Due at the beginning of class on 18 February 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Sto22, Chapter 2].

(1) A theorem of Serre shows that $\pi_i(S^n)$ for $i > 2$ is a finite abelian group, except for two classes of exceptions: $\pi_n(S^n) \cong \mathbb{Z}$ and $\pi_{4j-1}(S^{2j}) \cong \mathbb{Z} \oplus M$, where M is a torsion $\mathbb{Z}$-module. Use this to prove that the stable homotopy groups $\pi_i^s(S^0)$ are finite abelian for $i > 0$.

SOLUTION: Recall that the stable homotopy groups of S^0 are defined as follows:

$$\pi_i^s(S^0) = \mathrm{colim}_{n\to\infty}\pi_{i+n}(\Sigma^n S^0) = \mathrm{colim}_{n\to\infty}\pi_{i+n}(S^n)$$

Since $i > 0$, the fact that $\pi_n(S^n) \cong \mathbb{Z}$ is not relevant since it only impacts $\pi_0^s(S^0)$. Thus, we need only show that the colimit of each sequence containing $\pi_{4j-1}(S^{2j})$ is finite abelian. For $i > 0$, every sequence in the colimit $\pi_i^s(S^0)$ will eventually stabilize by the Freudenthal Suspension Theorem. Suppose this stabilization occurs at $\pi_{4j-1}(S^{2j})$. This would imply:

$$\pi_{4j-1}(S^{2j}) \cong \pi_{4j}(S^{2j+1})$$

But $\pi_{4j}(S^{2j+1})$ is finite abelian by Serre's Theorem, so stabilization cannot occur at $\pi_{4j-1}(S^{2j})$. Hence, $\pi_i^s(S^0)$ must be finite abelian for $i > 0$.

(2) A pointed space X is *well-based* if the inclusion of the basepoint is a cofibration. Let $f\colon X \to Y$ be a pointed map of well-based spaces.

(a) Let $\mathrm{cof}(f)$ be the homotopy cofiber of f. Prove that the homotopy cofiber of $Y \to \mathrm{cof}(f)$ is homotopy equivalent to ΣX.

(b) Prove the dual statement: if $\mathrm{fib}(f)$ is the homotopy fiber of f, then the homotopy fiber of $\mathrm{fib}(f) \to X$ is homotopy equivalent to ΩY.

SOLUTION:

(a) Consider the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \mathrm{cof}(f) \arrow[r] & \mathrm{cof}(Y \to \mathrm{cof}(f)) \end{tikzcd}$$

Both squares are individually homotopy pushouts. Thus, by a version of the pasting lemma, the rectangle is a homotopy pushout.

(To see this explicitly, recall that homotopy pushouts are computed by cofibrant replacement and literal pushout. Cofibrantly replace the top left cospan to compute $\mathrm{cof}(f)$ by literal pushout. Then, cofibrantly replace the entire square, along with the map to the point. The square is still a pushout after cofibrant replacement and now $\mathrm{cof}(Y \to \mathrm{cof}(f))$ can be computed by literal pushout. Now the literal pasting lemma yields that the entire diagram is a literal pushout, which proves the homotopy pasting lemma.)

Since the pushout of X mapping to points is the suspension of X, this demonstrates that $\mathrm{cof}(Y \to \mathrm{cof}(f))$ is homotopy equivalent to ΣX. The well-pointedness assumption guarantees that this homotopy pushout is the same as the smash product $S^1 \wedge X$, but if you take the pushout as the definition of suspension, the well-pointedness assumtpion is not needed.

(b) Prove the dual statement: if $\mathrm{fib}(f)$ is the homotopy fiber of f, then the homotopy fiber of $\mathrm{fib}(f) \to X$ is homotopy equivalent to ΩY.

Dually, consider the diagram

$$\begin{tikzcd} \mathrm{fib}(\mathrm{fib}(f) \to X) \arrow[r] \arrow[d] & \mathrm{fib}(f) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Both squares are homotopy pullbacks, so the rectangle is a homotopy pullback by the same reasoning to the above. But since the pushout of points along Y is the loop-space of Y, this demonstrates that $\mathrm{fib}(\mathrm{fib}(f) \to X)$ is homotopy equivalent to ΩY.

