Due at the beginning of class on 30 January 2024

NAME: SOLUTIONS

- Your answers should be neatly written and logically organized.
- You may collaborate on solving the problems, but the solutions you turn in should be your own.
- You may use any resource you find online (or elsewhere), but you must cite any resource you use.

Reading: Read §2.2 in [Rie14] and §1.5 in [Mal23].

(1) Let $\mathcal C$ be a homotopical category, and let $\mathcal I$ be any category. The category $\text{Fun}(\mathcal I,\mathcal C)$ of functors from $\mathcal I$ to $\mathcal C$ becomes a homotopical category with weak equivalences defined object-wise. By choosing a homotopical category $\mathcal C$ and a category $\mathcal I$, show that the limit functor

lim:
$$\operatorname{Fun}(\mathfrak{I},\mathfrak{C}) \to \mathfrak{C}$$
, $F \mapsto \lim F$

is not a homotopical functor.

SOLUTION: Plenty of examples. Homotopy pullbacks are not pullbacks. $S^1 \times_* S^1$ is a torus, but $S^1 \times_{D^2} S^1$ is homeomorphic to S^1 .

(2) Let \mathcal{C} be a homotopical category. Prove that for any discrete (the only morphisms are identities) category \mathcal{I} with finitely many morphisms, $\mathsf{ho}(\mathcal{C})^{\mathcal{I}}$ is equivalent to $\mathsf{ho}(\mathcal{C}^{\mathcal{I}})$, where $\mathcal{C}^{\mathcal{I}}$ has weak equivalences defined pointwise. Prove that finite products in $\mathsf{ho}(\mathcal{C})$ are homotopy products and finite coproducts in $\mathsf{ho}(\mathcal{C})$ are homotopy coproducts.

SOLUTION:

You need to assume that $\mathfrak I$ is finite (or maybe a model category), otherwise you get bad behavior. For example, if $\mathfrak C$ has one object and two morphisms $\mathfrak f$ and $\mathfrak g$ and you invert $\mathfrak g$. Let $\mathfrak I$ be the integers. In $\mathsf{ho}(\mathfrak C)^{\mathfrak I}$ has morphisms of the form $(\mathfrak f, \mathfrak f \mathfrak g^{-1} \mathfrak f, \mathfrak f \mathfrak g^{-1} \mathfrak f \mathfrak g^{-1} \mathfrak f, \ldots)$, but this is not a morphism in $\mathsf{ho}(\mathfrak C^{\mathfrak I})$. The problem is that there are different lengths of zigzags at different objects of $\mathfrak I$.

Let $L: \mathcal{C} \to \text{ho } \mathcal{C}$ be the localization functor. Then $L^{\mathcal{I}}: \mathcal{C}^{\mathcal{I}} \to (\text{ho } \mathcal{C})^{\mathcal{I}}$ is a functor which sends weak equivalences to isomorphisms. This yields a functor $\text{ho}(\mathcal{C}^{\mathcal{I}}) \to (\text{ho } \mathcal{C})^{\mathcal{I}}$. By construction, this functor is bijective on objects. Further, it is faithful, for it sends a zig-zag of tuples of morphisms to the corresponding tuple of zig-zags of morphisms.

We need additional assumptions to guarantee that this functor is full. If $\mathfrak I$ is finite, then the supremum of lengths of zig-zags accuring in an $\mathfrak I$ -indexed tuple is finite, and thus arises from a zig-zag of tuples. Otherwise, if $\mathfrak C$ is a model category (or more generally has an $\mathfrak n$ -step calculus of fractions), the supremum of lengths accuring in a zig-zag of $\mathfrak I$ -indexed tuples is again finite, and thus arises from a zig-zag of tuples.

Assume this functor is full. Then, $ho(\mathfrak{C}^{\mathfrak{I}}) \to (ho\,\mathfrak{C})^{\mathfrak{I}}$ is an equivalence (even an isomorphism, although it is forbidden to acknowledge that some functors are isomorphisms).

Assume that \mathcal{C} has homotopy products and ho \mathcal{C} has products. Then, the homotopy product functor $ho(\mathcal{C}^{\mathfrak{I}}) \to ho \, \mathcal{C}$ is the right derived functor of the product $\mathcal{C}^{\mathfrak{I}} \to \mathcal{C}$. By the previous problem set, this is the adjoint of the left derived functor of the diagonal, $\mathcal{C} \to \mathcal{C}^{\mathfrak{I}}$. Since the diagonal is already homotopical, its left derived functor is again the diagonal $ho(\mathcal{C})^{\mathfrak{I}} \to ho \, \mathcal{C}$ after identifying $ho(\mathcal{C}^{\mathfrak{I}})$ and $(ho\,\mathcal{C})^{\mathfrak{I}}$. But the adjoint to this diagonal is exactly the homotopy product in ho \mathcal{C} . See also [Rie14, Remark 6.3.1 and footnote 3 therein].

