Cellular Spectra NAME: ____________

Due at the beginning of class on 18 March 2025

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Section 2.6].

(1) Recall that the *free spectrum functor* $F_n \colon \mathcal{T}op_* \to \mathcal{S}p$ can be described by $F_n K \simeq \Sigma^{-n}\Sigma^\infty K$ for any $n \in \mathbb{Z}$.

(a) When $n \geq 0$, prove that F_n is left adjoint to evaluation $\mathrm{ev}_n \colon \mathcal{S}p \to \mathcal{T}op_*$, where ev_n is the functor that takes the n-th space of a spectrum: $\mathrm{ev}_n X = X_n$.

(b) Does F_n have a right adjoint when $n < 0$?

(2) Consider the homotopy pushout/pullback square of spectra:

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} \\ A & \xrightarrow{g} & Y. \end{array}$$

Prove that there is a Mayer–Vietoris-type long exact sequence of spectra:

$$\cdots \to \pi_{n+1}Y \to \pi_n X \to \pi_n A \oplus \pi_n B \to \pi_n Y \to \pi_{n-1}X \to \cdots$$

(3) A spectrum X is called n-connect**ed** if $\pi_i X = 0$ for $i \leq n$, or n-connect**ive** if $\pi_i X = 0$ for $i < n$. Let $X \to Y \to Z$ be a cofiber/fiber sequence of spectra.

(a) Prove that if X and Z are n-connected, then so is Y.

(b) What can you say about connectivity of Z if X and Y are n-connected? What can you say about connectivity of X if Y and Z are n-connected?

(4) Prove that the following three conditions are equivalent:

(a) X is a *finite spectrum*, i.e. X is stably equivalent to a cellular spectrum with finitely many stable cells.

(b) X is stably equivalent to a free spectrum $F_k K \simeq \Sigma^{-k}\Sigma^\infty K$ for a finite cell complex K.

(c) X is bounded below and the direct sum of the homology groups $\bigoplus_k H_k(X;\mathbb{Z})$ is finitely generated as an abelian group.

References

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.