DR. DAVID MEHRLE
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LEARNING OBJECTIVES

- Setup a triple integral for a given solid in $\mathbb{R}^3$
- Switch the order of integration for a triple integral

REVIEW

- Double integrals
- Finding inverse functions

ANNOUNCEMENTS

- Enjoy your fall break!

Previously

In Calculus II, you learned that $\int_a^b f(x)\,dx$ computes the signed area between $y = f(x)$ and the x-axis over the interval $[a, b]$.

Last week, we learned that $\iint_R f(x, y)\,dA$ computes the signed volume between $z = f(x, y)$ and the xy-plane over the region R.

Question

What does a triple integral $\iiint_S f(x, y, z)\,dV$ compute?

- Recall that $\iint_R 1\,dA$ is the area of R. So $\iiint_S 1\,dV$ is the volume of the solid S.
- If $\rho(x, y, z)$ is density, the triple integral $\iiint_S \rho(x, y, z)\,dV$ computes mass.
- moment of inertia, gravitational potential, electric charge, etc.

EXAMPLE

Find the mass of the rectangular box bounded by $1 \leq x \leq 3$, $0 \leq y \leq 1$, and $2 \leq z \leq 4$ with density $\rho(x, y, z) = xyz\,\mathrm{kg/m^3}$.

The mass of the box is the integral of the density, but now we need to integrate over all three of x, y, and z!

$$
\begin{aligned}
M &= \int_1^3 \int_0^1 \int_2^4 xyz\,dz\,dy\,dx \\
&= \int_1^3 \int_0^1 \left[\frac{xyz^2}{2}\right]_{z=2}^{z=4} dy\,dx \\
&= \int_1^3 \int_0^1 6xy\,dy\,dx \\
&= \int_1^3 \left[3xy^2\right]_{y=0}^{y=1} dx \\
&= \int_1^3 3x\,dx = \left[\frac{3x^2}{2}\right]_{x=1}^{x=3} = 13\,\mathrm{kg}
\end{aligned}
$$

Did the order of integration matter on the previous slide? Of course not:

Fubini's Theorem

If $f(x, y, z)$ is continuous over a rectangular solid S with $a \leq x \leq b$, $c \leq y \leq d$, $r \leq z \leq s$, then the triple integral of $f(x, y, z)$ over S can be computed in any order:

$$\iiint_S f\, dV = \int_r^s \int_c^d \int_a^b f\, dx\, dy\, dz = \int_a^b \int_r^s \int_c^d f\, dy\, dz\, dx = \ldots$$

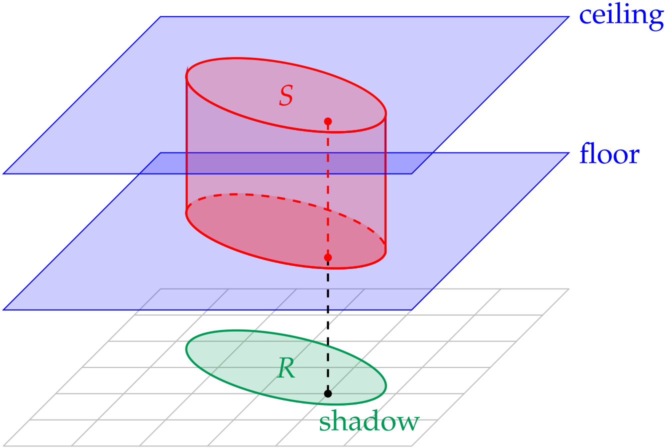
If the region is not rectangular, you have to change the bounds, but you can still integrate in any order.

Volume

The volume of a solid S in xyz-space is computed by the integral

$$\text{Volume}(S) = \iiint_S 1\, dV$$

How to find the bounds of a triple integral



A triple integral is a double integral over it's shadow R in the plane $z = 0$ of a single integral from the floor to the ceiling.

$$\iiint_S f(x,y,z)\,dV = \iint_R \left(\int_{\text{floor}}^{\text{ceiling}} f(x,y,z)\,dz \right) dA$$

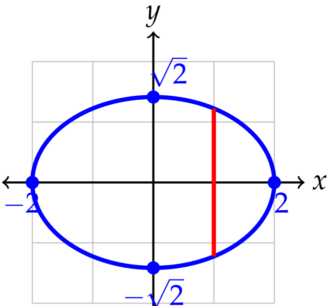
Example

The solid bounded above by $z = 8 - x^2 - y^2$ and bounded below by $z = x^2 + 3y^2$ has density $\rho(x, y, z) = x^2 + y^2$. Express the mass of this solid as a triple integral.

