GENERALIZED COHOMOLOGY NAME: SOLUTIONS

Due at the beginning of class on 25 February 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Hat02, Sections 4.3 and 4.E]. For K-theory, [Hat17, Chapter 2] is a good reference, or [May99, Chapter 24]. I also like [Zak24], because this is where I learned this stuff. However, you don't need to know a ton about K-theory to do these problems.

(1) Let X be any space. Prove that $QX := \operatorname{colim}_n \Omega^n \Sigma^n X$ is an infinite loopspace.

SOLUTION: The sets $\{n \mid n \geq k\}$ are cofinal, so

$$QX := \operatorname*{colim}_{n \geq 0} \Omega^n \Sigma^n X \cong \operatorname*{colim}_{n \geq k} \Omega^n \Sigma^n X \cong \operatorname*{colim}_{n \geq k} \Omega^k \Omega^{n-k} \Sigma^{n-k} \Sigma^k X.$$

Then, since S^k is compact and the maps in the colimit are closed inclusions, it follows that

$$QX \cong \Omega^k \operatorname*{colim}_{n \geq k} \Omega^{n-k} \Sigma^{n-k} \Sigma^k X = \Omega^k \operatorname*{colim}_{n \geq 0} \Omega^n \Sigma^n \Sigma^k X = \Omega^k Q \Sigma^k X.$$

This witnesses QX as an infinite loopspace.

(2) Let E be an infinite loopspace. Give an example of structure/conditions on E that guarantees the associated generalized cohomology theory $E^*(X) := \bigoplus_i [X, E_i]$ has the structure of a graded commutative ring.

SOLUTION: $E^*(X)$ is already a graded abelian group. In order for $E^*(X)$ to have a graded multiplication, there must be a map

$$E^*(X)^{\otimes 2} = \bigoplus_{i+j=*} E^i(X) \otimes E^j(X) \to E^*(X)$$

which is determined by a collection of maps $E^i(X) \otimes E^j(X) \to E^{i+j}(X)$ or equivalently $[X, E_i] \otimes [X, E_j] \to [X, E_{i+j}]$. Consider the natural map

$$[X, E_i] \otimes [X, E_j] \to [X, E_i \wedge E_j] \colon f \otimes g \mapsto X \xrightarrow{\Delta} X \wedge X \xrightarrow{f \wedge g} E_i \wedge E_j.$$

Using this, a graded multiplication on $E^*(X)$ can be induced from maps $E_i \wedge E_j \to E_{i+j}$. In order for the multiplication to distribute over addition, these must be maps of infinite loopspaces. The unit is determined by a map $\mathbb{Z} \to E^0(X) = [X, E_0]$ which can be naturally obtained by any element of E_0 i.e. a map $S^0 \to E_0$. In order for the multiplication to be associative, graded commutative, unital, certain diagrams must commute (up to homotopy). In particular,

$$\begin{tikzcd}
E_i \wedge E_j \wedge E_k \arrow[r] \arrow[d] & E_{i+j} \wedge E_k \arrow[d] \\
E_i \wedge E_{j+k} \arrow[r] & E_{i+j+k}
\end{tikzcd}$$

for associativity,

$$\begin{tikzcd}
E_n \arrow[r, "\cong"] \arrow[dr, "="'] & S^0 \wedge E_n \arrow[r] & E_0 \wedge E_n \arrow[dl] \\
& E_{0+n} &
\end{tikzcd}$$

for unitality, and

$$\begin{tikzcd}
E_i \wedge E_j \arrow[rr] \arrow[dr, "tw"'] & & E_{i+j} \\
& E_j \wedge E_i \arrow[ur] &
\end{tikzcd}$$

where the symmetry map for the smash product has the appropriate sign. I have been lax with the details and omitted structure isomorphisms, and this is not really a description of ring spectra anyway. We will see what the correct notions are later when we discuss ring spectra.

(3) Show that the infinite unitary group U is connected as a topological space. Use this to compute $\widetilde{K}^i(S^n)$ for all i and n.

