Operads   Name: _____

Due at the beginning of class on 29 April 2024

- *Your answers should be neatly written and logically organized.*

- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*

- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: Good introductions to operads include the following: [Bel17, Bra17, Sta04, Sar17]. Pick whichever reference that you find most accessible. The survey article [Man22] is a much more comprehensive overview of operads in stable homotopy theory, but it is also much more technical.

(1) Let X be an E_k-algebra in $\mathcal{Top}_*$ for some $k \geq 2$. Show that $\pi_0(X)$ is a commutative monoid and $\pi_1(X)$ is an abelian group.

(2) Let $\widehat{\mathcal{Sp}}_{\geq 0}$ be the full subcategory of the $\widehat{\mathcal{Sp}}$ on the connective spectra.

  (a) Show that there is an adjunction $\pi_0 \colon \widehat{\mathcal{Sp}}_{\geq 0} \leftrightarrows \mathcal{Ab} \colon H$.

  (b) Let A be an abelian group. Show that the endomorphism operads $\mathrm{End}_{\widehat{\mathcal{Sp}}}(HA)$ and $\mathrm{End}_{\mathcal{Ab}}(A)$ are weakly equivalent as operads in $\mathcal{Top}_*$.

  (c) Show that the structure of an $\mathcal{O}$-algebra on HA is equivalent to the structure of an $\pi_0(\mathcal{O})$-algebra on A, where $\pi_0(\mathcal{O})$ is the operad in $(\mathcal{Set}, \times, \{*\})$ constructed from $\mathcal{O}$ by taking $\pi_0(\mathcal{O})(n) = \pi_0(\mathcal{O}(n))$.

  (d) If A is a commutative ring, show that HA is an E_∞-algebra.

(3) Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \odot, J)$ be symmetric monoidal categories enriched in $\mathcal{Top}_*$, and let $\mathcal{O}$ be an operad of pointed spaces. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a lax symmetric monoidal functor.

  (a) If X is an $\mathcal{O}$-algebra in $\mathcal{C}$, show that $F(X)$ is an $\mathcal{O}$-algebra in $\mathcal{D}$.

  (b) Show that the suspension spectrum of a topological monoid is an E_1-ring spectrum.

  (c) If A is a commutative ring, show that HA is an E_∞-ring spectrum. Give a different proof than your solution to problem 2(d).

References

[Bel17]   Eva Belmont.   A quick introduction to operads.   https://mathweb.ucsd.edu/~ebelmont/operads-talk.pdf, 2017.

[Bra17]   Tai-Danae Bradley.   What is an operad?   https://www.math3ma.com/blog/what-is-an-operad-part-1, 2017.

[Man22]   Michael A. Mandell. Operads and operadic algebras in homotopy theory. In *Stable categories and structured ring spectra*, volume 69 of *Math. Sci. Res. Inst. Publ.*, pages 183–247. Cambridge Univ. Press, Cambridge, 2022.

[Sar17]   Maru Sarazola.   Loop spaces and operads.   https://sites.google.com/view/msarazola/notes-and-other-resources, 2017.

[Sta04]   Jim Stasheff. What is ... an operad? *Notices Amer. Math. Soc.*, 51(6):630–631, 2004.