In this question, the floor and the ceiling are clear. But the bounds for x and y are less clear. To figure this out, ask: where do the surfaces intersect?

$$8 - x^2 - y^2 = x^2 + 3y^2 \implies 4 = x^2 + 2y^2$$

They intersect in an ellipse $4 = x^2 + 2y^2$.



Top half: $y = +\sqrt{2 - \frac{x^2}{2}}$

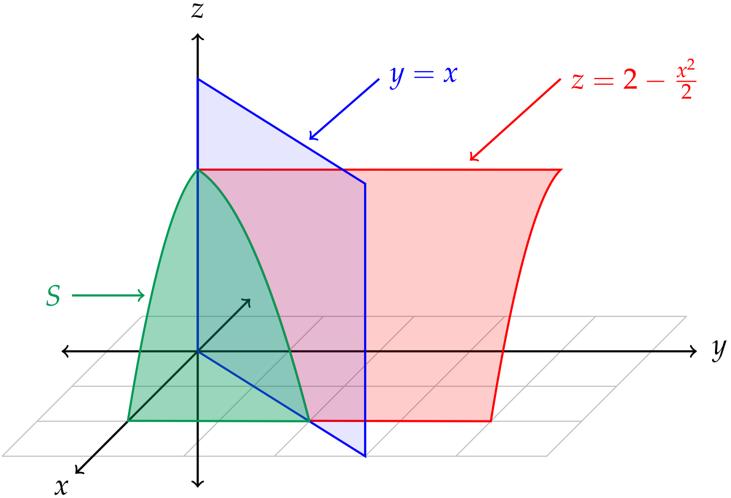
Bottom half: $y = -\sqrt{2 - \frac{x^2}{2}}$

$$V = \int_{-2}^{2} \int_{-\sqrt{2 - \frac{x^2}{2}}}^{\sqrt{2 - \frac{x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} x^2 + y^2 \, dz \, dy \, dx$$

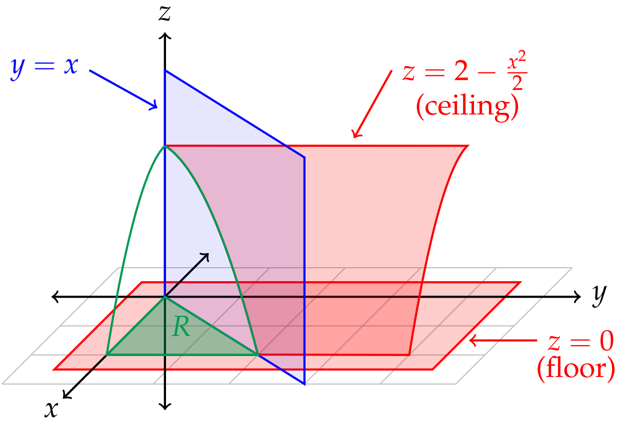
Switching Bounds

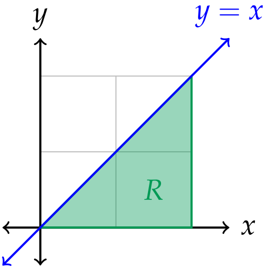
Example

Compute the volume of the solid S in the first octant bounded by $y = x$ and $z = 2 - \frac{x^2}{2}$



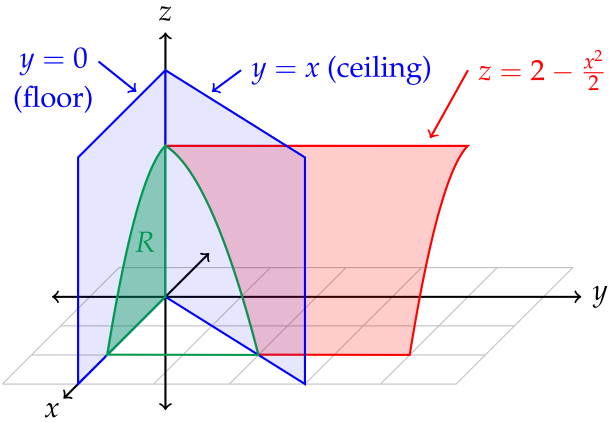
$$V = \iiint 1\,dz\,dy\,dx$$

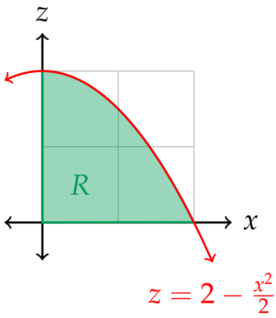




$$V = \iint_R \left(\int_0^{2-\frac{x^2}{2}} 1\,dz \right) dA = \int_0^2 \int_0^x \int_0^{2-\frac{x^2}{2}} 1\,dz\,dy\,dx$$