This problem is [Mal23, Exercise 16 in Chapter 1]. You can probably also find it (without any category theory) in Hatcher.

(3) Let $f \colon X \to Y$ be a map between simply connected spaces such that $f_* \colon H_i(X) \to H_i(Y)$ is an isomorphism for $i \leq n$. We will show that f is an n-connected map.

(a) Let C be the homotopy cofiber of f, and let F be the homotopy fiber of $Y \to C$. Use the Hurewicz theorem to show that C is n-connected and $F \to Y$ is an n-connected map.

Solution: The cofiber C of f is the homotopy pushout of $* \leftarrow X \xrightarrow{f} Y$, giving a Mayer-Vietoris exact sequence

$$\ldots \to H_i(X) \xrightarrow{f_*} H_i(Y) \to H_i(C) \xrightarrow{\partial} H_{i-1}(X) \xrightarrow{f_*} H_{i-1}(Y) \to \ldots$$

The map $f_* : H_i(X) \to H_i(Y)$ is an isomorphism for $i \leq n$. By exactness, $H_i(C)$ is then 0 for $i \leq n$. Furthermore, since X and Y are 1-connected, C is 1-connected as well. Assuming $n > 1$, the Hurewicz Theorem can be used to assert that $\pi_2(C) \cong H_2(C) = 0$. C is then 2-connected and we can iterate the argument to get $\pi_3(C) \cong H_3(C)$. This may proceed until we reach $\pi_n(C) \cong H_n(C) = 0$, and hence C is n-connected. To show that $F \to Y$ is an n-connected map, consider the long exact sequence in homotopy induced by the fiber sequence $F \to Y \to C$:

$$\ldots \to \pi_{i+1}(Y) \to \pi_{i+1}(C) \xrightarrow{\partial} \pi_i(F) \to \pi_i(Y) \to \pi_i(C) \to \ldots$$

We have $\pi_i(C) = 0$ for $i \leq n$, so $\pi_i(F) \to \pi_i(Y)$ is an isomorphism in that range, showing that the map $F \to Y$ is n-connected.

(b) Use the Blakers–Massey theorem to show that $X \to F$ is at least 2-connected.

Solution:

We have the following diagram:

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& F \arrow[r, "p"] \arrow[d] & Y \arrow[d, "q"] \\
& * \arrow[r] & C
\end{tikzcd}$$

where the outer square is a homotopy pushout and the inner square is a homotopy pullback. Thus, the Blakers-Massey theorem can be applied to the induced map $X \to F$.

Since X is 1-connected, the map g is at least 2-connected. Furthermore, since Y is also 1-connected, the map f is at least 1-connected. Hence, by the Blakers-Massey theorem, the map $X \to F$ is at least (2+1-1=2)-connected.

(c) Show that f is at least 2-connected. Iterate your argument from part (b) to show that f is n-connected.

SOLUTION: We can write f as a composite:

$$f\colon X \underset{2}{\longrightarrow} \mathrm{fib}(g) \underset{n}{\longrightarrow} Y;$$

the connectivity is drawn under the arrows. f is the composite of a 2-connected map and an n-connected map, so f must be 2-connected.

Now we can repeat the argument of the previous part: by Blakers–Massey, f is 2-connected and $X \to *$ is 2-connected, so $X \to \mathrm{fib}(g)$ must be at least $(2+2-1=3)$-connected. Then f is the composite of a 3-connected map and an n-connected map, so it must be 3-connected. Rinse and repeat to conclude f is n-connected.

(4) Let $X_0 \to X_1 \to X_2 \to \cdots$ be a sequence of spaces. Prove that $\Omega \operatorname{hocolim}_i X_i \simeq \operatorname{hocolim}_i \Omega X_i$. Use this to show that homotopy groups commute with sequential homotopy colimits.

SOLUTION: This relies on the fact that S^1 is compact, and that we are computing the homotopy colimit. This is a particularly thorny problem, and if you really get into the weeds you'll find yourself questioning what compactness even means. (Compact objects in $\mathcal{T}op$ vs compact spaces – they're different! And S^1 is only the latter.)

