Prelim 2 is Tuesday, 7 November from 7:30pm to 9pm. It covers sections 7.7, 7.8, 7.9, 8.1, 8.2, 8.3, 8.5, 8.7, 8.8, 9.1, 9.4.

Warning: These problems are by no means a comprehensive representation of the material that might appear on the exam. That is, there may be topics not covered by these problems that you are still responsible for knowing. Let these problems be a supplement to your preparation for the exam, but be sure to review other sources (e.g. your notes, homework assignments, and the textbook) as well.

(1) Find the derivatives of the following functions.

(a) $f(t) = (\sin^2 t)^t$

SOLUTION: Let $y = (\sin^2 t)^t$. Then $\ln(y) = 2t \ln(\sin t)$. Now use implicit differentiation.

$$\frac{1}{y}\frac{dy}{dt} = 2\ln(\sin(t)) + \frac{2t}{\sin(t)}\cos(t)$$

$$\frac{dy}{dt} = \left(2\ln(\sin(t)) + \frac{2t\cos(t)}{\sin(t)}\right)y$$

$$\frac{dy}{dt} = \left(2\ln(\sin(t)) + \frac{2t\cos(t)}{\sin(t)}\right)(\sin^2 t)^t$$

(b) $g(t) = \tanh^{-1}(e^x)$

SOLUTION: $g'(t) = \frac{e^x}{1-e^{2x}}$

(c) $h(t) = \sqrt{t^2 - 1}\sinh^{-1} t$

SOLUTION: $h'(t) = \frac{x\,\text{arsinh}\,(e^x)}{\sqrt{x^2-1}} + \frac{\sqrt{x^2-1}e^x}{\sqrt{e^{2x}+1}}$

(2) Find the following definite or indefinite integrals.

(a) $\int (\sin x)(\cosh x)\,dx$

SOLUTION: Integrate by parts twice. This is one of those problems that goes in a circle.

$$\int (\sin x)(\cosh x)\,dx = \frac{\sin(x)\sinh(x) - \cos(x)\cosh(x)}{2} + C$$

(b) $\int \frac{dt}{\cosh^2 t + \sinh^2 t}$

SOLUTION: Use the identity $\cosh^2(t) - \sinh^2(t) = 1$ and solve for $\cosh^2(t)$. Then use $\sinh(t) = \frac{\tanh(x)}{\text{sech}(x)}$

$$\int \frac{dt}{\cosh^2 t + \sinh^2 t} = \int \frac{dt}{1 + 2\sinh^2(t)}$$

$$= \int \text{sech}^2(t)\frac{1}{\tanh^2(t) + 1}\,dt$$

Then let $u = \tanh(t)$, so $du = \text{sech}^2(t)\,dt$. Then

$$\int \frac{dt}{\cosh^2 t + \sinh^2 t} = \int \frac{dt}{1 + 2\sinh^2(t)}$$
$$= \int \text{sech}^2(t) \frac{1}{\tanh^2(t) + 1}\,dt$$
$$= \int \frac{1}{u^2 + 1}\,du$$
$$= \arctan(u) + C$$
$$= \arctan(\tanh^{-1}(t)) + C$$

(c) $\int \dfrac{dx}{x + x^{-1}}$

SOLUTION: Multiply numerator and denominator by x to get

$$\int \frac{dx}{x + x^{-1}} = \int \frac{x}{x^2 + 1}\,dx$$

Then let $u = x^2 + 1$, so $du = 2x\,dx$. This becomes

$$\int \frac{dx}{x + x^{-1}} = \int \frac{x}{x^2 + 1}\,dx$$
$$= \frac{1}{2} \int \frac{1}{u}\,du$$
$$= \frac{1}{2} \ln(u) + C$$
$$= \frac{1}{2} \ln(x^2 + 1) + C$$

(d) $\int \dfrac{dx}{x(x^2 - 1)^{3/2}}$

SOLUTION: Let $x = \sec(t)$, so $dx = \sec(t)\tan(t)\,dt$. Then

$$\int \frac{dx}{x(x^2 - 1)^{3/2}} = \int \frac{\sec t \tan t \, dt}{\sec t(\sec^2 t - 1)^{3/2}}$$
$$= \int \frac{\tan t}{\tan^3 t}\,dt$$
$$= \int \frac{1}{\tan^2 t}\,dt$$
$$= \int \cot^2(t)\,dt$$
$$= \int \csc^2(t) - 1\,dt$$
$$= -\cot t + t + C$$
$$= \frac{-1}{\sqrt{x^2 + 1}} + \sec^{-1}(x) + C$$