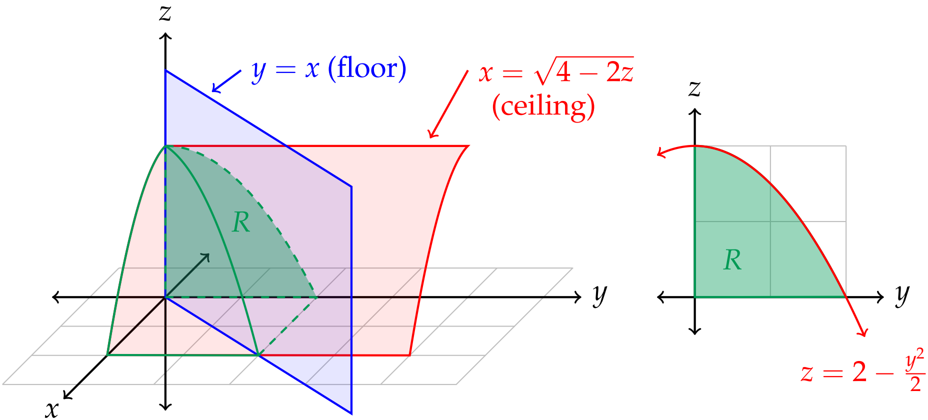
$$V = \iiint 1\, dy\, dx\, dz$$





$$V = \iint_R \left(\int_0^x 1\, dy \right)\, dA = \int_0^2 \int_0^{\sqrt{4-2z}} \int_0^x 1\, dy\, dx\, dz$$

$$V = \iiint 1\,dx\,dz\,dy$$



$$V = \iint_R \left(\int_0^x 1\,dx \right)\,dA = \int_0^2 \int_0^{2-\frac{y^2}{2}} \int_y^{\sqrt{4-2z}} 1\,dx\,dz\,dy$$

ATTENDANCE QUESTION saunter

Which integral computes the volume of the region below the plane $z = y$ and above the triangle in the xy-plane with vertices $(2, 0)$, $(5, 0)$ and $(5, 6)$?

(a) $\int_0^5 \int_1^x \int_0^y 1 \, dz \, dy \, dx$

(b) $\int_2^5 \int_0^{2x} \int_0^z 1 \, dz \, dy \, dx$

(c) $\int_2^5 \int_0^{2x-4} \int_0^y 1 \, dz \, dy \, dx$

(d) $\int_0^5 \int_0^{2x+4} \int_0^y 1 \, dz \, dy \, dx$

SUMMARY

- Triple integrals are used to compute volume, mass, moment of inertia, etc.

- **Fubini's Theorem:** if $f(x,y,z)$ is continuous, the triple integral $\iiint_S f(x,y,z)\,dV$ can be computed in any order.

- Triple integrals are double integrals of single integrals: project the solid S to some region R in a coordinate plane $u = 0$, and then

$$\iiint_S f(x,y,z)\,dV = \iint_R \left(\int_{\text{floor}}^{\text{ceiling}} f(x,y,z)\,du \right) dA.$$

  - If the innermost integral is du, the region R will be in the plane $u = 0$ (where u is one of x,y, or z).

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LEARNING OBJECTIVES

- Plot points and functions in cylindrical coordinates
- Compute triple integrals in cylindrical coordinates

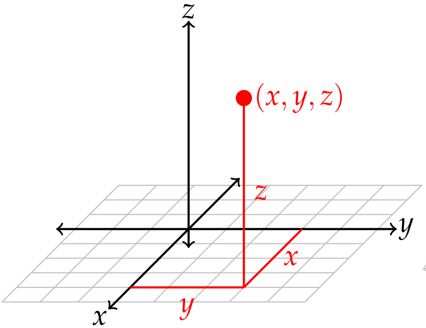
REVIEW

- Polar coordinates

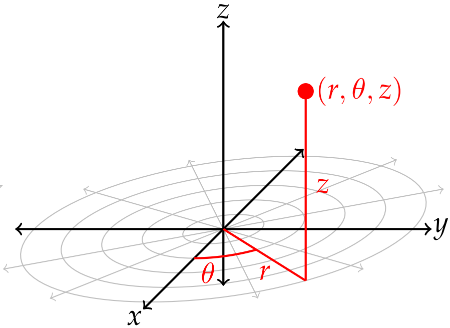
ANNOUNCEMENTS

- Webwork B7 due tonight

Cylindrical Coordinates



Rectangular



Cylindrical

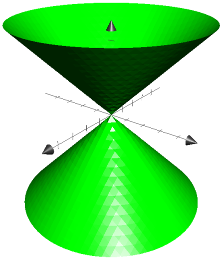
$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z$$

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$
$$z = z$$

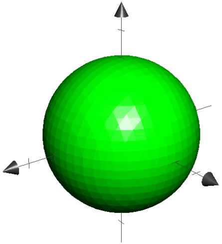
Cylindrical coordinates uses polar coordinates in the xy-plane and keeps the z-coordinate for height.

