SEQUENTIAL SPECTRA NAME: SOLUTIONS

Due at the beginning of class on 4 March 2025

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Sections 2.1, 2.2, and 2.3].

(1) Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$ be a sequence of spectra. Show that the homotopy colimit [Mal23, Definition 2.3.17] of this sequence commutes with stable homotopy groups, in the sense that

$$\pi_k\left(\operatorname{hocolim}_n X_n\right) = \operatorname*{colim}_n \pi_k(X_n).$$

SOLUTION: We'll start with the right hand side and produce a chain of isomorphisms leading to the left hand side. Let $X_{n,m}$ be the m-th space in the spectrum X_n.

$$\begin{aligned}
\operatorname*{colim}_n \pi_k(X_n) &= \operatorname*{colim}_n \operatorname*{colim}_m \pi_{k+m}(X_{n,m}) && \text{definition of } \pi_k \text{ of a spectrum}\\
&\cong \operatorname*{colim}_m \operatorname*{colim}_n \pi_{k+m}(X_{n,m}) && \text{colimits commute with colimits}\\
&\cong \operatorname*{colim}_m \pi_{k+m}(\operatorname{hocolim}_n X_{n,m}) && \text{the same fact for spaces (PSet 3, Q4)}\\
&\cong \pi_k(\operatorname{hocolim}_n X_n) && \text{definition of } \pi_k(\operatorname{hocolim}_n X_n)
\end{aligned}$$

(2) Eilenberg-MacLane spectra are characterized by their homotopy groups: if any other spectrum X satisfies $\pi_i X = 0$ for $i \neq 0$, then $X \simeq H(\pi_0 X)$.

(a) Prove that $H(\mathbb{Z}/p)$ is stably equivalent to the homotopy cofiber of the map obtained by applying the functor H to $p\colon \mathbb{Z} \to \mathbb{Z}$.

SOLUTION: There is a homotopy cofiber sequence in spectra:

$$H\mathbb{Z} \xrightarrow{Hp} H\mathbb{Z} \to \operatorname{cof}(Hp)$$

We want to show that the cofiber $Q := \operatorname{cof}(Hp)$ has a single nonzero homotopy group concentrated in degree zero, and that this homotopy group is $\mathbb{Z}/p$. To do so, we break up the homotopy cofiber sequence of spectra into homotopy cofiber sequences of pointed spaces:

$$K(\mathbb{Z}, n) \xrightarrow{p} K(\mathbb{Z}, n) \to Q_n.$$

Ideally, we could compute the homotopy groups of Q_n from a long exact sequence in homotopy, but this is a cofiber sequence and not a fiber sequence. The idea now is to apply Blakers–Massey to get a fiber sequence to which we can apply the LES in homotopy.

Rewrite this cofiber sequence as a homotopy pushout of pointed spaces

$$\begin{tikzcd}
K(\mathbb{Z}, n) \arrow[r, "p"] \arrow[d] & K(\mathbb{Z}, n) \arrow[d] \\
* \arrow[r] & Q_n
\end{tikzcd}$$

Then we can take the homotopy pullback R of $* \to Q_n \leftarrow K(\mathbb{Z},n)$ and apply Blakers–Massey: the map $p\colon K(\mathbb{Z},n) \to K(\mathbb{Z},n)$ is $(n-1)$-connected, and the map $K(\mathbb{Z},n) \to *$ is n-connected, so the canonical map $K(\mathbb{Z},n) \to R$ is $(n-1)+n-1 = (2n-2)$-connected.

$$\begin{tikzcd}
K(\mathbb{Z},n) \arrow[rr, "p"] \arrow[dd] \arrow[dr, dashed] & & K(\mathbb{Z},n) \arrow[dd] \\
& R \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\ulcorner h", very near start] & \\
* \arrow[rr] & & Q_n
\end{tikzcd}$$

Then we have a homotopy fiber sequence

$$R \to K(\mathbb{Z},n) \to Q_n,$$

and $\pi_k(R) \cong \pi_k(K(\mathbb{Z},n))$ for $0 \le k < 2n-2$. Thus, we learn that

$$\pi_k(Q_n) = \begin{cases} \mathbb{Z}/p & (k = n) \\ 0 & (0 \le k < n) \\ 0 & (n < k < 2n-2) \\ ? & (k \ge 2n-2) \end{cases}$$

We can use this to compute the spectrum homotopy of $Q = \operatorname{cof}(H\mathbb{Z} \xrightarrow{Hp} H\mathbb{Z})$

$$\pi_k Q = \operatorname*{colim}_n \pi_{k+n} Q_n$$

When $k = 0$, we learn that $\pi_k Q = \mathbb{Z}/p$, and when $k \neq 0$, we learn that $\pi_k Q = 0$. Therefore, Q has a single nonzero homotopy group concentrated in degree zero, so $Q \simeq H(\mathbb{Z}/p)$.

(b) The *rationalization* $\mathbb{S}_{\mathbb{Q}}$ of the sphere spectrum $\mathbb{S}$ is the homotopy colimit of the diagram

$$\mathbb{S} \xrightarrow{1} \mathbb{S} \xrightarrow{2} \mathbb{S} \xrightarrow{3} \mathbb{S} \xrightarrow{4} \mathbb{S} \xrightarrow{5} \mathbb{S} \to \cdots$$

Prove that $\mathbb{S}_{\mathbb{Q}}$ is stably equivalent to $H\mathbb{Q}$.