SOLUTION: You can show that each U_n is path connected though paths of unitary matrices, and this holds for U too. To compute K-theory of spheres, note that $\widetilde{K}^i(S^n) \cong \widetilde{K}^0(\Sigma^i S^n) \cong \widetilde{K}^0(S^{n+i})$, so we just have to compute $\widetilde{K}^0(S^n)$.

$$\widetilde{K}^0(S^n) = [S^n, \mathbb{Z} \times BU] = [\Sigma^n S^0, \mathbb{Z} \times BU] \cong [S^0, \Omega^n(\mathbb{Z} \times BU)] = \begin{cases} [S^0, \mathbb{Z} \times BU] & n \text{ is even} \\ [S^0, U] & n \text{ is odd} \end{cases}$$

Then $\pi_0 U = 0$ since U is connected. And $\pi_0(\mathbb{Z} \times BU)$ has $\mathbb{Z}$ as the connected components, because BU is connected (this is true of any classifying space). So the answer is

$$\widetilde{K}^0(S^n) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

(4) Let A be an abelian group. A *cohomology operation* is a natural transformation $\widetilde{H}^m(-;A) \to \widetilde{H}^n(-;A)$. The set of all (stable) cohomology operations forms a ring, called the *Steenrod algebra*, whose product is composition of operations.

(a) For fixed m and n, prove that the set of all cohomology operations $\theta \colon \widetilde{H}^m(-;A) \Rightarrow \widetilde{H}^n(-;A)$ is in bijection with $H^n(K(A,m);A)$.

SOLUTION: Recall that reduced cohomology $\widetilde{H}(-;A)$ is represented by $[-, K(A,n)]$. By the Yoneda Lemma, there is a natural isomorphism:

$$\mathrm{Nat}([-, K(A,m)], [-, K(A,n)]) \cong [K(A,m), K(A,n)] \cong \widetilde{H}^n(K(A,m);A)$$

(b) Prove that there are no nontrivial cohomology operations that decrease degree.

SOLUTION: By part (a), it suffices to compute $\widetilde{H}^m(K(A,n);A)$, where $m < n$. The statement is trivial if $n = 0$, so assume $n > 0$. Let's handle the $n > 1$ case before considering $n = 1$. Since $K(A,n)$ is $(n-1)$-connected, by the Hurewicz theorem we have that $\widetilde{H}^m(K(A,n);A) \cong \pi_m(K(A,n)) \cong 0$ for all $m < n$. The Universal Coefficient Theorem gives an isomorphism:

$$\widetilde{H}^k(K(A,n);A) \cong \mathrm{Ext}(H_{k-1}(A;\mathbb{Z}), A) \oplus \mathrm{Hom}(H_k(A;\mathbb{Z}, A)$$

So $H^k(K(A,n);A) = 0$ for $0 < k < n$, and hence $H^m(K(A,n);A) = \widetilde{H}^m(K(A,n);A) = 0$ (since $m > 0$). Therefore, there is a unique natural transformation that decreases degree and it is necessarily the trivial natural transformation.

Now consider the $n = 1$ case:

$$\begin{aligned} H^0(K(A,1),A) &= F(\pi_0(K(A,1),A) \\ &= F(*,A) \\ &= A \end{aligned}$$

If the map $H^n(X;A) \longrightarrow H^0(X;A) = A$ sent $x \mapsto a$, it would not be additive. So it must be trivial.

(c) For $m \geq 1$, prove that the set of cohomology operations $\widetilde{H}^m(-;A) \to \widetilde{H}^m(-;A)$ which preserve degree are in bijection with the abelian group $\mathrm{Hom}(A, A)$.

SOLUTION: By part (a), cohomology operations which preserve degree are in bijection with $\widetilde{H}^n(K(A,n);A)$. Again by the Hurewicz theorem, $H_m(K(A,n);A) \cong 0$ for $0 < m < n$ and $H_n(K(A,n);A) \cong \pi_n(K(A,n)) \cong A$. Thus, the Universal Coefficient Theorem gives an isomorphism:

$$\begin{aligned} \widetilde{H}^n(K(A,n);A) &\cong \mathrm{Hom}(H_n(K(A,n);\mathbb{Z}), A) \\ &\cong \mathrm{Hom}(A, A) \end{aligned}$$

References

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge University Press, 2002.

[Hat17] Allen Hatcher. Vector Bundles and K-theory. `https://pi.math.cornell.edu/~hatcher/VBKT/VBpage.html`, 2017.

[May99] J. P. May. *A concise course in algebraic topology*. Chicago, IL: University of Chicago Press, 1999.

[Zak24] Inna Zakharevich. K-theory and Characteristic Classes: A homotopical perspective. `https://pi.math.cornell.edu/~zakh/book.pdf`, 2024.