OPERADS NAME: SOLUTIONS

Due at the beginning of class on 29 April 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: Good introductions to operads include the following: [Bel17, Bra17, Sta04, Sar17]. Pick whichever reference that you find most accessible. The survey article [Man22] is a much more comprehensive overview of operads in stable homotopy theory, but it is also much more technical.

(1) Let X be a grouplike E_k-algebra in $\mathcal{T}op_*$ for some $k \geq 2$. Show that $\pi_0(X)$ and $\pi_1(X)$ are abelian groups.

SOLUTION: You can check this explicitly using the definition of an algebra over an operads, but it's a one-liner using the recognition principle: if X is an E_k-algebra, then $X \simeq \Omega^k Y$ for some space Y. Hence, $\pi_1(X) \cong \pi_1(\Omega^k Y) \cong \pi_{1+k} Y$ and $\pi_0(X) \cong \pi_k Y$. Since $k \geq 2$, these are abelian groups.

(2) Let $\widehat{\mathcal{S}p}_{\geq 0}$ be the full subcategory of $\widehat{\mathcal{S}p}$ on the connective spectra.

This question comes from [Law20, Section 1.3.3].

(a) Show that there is an adjunction $\pi_0\colon \mathrm{ho}(\widehat{\mathcal{S}p}_{\geq 0}) \leftrightarrows \mathcal{A}b\colon H$.

SOLUTION: Recall that the morphisms in $\mathrm{ho}(\widehat{\mathcal{S}p})$ are the homotopy classes of maps $[X, Y]$. The maps are the same for connective spectra because it's a full subcategory. We have:

$$[X, HA] \cong H^0(X; A)$$

by Brown Representability, where $H^0(X; A)$ is the zeroth cohomology of the spectrum X with coefficients in A. By the Universal Coefficient Theorem and the fact that X is connective, we have

$$[X, HA] \cong H^0(X; A) \cong \mathrm{Hom}_{\mathcal{A}b}(H^0(X; \mathbb{Z}), A).$$

Then by the Hurewicz theorem, we have

$$[X, HA] \cong H^0(X; A) \cong \mathrm{Hom}_{\mathcal{A}b}(H^0(X; \mathbb{Z}), A) \cong \mathrm{Hom}_{\mathcal{A}b}(\pi_0 X, A).$$

This chain of isomorphisms is the required chain of isomorphisms to realize the adjunction.

Alternatively, one can build the unit and the counit of the adjunction and explicitly check the triangular identities. The unit of the adjunction is the map from X to its' zeroth Postnikov section $X \to P_0 X$. Because X is connective, $P_0 X \cong H(\pi_0 X)$. The counit of the adjunction is the identity homomorphism $\pi_0 HA \to A$. The triangular identities follow from $H\pi_0 HA \simeq HA$ and $\pi_0 H \pi_0 X \cong \pi_0 X$.

(b) Let A be an abelian group. Show that the endomorphism operads $\mathrm{End}_{\widehat{\mathcal{S}p}}(HA)$ and $\mathrm{End}_{\mathcal{A}b}(A)$ are weakly equivalent as operads in $\mathcal{T}op_*$.

SOLUTION: Let $\mathrm{Map}(X, Y)$ denote the space of maps between two spectra [Mal23, Definition 2.3.12]. Recall that $\pi_0 \mathrm{Map}(X, Y) = [X, Y]$. We want to show that $\mathrm{Map}(X, HA)$ is equivalent to the discrete space $\mathrm{Hom}_{\mathcal{A}b}(\pi_0 X, A)$ when X is a connective spectrum.

Using the adjunction from the previous part:

$$\pi_0 \mathrm{Map}(X, HA) = [HA^{\wedge n}, HA] \cong \mathrm{Hom}(\pi_0(HA^{\wedge n}), A) \cong \mathrm{Hom}(A^{\otimes n}, A)$$

You need to know that $\pi_0(Y^{\wedge n}) \cong (\pi_0 Y)^{\otimes n}$, but this is true for connective spectra. One can see this from the Künneth spectral sequence or the fact that $\pi_n(HA \wedge HB) \cong \mathrm{Tor}_n(A, B)$.

