STABILITY NAME: ________________________

Due at the beginning of class on 11 March 2025

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Sections 2.4 and 2.5].

(1) Consider the commuting square of spectra

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

(a) Prove that this square is a homotopy pushout if and only if the induced map of homotopy cofibers $\mathrm{cof}(f) \to \mathrm{cof}(g)$ is a stable equivalence.

(b) Use this fact to prove that a commuting square of spectra is a homotopy pullback if and only if it is a homotopy pushout. You may use the fact that a sequence of spectra is a cofiber sequence if and only if it is a fiber sequence.

(2) An *semiadditive category* is a category $\mathcal{A}$ with a zero object 0 that admits all finite products and coproducts, such that the canonical morphism $X \amalg Y \to X \times Y$ is an isomorphism.

(a) Show that $\mathrm{ho}(\mathcal{S}p)$ is a semiadditive category.

An *additive functor* between semiadditive categories is a functor that preserves the zero object and preserves finite products/coproducts.

(b) Is the Eilenberg–MacLane spectrum functor $H\colon \mathcal{A}b \to \mathrm{ho}(\mathcal{S}p)$ an additive functor?

(3) Let $k \geq 0$. Define the *shift functor* $\mathrm{sh}_k\colon \mathcal{S}p \to \mathcal{S}p$ by $\mathrm{sh}_k(X)_n = X_{k+n}$.

(a) Prove that there is a natural stable equivalence $\Sigma \simeq \mathrm{sh}_1$.

(b) Define functors sh_k for $k < 0$. Prove that $\mathrm{sh}_{-1} \simeq \Omega$.

REFERENCES

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.