K_0 and Wall's Finiteness Obstruction

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Let X be a (compactly generated, weakly Hausdorff) topological space. To understand this space for the purposes of homotopy theory, we want to see it combinatorially as a CW complex. Fortunately, every space has a CW approximation.

Theorem 0.1 (CW Approximation, [Hat02, Proposition 4.13])**.** *For any space X, there is a CW complex Z and a weak homotopy equivalence $Z \xrightarrow{\sim} X$.*

Moreover, the CW complex Z is unique up to homotopy equivalence and can be chosen functorially in X. For example, we might take $Z = |\mathrm{Sing}(X)|$ to be the geometric realization of the singular simplicial set for X. This approach, however, leaves some things to be desired:

(1) What if we want X to be not just weakly homotopy equivalent to a CW complex, but actually homotopy equivalent to a CW complex?

(2) The space $|\mathrm{Sing}(Z)|$ is usually quite large. Can we control the size of the CW approximation? When can we approximate X by a finite CW complex? When is X homotopy equivalent to a finite CW complex?

For the first question, consider the following.

Definition 0.2. We say that a space X is **dominated** by a space Y if X is a retract of Y, i.e. there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to the identity on X.

The following corollary to CW approximation gives an approach the first of these questions.

Corollary 0.3 ([Ros05, Paragraph before Theorem 1])**.** *If X is dominated by a CW complex, then X is homotopy equivalent to a CW complex.*

This corollary suggests that we should begin by considering spaces that are dominated by a finite CW complex.

Definition 0.4. We say that a space X is **finitely dominated** if it is dominated by a finite CW complex.

Example 0.5 ([Lur14, Lecture 2, Exercise 2])**.** Finitely dominated spaces aren't so hard to produce. If a space X is a compact (topological) manifold dominated by a CW complex Y via $f\colon X \to Y$, then the image of f is contained in a finite subcomplex of Y. In this case, X is finitely dominated.

Moreover, finitely dominated spaces have a few nice properties:

Proposition 0.6 ([Lur14, Lecture 2, Lemma 6]). *Let X be a finitely dominated space. Then $\pi_0(X)$ is finite and $\pi_1(X)$ is finitely presented.*

Proof. If X is dominated by a finite CW-complex Y, then $\pi_0(Y)$ is finite and $\pi_1(Y)$ is finitely presented. The same is true of $\pi_0(X)$ and $\pi_1(X)$, since these are retracts of $\pi_0(Y)$ and $\pi_1(Y)$ respectively. □

In light of this new definition, let's make the second question a little more precise:

Question 0.7. When is a finitely dominated space X homotopy equivalent to a finite CW complex?

To answer this question, we're going to take a detour into K-theory. We will see that we can quantify the answer to this question with an element of reduced $K_0(\mathbb{Z}[\pi_1 X])$ related to the Euler characteristic.

1 The Grothendieck Group K_0

To introduce the flavor of algebraic K-theory, we're going to introduce K_0 as a functor from exact categories to abelian groups. There is another way to define K_0 of rings using the group completion of monoids, but higher algebraic K-theory is best approached from this categorical perspective. Roughly speaking, an exact category is an additive category with a class of short exact sequences.

Definition 1.1 ([Qui73, §2]). An **exact category** $(\mathcal{C}, \mathcal{E})$ is a pair of an additive category $\mathcal{C}$ and a class $\mathcal{E}$ of "short exact sequences" in $\mathcal{C}$ of the form

$$A \hookrightarrow B \twoheadrightarrow C$$

If $A \hookrightarrow B$ occurs as the first morphism of a sequence in $\mathcal{E}$, we call it an **admissible monomorphism**. If $B \twoheadrightarrow C$ occurs as the second morphism in a sequence in $\mathcal{E}$, we call it an **admissible epimorphism**. These data must satisfy the following axioms:

(E1) $\mathcal{E}$ is closed under isomorphisms and contains the "split short exact sequences:"

$$A \hookrightarrow A \oplus C \twoheadrightarrow C.$$

(E2) Admissible monomorphisms are closed under pushout and composition. Admissible epimorphisms are closed under pullback and composition.

(E3) Admissible monomorphisms are kernels of the corresponding admissible epimorphisms, and dually.

