IMPLICIT DIFFERENTIATION

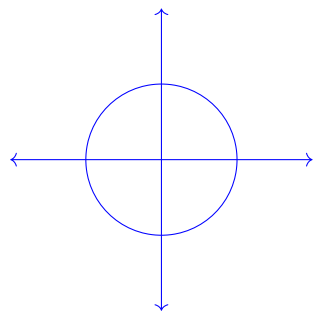
Math 1110
March 10, 2017

(1) In your own words, what is implicit differentiation and why is it useful?

SOLUTION: Answers may vary. Implicit differentiation is the technique of finding the slope of the tangent line to an implicitly defined curve (like $x^2 + y^2 = r^2$). It is useful because the usual techniques of finding a derivative fail when we cannot solve for y.

(2) Consider the equation of a circle of radius one: $x^2 + y^2 = 1$.

(a) Draw a picture of this circle.
SOLUTION:



(b) What would you guess the slope of the tangent line is to the circle at $x = \frac{1}{2}$. Why do you say this?
SOLUTION: The tangent line at the point $x = \frac{1}{2}$ will be perpendicular to the line through the origin and the point on the circle at $x = \frac{1}{2}$. To find the coordinates of the point on the circle with $x = \frac{1}{2}$ solve $\left(\frac{1}{2}\right)^2 + y^2 = 1$ for y. This gives $y = \pm\frac{\sqrt{3}}{2}$ (there are two points on the circle with x-coordinate $\frac{1}{2}$).
So the slope of the line through the origin and the point $(\frac{1}{2}, \frac{\pm\sqrt{3}}{2})$ is $\pm\sqrt{3}$. The slope of a line perpendicular to this one is therefore $\pm\frac{1}{\sqrt{3}}$

(c) Check your work by first finding a formula for $\frac{dy}{dx}$ and then finding the slope of the tangent line at $x = \frac{1}{2}$. Does your answer make sense with your picture? Why or why not?
SOLUTION:

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\ 2x + \frac{d}{dx}(y^2) &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \\ \implies \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

The coordinates of the points on the circle with $x = \frac{1}{2}$ are $(\frac{1}{2}, \frac{\pm\sqrt{3}}{2})$. So the slopes of the two tangent lines at that point are $\pm\frac{1}{\sqrt{3}}$. This agrees with what we guessed earlier.

(3) Use implicit differentiation to find $\frac{dy}{dx}$:

(a) $-y^2 = 1$

SOLUTION: $\frac{dy}{dx} = \begin{cases} 0 & y \neq 0 \\ \text{UND} & y = 0 \end{cases}$

(b) $\sqrt{x} - \sqrt{y} = 1$

SOLUTION: $\dfrac{dy}{dx} = \dfrac{\sqrt{y}}{\sqrt{x}}, x \neq 0$

(c) $2x^2y + 3xy^3 = 1$

SOLUTION: $\dfrac{dy}{dx} = \dfrac{-3y^3 - 4xy}{2x^2 + 9xy^2}$

(d) $(x-1)y^2 = x + 1$

SOLUTION: $\dfrac{dy}{dx} = \dfrac{1 - y^2}{2(x-1)y}$

(4) For each of the problems in the previous part, find the second derivative with respect to x. What is different or notable about this process?

(a) $-y^2 = 1$

SOLUTION: The second derivative is still zero, or undefined at $y = 0$.

(b) $\sqrt{x} - \sqrt{y} = 1$

SOLUTION: $\frac{d^2y}{dx^2} = \frac{-\sqrt{y}}{2\sqrt{x}^3} + \frac{1}{2x}$

(c) $2x^2y + 3xy^3 = 1$

SOLUTION: $\frac{d^2y}{dx^2} = \frac{12y(x)\left(4x^3 + 4x^2y(x)^2 + 27y(x)^6 + 18xy(x)^4\right)}{x^2\left(9y(x)^2 + 2x\right)^3}$

(d) $(x-1)y^2 = x+1$

SOLUTION: $\frac{d^2y}{dx^2} = \frac{y^2 - 1}{2(x-1)^2y}$

(5) Find the slope of the tangent line to the given curve at the given point.

(a) $xy^5 + yx^5 = 1$ at $(-1, 1)$

SOLUTION: $\dfrac{dy}{dx} = \dfrac{-(y^5 - 5yx^4)}{x^5 - 5xy^4}$ $\qquad$ $\left.\dfrac{dy}{dx}\right|_{(-1,1)} = -\dfrac{3}{2}$

(b) $\frac{1}{x^3} + \frac{1}{y^3} = 2$ at $(1, 1)$

SOLUTION: $\dfrac{dy}{dx} = -\dfrac{y^4}{x^4}$ $\qquad$ $\left.\dfrac{dy}{dx}\right|_{(1,1)} = -1$