SOLUTION: The rationalization of $\mathbb{S}$ is the spectrum

$$\mathbb{S}_{\mathbb{Q}} = \operatorname{hocolim}(\mathbb{S} \xrightarrow{1} \mathbb{S} \xrightarrow{2} \mathbb{S} \xrightarrow{3} \mathbb{S} \xrightarrow{4} \cdots).$$

Recall that stable homotopy groups commute with homotopy colimits (turning the homotopy colimits to colimits in $\mathcal{A}b$). So,

$$\pi_n(\mathbb{S}_{\mathbb{Q}}) = \operatorname{colim}(\pi_n\mathbb{S} \xrightarrow{1} \pi_n\mathbb{S} \xrightarrow{2} \pi_n\mathbb{S} \xrightarrow{3} \cdots)$$

This is isomorphic to $\pi_n\mathbb{S} \otimes \mathbb{Q}$. Recall from the last problem set that $\pi_n\mathbb{S}$ is a finite abelian group for $n > 0$ and $\pi_n\mathbb{S} = \mathbb{Z}$ for $n = 0$. Therefore, $\pi_n(\mathbb{S}_{\mathbb{Q}}) = \pi_n\mathbb{S} \otimes \mathbb{Q} = 0$ for $n > 0$ and $\pi_0(\mathbb{S}_{\mathbb{Q}}) = \mathbb{Z} \otimes \mathbb{Q} = \mathbb{Q}$. So this has one nonzero homotopy group, so $\mathbb{S}_{\mathbb{Q}} \simeq H\mathbb{Q}$.

See also [Mal23, Chapter 1, Exercise 39] and [Mal23, Example 2.5.30].

(3) Let $\mathrm{ev}_0\colon \mathcal{S}p \to \mathcal{T}op_*$ be the functor that evaluates a spectrum at its zeroth space: $\mathrm{ev}_0 X = X_0$.

(a) Prove that Σ^∞ is left adjoint to ev_0.

SOLUTION: We must show there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Sp}}(\Sigma^\infty X, Y) \cong \mathrm{Hom}_{\mathbf{Top}_*}(X, Y_0)$$

Recall that a map of spectra $\Sigma^\infty \to Y$ is a sequence of pointed maps $f_n : \Sigma^n X \to Y_n$ such that the following squares commute for all n:

$$\begin{tikzcd} \Sigma(\Sigma^n X) \arrow[r, "\Sigma f_n"] \arrow[d, "="'] & \Sigma Y_n \arrow[d] \\ \Sigma^{n+1} X \arrow[r, "f_{n+1}"] & Y_{n+1} \end{tikzcd}$$

This is naturally determined by the map $f_0 : X \to Y_0$. Denote the composite structure maps of Y by $y_n : \Sigma^n Y_0 \to Y_n$. Note that the diagram above forces $f_1 = y_1 \circ \Sigma f_0$. Suppose for induction that $f_n = y_n \circ \Sigma f_0$, then the above square commuting forces $f_{n+1} = y_{n+1} \circ \Sigma f_0$. So the map $\Sigma^\infty X \to Y$ uniquely determines the map $f_0 : X \to Y_0$ by the structure maps of Y. Going the other way, any map $g : X \to Y_0$ can naturally ascend to a map of spectra $\Sigma^\infty X \to Y$ by setting $g_0 = g$ and $g_n = y_n \circ \Sigma g_0$. These constructions are clearly natural in X and Y, so this completes the proof of the adjunction.

(b) For any spectrum X, let $\Omega^\infty X := \mathrm{ev}_0 RX$, where R is the fibrant replacement functor [Mal23, Proposition 2.2.9]. Use [Rie14, Exercise 2.2.15] to prove that there is an adjunction

$$\Sigma^\infty \colon \mathrm{ho}(\mathcal{T}\mathrm{op}_*) \leftrightarrows \mathrm{ho}(\mathcal{S}\mathrm{p}) \colon \Omega^\infty.$$

SOLUTION: First, observe that Σ^∞ is homotopical. For any space X, the stable homotopy groups of $\Sigma^\infty X$ are the homotopy groups of X, and if $X \xrightarrow{f} Y$ is a weak equivalence of spaces, the maps induced by $\Sigma^\infty f$ between the stable homotopy groups are exactly the isomomorphisms induced by f between the homotopy groups of X and Y. Hence, Σ^∞ is a stable equivalence.

Note that the above is true so long as our spaces are well-based, so that $S^1 \wedge X$ agrees with the homotopy pushout of $* \leftarrow X \to *$. If the spaces are not well based, then these need not agree. See this mathOverflow answer: [Law13].

A derived functor for ev_0 can be computed by pre-composing with a right deformation of Sp. In particular, the functor R in the exercise is such a deformation. ev_0 is homotopical on the full subcategory of Sp generated by the image of R: if $f : X \to Y$ is a stable equivalence between two Ω-spectra, then $f_0 : X_0 \to Y_0$ is a weak equivalence because $\pi_m \cong \pi_m(X_0)$, $\pi_m(Y) \cong \pi_m(Y_0)$ and since $\{f_m\}_{m\in\mathbb{N}}$ commute with the bonding maps, the isomorphisms $\pi_m(X) \to \pi_m(Y)$ induced by f is exactly the map induced by f_0 from $\pi_m X_0$ to $\pi_m(Y_0)$.

So $\Omega^\infty = ev_0 \circ R$ is a right derived functor for ev_0. By proposition 2.2.13 in Riehl, a derived functor computed thanks to a deformation is an absolute Kan extension. Therefore, by [Rie14, Exercise 2.2.15], Σ^∞ and Ω^∞ are an adjoint pair.

REFERENCES

[Law13] Tyler Lawson. Must a weak homotopy equivalence induce an isomorphism between stable homotopy groups? "https://mathoverflow.net/questions/148963/must-a-weak-homotopy-equivalence-induce-an-isomorphism-between-stable-homotopy-g", 2013.

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.