(E4) If an admissible epimorphism $A_0 \twoheadrightarrow A_2$ factors as $A_0 \to A_1 \to A_2$ and $A_1 \to A_2$ has a kernel, then $A_1 \to A_2$ is an admissible epimorphism. Dually, if an admissible monomorphism $C_0 \twoheadrightarrow C_2$ factors as $C_0 \to C_1 \to C_2$ and $C_0 \to C_1$ has a cokernel, then $C_0 \to C_1$ is an admissible monomorphism.

We will often abuse notation and write $\mathcal{C}$ instead of $(\mathcal{C}, \mathcal{E})$, the class of exact sequences being understood.

Definition 1.2. An **exact functor** $F\colon \mathcal{C} \to \mathcal{D}$ between exact categories is an additive functor $F\colon \mathcal{C} \to \mathcal{D}$ that carries short exact sequences in $\mathcal{C}$ to short exact sequences in $\mathcal{D}$.

Example 1.3. Any abelian category $\mathcal{A}$ becomes an exact category with $\mathcal{E}$ all exact sequences. Alternatively, we could take $\mathcal{E}$ only the split short exact sequences.

Definition 1.4. We say that a full additive subcategory $\mathcal{C}$ of an abelian category $\mathcal{C}$ is **closed under extensions** if for each short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$ with $A, C \in \mathrm{Ob}(\mathcal{C})$, then $B \in \mathrm{Ob}(\mathcal{C})$ as well.

Example 1.5 ([Wei13, II.7.0])**.** Any additive subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ that is closed under extensions is an exact category where $\mathcal{E}$ is the class of all sequences in $\mathcal{C}$ which are exact in $\mathcal{A}$. In fact, every exact category arises from an abelian category in this way.

Having defined exact categories, we can use these to define K_0.

Definition 1.6. Let $(\mathcal{C}, \mathcal{E})$ be an exact category. Then $K_0(\mathcal{C})$ is the abelian group with generators $[C]$, one for each $C \in \mathrm{Ob}(\mathcal{C})$ and relations $[B] = [A] + [C]$ for each short exact sequence:

$$A \hookrightarrow B \twoheadrightarrow C.$$

Example 1.7. Let R be a unital associative ring. The category $\mathbf{Mod}^{\mathrm{fg}}(R)$ of finitely generated R-modules is exact as a full subcategory of the abelian category $\mathbf{Mod}(R)$ closed under extensions. We define $G_0(R) := K_0(\mathbf{Mod}^{\mathrm{fg}}(R))$.

Let's prove that $G_0(\mathbb{Z}) \cong \mathbb{Z}$, following [Wei13, II.6.2.1]. We can see this because the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

shows that $[\mathbb{Z}/n\mathbb{Z}] = 0$, and then the fundamental theorem of finitely generated abelian groups shows that the class of any abelian group only depends on its torsion free part. Hence $G_0(\mathbb{Z})$ is generated by $\mathbb{Z}$. We can show that $[\mathbb{Z}]$ is not a torsion element in $G_0(\mathbb{Z})$ using the rank homomorphism

$$r\colon G_0(\mathbb{Z}) \to^{\cong} \mathbb{Z}$$

defined by $r([A]) = \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q})$. The rank homomorphism r in clearly surjective and sends the generator $[\mathbb{Z}]$ to 1, so is an isomorphism.

Example 1.8. Let R be a unital associative ring. The category $\mathbf{Proj}^{\mathrm{fg}}(R)$ of finitely generated projective R-modules is exact as a full subcategory of $\mathbf{Mod}(R)$ that is closed under extensions. We define $K_0(R) := K_0(\mathbf{Proj}^{\mathrm{fg}}(R))$. This is what you usually think of as $K_0(R)$. The sequence

$$0 \to 0 \to 0$$

shows that $[0]$ is the unit in K_0 and the sequence

$$0 \to A \xrightarrow{\cong} A'$$

shows that $[A] = [A']$ when A and A' are isomorphic. Finally, all short exact sequences of projective modules split, so every relation is of the form $[A \oplus B] = [A] + [B]$.

We can describe the group $K_0(R)$ in many cases:

(a) If R is a field, then any R-module is projective and any two R-module of the same dimension are isomorphic. Hence, $K_0(R) \cong \mathbb{Z}$.

(b) More generally, if R is a PID, then every finitely generated projective R-module is free and hence isomorphic to R^n for some n. Hence, $K_0(R) \cong \mathbb{Z}$.

(c) If A is a Dedekind domain, then $K_0(A) \cong \mathbb{Z} \oplus \mathrm{Cl}(A)$, where $\mathrm{Cl}(A)$ is its class group [Ros94, Theorem 1.4.12].

