Sequential Spectra   Name: ____________________

Due at the beginning of class on 27 February 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Sections 2.1, 2.2, 2.3, 2.4, and 2.5].

(1) Eilenberg-MacLane spectra are characterized by their homotopy groups: if any other spectrum X satisfies $\pi_i X = 0$ for $i \neq 0$, then $X \simeq H(\pi_0 X)$.

(a) Prove that $H(\mathbb{Z}/p)$ is the cofiber of the map obtained by applying the functor H to $p \colon \mathbb{Z} \to \mathbb{Z}$.

(b) Prove that the rationalization of $\mathbb{S}$ is stably equivalent to $H\mathbb{Q}$.

(c) Prove that the functor $H \colon \mathcal{A}b \to \mathcal{S}p$ is a pointed functor in the sense that it sends the zero object in $\mathcal{A}b$ to the zero object in $\mathcal{S}p$.

(d) Is $H \colon \mathcal{A}b \to \mathrm{ho}(\mathcal{S}p)$ an additive functor?

(2) Let $\mathrm{ev}_0 \colon \mathcal{S}p \to \mathcal{T}op_*$ be the functor that evaluates a spectrum at its zeroth space: $\mathrm{ev}_0 X = X_0$.

(a) Prove that Σ^∞ is left adjoint to ev_0.

(b) For any spectrum X, let $\Omega^\infty X = \mathrm{ev}_0 RX$, where R is the fibrant replacement functor [Mal23, Proposition 2.2.9]. Prove that there is an adjunction $\Sigma^\infty \colon \mathrm{ho}(\mathcal{T}op_*) \leftrightarrows \mathrm{ho}(\mathcal{S}p) \colon \Omega^\infty$.

(3) Consider the commuting square of spectra

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{g} & D \end{array}$$

(a) Prove that this square is a homotopy pushout if and only if the induced map of homotopy cofibers $\mathrm{cof}(f) \to \mathrm{cof}(g)$ is a stable equivalence.

(b) Use this fact to prove that a commuting square of spectra is a homotopy pullback if and only if it is a homotopy pushout. You may use the fact that a sequence of spectra is a cofiber sequence if and only if it is a fiber sequence.

(4) Let $k \geq 0$. Define the *shift functor* $\mathrm{sh}_k \colon \mathcal{S}p \to \mathcal{S}p$ by $\mathrm{sh}_k(X)_n = X_{k+n}$.

(a) Prove that there is a natural stable equivalence $\Sigma \simeq \mathrm{sh}_1$.

(b) Define functors sh_k for $k < 0$. Prove that $\mathrm{sh}_{-1} \simeq \Omega$.

References

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.