Homotopical Categories Name: ______________________

Due at the beginning of class on 23 January 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: Read §2.1 and §2.2 in [Rie14] or §B.1 in [HHR16].

(1) A class of morphisms $\mathcal{W}$ in a category $\mathcal{C}$ satisfies the *two-out-of-three property* if given any two composable morphisms f and g, if any two of f, g, and gf are in $\mathcal{W}$, then so is the third.

  (a) Prove that the class of weak equivalences $\mathcal{W}$ in a homotopical category $\mathcal{C}$ obeys the two-out-of-three property.

  (b) Is the two-out-of-three property equivalent to the two-out-of-six property?

(2) Let $\mathcal{C}$ be any category equipped with a collection of morphisms $\mathcal{W}$. We say that $\mathcal{W}$ is *saturated* if every morphism f in $\mathcal{C}$ which becomes an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$ is in $\mathcal{W}$. We say that a homotopical category is *saturated* if the class of weak equivalences is saturated (i.e. if f becomes an isomorphism in $\mathrm{ho}(\mathcal{C})$, then $f \in \mathcal{W}$).

  (a) Prove that if $\mathcal{W}$ is saturated, then $\mathcal{W}$ has the two-out-of-six property.

  (b) Give an example of a homotopical category that is *not* saturated.

  (c) (Optional) Show that the class of weak equivalences in a model category is saturated.

(3) Let $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon G$ be an adjoint pair of functors between homotopical categories $\mathcal{C}$ and $\mathcal{D}$. Prove that there is an adjunction $\mathrm{ho}(F)\colon \mathrm{ho}(\mathcal{C}) \leftrightarrows \mathrm{ho}(\mathcal{D})\colon \mathrm{ho}(G)$ between the homotopy functors $\mathrm{ho}(F)$ and $\mathrm{ho}(G)$.

(4) Let $\mathcal{C}$ be a homotopical category and let $L\colon \mathcal{C} \to \mathrm{ho}(\mathcal{C})$ be the localization functor.

  (a) Let $c \in \mathcal{C}$. Prove that any natural transformation $\mathcal{C}(c, -) \Rightarrow F$ factors through $\mathrm{ho}(\mathcal{C})(c, -)$, where $F\colon \mathcal{C} \to \mathbf{Set}$ is a homotopical functor.

  (b) Let $c \in \mathcal{C}$ be an object such that $\mathcal{C}(c, -)$ is a homotopical functor. Prove that the natural transformation $\mathcal{C}(c, -) \to \mathrm{ho}(\mathcal{C})(c, -)$ induced by L is a natural bijection.

References

[HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.

[Qui67] Daniel G. Quillen. *Homotopical algebra*, volume No. 43 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1967.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.