SEQUENTIAL SPECTRA NAME: SOLUTIONS

**Due at the beginning of class on 4 March 2025**

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

**Reading:** [Mal23, Sections 2.1, 2.2, and 2.3].

(1) Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$ be a sequence of spectra. Show that the homotopy colimit [Mal23, Definition 2.3.17] of this sequence commutes with stable homotopy groups, in the sense that

$$\pi_k\left(\operatorname{hocolim}_n X_n\right) = \operatorname*{colim}_n \pi_k(X_n).$$

SOLUTION: We'll start with the right hand side and produce a chain of isomorphisms leading to the left hand side. Let $X_{n,m}$ be the $m$-th space in the spectrum $X_n$.

$$\begin{aligned}
\operatorname*{colim}_n \pi_k(X_n) &= \operatorname*{colim}_n \operatorname*{colim}_m \pi_{k+m}(X_{n,m}) && \text{definition of } \pi_k \text{ of a spectrum}\\
&\cong \operatorname*{colim}_m \operatorname*{colim}_n \pi_{k+m}(X_{n,m}) && \text{colimits commute with colimits}\\
&\cong \operatorname*{colim}_m \pi_{k+m}(\operatorname{hocolim}_n X_{n,m}) && \text{the same fact for spaces (PSet 3, Q4)}\\
&\cong \pi_k(\operatorname{hocolim}_n X_n) && \text{definition of } \pi_k(\operatorname{hocolim}_n X_n)
\end{aligned}$$

(2) Eilenberg-MacLane spectra are characterized by their homotopy groups: if any other spectrum $X$ satisfies $\pi_i X = 0$ for $i \neq 0$, then $X \simeq H(\pi_0 X)$.

(a) Prove that $H(\mathbb{Z}/p)$ is stably equivalent to the homotopy cofiber of the map obtained by applying the functor $H$ to $p\colon \mathbb{Z} \to \mathbb{Z}$.

SOLUTION: Consider the long exact sequence in homotopy of the cofiber sequence

$$H\mathbb{Z} \xrightarrow{Hp} H\mathbb{Z} \to \operatorname{cof}(Hp)$$

Because the homotopy of $H\mathbb{Z}$ is concentrated in degree zero, this boils down to an exact sequence

$$0 \to \pi_1 \operatorname{cof}(Hp) \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \pi_0 \operatorname{cof}(Hp) \to 0$$

But $p\colon \mathbb{Z} \to \mathbb{Z}$ is injective, so $\pi_1 \operatorname{cof}(Hp) = 0$, and we have a short exact sequence with third term isomorphic to $\mathbb{Z}/p$. Hence, $\operatorname{cof}(Hp)$ has one nonzero homotopy group $H\mathbb{Z}/p$, so it is stably equivalent to $H(\mathbb{Z}/p)$.

(b) The *rationalization* $\mathbb{S}_\mathbb{Q}$ of the sphere spectrum $\mathbb{S}$ is the homotopy colimit of the diagram

$$\mathbb{S} \xrightarrow{1} \mathbb{S} \xrightarrow{2} \mathbb{S} \xrightarrow{3} \mathbb{S} \xrightarrow{4} \mathbb{S} \xrightarrow{5} \mathbb{S} \to \cdots$$

Prove that $\mathbb{S}_\mathbb{Q}$ is stably equivalent to $H\mathbb{Q}$.

SOLUTION: The rationalization of $\mathbb{S}$ is the spectrum

$$\mathbb{S}_\mathbb{Q} = \operatorname{hocolim}(\mathbb{S} \xrightarrow{1} \mathbb{S} \xrightarrow{2} \mathbb{S} \xrightarrow{3} \mathbb{S} \xrightarrow{4} \cdots).$$

Recall that stable homotopy groups commute with homotopy colimits (turning the homotopy colimits to colimits in $\mathcal{A}b$). So,

$$\pi_n(\mathbb{S}_{\mathbb{Q}}) = \operatorname{colim}(\pi_n\mathbb{S} \xrightarrow{1} \pi_n\mathbb{S} \xrightarrow{2} \pi_n\mathbb{S} \xrightarrow{3} \cdots)$$

This is isomorphic to $\pi_n\mathbb{S} \otimes \mathbb{Q}$. Recall from the last problem set that $\pi_n\mathbb{S}$ is a finite abelian group for $n > 0$ and $\pi_n\mathbb{S} = \mathbb{Z}$ for $n = 0$. Therefore, $\pi_n(\mathbb{S}_{\mathbb{Q}}) = \pi_n\mathbb{S} \otimes \mathbb{Q} = 0$ for $n > 0$ and $\pi_0(\mathbb{S}_{\mathbb{Q}}) = \mathbb{Z} \otimes \mathbb{Q} = \mathbb{Q}$. So this has one nonzero homotopy group, so $\mathbb{S}_{\mathbb{Q}} \simeq H\mathbb{Q}$.
See also [Mal23, Chapter 1, Exercise 39] and [Mal23, Example 2.5.30].