(e) $\int \dfrac{dx}{x^2 + 4x + 5}$

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SOLUTION: Complete the square.

$$\int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{(x+2)^2 + 1}$$

Then substitute $u = x + 2$, $du = dx$.

$$\int \frac{dx}{x^2 + 4x + 5} = \int \frac{du}{u^2 + 1} = \arctan(u) + C = \arctan(x+2) + C$$

(f) $\displaystyle\int \frac{dx}{x^2 + 4x - 5}$

SOLUTION: Complete the square, substitute $u = x + 2$, factor as a difference of squares, and then use partial fractions.

$$\begin{aligned}
\int \frac{dx}{x^2 + 4x - 5} &= \int \frac{dx}{(x+2)^2 - 9} \\
&= \int \frac{du}{u^2 - 9} \\
&= \int \frac{du}{(u-3)(u+3)} \\
&= \int \frac{1/6}{u-3} + \frac{-1/6}{u+3}\, du \\
&= \frac{1}{6} \ln|u-3| - \frac{1}{6} \ln|u+3| + C \\
&= \frac{1}{6} \ln|x+1| - \frac{1}{6} \ln|x+5| + C
\end{aligned}$$

(g) $\displaystyle\int_0^{\pi/2} \cot\theta\, d\theta$

SOLUTION: Because $\cot(0)$ is undefined, this integral is improper. So write

$$\int_0^{\pi/2} \cot\theta\, d\theta = \lim_{R \to 0} \int_R^{\pi/2} \cot(\theta)\, d\theta$$

To integrate, we can rewrite $\cot(\theta) = \cos(\theta)/\sin(\theta)$ and substitute $u = \sin\theta$, $du = \cos(\theta)\, d\theta$.

$$\begin{aligned}
\lim_{R \to 0} \int_R^{\pi/2} \cot(\theta)\, d\theta &= \lim_{R \to 0} \int_R^{\pi/2} \frac{\cos(\theta)}{\sin(\theta)}\, d\theta \\
&= \lim_{R \to 0} \int_R^1 \frac{1}{u}\, du \\
&= \lim_{R \to 0} \ln(u)\Big|_R^1 \\
&= 0 - \lim_{R \to 0} \ln(R)
\end{aligned}$$

Since $\lim_{R \to 0} \ln(R) = -\infty$, then this integral diverges.

(h) $\displaystyle\int_1^\infty \frac{dx}{(x-2)(2x+3)}$

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SOLUTION: This can be integrated using partial fractions. Here's the indefinite integral of the function.

$$\int \frac{dx}{(x-2)(2x+3)} = \int \frac{1/7}{x-2} + \frac{-2/7}{2x+3}\, dx$$

$$= \left(\frac{1}{7}\ln|x-2| - \frac{1}{7}\ln|2x+3| \right) + C$$

$$= \frac{1}{7}\ln\left| \frac{x-2}{2x+3} \right| + C$$

However, notice that there are two places where the integral is improper: when $x = 2$ and at the bounds where $x = \infty$. Therefore, we have to split the integral into three parts, each of which only has one improper bound.

$$\int_1^\infty \frac{dx}{(x-2)(2x+3)} = \int_1^2 \frac{dx}{(x-2)(2x+3)} + \int_2^5 \frac{dx}{(x-2)(2x+3)} + \int_5^\infty \frac{dx}{(x-2)(2x+3)}$$

$$= \lim_{R_1 \to 2} \frac{1}{7}\ln\left| \frac{R_1-2}{2R_1+3} \right| - \frac{1}{7}\ln\left(\frac{1}{2} \right)$$

$$+ \frac{1}{7}\ln\left| \frac{5-2}{2(5)+3} \right| - \lim_{R_2 \to 2} \frac{1}{7}\ln\left| \frac{R_2-2}{2R_2+3} \right|$$

$$+ \lim_{R_3 \to \infty} \frac{1}{7}\ln\left| \frac{R_3-2}{2R_3+3} \right| - \frac{1}{7}\ln\left| \frac{5-2}{2(5)+3} \right|$$

$$= \lim_{A \to 0} \frac{1}{7}\ln(A) - \frac{1}{7}\ln(1/2) + \frac{1}{7}\ln(3/13) - \lim_{B \to 0}\frac{1}{7}\ln(B) + \frac{1}{7}\ln(1/2) - \frac{1}{7}\ln(3/13)$$

Since the two limits in this last line are divergent, the whole integral diverges.

