\cong]{D} & H_{n-*}(M) \\ & & \cong \big\downarrow \text{htpy} \\ H^*_{\text{ct}}(\mathring{M}) & \xrightarrow[\cong]{D} & H_{n-*}(\mathring{M}) \end{array} \tag{24.4}$$

Our construction of fundamental classes shows there exists a unique class $[M, \partial M] \in H_n(M, \partial M)$ such that $[M, \partial M]_y$ generates $H_n(M, M \setminus y)$ for all $y \in \mathring{M}$.

Lemma 24.5. (a) The natural map $H_n(M, \partial M) \to H_{n-1}(\partial M)$ sends $[M, \partial M] \to [\partial M]$
(b) Cap product defines an isomorphism $H^p(M) \to H_{n-p}(M, \partial M)$ given by D.

Proof. We begin with (a). For Let $x \in \partial M$ and choose U a closed neighborhood of X and $V = U \cap \partial M$ and let $y \in \mathring{U}$. The class $[M, \partial M]$ is characterized by restricting to a generator of $H_n(M, M \setminus y)$ for any y. The class $[\partial M]$ is characterized by restrcting to a generator of $H_{n-1}(\partial M, \partial M \setminus x)$. We now consider the triple $(M, M \setminus \mathring{U}, M \setminus U)$, noting that $H_*(M, M \setminus U) \cong M_*(M, M) = 0$.

We use the following lemma, which we will not prove.

Lemma 24.6. Whenever $A \subset B \subset X$ is a triple, we have a long exact sequence

$$\cdots \to H_i(B, A) \to H_i(X, A) \to H_i(X, B) \to H_{i-1}(B, A) \to \cdots \tag{24.5}$$

and so we see that $H_*(M, M \setminus \mathring{U}) \cong H_{*-1}(M \setminus \mathring{U}, M \setminus U)$ from the long exact sequence.

Now, we note

$$\begin{array}{ccc} H_*(M, M \setminus \mathring{U}) & \xrightarrow[\cong]{\text{htpy}} & H_{n-1}(M \setminus \mathring{U}, (M \setminus \mathring{U}) \setminus x) \\ & & \cong \big\downarrow \text{Exc} \\ H_n(M, M \setminus \mathring{U}) \cong H_n(M, M \setminus y) & \xrightarrow{\cong} & H_{n-1}(\partial M, \partial M \setminus x) \end{array} \tag{24.6}$$

and chase orientation classes throughout.

For part (b), we consider the diagram

$$\begin{tikzcd} H^{p-1}(\partial M) \arrow[r] \arrow[d, "D"] & H^p(M,\partial M) \arrow[r] \arrow[d, "D"] & H^p(M) \arrow[r] \arrow[d, "D", "(*)"'] & H^p(\partial M) \arrow[d, "D"] \\ H_{n-p+1}(\partial M) \arrow[r] & H_{n-p}(M) \arrow[r] & H_{n-p}(M,\partial M) \arrow[r] & H_{n-p-1}(\partial M) \end{tikzcd} \tag{24.7}$$

while part (a) shows that the diagram commutes and the five lemma shows that (*) is an isomorphism. $\square$

Our second digression is that we have Chern classes as characteristic classes of smooth manifolds. We want invariants of real bundles such as the tangent bundle.

Definition 24.2. If $F \to X$ is a real vector bundle, then the i^{th} *Pontragyin class* of F, denoted $p_i(F) \in H^{4i}(X, \mathbf{Z})$, is given by

$$p_i(F) \ = \ (-1)^i c_{2i}(F \otimes \mathbf{C}) \tag{24.8}$$

Remark 24.7. If $E \to X$ is a complex vector bundle and $\overline{E} \to X$ is the bundle with the opposite complex structure, then $c_i(\overline{E}) = (-1)^i c_i(E)$. To show this, check it for a line bundle and use the splitting principle. If F is a real vector bundle, the $F \otimes \mathbf{C} \cong \overline{F \otimes \mathbf{C}}$ and so odd classes of $F \otimes \mathbf{C}$ are 2-torsion.

Definition 24.3. If M^{4n} is a closed smooth manifold and $I = i_1 + \cdots + i_n$ is a partition, of n, we get a number

$$P_I(M) \ = \ \int_M p_{i_1}(TM) \smile \cdots \smile p_{i_n}(TM) \tag{24.9}$$

which depends on an orientation of M.

Proposition 24.8. If $M^{4n} = \partial \omega^{4n+1}$, then $P_I(M) = 0$ for all I.

Proof. Suppose $M^{4n} = \partial \omega^{4n+1}$. We consider

$$\begin{tikzcd} H^{4n}(M) \arrow[r, "\partial^*"] \arrow[d, "D"] & H^{4n+1}(\omega, \partial\omega = M) \arrow[d, "D"] \\ H_0(M) \arrow[r, "\cong"] & H_0(\omega) \end{tikzcd} \tag{24.10}$$

Then, $\int_M \phi = \langle \phi, [M] \rangle = \langle \partial^* \phi, [\omega, \partial \omega] \rangle = \int_\omega \partial^* \phi$ where the central equality holds by our previous lemma.

But, $T\omega | \partial \omega = T\partial\omega \oplus \mathbf{R} = TM \oplus R$ and $\mathbf{R}$ is a trivial line bundle.

Since $c(E \oplus \mathbf{C}) = c(E)$, one checks $P_I(T\omega|_M) = p(TM)$. But,

$$\begin{aligned} \int_M p_{i_1}(TM) \smile \cdots \smile p_{i_r}(TM) \ &= \ \int_\omega \partial^*(p_{i_1}(TM) \smile \cdots \smile p_{i_r}(TM)) \\ &= \ \int_\omega \partial^*(\iota^* p_{i_1}(T\omega) \smile \cdots \smile i^* p_{i_r}(T\omega)) \end{aligned} \tag{24.11}$$

But, ∂^*, ι^* are consecutive terms in a long exact sequence. $\square$

Theorem 24.9 (Thom)**.** The converse holds, working over $\mathbf{Q}$. The p_I defined homomorphisms $p_I : \Omega_{4n} \to \mathbf{Z}$ which are linearly independent and rationally classify manifolds up to cobordism. Thus,

$$\Omega_* \otimes \mathbf{Q} \ = \ \begin{cases} \mathbf{Q}^{\pi(n)} & \pi(n) = \text{no of partitions of } 4n, \text{ if } * = 4n \\ 0 & \text{o/w} \end{cases} \tag{24.12}$$

This Theorem was enormously unexpected. It was a huge success for the study of vector bundles.