EXAMPLE

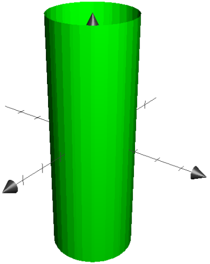
Write equations in cylindrical coordinates for each of the following surfaces.



$r = z$



$r^2 + z^2 = c^2$



$r = c$

BASIC SHAPES IN CYLINDRICAL COORDINATES

The basic shapes in cylindrical coordinates are:

- r is constant: cylinder
- θ is constant: plane containing the z-axis
- z is constant: horizontal plane

EXAMPLE

In cylindrical coordinates, describe the solid S in $\mathbb{R}^3$ bounded by the cylinder $x^2 + (y-1)^2 = 1$, the paraboloid $z = x^2 + y^2$, and the xy-plane.

The cylinder is:

$$x^2 + (y-1)^2 = 1 \implies (r\cos\theta)^2 + (r\sin\theta - 1)^2 = 1$$

Simplifying, this becomes $r(r - 2\sin\theta) = 0$. Since $r = 0$ is just the z-axis, this cylinder is described by $r = 2\sin\theta$.

The paraboloid is $z = r^2$, and the xy-plane is $z = 0$.

This is the region bounded below by $z = 0$ and above $z = r^2$ and bounded laterally by $r = 2\sin\theta$.

$$\begin{aligned} 0 \leq& z \leq r^2 \\ 0 \leq& r \leq 2\sin\theta \\ 0 \leq& \theta \leq \pi \end{aligned}$$

(We only need $0 \leq \theta \leq \pi$, because when $\theta = \pi$, $r = 0$.)

Integration in cylindrical coordinates

Cylindrical Triple Integral

$$\iiint_S f(x,y,z)\,dV = \iiint_S f(r\cos(\theta), r\sin(\theta), z)\,r\,dr\,d\theta\,dz$$

Example

Write an integral in cylindrical coordinates to compute the volume of a cylinder with radius 1 and height 1.

In cylindrical coordinates, this cylinder (and its interior) is $0 \le r \le 1$, $0 \le \theta \le 2\pi$, and $0 \le z \le 1$. So

$$V = \int_0^1 \int_0^{2\pi} \int_0^1 r\,dr\,d\theta\,dz$$

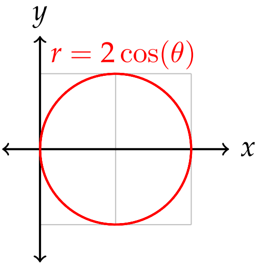
EXAMPLE

Use a triple integral to compute the volume of the solid S bounded above by the paraboloid $z = 4 - x^2 - y^2$, bounded below by $z = 0$, and bounded laterally by $y = 0$ and $x^2 + y^2 = 2x$.

Let's first convert this to polar coordinates.

- Paraboloid: $z = 4 - x^2 - y^2 \implies z = 4 - r^2$.
- $x^2 + y^2 = 2x \implies r^2 = 2r\cos(\theta) \implies r = 2\cos(\theta)$.

This is an off-center cylinder. In the plane $z = 0$ it looks like:

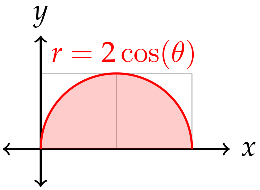


However, since we must have $y \geq 0$, we should restrict the domain:

$$0 \leq \theta \leq \pi/2$$

$$0 \leq r \leq 2\cos\theta$$

With the domains properly restricted, the shadow in the xy-plane looks like this:



Above each point in this shadow, the z-coordinate of the solid is:

$$0 \leq z \leq 4 - r^2$$

Therefore, the volume of this solid is:

$$V = \int_0^{\pi/2} \int_0^{2\cos\theta} \int_0^{4-r^2} r \, dz \, dr \, d\theta$$

Note: the non-cylindrical version of this integral requires trig sub and is much much harder!