(i) $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2}$

SOLUTION:

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \lim_{R \to \infty} \int_{-R}^R \frac{dx}{1+x^2}\, dx$$

$$= \lim_{R \to \infty} \arctan(x) \Big|_{-R}^R$$

$$= \lim_{R \to \infty} \arctan(R) - \lim_{R \to \infty} \arctan(-R)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2} \right)$$

$$= \pi$$

(3) Calculate the following limits.

(a) $\displaystyle\lim_{n \to \infty} \left(1 + \frac{4}{n} \right)^n$

SOLUTION: Recall that $e = \lim_{n \to 0}(1+x)^{1/x}$. So in the limit we're asked to compute, let $u = 4/n$. Then $n = 4/u$, so we have

$$\lim_{n \to \infty} \left(1 + \frac{4}{n} \right)^n = \lim_{u \to 0}(1+u)^{4/u} = e^4$$

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(b) $\lim\limits_{t\to\infty} \dfrac{\ln(e^t+1)}{t}$ SOLUTION: Both the numerator and denominator go to infinity as $u\to\infty$, so we can use L'Hopital's rule. We need to use it twice.

$$\lim_{t\to\infty}\frac{\ln(e^t+1)}{t}=\lim_{t\to\infty}\frac{e^t/(e^t+1)}{1}=\lim_{t\to\infty}\frac{e^t}{e^t}=1$$

(c) $\lim\limits_{x\to 0}\dfrac{\tanh x-\sinh x}{\sin x-x}$

SOLUTION: The numerator and denominator are both zero when we plug in $x=0$, so again we use L'Hopital's rule. We need to use it three times.

$$\lim_{x\to 0}\frac{\tanh x-\sinh x}{\sin x-x}=\lim_{x\to 0}\frac{\operatorname{sech}^2(x)-\cosh(x)}{\cos(x)-1}$$
$$=\lim_{x\to 0}\frac{-2\operatorname{sech}^2(x)\tanh(x)-\cosh(x)}{-\sin(x)}$$
$$=\lim_{x\to 0}\frac{4\operatorname{sech}^2(x)\tanh^2(x)-\sinh(x)-2\operatorname{sech}^4(x)}{-\cos(x)}$$

At this stage, we know that $\cos(0)=1$, $\operatorname{sech}(0)=1/\cosh(0)=1$, $\tanh(0)=\sinh(0)/\cosh(0)=0$, and $\sinh(0)=0$. So we can just plug in $x=0$ to get a final answer of 2.

(4) (a) Find the arc length of $y=\cosh x$ over the interval $[a,b]$.
SOLUTION: Plug $y=\cosh(x)$ into the arc length formula.

$$\int_a^b\sqrt{1+(y')^2}\,dx=\int_a^b\sqrt{1+\sinh^2(x)}\,dx$$
$$=\int_a^b\sqrt{\cosh^2(x)}\,dx$$
$$=\int_a^b\cosh(x)\,dx$$
$$=\sinh(b)-\sinh(a)$$

(b) Find the surface area obtained by rotating $y=\sin x$ about the x-axis for $0\le x\le\pi$.
SOLUTION: Plug $y=\sin(x)$ into the surface area formula.

$$\int_0^\pi 2\pi y\sqrt{1+(y')^2}\,dx=\int_0^\pi 2\pi\sin(x)\sqrt{1+\cos^2(x)}\,dx$$