(3) Let $\mathrm{ev}_0 \colon \mathcal{S}p \to \mathcal{T}op_*$ be the functor that evaluates a spectrum at its zeroth space: $\mathrm{ev}_0 X = X_0$.

(a) Prove that $\Sigma^\infty$ is left adjoint to $\mathrm{ev}_0$.
Solution: We must show there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Sp}}(\Sigma^\infty X, Y) \cong \mathrm{Hom}_{\mathbf{Top}_*}(X, Y_0)$$

Recall that a map of spectra $\Sigma^\infty \to Y$ is a sequence of pointed maps $f_n : \Sigma^n X \to Y_n$ such that the following squares commute for all n:

$$\begin{tikzcd} \Sigma(\Sigma^n X) \arrow[r, "\Sigma f_n"] \arrow[d, "="'] & \Sigma Y_n \arrow[d] \\ \Sigma^{n+1} X \arrow[r, "f_{n+1}"] & Y_{n+1} \end{tikzcd}$$

This is naturally determined by the map $f_0 : X \to Y_0$. Denote the composite structure maps of Y by $y_n : \Sigma^n Y_0 \to Y_n$. Note that the diagram above forces $f_1 = y_1 \circ \Sigma f_0$. Suppose for induction that $f_n = y_n \circ \Sigma f_0$, then the above square commuting forces $f_{n+1} = y_{n+1} \circ \Sigma f_0$. So the map $\Sigma^\infty X \to Y$ uniquely determines the map $f_0 : X \to Y_0$ by the structure maps of Y. Going the other way, any map $g : X \to Y_0$ can naturally ascend to a map of spectra $\Sigma^\infty X \to Y$ by setting $g_0 = g$ and $g_n = y_n \circ \Sigma g_0$. These constructions are clearly natural in X and Y, so this completes the proof of the adjunction.

(b) For any spectrum X, let $\Omega^\infty X := \mathrm{ev}_0 RX$, where R is the fibrant replacement functor [Mal23, Proposition 2.2.9]. Use [Rie14, Exercise 2.2.15] to prove that there is an adjunction

$$\Sigma^\infty \colon \mathrm{ho}(\mathcal{T}op_*) \leftrightarrows \mathrm{ho}(\mathcal{S}p) \colon \Omega^\infty.$$

Solution: First, observe that $\Sigma^\infty$ is homotopical. For any space X, the stable homotopy groups of $\Sigma^\infty X$ are the homotopy groups of X, and if $X \xrightarrow{f} Y$ is a weak equivalence of spaces, the maps induced by $\Sigma^\infty f$ between the stable homotopy groups are exactly the isomorphisms induced by f between the homotopy groups of X and Y. Hence, $\Sigma^\infty$ is a stable equivalence.
Note that the above is true so long as our spaces are well-based, so that $S^1 \wedge X$ agrees with the homotopy pushout of $* \leftarrow X \to *$. If the spaces are not well based, then these need not agree. See this mathOverflow answer: [Law13].
A derived functor for $ev_0$ can be computed by pre-composing with a right deformation of Sp. In particular, the functor R in the exercise is such a deformation. $ev_0$ is homotopical on the full subcategory of Sp generated by the image of R: if $f : X \to Y$ is a stable equivalence between two $\Omega$-spectra, then $f_0 : X_0 \to Y_0$ is a weak equivalence because $\pi_m \cong \pi_m(X_0)$, $\pi_m(Y) \cong \pi_m(Y_0)$ and since $\{f_m\}_{m\in\mathbb{N}}$ commute with the bonding maps, the isomorphisms $\pi_m(X) \to \pi_m(Y)$ induced by f is exactly the map induced by $f_0$ from $\pi_m X_0$ to $\pi_m(Y_0)$.
So $\Omega^\infty = ev_0 \circ R$ is a right derived functor for $ev_0$. By proposition 2.2.13 in Riehl, a derived functor computed thanks to a deformation is an absolute Kan extension. Therefore, by [Rie14, Exercise 2.2.15], $\Sigma^\infty$ and $\Omega^\infty$ are an adjoint pair.

## REFERENCES


[Law13] Tyler Lawson. Must a weak homotopy equivalence induce an isomorphism between stable homotopy groups? "https://mathoverflow.net/questions/148963/must-a-weak-homotopy-equivalence-induce-an-isomorphism-between-stable-homotopy-g", 2013.

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.