We also have:

$$\begin{aligned}
\pi_n \operatorname{Map}(X, HA) &\cong \pi_0 \Omega^n \operatorname{Map}(X, HA) \\
&\cong \pi_0 \operatorname{Map}(\Sigma^n X, HA) \\
&\cong [\Sigma^n X, HA] \\
&\cong \operatorname{Hom}(\pi_0(\Sigma^n X), A) \\
&\cong \operatorname{Hom}(0, A) = 0.
\end{aligned}$$

Therefore, $\operatorname{Map}(X, HA)$ is homotopy equivalent to the discrete space $\operatorname{Hom}(\pi_0 X, A)$. In particular, $\operatorname{Map}(HA^{\wedge n}, HA)$ is homotopy equivalent to the discrete space $\operatorname{Hom}(A^{\otimes n}, A)$, so the spaces in the operads $\mathrm{End}_{\widehat{\mathcal{S}p}}(HA)$ and $\mathrm{End}_{\mathcal{A}b}(A)$ are equivalent.

Since the Eilenberg–MacLane functor $H\colon \mathcal{A}b \to \mathrm{ho}(\widehat{\mathcal{S}p})$ is fully faithful, the operad composition in both operads are the same.

(c) Show that the structure of an $\mathcal{O}$-algebra on HA is equivalent to the structure of an $\pi_0(\mathcal{O})$-algebra on A, where $\pi_0(\mathcal{O})$ is the operad in $(\mathcal{S}et, \times, \{*\})$ constructed from $\mathcal{O}$ by taking $\pi_0(\mathcal{O})(n) = \pi_0(\mathcal{O}(n))$.

SOLUTION: An $\mathcal{O}$-algebra structure on HA is a map of operads $\mathcal{O} \to \mathrm{End}_{\widehat{\mathcal{S}p}}(HA)$. By the previous part, this is equivalently a map $\mathcal{O} \to \mathrm{End}_{\mathcal{A}b}(HA)$. Since the target is discrete, this is equivalent to a function $\pi_0(\mathcal{O}) \to \mathrm{End}_{\mathcal{A}b}(HA)$, which is a $\pi_0(\mathcal{O})$-algebra structure on A.

One can also check that the structure maps of an operad are preserved.

(d) If A is a commutative ring, show that HA is an E_∞-algebra.

SOLUTION: Let $\mathcal{O}$ be an E_∞-operad. Then $\pi_0(\mathcal{O})$ is the commutative operad, and A is a $\pi_0(\mathcal{O})$-algebra. By the previous part, HA is an $\mathcal{O}$-algebra and $\mathcal{O}$ is an E_∞-operad.

(3) Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \odot, J)$ be symmetric monoidal categories enriched in $\mathcal{T}op_*$, and let $\mathcal{O}$ be an operad of pointed spaces. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a lax symmetric monoidal functor.

(a) If X is an $\mathcal{O}$-algebra in $\mathcal{C}$, show that $F(X)$ is an $\mathcal{O}$-algebra in $\mathcal{D}$.

SOLUTION: Given an $\mathcal{O}$-algebra in $\mathcal{C}$, we have a morphism of operads $\mathcal{O} \to \mathrm{End}_{\mathcal{C}}(X)$. To get an $\mathcal{O}$-algebra structure on $F(X)$ in $\mathcal{D}$, we must build an operad morphism $\mathcal{O} \to \mathrm{End}_{\mathcal{D}}(F(X))$. Thus, it suffices to define an operad morphism $\mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{D}}(F(X))$, and composing with the map $\mathcal{O} \to \mathrm{End}_{\mathcal{C}}(X)$ gives the $\mathcal{O}$-algebra structure on D.

