GENERALIZED COHOMOLOGY NAME: _____

Due at the beginning of class on 20 February 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Hat02, Sections 4.3 and 4.E]. For K-theory, [Hat17, Chapter 2] is a good reference, or [May99, Chapter 24]. I also like [Zak24], because this is where I learned this stuff. However, you don't need to know a ton about K-theory to do these problems.

(1) Let X be any space. Prove that $QX := \operatorname{colim}_n \Omega^n \Sigma^n X$ is an infinite loopspace.

(2) Let E be an infinite loopspace. Give an example of structure/conditions on E that guarantees the associated generalized cohomology theory $E^*(X) := \bigoplus_i [X, E_i]$ has the structure of a graded commutative ring.

(3) Show that the infinite unitary group U is connected as a topological space. Use this to compute $\widetilde{K}^i(S^n)$ for all i and n.

(4) Let A be an abelian group. A *cohomology operation* is a natural transformation $\widetilde{H}^m(-;A) \to \widetilde{H}^n(-;A)$. The set of all (stable) cohomology operations forms a ring, called the *Steenrod algebra*, whose product is composition of operations.

   (a) For fixed m and n, prove that the set of all cohomology operations $\theta\colon \widetilde{H}^m(-;A) \Rightarrow \widetilde{H}^n(-;A)$ is in bijection with $H^n(K(A,m);A)$.

   (b) Prove that there are no nontrivial cohomology operations that decrease degree.

   (c) For $m \geq 1$, prove that the set of cohomology operations $\widetilde{H}^m(-;A) \to \widetilde{H}^m(-;A)$ which preserve degree are in bijection with the abelian group $\operatorname{Hom}(A,A)$.

REFERENCES

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge University Press, 2002.

[Hat17] Allen Hatcher. Vector Bundles and K-theory. https://pi.math.cornell.edu/~hatcher/VBKT/VBpage.html, 2017.

[May99] J. P. May. *A concise course in algebraic topology*. Chicago, IL: University of Chicago Press, 1999.

[Zak24] Inna Zakharevich. K-theory and Characteristic Classes: A homotopical perspective. https://pi.math.cornell.edu/~zakh/book.pdf, 2024.