Let's evaluate this integral.

$$V = \int_0^{\pi/2} \int_0^{2\cos\theta} \int_0^{4-r^2} r\,dz\,dr\,d\theta = \int_0^{\pi/2} \int_0^{2\cos\theta} \Big[rz \Big]_{z=0}^{z=4-r^2} dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} 4r - r^3\,dr\,d\theta = \int_0^{\pi/2} \Big[2r^2 - \tfrac{1}{4}r^4 \Big]_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_0^{\pi/2} 8\cos^2\theta - 4\cos^4\theta\,d\theta$$

To evaluate this integral, we use the power reducing identity:

$$\cos^2(\theta) = \tfrac{1}{2}(1 + \cos(2\theta))$$

Squaring this and reducing, we get

$$\cos^4(\theta) = \tfrac{1}{4}\left(1 + 2\cos(2\theta) + \tfrac{1}{2}\left(1 + \cos(4\theta)\right)\right)$$

Therefore,

$$\int_0^{\pi/2} 8\cos^2\theta - 4\cos^4\theta\,d\theta = \int_0^{\pi/2} \frac{5}{2} + 2\cos(2\theta) - \frac{1}{2}\cos(4\theta)\,d\theta = \boxed{\frac{5\pi}{4}}$$

EXAMPLE

Evaluate the integral

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} x^2 + y^2 \, dz \, dy \, dx$$

This integral is easier to do in cylindrical coordinates. Let's figure out what the solid of integration looks like.

$$-\sqrt{2} \leq x \leq \sqrt{2}$$
$$-\sqrt{2 - x^2} \leq y \leq \sqrt{2 - x^2}$$
$$\sqrt{x^2 + y^2} \leq z \leq \sqrt{4 - x^2 - y^2}$$

Ignoring the z-bound for a second to figure out the shadow in the xy-plane, notice that the y bound is $y^2 = 2 - x^2$, or $x^2 + y^2 = 2$. Since x ranges from $-\sqrt{2}$ to $\sqrt{2}$, the shadow in the xy-plane is full circle of radius $\sqrt{2}$.

So the shadow in the xy-plane becomes in cylindrical coordinates:

$$0 \leq r \leq \sqrt{2}, \quad 0 \leq \theta \leq 2\pi$$

We just have to figure out the bounds for z. The floor is the cone:

$$z = \sqrt{x^2 + y^2} \implies z = r$$

And the ceiling is the top half of a sphere:

$$z = \sqrt{4 - x^2 - y^2} \implies z = \sqrt{4 - r^2} \implies z^2 + r^2 = 4$$

Therefore, the integral is in cylindrical coordinates:

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} x^2 + y^2 \, dz \, dy \, dx$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r^2 \, r \, dz \, dr \, d\theta$$

Let's evaluate this integral.

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} r^3\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^{\sqrt{2}}(\sqrt{4-r^2}-r)r^3\,dr\,d\theta \\
&= 2\pi\left(\int_0^{\sqrt{2}} r^3\sqrt{4-r^2}-r^4\,dr\right) \\
&= 2\pi\int_0^{\sqrt{2}} r^3\sqrt{4-r^2}\,dr - 2\pi\int_0^{\sqrt{2}} r^4\,dr \\
&= 2\pi\int_0^{\sqrt{2}} r^3\sqrt{4-r^2}\,dr - 2\pi\left[\frac{r^5}{5}\right]_0^{\sqrt{2}} \\
&= 2\pi\int_0^{\sqrt{2}} r^3\sqrt{4-r^2}\,dr - \frac{8\pi\sqrt{2}}{5}
\end{aligned}$$

To evaluate this last remaining integral, substitute $u = 4 - r^2$, $du = -2r\,dr$. This also gives $r^2 = 4 - u$.

$$
\begin{aligned}
2\pi \int_0^{\sqrt{2}} r^3 \sqrt{4 - r^2}\, dr &= 2\pi \int_4^2 (4-u)\sqrt{u}\frac{du}{-2} \\
&= -\pi \int_4^2 4\sqrt{u} - u^{3/2}\, du \\
&= -\pi \left[\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5} \right]_{u=4}^{u=2} \\
&= -\pi \left(\left(\frac{4\sqrt{2}}{3} - \frac{8\sqrt{2}}{5} \right) - \left(\frac{16}{3} - \frac{64}{5} \right) \right) \\
&= -\frac{4\pi\sqrt{2}}{3} + \frac{8\pi\sqrt{2}}{5} + \frac{16\pi}{3} - \frac{64\pi}{5}
\end{aligned}
$$

So the final answer is:

$$
\boxed{\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r^3\, dz\, dr\, d\theta = -\frac{4\pi\sqrt{2}}{3} + \frac{16\pi}{3} - \frac{64\pi}{5}}
$$

Attendance Question stroll

Express the integral

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{x^2+y^2+1} xz^2 dzdydx$$

in cylindrical coordinates.