To that end, note that the n-th space of $\mathrm{End}_{\mathcal{C}}(X)$ is

$$\mathcal{C}(X^{\otimes n}, X).$$

The functor F takes $f\colon X^{\otimes n} \to X$ to $F(f)\colon F(X^{\otimes n}) \to F(X)$. Because F is lax symmetric monoidal, we may then precompose with $\eta_n\colon F(X)^{\odot n} \to F(X^{\otimes n})$ to get

$$F(X)^{\odot n} \xrightarrow{\eta_n} F(X^{\otimes n}) \xrightarrow{F(f)} F(X),$$

and this defines a map

$$\Psi_n\colon \mathrm{End}_{\mathcal{C}}(X)(n) = \mathcal{C}(X^{\otimes n}, X) \to \mathcal{D}(F(X)^{\odot n}, F(X)) = \mathrm{End}_{\mathcal{D}}(F(X))(n).$$

Note that when n is zero, we must use the map $J \to F(I)$ from the unit of $\mathcal{D}$ to the functor F applied to the unit of $\mathcal{C}$.

Note that the unit of the operad $\mathrm{End}_{\mathcal{C}}(X)$ is $\mathrm{id}_X \in \mathcal{C}(X,X)$, and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, so Ψ preserves units of these operads.

It remains to show that this commutes with the structure maps of the operads. For brevity, let $\mathcal{P} = \mathrm{End}_{\mathcal{C}}(X)$ and let $\mathcal{Q} = \mathrm{End}_{\mathcal{D}}(F(X))$. This amounts to showing that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{P}(n) \times \mathcal{P}(i_1) \times \cdots \times \mathcal{P}(i_n) \arrow[r, "\circ"] \arrow[d, "\Psi_n \times \Psi_{i_1} \times \cdots \times \Psi_{i_n}"'] & \mathcal{P}(i_1 + \ldots + i_n) \arrow[d, "\Psi_{i_1+\ldots+i_n}"] \\
\mathcal{Q}(n) \times \mathcal{Q}(i_1) \times \cdots \times \mathcal{Q}(i_n) \arrow[r, "\circ"] & \mathcal{Q}(i_1 + \ldots + i_n)
\end{tikzcd}$$

To that end, let $f \colon X^{\otimes n} \to X$ and $f_j \colon X^{\otimes i_j} \to X$. We compute:

$$\begin{aligned}
\Psi(f) \circ \big(\Psi(f_1) \odot \cdots \odot \Psi(f_n)\big) &= (F(f) \circ \eta_n) \circ \big((F(f_1) \circ \eta_{i_1}) \odot \cdots \odot (F(f_n) \circ \eta_{i_n})\big) \\
&= F(f) \circ \big(F(f_1) \otimes \cdots \otimes F(f_n) \circ \eta_{i_1+\cdots+i_n}\big) \\
&= F\big(f \circ f_1 \otimes \cdots \otimes f_n\big) \circ \eta_{i_1+\cdots+i_n} \\
&= \Psi(f \circ f_1 \otimes \cdots \otimes f_n).
\end{aligned}$$

The first line is the definition of Ψ, the second is the symmetric monoidal properties of F, the third is naturality of η, and the fourth is the definition of Ψ again.

(b) Show that the suspension spectrum of a topological monoid is an E_1-ring spectrum.

SOLUTION: A topological monoid is in particular an E_1-algebra in $\mathcal{T}op_*$, so by part (a) and the fact that Σ^∞ is strong monoidal, a topological monoid becomes an E_1-ring spectrum.

(c) If A is a commutative ring, show that HA is an E_∞-ring spectrum. Give a different proof than your solution to problem 2(d).

SOLUTION: A commutative ring A is an E_∞-algebra in $\mathcal{A}b$, so by part (a) and the fact that H is lax monoidal, HA is an E_∞-algebra in $\widehat{\mathcal{S}p}$.

References

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