THE SMASH PRODUCT NAME: ____________________

Due at the beginning of class on 19 March 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Sections 4.1 and 4.3] and the introduction to [Lew91].

(1) Define spectrum structure maps for a "fake smash product" $\wedge_{\text{fake}} \colon \mathcal{S}p \times \mathcal{S}p \to \mathcal{S}p$ with n-th space

$$(X \wedge_{\text{fake}} Y)_n = X_n \wedge Y_n.$$

Argue that, with your choice of structure maps, $(\mathcal{S}p, \wedge_{\text{fake}}, \mathbb{S})$ is *not* a symmetric monoidal category.

(2) (a) Let $\mathcal{C}$ be a category that is both additive and symmetric monoidal with product $\otimes$ and unit I. Assume that $\otimes$ preserves coproducts in each variable separately. Prove that the product induces bilinear maps

$$\mathcal{C}(A, B) \times \mathcal{C}(X, Y) \to \mathcal{C}(A \otimes X, B \otimes Y)$$

(b) Show that there is no natural transformation in the stable homotopy category $\delta \colon X \to X \wedge X$ that agrees with the diagonal on suspension spectra

$$\Sigma^\infty K \xrightarrow{\Sigma^\infty \Delta} \Sigma^\infty (K \wedge K) \cong \Sigma^\infty K \wedge \Sigma^\infty K.$$

Hint: argue by contradiction and apply δ to $\mathbb{S} \xrightarrow{2} \mathbb{S}$.

(3) Assume that we are given a closed symmetric monoidal category of spectra $(\widehat{\mathcal{S}p}, \wedge, \mathbb{S})$ such that $\wedge$ preserves colimits and weak equivalences in each variable separately, and for any pointed space K, $\Sigma^\infty K \wedge X \cong K \wedge X$. The right adjoint to $X \wedge -$ is the function spectrum $F(X, -)$.

(a) Prove that for any integer $n \in \mathbb{Z}$, the smash product $\mathbb{S}^n \wedge X$ is stably equivalent to $\Sigma^n X$.

(b) Prove that for any integer $n \in \mathbb{Z}$, the function spectrum $F(\mathbb{S}^n, X)$ is stably equivalent to $\Sigma^{-n} X$.

(4) Let X be an $H\mathbb{F}_p$-module spectrum. What does this imply about the homotopy groups of X? Can you say anything about the homology groups?

REFERENCES

[bip] nLab: biproduct. "https://ncatlab.org/nlab/show/biproduct".

[Lew91] L. Gaunce Lewis, Jr. Is there a convenient category of spectra? *J. Pure Appl. Algebra*, 73(3):233–246, 1991.

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.