(a) $\int_0^{\pi} \int_0^{2} \int_0^{r^2+1} xz^2\, r\, dz\, dr\, d\theta$

(b) $\int_0^{\pi/2} \int_0^{2} \int_0^{r^2+1} r\cos(\theta) z^2\, r\, dz\, dr\, d\theta$

(c) $\int_0^{\pi/2} \int_0^{2} \int_0^{r^2+1} r\cos(\theta) z^2\, dz\, dr\, d\theta$

(d) $\int_0^{\pi/2} \int_0^{4} \int_0^{r^2+1} r\cos(\theta) z^2\, r\, dz\, dr\, d\theta$

SUMMARY

- Cylindrical coordinates (r, θ, z) are the 3D version of polar coordinates.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad z = z$$

- The basic shapes in cylindrical coordinates are:
  - Constant r: cylinder
  - Constant θ: plane containing the z-axis
  - Constant z: horizontal plane
- Triple integrals in cylindrical coordinates replace $dx\,dy$ with $r\,dr\,d\theta$, like with polar coordinates.

$$\iiint_S f(x, y, z)\,dV = \iiint_S f(r\cos(\theta), r\sin(\theta), z)\,r\,dr\,d\theta\,dz$$

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LEARNING OBJECTIVES

- Plot points and functions in spherical coordinates
- Compute triple integrals in spherical coordinates

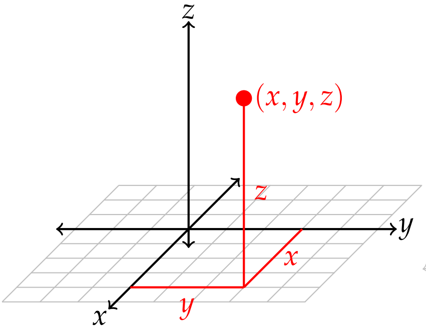
REVIEW

- Polar coordinates

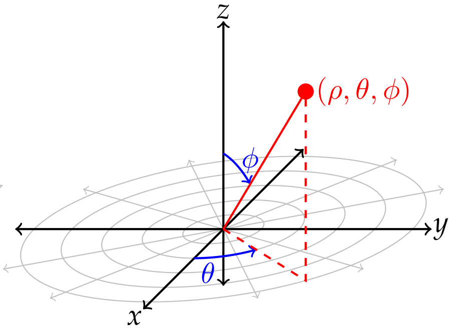
ANNOUNCEMENTS

- Webwork B8 due tonight

Spherical Coordinates



Rectangular

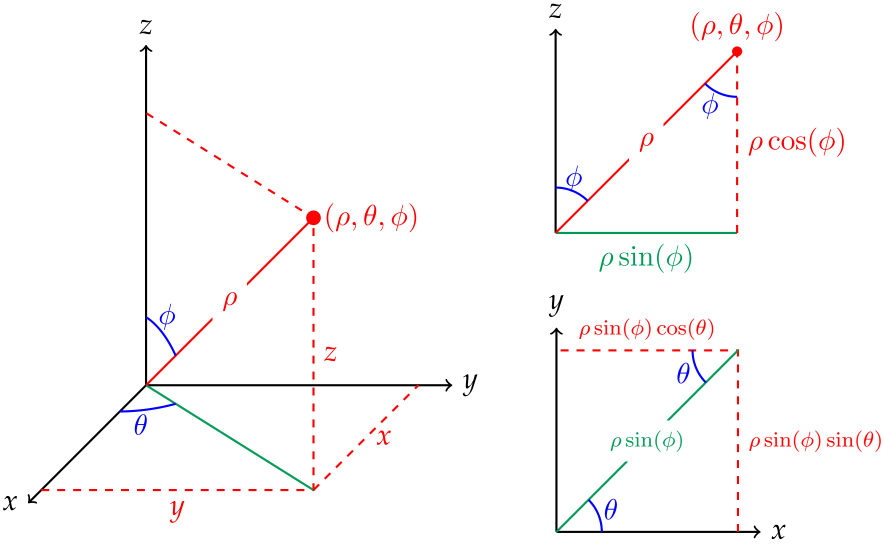


Spherical

- ρ distance from the origin, $\rho \geq 0$.
- θ angle from positive x-axis, $0 \leq \theta \leq 2\pi$
- ϕ angle from positive z-axis, $0 \leq \phi \leq \pi$

