Sequential Spectra Talk Outline

Read [Mal23, Sections 2.1, 2.2, 2.4], but skip Section 2.1.4 on Thom spectra and skip [Mal23, Section 2.3]. We will talk about that next week.

(1) Motivations

(a) Remind us about two big theorems that we have seen so far: the Brown representability theorem and the Freudenthal Suspension Theorem.

(b) The Freudenthal suspension theorem suggests that it is interesting and fruitful to study the sequence of spaces $X, \Sigma X, \Sigma^2 X, \ldots$. We can use these to talk about stable homotopy groups, which can be easier to study than (unstable) homotopy groups.

(c) The Brown representability theorem suggests that it's interesting to study infinite loopspaces: these are spaces X with infinitely many deloopings $X \simeq \Omega X_1 \simeq \Omega^2 X_2 \simeq \ldots$ These spaces define generalized cohomology theories, which are very useful tools in algebraic topology.

(d) There is an adjunction between the two functors Σ and Ω here so these concepts should be related. This motivates the definition of spectra.

(2) Definitions

(a) Define spectra and maps of spectra [Mal23, Definition 2.1.1]. The category of spectra and the maps described above is denoted $\mathcal{S}p$.

(b) Note that the bonding maps $\sigma_i \colon \Sigma X_i \to X_{i+1}$ have adjoints $\widetilde{\sigma_i} \colon X_i \to \Omega X_{i+1}$, and we could equivalently define a spectrum using the adjoint bonding maps.

(c) Define the homotopy groups of a spectrum [Mal23, Definition 2.1.2]. Make sure to note that we have negative homotopy groups now: a spectrum X has negative homotopy groups $\pi_{-1}X$, $\pi_{-2}X, \ldots$

(d) Define a stable equivalences [Mal23, Definition 2.1.10] of spectra. Stable equivalences satisfy the 2-out-of-6 property, (cf. [Rie14, Digression 2.1.5]), so we can invert them. The category $\mathrm{ho}(\mathcal{S}p)$ that we get by inverting the stable equivalences is called the *stable homotopy category*.

(e) Give a couple examples of spectra:

- [Mal23, Example 2.1.7] The suspension spectrum of a pointed space X is the sequence of spaces $X, \Sigma X, \Sigma^2 X, \ldots$ The bonding maps are identities $\mathrm{id} \colon \Sigma(\Sigma^i X) \to \Sigma^{i+1} X$. The suspension spectrum of a space is denoted $\Sigma^\infty X$, or $\Sigma^\infty_+ Y := \Sigma^\infty(Y_+)$ for an unpointed space Y (first add a disjoint basepoint).
  The homotopy groups of a suspension spectrum $\Sigma^\infty X$ are the stable homotopy groups of the space X.
- [Mal23, Example 2.1.6] A particularly important example of a suspension spectrum is the *sphere spectrum* $\mathbb{S} := \Sigma^\infty S^0$. This is the sequence of spheres $S^0, S^1, S^2, \ldots$.
  The homotopy groups of the sphere spectrum are called the *stable stems*. They are the stable homotopy groups of S^0, often written $\pi_i^s := \pi_i \mathbb{S}$.
- [Mal23, Example 2.1.12] The *zero spectrum* $\Sigma^\infty(*)$ is the suspension spectrum of a point. All of its homotopy groups are zero. Depending on the author, this is alternatively denoted as $*$ or 0. Explain the existence zero maps between spectra.
- [Mal23, Example 2.2.2] We have already seen the example of Eilenberg–MacLane spaces as infinite loopspaces. We can use the maps adjoint to the equivalences $K(A, n) \simeq \Omega K(A, n+1)$ to define an *Eilenberg–MacLane spectra* HA, which are given by the sequence of spaces

$K(A,0),\ K(A,1), K(A,2),\ \ldots$ The functor $H\colon \mathcal{A}b \to \mathcal{S}p$ takes an abelian group to the Eilenberg–MacLane spectrum HA. We use the letter H because the infinite loopspace represents cohomology with coefficients in A.

The homotopy groups of HA are given by

$$\pi_i HA = \begin{cases} A & (i = 0) \\ 0 & (i \neq 0) \end{cases}$$

- [Mal23, Example 2.2.3] Another example is the complex K-theory spectrum, usually written KU[1]. This is given by the sequence of spaces $\mathbb{Z} \times BU$, U, $\mathbb{Z} \times BU$, U, ... and the structure maps adjoint to the equivalences $\Omega U \simeq \mathbb{Z} \times BU$ and $\Omega(\mathbb{Z} \times BU) \simeq U$.

  The homotopy groups of KU are given by

$$\pi_i KU = \begin{cases} \mathbb{Z} & (i \text{ is even}) \\ 0 & (i \text{ is odd}) \end{cases}$$

(3) Ω-spectra

(a) Define Ω-spectra [Mal23, Definition 2.2.1].

(b) Explain which of the examples above are Ω-spectra.

(c) State the proposition that every spectrum is stably equivalent to an Ω-spectrum [Mal23, Proposition 2.2.9]. In our language, the functor R that takes a spectrum X and replaces it by a stably equivalent Ω-spectrum RX is a right deformation of the category $\mathcal{S}p$ (dual to [Rie14, Definition 2.2.1]).

(d) Give an indication of how to prove this proposition. You don't need to go into all of the gory details, but you should construct the spectrum RX.

(e) Define the functor $\Omega^\infty\colon \mathcal{S}p \to \mathcal{T}op_*$ [Mal23, Definition 2.2.11].

(f) State the proposition that Ω^∞ is right adjoint to Σ^∞. You should not prove this proposition; we will do so on the homework. Explain that $\Omega^\infty\Sigma^\infty X = QX$.

(4) Stability

(a) Define the functors $\Sigma\colon \mathcal{S}p \to \mathcal{S}p$ and $\Omega\colon \mathcal{S}p \to \mathcal{S}p$ [Mal23, Definitions 2.3.6 and 2.3.8].

(b) State [Mal23, Proposition 2.1] and prove it.

(c) State [Mal23, Corollary 2.4.5] and prove it. The consequence is that Σ and Ω are inverse functors up to stable equivalence.

(d) Define fiber and cofiber sequences of spectra following [Mal23, Definition 2.4.6]. The homotopy fiber and cofiber of a map of spectra can be taken levelwise.

(e) Prove that a sequence is a fiber sequence of spectra if and only if it is a cofiber sequence of spectra [Mal23, Proposition 2.4.12]. This is the big theorem here!

(f) Explain that both fiber and cofiber sequences yield the long exact sequence in homotopy. You may need to prove these statements to prove that cofiber and fiber sequences agree.

(g) As a corollary, show that a square is a homotopy pullback of spectra if and only if it is a homotopy pushout [Mal23, Corollary 2.4.15]. You don't need to prove the lemma, you can just state it. Point out that this is like Blakers–Massey, but without any assumptions!

(h) State other corollaries as you have time or interest in doing.

[1]The U is for "unitary," to distinguish this from KO, where O is for "orthogonal". KO is like KU, but for real vector bundles instead of complex vector bundles.

References

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.