Remark 1.9 (Eilenberg swindle)**.** What if we don't restrict to finitely generated R-modules? Let R^∞ be a free R-module on a countably infinite basis. Then $R \oplus R^\infty \cong R^\infty$. For any countably generated projective R-module P, write P as a direct summand of a free module R^n by $P \oplus Q = R^n$, possibly with $n = \infty$. Then

$$P \oplus R^\infty \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \ldots \cong R^n \oplus R^n \oplus \ldots \cong R^\infty.$$

Therefore, $[P] = 0 \in K_0(\mathbf{Proj}(R))$. Hence, $K_0(\mathbf{Proj}(R)) = 0$.

Example 1.10 ([Wei13, II.7.1.2])**.** If X is a topological space, then the category $\mathbf{Vect}_{\mathbb{C}}(X)$ of finite-dimensional complex vector bundles over X is an exact category, and we may define

$$K^0(X) := K_0(\mathbf{Vect}_{\mathbb{C}}(X)).$$

This is the **topological K-theory of** X. The Serre–Swan theorem asserts that $K^0(X) \cong K_0(R)$, where R is the ring of continuous functions $X \to \mathbb{C}$.

If $F\colon \mathcal{C} \to \mathcal{D}$ is an exact functor between exact categories, let $K_0(F)\colon K_0(\mathcal{C}) \to K_0(\mathcal{D})$ be the homomorphism of abelian groups defined by $[C] \mapsto [F(C)]$. Because F sends exact sequences in $\mathcal{C}$ to exact sequences in $\mathcal{D}$, this is well-defined. With this construction, K_0 becomes a functor from the category of exact categories and functors to the category of abelian groups.

In particular, if $R \to S$ is a ring map, there is an exact functor from $\mathbf{Proj}^{\mathrm{fg}}(R) \to \mathbf{Proj}^{\mathrm{fg}}(S)$ by extension of scalars. This yields an abelian group homomorphism $K_0(R) \to K_0(S)$.

For any ring R, there is a ring homomorphism $\mathbb{Z} \to R$ sending 1 to the unit of R. This yields an abelian group homomorphism $\mathbb{Z} \cong K_0(\mathbb{Z}) \to K_0(R)$ whose image is the subgroup of $K_0(R)$ generated by the finitely generated free R-modules. In fact, the homomorphism may be described as $n \mapsto [R^n]$.

Definition 1.11. The **reduced K-theory** $\widetilde{K}_0(R)$ of R is the quotient of $K_0(R)$ by the image of the homomorphism $\mathbb{Z} \to K_0(R)$ induced from $\mathbb{Z} \to \mathbb{R}$.

Finally, the functor K_0 satisfies a universal property: it is the universal additive function on an exact category.

Definition 1.12. An **additive function** $f\colon \mathrm{Ob}(\mathcal{C}) \to \Gamma$ is a function from the objects of an exact category $\mathcal{C}$ to an abelian group Γ such that $f(B) = f(A) + f(C)$ for every short exact sequence $A \hookrightarrow B \twoheadrightarrow C$ in $\mathcal{C}$.

Theorem 1.13 (Universal property of K_0, [Wei13, II.6.1.2]). *Any additive function $f\colon \mathrm{Ob}(\mathcal{C}) \to \Gamma$ induces a unique group homomorphism $\overline{f}\colon K_0(\mathcal{C}) \to \Gamma$, with $\overline{f}([C]) = f(C)$ for all $X \in \mathrm{Ob}(\mathcal{C})$.*

$$\begin{tikzcd} \mathrm{Ob}(\mathcal{C}) \arrow[r, "f"] \arrow[dr] & \Gamma \\ & K_0(\mathcal{C}) \arrow[u, "\overline{f}"'] \end{tikzcd}$$

This universal property also shows that $K_0(\mathcal{C})$ is in some sense the universal receiver of generalized Euler characteristics.

Definition 1.14. Let $C_\bullet$ be a bounded chain complex of objects in an abelian category $\mathcal{A}$. The **Euler characteristic** of $C_\bullet$ is the element

$$\chi(C_\bullet) = \sum_i (-1)^i [C_i] \in K_0(A).$$

The **reduced Euler characteristic** is the composition of χ with the quotient homomorphism $K_0 \to \widetilde{K}_0$, and is denoted $\widetilde{\chi}$.