- (3) A *coequalizer* is the colimit of a diagram of shape $\bullet \Rightarrow \bullet$ in a category.
 - (a) Prove that the data of the coequalizer of two parallel morphisms $A \xrightarrow{f \\ g} B$ is equivalent to the data of the pushout of the diagram

$$A \stackrel{\nabla}{\longleftarrow} A \coprod A \stackrel{(f,g)}{\longrightarrow} B$$

where $\nabla \colon A \coprod A \to A$ is the fold map.

SOLUTION: We will show that each object has the universal property of the other. First, assume that P is a pushout in the diagram below

$$\begin{array}{ccc}
A \coprod A \xrightarrow{(f,g)} B \\
\downarrow \nabla & & \downarrow p \\
A \xrightarrow{q} & P
\end{array}$$

First, claim that p equalizes f and g. To see this, consider the composite $p(f,g)i_1$, where $i_1: A \to A \coprod A$ is the left injection into the coproduct. We have $(f,g)i_1 = f$, and $\nabla i_1 = id_A$. Then:

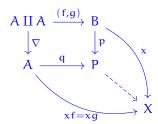
$$pf = p(f, g)i_1 = q\nabla i_1 = q$$

Similarly, $(f, g)i_2 = g$ and $\nabla i_2 = id_A$, so:

$$pg = p(f, g)i_2 = q\nabla i_2 = q$$

So pf = pg. So p equalizes f and g. It remains to be seen that it is the universal morphism that does so.

Let $x: B \to X$ be any morphism such that xf = xg. Then $x(f,g)i_1 = xf = xg = x(f,g)i_2$, which is the same as $x(f,g) = xf\nabla = xg\nabla$. Then we can draw a commuting diagram



and fill it in with the dashed arrow, which shows that there is a unique morphism $P \to X$ exhibiting P as the coequalizer of f and g.

The converse is similar.

(b) Use part (a) to describe the homotopy coequalizer of two maps in the category **Top** of (unpointed) topological spaces¹.

SOLUTION: The homotopy coequalizer of two parallel maps f, q: $X \to Y$ is the mapping torus

$$X \times [0,1] \sqcup Y/\sim$$

where ~ identifies

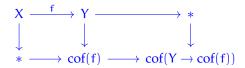
$$(x, 0) \sim f(x) \text{ and } (x, 1) \sim g(x).$$

¹To be precise, we assume all spaces are compactly generated and weakly Hausdorff.

- (4) A pointed space X is *well-based* if the inclusion of the basepoint is a cofibration. Let $f: X \to Y$ be a pointed map of well-based spaces.
 - (a) Let cof(f) be the homotopy cofiber of f. Prove that the homotopy cofiber of $Y \to cof(f)$ is homotopy equivalent to ΣX .
 - (b) Prove the dual statement: if fib(f) is the homotopy fiber of f, then the homotopy fiber of $fib(f) \to X$ is homotopy equivalent to ΩY .

SOLUTION:

(a) Consider the diagram



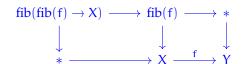
Both squares are individually homotopy pushouts. Thus, by a version of the pasting lemma, the rectangle is a homotopy pushout.

(To see this explicitly, recall that homotopy pushouts are computed by cofibrant replacement and literal pushout. Cofibrantly replace the top left cospan to compute cof(f) by literal pushout. Then, cofibrantly replace the entire square, along with the map to the point. The square is still a pushout after cofibrant replacement and now $cof(Y \rightarrow cof(f))$ can be computed by literal pushout. Now the literal pasting lemma yields that the entire diagram is a literal pushout, which proves the homotopy pasting lemma.)

Since the pushout of X mapping to points is the suspension of X, this demonstrates that $cof(Y \to cof(f))$ is homotopy equivalent to ΣX . The well-pointedness assumption guarantees that this homotopy pushout is the same as the smash product $S^1 \wedge X$, but if you take the pushout as the definition of suspension, the well-pointedness assumption is not needed.

(b) Prove the dual statement: if fib(f) is the homotopy fiber of f, then the homotopy fiber of $fib(f) \to X$ is homotopy equivalent to ΩY .

Dually, consider the diagram



Both squares are homotopy pullbacks, so the rectangle is a homotopy pullback by the same reasoning to the above. But since the pushout of points along Y is the loop-space of Y, this demonstrates that $fib(fib(f) \to X)$ is homotopy equivalent to ΩY .

This problem is [Mal23, Exercise 16 in Chapter 1]. You can probably also find it (without any category theory) in Hatcher.

REFERENCES

- [Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.
- [Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.