Recall that to compute $\operatorname{hocolim}_i X_i$, we cofibrantly replace the sequence X_i with a sequence Y_i and the $\operatorname{hocolim}_i X_i \simeq \operatorname{colim}_i Y_i$. The standard choice is to pick Y_i to be the partial mapping telescopes, and the maps in the sequence of partial mapping telescopes are closed inclusions.

Since $\Omega = F(S^1, -)$ preserves homotopy, the ΩY_i are equivalent to the ΩX_i. Further, the maps between the ΩY_i's are also closed inclusions (ΩY_i can be identified with the functions into Y_{i+1} that happen to land in Y_i). Therefore, $\operatorname{hocolim}_i \Omega X_i \simeq \operatorname{colim}_i \Omega Y_i$. Thus, it suffices to show that $\Omega \operatorname{colim}_i Y_i \simeq \operatorname{colim}_i \Omega Y_i$.

Let $Y = \operatorname{colim}_i Y_i = \bigcup_i Y_i$. There is a map $\operatorname{colim}_i \Omega Y_i \to \Omega Y$ induced by the universal property of the colimit. Concretely, it takes a function (or an equivalence class of functions) into a partial mapping telescope and interprets it as a function into the mapping telescope Y.

It is a standard exercise in point set topology that this is a homeomorphism. More generally, if K is compact,

$$\operatorname*{colim}_i F(K, Y_i) \to F(K, Y)$$

is a homeomorphism. We give two arguments:

(a) To produce the inverse, recognize that the $Y_i \setminus X_i$'s form an open cover of Y, so their preimages cover K. Since K is compact, given any map $K \to Y$, only finitely many of these preimages are needed to cover K, and therefore K lands in only finitely many of the partial mapping telescopes. If we take Y_i to be the large,st such partial mapping telescope, we get a map $K \to Y_i$. It is straight-forward to show that this is inverse to the canonical map.

(b) It is not too hard to see that this map is an injection, and that it is a surjection if and only if the natural map $K \to Y$ factors through Y_i for some i.

So it remains to be seen that $f \colon K \to Y$ factors through one of the X_i. We are still assuming that these $Y_i \to Y_{i+1}$ are closed inclusions. It is now important that these spaces are CGWH. Assume for the sake of contradiction that $f \colon S^1 \to Y$ does not factor through any of the Y_i. Then there is $y_0 \in f(K)$ such that $y_0 \notin Y_0$. However, by properties of the colimit, there is $a_0 \in \mathbb{N}$ such that $y_0 \in Y_{a_0}$. Choose $y_1 \in f(S^1)$ such that $y_1 \notin X_{a_0}$. There exists $a_1 \in \mathbb{N}$ such that $y_1 \in Y_{a_1}$. Choose $y_2 \in f(K)$ such that $y_2 \notin Y_{a_1}$. Continue inductively to find a sequence of points $y_0, y_1, y_2, \ldots$ such that $y_i \in f(K) \setminus Y_{a_{i-1}}$. Consider the set $Z = \{y_0, y_1, y_2, \ldots\} \subseteq f(S^1)$. A subset $K \subseteq Z$ is closed if and only if $K \cap Y_i$ is closed for all i. By construction, $K \cap Y_i$ is finite and therefore closed by weak Hausdorff (in fact, T_1 would be enough). So any subset of Z is closed in Y, including Z itself.

Now consider $f^{-1}(Z) \subseteq S^1$. This is the preimage of a closed set, so closed itself. Therefore, $f^{-1}(Z)$ is compact, as a closed subset of a compact space. But by the above, it is also a discrete topological space since every subset is closed. But any discrete compact space is finite. This is a contradiction, since $f^{-1}(Z)$ is at least countable.

REFERENCES

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. `http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf`, October 2023.

[Sto22] Bruno Stonek. Introduction to stable homotopy theory. `https://bruno.stonek.com/stable-homotopy-2022/stable-online.pdf`, July 2022.