Sequential Spectra 1 Talk Outline

Read [Mal23, Sections 2.1, 2.2, 2.3], but skip Section 2.1.4 on Thom spectra.

Outline

(1) Motivations

  (a) Remind us about two big theorems that we have seen so far: the Brown representability theorem and the Freudenthal Suspension Theorem.
  (b) The Freudenthal suspension theorem suggests that it is interesting and fruitful to study the sequence of spaces $X, \Sigma X, \Sigma^2 X, \ldots$. We can use these to talk about stable homotopy groups, which can be easier to study than (unstable) homotopy groups.
  (c) The Brown representability theorem suggests that it's interesting to study infinite loopspaces: these are spaces X with infinitely many deloopings $X \simeq \Omega X_1 \simeq \Omega^2 X_2 \simeq \ldots$ These spaces define generalized cohomology theories, which are very useful tools in algebraic topology.
  (d) There is an adjunction between the two functors Σ and Ω here so these concepts should be related. This motivates the definition of spectra.

(2) Definitions

  (a) Define spectra and maps of spectra [Mal23, Definition 2.1.1]. The category of spectra and the maps described above is denoted $\mathcal{S}p$.
  (b) Note that the bonding maps $\sigma_i \colon \Sigma X_i \to X_{i+1}$ have adjoints $\widetilde{\sigma_i} \colon X_i \to \Omega X_{i+1}$, and we could equivalently define a spectrum using the adjoint bonding maps.
  (c) Define the homotopy groups of a spectrum [Mal23, Definition 2.1.2]. Make sure to note that we have negative homotopy groups now: a spectrum X has negative homotopy groups $\pi_{-1}X$, $\pi_{-2}X, \ldots$
  (d) Define a stable equivalences [Mal23, Definition 2.1.10] of spectra. Stable equivalences satisfy the 2-out-of-6 property, (cf. [Rie14, Digression 2.1.5]), so we can invert them. The category $\mathrm{ho}(\mathcal{S}p)$ that we get by inverting the stable equivalences is called the *stable homotopy category*.
  (e) Give a couple examples of spectra:
    - [Mal23, Example 2.1.7] The suspension spectrum of a pointed space X is the sequence of spaces $X, \Sigma X, \Sigma^2 X, \ldots$ The bonding maps are identities $\mathrm{id} \colon \Sigma(\Sigma^i X) \to \Sigma^{i+1} X$. The suspension spectrum of a space is denoted $\Sigma^\infty X$, or $\Sigma^\infty_+ Y := \Sigma^\infty(Y_+)$ for an unpointed space Y (first add a disjoint basepoint).
      The homotopy groups of a suspension spectrum $\Sigma^\infty X$ are the stable homotopy groups of the space X.
    - [Mal23, Example 2.1.6] A particularly important example of a suspension spectrum is the *sphere spectrum* $\mathbb{S} := \Sigma^\infty S^0$. This is the sequence of spheres $S^0, S^1, S^2, \ldots$.
      The homotopy groups of the sphere spectrum are called the *stable stems*. They are the stable homotopy groups of S^0, often written $\pi_i^s := \pi_i \mathbb{S}$.
    - [Mal23, Example 2.1.12] The *zero spectrum* $\Sigma^\infty(*)$ is the suspension spectrum of a point. All of its homotopy groups are zero. Depending on the author, this is alternatively denoted as $*$ or 0. Explain the existence zero maps between spectra.

