

Due at the beginning of class on 6 February 2024

- Your answers should be neatly written and logically organized.
- You may collaborate on solving the problems, but the solutions you turn in should be your own.
- You may use any resource you find online (or elsewhere), but you must cite any resource you use.

Reading: [Sto22, Chapter 2].

- (1) A theorem of Serre shows that $\pi_k(S^n)$ for $i > 2$ is a finite abelian group, except for two classes of exceptions: $\pi_n(S^n) \cong \mathbb{Z}$ and $\pi_{4j-1}(S^{2j}) \cong \mathbb{Z} \oplus M$, where M is a torsion \mathbb{Z} -module. Use this to prove that the stable homotopy groups $\pi_k^s(S^0)$ are finite abelian for $i > 0$.
- (2) Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category. A *monoid* in \mathcal{C} is an object M together with morphisms $m: M \otimes M \rightarrow M$ and $i: 1 \rightarrow M$ such that the following diagrams commute:

$$\begin{array}{ccc} M \otimes M \otimes M & \xrightarrow{m \otimes 1} & M \otimes M \\ \downarrow 1 \otimes m & & \downarrow m \\ M \otimes M & \xrightarrow{m} & M \end{array}$$

$$\begin{array}{ccccc} 1 \otimes M & \xrightarrow{i \otimes \text{id}} & M \otimes M & \xleftarrow{\text{id} \otimes i} & M \otimes 1 \\ & \searrow \cong & \downarrow m & \swarrow \cong & \\ & & M & & \end{array}$$

A *morphism of monoids* $f: M \rightarrow N$ is one that commutes with the structure morphisms:

$$\begin{array}{ccc} M \otimes M & \xrightarrow{f \otimes f} & N \otimes N \\ \downarrow m & & \downarrow m \\ M & \xrightarrow{f} & N \end{array}$$

$$\begin{array}{ccc} 1 & \xrightarrow{i} & M \\ \searrow i & & \downarrow f \\ & & N \end{array}$$

(M, m, i) is a *commutative monoid* if $f = f \circ s$, where $s: M \otimes M \rightarrow M \otimes M$ is the symmetry in \mathcal{C} .

- Let M and C be objects in \mathcal{C} . Prove that if either M is a monoid or C is a comonoid, then $\mathcal{C}(C, M)$ is a monoid in the ordinary sense: a set with an associative and unital operation.
 - Let M in \mathcal{C} be a monoid in two different ways: (M, m, i) and (M, n, j) . Further assume that m and n are morphisms of monoids. Prove that M is a commutative monoid and the two structures are the same.
 - For any spaces X and Y , prove that $[X, \Omega^2 Y]$ and $[\Sigma X, \Omega Y]$ are abelian groups.
- (3) Let $f: X \rightarrow Y$ be a map between simply connected spaces such that $f_*: H_i(X) \rightarrow H_i(Y)$ is an isomorphism for $i \leq n$. We will show that f is an n -connected map.
 - Let C be the homotopy cofiber of f , and let F be the homotopy fiber of $Y \rightarrow C$. Use the Hurewicz theorem to show that C is n -connected and $F \rightarrow Y$ is an n -connected map.
 - Use the Blakers–Massey theorem to show that $X \rightarrow F$ is at least 2-connected.
 - Show that f is at least 2-connected. Iterate your argument from part (b) to show that f is n -connected.
 - (4) Let $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$ be a sequence of spaces. Prove that $\Omega \text{hocolim}_i X_i \simeq \text{hocolim}_i \Omega X_i$. Use this to show that homotopy groups commute with sequential homotopy colimits.

REFERENCES

- [Sto22] Bruno Stonek. Introduction to stable homotopy theory. <https://bruno.stonek.com/stable-homotopy-2022/stable-online.pdf>, July 2022.