HOMOTOPY LIMITS AND COLIMITS NAME: SOLUTIONS

Due at the beginning of class on 30 January 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: Read §2.2 in [Rie14] and §1.5 in [Mal23].

(1) Let $\mathcal{C}$ be a homotopical category, and let $\mathcal{J}$ be any category. The category $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ of functors from $\mathcal{J}$ to $\mathcal{C}$ becomes a homotopical category with weak equivalences defined object-wise. By choosing a homotopical category $\mathcal{C}$ and a category $\mathcal{J}$, show that the limit functor

$$\lim\colon \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C}, \qquad F \mapsto \lim F$$

is *not* a homotopical functor.

SOLUTION: Plenty of examples. Homotopy pullbacks are not pullbacks. $S^1 \times_* S^1$ is a torus, but $S^1 \times_{D^2} S^1$ is homeomorphic to S^1.

(2) Let $\mathcal{C}$ be a homotopical category. Prove that for any discrete (the only morphisms are identities) category $\mathcal{J}$ with finitely many morphisms, $\mathrm{ho}(\mathcal{C})^{\mathcal{J}}$ is equivalent to $\mathrm{ho}(\mathcal{C}^{\mathcal{J}})$, where $\mathcal{C}^{\mathcal{J}}$ has weak equivalences defined pointwise. Prove that finite products in $\mathrm{ho}(\mathcal{C})$ are homotopy products and finite coproducts in $\mathrm{ho}(\mathcal{C})$ are homotopy coproducts.

SOLUTION:

You need to assume that $\mathcal{J}$ is finite (or maybe a model category), otherwise you get bad behavior. For example, if $\mathcal{C}$ has one object and two morphisms f and g and you invert g. Let $\mathcal{J}$ be the integers. In $\mathrm{ho}(\mathcal{C})^{\mathcal{J}}$ has morphisms of the form $(f, fg^{-1}f, fg^{-1}fg^{-1}f, \ldots)$, but this is not a morphism in $\mathrm{ho}(\mathcal{C}^{\mathcal{J}})$. The problem is that there are different lengths of zigzags at different objects of $\mathcal{J}$.

Let $L\colon \mathcal{C} \to \mathrm{ho}\,\mathcal{C}$ be the localization functor. Then $L^{\mathcal{J}}\colon \mathcal{C}^{\mathcal{J}} \to (\mathrm{ho}\,\mathcal{C})^{\mathcal{J}}$ is a functor which sends weak equivalences to isomorphisms. This yields a functor $\mathrm{ho}(\mathcal{C}^{\mathcal{J}}) \to (\mathrm{ho}\,\mathcal{C})^{\mathcal{J}}$. By construction, this functor is bijective on objects. Further, it is faithful, for it sends a zig-zag of tuples of morphisms to the corresponding tuple of zig-zags of morphisms.

We need additional assumptions to guarantee that this functor is full. If $\mathcal{J}$ is finite, then the supremum of lengths of zig-zags accuring in an $\mathcal{J}$-indexed tuple is finite, and thus arises from a zig-zag of tuples. Otherwise, if $\mathcal{C}$ is a model category (or more generally has an n-step calculus of fractions), the supremum of lengths accuring in a zig-zag of $\mathcal{J}$-indexed tuples is again finite, and thus arises from a zig-zag of tuples.

Assume this functor is full. Then, $\mathrm{ho}(\mathcal{C}^{\mathcal{J}}) \to (\mathrm{ho}\,\mathcal{C})^{\mathcal{J}}$ is an equivalence (even an isomorphism, although it is forbidden to acknowledge that some functors are isomorphisms).

Assume that $\mathcal{C}$ has homotopy products and $\mathrm{ho}\,\mathcal{C}$ has products. Then, the homotopy product functor $\mathrm{ho}(\mathcal{C}^{\mathcal{J}}) \to \mathrm{ho}\,\mathcal{C}$ is the right derived functor of the product $C^{\mathcal{J}} \to \mathcal{C}$. By the previous problem set, this is the adjoint of the left derived functor of the diagonal, $\mathcal{C} \to \mathcal{C}^{\mathcal{J}}$. Since the diagonal is already homotopical, its left derived functor is again the diagonal $\mathrm{ho}(\mathcal{C})^{\mathcal{J}} \to \mathrm{ho}\,\mathcal{C}$ after identifying $\mathrm{ho}(\mathcal{C}^{\mathcal{J}})$ and $(\mathrm{ho}\,\mathcal{C})^{\mathcal{J}}$. But the adjoint to this diagonal is exactly the homotopy product in $\mathrm{ho}\,\mathcal{C}$. See also [Rie14, Remark 6.3.1 and footnote 3 therein].

(3) A *coequalizer* is the colimit of a diagram of shape $\bullet \rightrightarrows \bullet$ in a category.

(a) Prove that the data of the coequalizer of two parallel morphisms $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is equivalent to the data of the pushout of the diagram

$$A \xleftarrow{\nabla} A \amalg A \xrightarrow{(f,g)} B,$$

where $\nabla \colon A \amalg A \to A$ is the fold map.