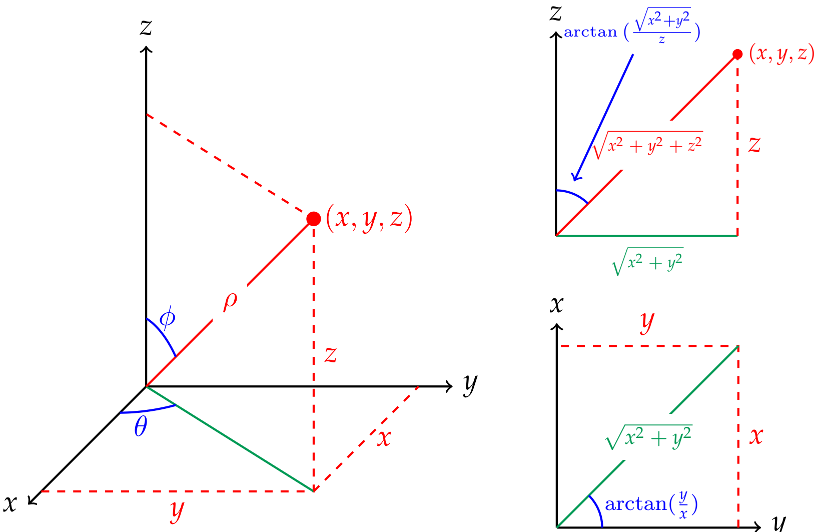
Converting coordinates

Given spherical coordinates for a point (ρ, θ, ϕ), how do you find its rectangular coordinates?



Converting coordinates

Given rectangular coordinates for a point (x, y, z), how do you find its spherical coordinates?



Converting coordinates

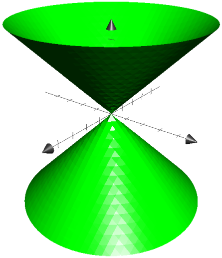
Rectangular to Spherical

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$
$$\phi = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

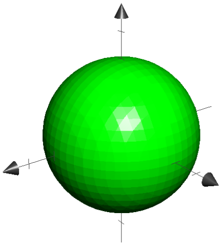
Spherical to Rectangular

$$x = \rho\cos(\theta)\sin(\phi)$$
$$y = \rho\sin(\theta)\sin(\phi)$$
$$z = \rho\cos(\phi)$$

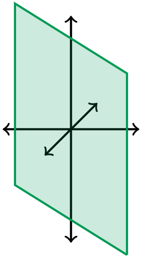
Basic Shapes



$\phi =$ constant



$\rho =$ constant



$\theta =$ constant

Basic Shapes in Cylindrical Coordinates

The basic shapes in cylindrical coordinates are:

- ρ is constant: sphere
- θ is constant: plane containing the z-axis
- ϕ is constant: cone

EXAMPLE

Describe the equation $x^2 + y^2 = 4$ in spherical coordinates.

Replace x and y by their equations in spherical coordinates:

$$\begin{aligned}
(\rho \cos(\theta) \sin(\phi))^2 + (\rho \sin(\theta) \sin(\phi))^2 &= 4 \\
\rho^2 \cos^2(\theta) \sin^2(\phi) + \rho^2 \sin^2(\theta) \sin^2(\phi) &= 4 \\
\rho^2 \sin^2(\phi)(\cos^2(\theta) + \sin^2(\theta)) &= 4 \\
\rho^2 \sin^2(\phi) &= 4 \\
\rho \sin(\phi) &= 2
\end{aligned}$$

Alternatively, we could use $\rho^2 = x^2 + y^2 + z^2$, so $x^2 + y^2 = \rho^2 - z^2$

$$\begin{aligned}
x^2 + y^2 &= 4 \\
\rho^2 - z^2 &= 4 \\
\rho^2 - (\rho \cos(\phi))^2 &= 4 \\
\rho^2(1 - \cos^2(\phi)) &= 4 \\
\rho^2 \sin^2(\phi) &= 4
\end{aligned}$$

Attendance Question rhythm

Express the surface $x^2 - 2x + y^2 + z^2 = 0$ as an equation in spherical coordinates.

(a) $\rho^2 - 2\rho\cos(\theta) + z^2 = 0$

(b) $\sin(\phi)\cos(\theta) = 1$

(c) $\rho = 2\cos(\theta)\sin(\phi)$

(d) $\rho^2 - 2\rho\cos(\phi) = 0$

(e) None of the above.

Integrals in Spherical Coordinates

$$\iiint_S f(x, y, z)\, dV = \iint_S f(\rho, \theta, \phi)\rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$

Example

Setup an integral in spherical coordinates to compute the volume of a solid ball of radius 1.