Proposition 1.15 ([Wei13, II.6.6]). *If $C_\bullet$ is a bounded complex in an abelian category $\mathcal{A}$, then its Euler characteristic depends only on its homology:*

$$\chi(C_\bullet) = \sum_i (-1)^i [H_i(C_\bullet)]$$

In fact, this shows that the Euler characteristic is well-defined for the complexes which are only **homologically bounded**, i.e. those with only finitely many nonzero homology groups. Moreover, the Euler characteristic defines a homomorphism $\chi\colon K_0(\mathbf{Ch}^b(R)) \to K_0(R)$ from K_0 of the category of bounded chain complexes of R-modules to $K_0(R)$. In particular, if

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a short exact sequence of complexes, then

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

Using the universal property of K_0, we can show that $\chi\colon K_0(\mathbf{Ch}^b(R)) \to K_0(R)$ is an isomorphism [Wei13, II.9.2.2].

2 Wall's Finiteness Obstruction

Recall that we were trying to answer question 0.7: when is a finitely dominated topological space homotopy equivalent to a finite CW complex? Recall that finitely dominated means that X is the retract of a finite CW complex Y.

Definition 2.1. Let X be a finitely dominated topological space with universal cover $\widetilde{X}$. Let $\pi_1(X)$ act on $\widetilde{X}$ by deck transformations, so that the chain complex $C_*(\widetilde{X})$ becomes a complex of $\mathbb{Z}[\pi_1 X]$-modules. **Wall's finiteness obstruction** is the reduced Euler characteristic of the complex $C_*(\widetilde{X})$:

$$w(X) := \widetilde{\chi}\left(C_*(\widetilde{X})\right) \in \widetilde{K}_0(\mathbb{Z}[\pi_1 X]).$$

This is well-defined because $\widetilde{\chi}(C_*(\widetilde{X}))$ depends only on the homology of $\widetilde{X}$. Wall proved [Ros05, Theorem 1] that when $\widetilde{X}$ is finitely dominated, its homology consists only of finitely generated projective $\mathbb{Z}[\pi_1 X]$-modules.

Theorem 2.2 (Wall, [Ros05, Theorem 1])**.** *Let X be a finitely dominated space. Then X is homotopy-equivalent to a finite CW-complex if and only if $w(X) \in \widetilde{K}_0(R)$ vanishes.*

The following theorem shows that the reduced Euler characteristic detects the difference between a chain complex of projective modules and a chain complex of free ones. This is the key K-theoretic component that Wall proved; the above is merely its translation into the context of the question at the beginning of the talk.

Theorem 2.3 (Wall, [Ros94, Theorem 1.7.12])**.** *Let $C_\bullet$ be a chain complex of finitely generated projective R-modules. Then $C_\bullet$ is homotopy equivalent to a chain complex of finitely generated free R-modules if and only if the image of $\chi(C_\bullet)$ in $\widetilde{K}_0(R)$ vanishes.*

Corollary 2.4. *Let X be a finitely dominated space which is simply connected. Then X has the homotopy type of a finite CW complex.*

Proof. In this case, $w(X)$ is an alternating sum of finitely generated projective $\mathbb{Z}$ modules. Every finitely generated projective $\mathbb{Z}$-module is free, so $w(X)$ lies the kernel of $K_0 \to \widetilde{K}_0$. Then apply Wall's theorem. □

More generally, the previous corollary is true when $\mathbb{Z}[\pi_1 X]$ is a PID.

References

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[Lur14] Jacob Lurie. Algebraic K-Theory and Manifold Topology (Math 281). `http://www.math.harvard.edu/~lurie/281.html`, 2014.

[Qui73] Daniel Quillen. Higher algebraic K-theory: I. In H. Bass, editor, *Higher K-Theories*, Lecture Notes in Mathematics, pages 85–147. Springer Berlin Heidelberg, 1973.

[Ros94] Jonathan Rosenberg. *Algebraic K-Theory and Its Applications*. Graduate Texts in Mathematics. Springer-Verlag, New York, 1994. https://www.springer.com/gp/book/9780387942483.

[Ros05] Jonathan Rosenberg. K-Theory and Geometric Topology. In Eric M. Friedlander and Daniel R. Grayson, editors, *Handbook of K-Theory*, pages 577–610. Springer Berlin Heidelberg, Berlin, Heidelberg, 2005. https://doi.org/10.1007/978-3-540-27855-9_12.

[Wei13] Charles Weibel. *The K-Book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, 2013.