SOLUTION: We will show that each object has the universal property of the other. First, assume that P is a pushout in the diagram below

$$\begin{tikzcd} A \amalg A \arrow[r, "{(f,g)}"] \arrow[d, "\nabla"] & B \arrow[d, "p"] \\ A \arrow[r, "q"] & P \end{tikzcd}$$

First, claim that p equalizes f and g. To see this, consider the composite $p(f,g)i_1$, where $i_1 : A \to A \amalg A$ is the left injection into the coproduct. We have $(f,g)i_1 = f$, and $\nabla i_1 = \mathrm{id}_A$. Then:

$$pf = p(f,g)i_1 = q\nabla i_1 = q$$

Similarly, $(f,g)i_2 = g$ and $\nabla i_2 = \mathrm{id}_A$, so:

$$pg = p(f,g)i_2 = q\nabla i_2 = q$$

So $pf = pg$. So p equalizes f and g. It remains to be seen that it is the universal morphism that does so.

Let $x \colon B \to X$ be any morphism such that $xf = xg$. Then $x(f,g)i_1 = xf = xg = x(f,g)i_2$, which is the same as $x(f,g) = xf\nabla = xg\nabla$. Then we can draw a commuting diagram

$$\begin{tikzcd} A \amalg A \arrow[r, "{(f,g)}"] \arrow[d, "\nabla"] & B \arrow[d, "p"] \arrow[ddr, bend left, "x"] & \\ A \arrow[r, "q"] \arrow[drr, bend right, "xf=xg"'] & P \arrow[dr, dashed] & \\ & & X \end{tikzcd}$$

and fill it in with the dashed arrow, which shows that there is a unique morphism $P \to X$ exhibiting P as the coequalizer of f and g.

The converse is similar.

(b) Use part (a) to describe the homotopy coequalizer of two maps in the category **Top** of (unpointed) topological spaces[1].

SOLUTION: The homotopy coequalizer of two parallel maps $f, g \colon X \to Y$ is the mapping torus

$$X \times [0,1] \sqcup Y / \sim$$

where $\sim$ identifies

$$(x,0) \sim f(x) \text{ and } (x,1) \sim g(x).$$

[1]To be precise, we assume all spaces are compactly generated and weakly Hausdorff.

(4) A pointed space X is *well-based* if the inclusion of the basepoint is a cofibration. Let $f\colon X \to Y$ be a pointed map of well-based spaces.

(a) Let $\mathrm{cof}(f)$ be the homotopy cofiber of f. Prove that the homotopy cofiber of $Y \to \mathrm{cof}(f)$ is homotopy equivalent to ΣX.

(b) Prove the dual statement: if $\mathrm{fib}(f)$ is the homotopy fiber of f, then the homotopy fiber of $\mathrm{fib}(f) \to X$ is homotopy equivalent to ΩY.

SOLUTION:

(a) Consider the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \mathrm{cof}(f) \arrow[r] & \mathrm{cof}(Y \to \mathrm{cof}(f)) \end{tikzcd}$$

Both squares are individually homotopy pushouts. Thus, by a version of the pasting lemma, the rectangle is a homotopy pushout.

(To see this explicitly, recall that homotopy pushouts are computed by cofibrant replacement and literal pushout. Cofibrantly replace the top left cospan to compute $\mathrm{cof}(f)$ by literal pushout. Then, cofibrantly replace the entire square, along with the map to the point. The square is still a pushout after cofibrant replacement and now $\mathrm{cof}(Y \to \mathrm{cof}(f))$ can be computed by literal pushout. Now the literal pasting lemma yields that the entire diagram is a literal pushout, which proves the homotopy pasting lemma.)

Since the pushout of X mapping to points is the suspension of X, this demonstrates that $\mathrm{cof}(Y \to \mathrm{cof}(f))$ is homotopy equivalent to ΣX. The well-pointedness assumption guarantees that this homotopy pushout is the same as the smash product $S^1 \wedge X$, but if you take the pushout as the definition of suspension, the well-pointedness assumtpion is not needed.

(b) Prove the dual statement: if $\mathrm{fib}(f)$ is the homotopy fiber of f, then the homotopy fiber of $\mathrm{fib}(f) \to X$ is homotopy equivalent to ΩY.

Dually, consider the diagram

$$\begin{tikzcd} \mathrm{fib}(\mathrm{fib}(f) \to X) \arrow[r] \arrow[d] & \mathrm{fib}(f) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Both squares are homotopy pullbacks, so the rectangle is a homotopy pullback by the same reasoning to the above. But since the pushout of points along Y is the loop-space of Y, this demonstrates that $\mathrm{fib}(\mathrm{fib}(f) \to X)$ is homotopy equivalent to ΩY.

This problem is [Mal23, Exercise 16 in Chapter 1]. You can probably also find it (without any category theory) in Hatcher.

REFERENCES

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. `http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf`, October 2023.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.