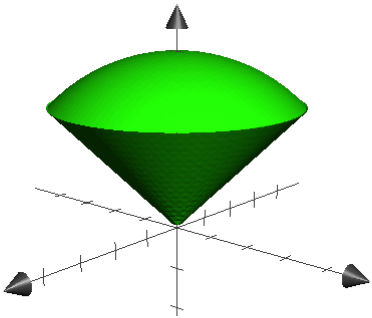
A solid ball in spherical coordinates is described by $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$.

$$V = \int_0^{\pi} \int_0^{2\pi} \int_0^{1} \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$

Example

Find the volume inside the sphere $x^2 + y^2 + z^2 = 4$ and inside the cone $x^2 + y^2 = z^2$.

This shape is an ice cream cone.



The sphere is $\rho^2 = 4$, or $\rho = 2$

The cone is $\phi = \frac{\pi}{4}$

So the shape is described by:

$$0 \le \rho \le 2$$
$$0 \le \phi \le \frac{\pi}{4}$$
$$0 \le \theta \le 2\pi$$

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^2 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

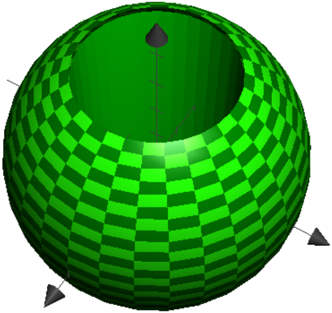
Let's evaluate this integral.

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^2 \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta &= 2\pi\int_0^{\frac{\pi}{4}}\int_0^2 \rho^2\sin(\phi)\,d\rho\,d\phi \\
&= 2\pi\int_0^{\frac{\pi}{4}}\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=2}\sin(\phi)\,d\phi \\
&= 2\pi\int_0^{\frac{\pi}{4}}\frac{8}{3}\sin(\phi)\,d\phi \\
&= \frac{16\pi}{3}\Big[-\cos(\phi)\Big]_{\phi=0}^{\phi=\frac{\pi}{4}} \\
&= \frac{16\pi}{3}\left(1-\cos\left(\tfrac{\pi}{4}\right)\right) \\
&= \boxed{\frac{16\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)}
\end{aligned}$$

EXAMPLE

A cylindrical hole of radius 1 is drilled through a sphere of radius 2. Find the volume of the resulting shape.

This shape is a bead.



The sphere is $\rho = 2$

The cylinder is $\rho \sin(\phi) = 1$ (see earlier example)

These intersect when:

$$2\sin(\phi) = 1 \implies \phi = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$V = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{\csc(\phi)}^{2} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{\csc(\phi)}^{2}\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta = 2\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{\csc(\phi)}^{2}\rho^2\sin(\phi)\,d\rho\,d\phi$$

$$= 2\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\left[\frac{\rho^3}{3}\right]_{\rho=\csc(\phi)}^{\rho=2}\sin(\phi)\,d\phi$$

$$= 2\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\left(\frac{8}{3}-\frac{\csc^3(\phi)}{3}\right)\sin(\phi)\,d\phi$$

$$= \frac{16\pi}{3}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\sin(\phi)\,d\phi - \frac{2\pi}{3}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\csc^3(\phi)\sin(\phi)\,d\phi$$

$$= \frac{16\pi}{3}\Big[-\cos(\phi)\Big]_{\phi=\frac{\pi}{6}}^{\phi=\frac{5\pi}{6}} - \frac{2\pi}{3}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\csc^2(\phi)\,d\phi$$

$$= \frac{16\pi}{3}\left(-\cos\left(\frac{5\pi}{6}\right)+\cos\left(\frac{\pi}{6}\right)\right) - \frac{2\pi}{3}\Big[-\cot(\phi)\Big]_{\phi=\frac{\pi}{6}}^{\phi=\frac{5\pi}{6}}$$

$$= \frac{16\pi}{3}\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right) - \frac{2\pi}{3}\left(\sqrt{3}+\sqrt{3}\right) = \boxed{4\pi\sqrt{3}}$$

SUMMARY

- Spherical coordinates keep the θ from cylindrical coordinates, but everything else is different.
  - ρ distance from the origin, $\rho \geq 0$.
  - θ angle from positive x-axis, $0 \leq \theta \leq 2\pi$
  - ϕ angle from positive z-axis, $0 \leq \phi \leq \pi$
- To convert between rectangular and spherical coordinates,

$$x = \rho\cos(\theta)\sin(\phi), \quad y = \rho\sin(\theta)\sin(\phi), \quad z = \rho\cos(\phi)$$

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad \phi = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

- Integrals in spherical coordinates can be calculated using

$$\iiint_S f(x, y, z)\, dV = \iint_S f(\rho, \theta, \phi)\rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$