- [Mal23, Example 2.2.2] We have already seen the example of Eilenberg–MacLane spaces as infinite loopspaces. We can use the maps adjoint to the equivalences $K(A, n) \simeq \Omega K(A, n+1)$ to define an *Eilenberg–MacLane spectra* HA, which are given by the sequence of spaces $K(A, 0)$, $K(A, 1), K(A, 2), \ldots$ The functor $H\colon \mathcal{A}b \to \mathcal{S}p$ takes an abelian group to the Eilenberg–MacLane spectrum HA. We use the letter H because the infinite loopspace represents cohomology with coefficients in A.
  The homotopy groups of HA are given by
  $$\pi_i HA = \begin{cases} A & (i = 0) \\ 0 & (i \neq 0) \end{cases}$$
- [Mal23, Example 2.2.3] Another example is the complex K-theory spectrum, usually written KU[1]. This is given by the sequence of spaces $\mathbb{Z} \times BU$, U, $\mathbb{Z} \times BU$, U, ... and the structure maps adjoint to the equivalences $\Omega U \simeq \mathbb{Z} \times BU$ and $\Omega(\mathbb{Z} \times BU) \simeq U$.
  The homotopy groups of KU are given by
  $$\pi_i KU = \begin{cases} \mathbb{Z} & (i \text{ is even}) \\ 0 & (i \text{ is odd}) \end{cases}$$

(3) Ω-spectra

(a) Define Ω-spectra [Mal23, Definition 2.2.1].

(b) Explain which of the examples above are Ω-spectra.

(c) State the proposition that every spectrum is stably equivalent to an Ω-spectrum [Mal23, Proposition 2.2.9]. In our language, the functor R that takes a spectrum X and replaces it by a stably equivalent Ω-spectrum RX is a right deformation of the category $\mathcal{S}p$ (dual to [Rie14, Definition 2.2.1]).

(d) Give an indication of how to prove this proposition. You don't need to go into all of the gory details, but you should construct the spectrum RX.

(e) Define the functor $\Omega^\infty\colon \mathcal{S}p \to \mathcal{T}op_*$ [Mal23, Definition 2.2.11].

(f) State the proposition that Ω^∞ is right adjoint to Σ^∞. You should not prove this proposition; we will do so on the homework. Explain that $\Omega^\infty\Sigma^\infty X = QX$.

(4) Operations on Spectra

(a) Define the coproduct $X \vee Y$ and product $X \times Y$ of spectra X and Y [Mal23, Definition 2.3.1]. Give examples to show how these notions interact with suspension spectra and Eilenberg–MacLane spectra [Mal23, Example 2.3.2].

(b) Define the shift operator sh_d and demonstrate how it interacts with homotopy groups [Mal23, Definition 2.3.4].

(c) Define the smash product $K \wedge X$ of a space K with a spectrum X [Mal23, Definition 2.3.6]. Make special note that the suspensions go on the left here, whereas the suspensions for the bonding maps in a spectrum go on the right. Give the examples of $\Sigma X := S^1 \wedge X$ and $\Sigma^\infty K \cong K \wedge S$ (note that this is an isomorphism of spectra, not an equality!).

(d) Define the function spectrum $F(K, X)$ of a space K with a spectrum X [Mal23, Definition 2.3.8]. As a special case, define the loops of a spectrum ΩX.

[1]The U is for "unitary," to distinguish this from KO, where O is for "orthogonal". KO is like KU, but for real vector bundles instead of complex vector bundles.

(e) State the fact that $\Sigma \dashv \Omega$ as functors $\mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}$. You need not prove this fact; it's an exercise in the book [Mal23, Exercise 2.15].

(f) Define the homotopy pullback [Mal23, Before Example 2.3.22] and homotopy pushout [Mal23, Definition 2.3.17] of spectra. Don't bother to define the general homotopy limits and colimits of spectra, nor general limits and colimits of spectra (only if you have time).

(g) As special cases of homotopy pullbacks and pushouts, define homotopy fibers and homotopy cofibers of maps of spectra.

(h) As a preview for next week, we will show that homotopy pushouts and homotopy pullbacks of spectra agree, and therefore fiber sequences and cofiber sequences of spectra are the same. This is the phenomenon of *stability*.

(i) Don't talk about the other constructions in [Mal23, Section 3]. We will talk about them when we need them, but we don't need them right now.

References

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. `http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf`, October 2023.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.