Let $u=\cos(x)$, so $du=-\sin(x)\,dx$. Then we get

$$-2\pi\int_1^{-1}\sqrt{1+u^2}\,du$$

Now let $u=\tan(\theta)$, so $du=\sec^2(\theta)\,d\theta$. Hence, this integral becomes

$$2\pi\int_{-\pi/4}^{\pi/4}\sqrt{1+\tan^2(\theta)}\sec^2\theta\,d\theta=2\pi\int_{-\pi/4}^{\pi/4}\sec^3(\theta)\,d\theta$$
$$=2\pi\frac{1}{2}\left(\ln|\tan(\theta)+\sec(\theta)|+\sec\theta\tan\theta\right)\Big|_{-\pi/4}^{\pi/4}$$
$$=\pi\left(\ln|1+\sqrt{2}|+\sqrt{2}\right)-\pi\left(\ln|-1+\sqrt{2}|-\sqrt{2}\right)$$
$$=2\sqrt{2}\pi+\pi\ln\left(\sqrt{2}+1\right)-\pi\ln\left(\sqrt{2}-1\right)$$

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(c) Show that the arc length of $y = \ln(f(x))$ for $a \le x \le b$ is

$$\int_a^b \frac{\sqrt{f(x)^2 + f'(x)^2}}{f(x)} \, dx$$

SOLUTION: For this, we need only plug in $y = \ln(f(x))$ into the arc length formula. First, notice that $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$. Therefore, the arc length is

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_a^b \sqrt{1 + \left(\frac{f'(x)}{f(x)}\right)^2} \, dx = \int_a^b \sqrt{\frac{f(x)^2}{f(x)^2} + \frac{f'(x)^2}{f(x)^2}} \, dx = \int_a^b \frac{\sqrt{f(x)^2 + f'(x)^2}}{f(x)} \, dx$$

(5) Let R be the region under the graph of $y = 1/(x+1)$ for $0 \le x < \infty$. Which of the following quantities is finite? Calculate the ones that are.

(a) The area of R.

SOLUTION: The area of R is the integral of y from 0 to ∞.

$$\int_0^\infty \frac{1}{x+1} \, dx = \lim_{a \to \infty} \int_0^a \frac{1}{x+1} \, dx = \lim_{x \to \infty} \ln|x+1| \Big|_0^a = \lim_{a \to \infty} \ln|a+1| - \ln(1)$$

This diverges because $\lim_{a \to \infty} \ln(a+1) = \infty$ and $\ln(1) = 0$.

(b) The volume of the solid obtained by rotating R about the x-axis.

SOLUTION: Using the disk method, this volume is

$$\int_0^\infty \frac{\pi}{(x+1)^2} \, dx = \frac{-\pi}{x+1} \Big|_0^\infty = \pi$$

(c) The volume of the solid obtained by rotating R about the y-axis.

SOLUTION: Using the shell method, this volume is

$$\int_0^\infty \frac{2\pi x}{x+1} \, dx = 2\pi \int_0^\infty \frac{x}{x+1} \, dx$$

Substitute $u = x + 1$, $du = dx$.

$$2\pi \int_1^\infty \frac{u-1}{u} \, du = 2\pi \int_1^\infty 1 - \frac{1}{u} \, du = 2\pi (u - \ln|u|) \Big|_1^\infty$$

This diverges.

(6) Let

$$F(x) = x\sqrt{x^2 - 1} - 2 \int_1^x \sqrt{t^2 - 1} \, dt.$$

Prove that $F(x)$ and $\cosh^{-1} x$ differ by a constant by showing that their derivatives are the same for all x. Then show that the constant must be zero by evaluating at $x = 1$, so that the functions $F(x)$ and $\cosh^{-1} x$ are in fact equal.

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SOLUTION: We have to show that $F(x)$ and $\cosh^{-1}(x)$ have the same derivatives. The problem tells us to take derivatives, so we take derivatives.

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(x\sqrt{x^2-1}\right) - 2\frac{d}{dx}\int_1^x \sqrt{t^2-1}\,dt$$

$$= \sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}} - 2\sqrt{x^2-1}$$

$$= \frac{x^2}{\sqrt{x^2-1}} - \sqrt{x^2-1}$$

$$= \frac{x^2}{\sqrt{x^2-1}} - \frac{x^2-1}{\sqrt{x^2-1}}$$

$$= \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$$

So these two have the same derivatives. Hence, $F(x)$ and $\cosh^{-1}(x)$ must differ by a constant. That constant is zero, since $F(1) = 0$, and $\cosh^{-1}(1) = 0$.