Notes on Hesselholt–Madsen "On the K-theory of finite algebras over Witt vectors of perfect fields"

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Abstract

This document is notes that I made to give a talk on Hesselholt–Madsen's paper [HM97] for the Electronic Computational Homotopy Theory Kan seminar in Fall 2019. Please let me know if you find any mistakes!

1 Introduction

This paper made several contributions to the theory of trace methods that would each make it very influential.

(1) Hesselholt and Madsen rewrite the theory of topological Hochschild and topological cyclic homology in terms of orthogonal spectra. In the process, they introduce cyclotomic spectra, which become the framework for topological cyclic homology thereafter.

(2) They compute $\mathrm{TC}(k)^{\wedge}_{p}$ as a spectrum, where k is a perfect field of characteristic p. It's connective cover can be identified with the Witt vectors of k. The Witt vectors is an object that already has arithmetic significance.

(3) They prove that for any $W(k)$-algebra A that is finitely generated as a $W(k)$-module,

$$\mathrm{TC}(A)^{\wedge}_{p}[0,\infty) \simeq \mathrm{K}(A)^{\wedge}_{p},$$

where $[0,\infty)$ denotes the connective cover. Note that they don't actually compute $\pi_* \mathrm{TC}(A)$ unless $A = k$.

(4) They compute $\mathrm{TC}(k[x]/x^2)$ using so-called pointed monoid algebras. This is then applied to compute $\mathrm{K}(k[x]/x^2)$. Note that here, no p-completion is required.

I'll try to cover the first two of these in the talk, with some of the third if there's time. I will not talk about the fourth, but see Martin Speirs's papers [Spe19a, Spe19b] if interested. His papers compute $\mathrm{K}(k[x]/x^n)$ or more general quotients using the new framework for TC from [NS18].

2 The Witt Vectors

Let A be an $\mathbb{F}_p$-algebra.

Definition 2.1. The ring of p**-typical Witt vectors** of A is the set $W(A) = \prod_{i=0}^{\infty} A$ of infinite sequences in A with ring structure defined by the property that the **ghost map**

$$\begin{aligned} W(A) &\longrightarrow \prod_{i=0}^{\infty} A \\ (a_0, a_1, a_2, \ldots) &\longmapsto (w_0, w_1, w_2, \ldots) \end{aligned}$$

is a homomorphism of rings, where

$$\begin{aligned} w_0 &= a_0 \\ w_1 &= a_0^p + p a_1 \\ w_2 &= a_0^{p^2} + p a_1^p + p^2 a_2 \\ &\vdots \\ w_n &= \sum_{i=0}^{n} p^i a_i^{p^{n-i}}. \end{aligned}$$

It is a theorem that there is a unique ring structure on $W(A)$ that makes this into a homomorphism. Indeed, the addition and multiplication looks pretty wacky:

$$\begin{aligned} (a_0, a_1, a_2, \ldots) + (b_0, b_1, b_2, \ldots) &= \left(a_0 + b_0, a_1 + b_1 + \frac{a_0^p + b_0^p - (a_0 + b_0)^p}{p}, \cdots \right) \\ (a_0, a_1, a_2, \ldots) \cdot (b_0, b_1, b_2, \ldots) &= (a_0 b_0, a_0^p b_1 + a_1^p b_0 + p a_1 b_1, \ldots) \end{aligned}$$

Note that in the zeroth coordinate, we just have a copy of A. In fact, there is a multiplicative homomorphism $\omega\colon A \to W(A)$ called the **Teichmuller character**. There are two other important operators

$$\begin{aligned} F\colon W(A) &\to W(A) & F(a_0, a_1, a_2, \ldots) &= (a_0^p, a_1^p, a_2^p, \ldots), \\ V\colon W(A) &\to W(A) & V(a_0, a_1, a_2, \ldots) &= (0, a_0, a_1, \ldots). \end{aligned}$$

F is called the **Frobenius** and V is called the Verschiebung. F is a ring homomorphism on $W(A)$ and V is an additive homomorphism. These satisfy the relations:

$$\begin{aligned} VF &= p \\ FV &= p \\ x \cdot V(y) &= V(F(x) \cdot y) \end{aligned}$$

Definition 2.2. For each $n \geq 1$, define the ring $W_n(A)$ of **truncated** p**-typical Witt vectors of length** n by

$$W_n(A) := {}^{W(A)}\!/_{V^n W(A)}.$$

Note that $V^nW(A)$ is an ideal by the relations above. Elements of this ring are sequences $(a_0, a_1, \dots, a_{n-1})$ of elements of A of length n with addition and multiplication given as before.

Note that the infinite sequences are just the inverse limit of the finite sequences, so

$$W(A) = \varprojlim_n W_n(A),$$

with the limit taken over the restriction homomorphisms $R\colon W_n(A) \to W_{n-1}(A)$ given by

$$(a_0, \dots, a_{n-1}) \mapsto (a_0, \dots, a_{n-2}).$$

Note that the restriction map commutes with F and V.

Example 2.3. If $A = \mathbb{F}_p$, then the ring $W(\mathbb{F}_p)$ of p-typical Witt vectors of $\mathbb{F}_p$ is isomorphic to $\mathbb{Z}_p$, the p-adic integers. Any p-adic integer a may be written as a power series in p,

$$a = a_0 + a_1 p + a_2 p^2 + \dots = \sum_{i=0}^{\infty} a_i p^i,$$

with coefficients $a_i \in \mathbb{F}_p$. The isomorphism is given by

$$(a_0, a_1, a_2, \dots) \mapsto \sum_{i=0}^{\infty} a_i p^i.$$

With this presentation, I claim that the Frobenius endomorphism is given by the identity. Indeed, by Fermat's Little Theorem $x^p \equiv x \pmod{p}$, so we have

$$F(a_0 + a_1 p + a_2 p^2 + \dots) = a_0^p + a_1^p p + a_2^p p^2 + \dots = a_0 + a_1 p + a_2 p^2 + \dots.$$

The Verschiebung endomorphism is given by multiplication by p.

$$V(a_0 + a_1 p + a_2 p^2 + \dots) = a_0 p + a_1 p^2 + a_2 p^3 + \dots.$$

Therefore, the truncations $W_n(\mathbb{F}_p)$ of $\mathbb{F}_p$ are isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$:

$$W_n(\mathbb{F}_p) = {W(\mathbb{F}_p)}/{V^nW(\mathbb{F}_p)} \cong {\mathbb{Z}_p}/{p^n\mathbb{Z}_p} \cong {\mathbb{Z}}/{p^n\mathbb{Z}}.$$

The restriction homomorphism $R\colon W_n(\mathbb{F}_p) \to W_{n-1}(\mathbb{F}_p)$ can then be realized as the quotient homomorphism

$$\begin{aligned} R\colon {\mathbb{Z}}/{p^n\mathbb{Z}} &\longrightarrow {\mathbb{Z}}/{p^{n-1}\mathbb{Z}} \\ n + p^n\mathbb{Z} &\longmapsto n + p^{n-1}\mathbb{Z} \end{aligned}$$

Example 2.4. If $A = \mathbb{F}_{p^n}$ is a finite field of prime power order, then the ring $W(\mathbb{F}_{p^n})$ of p-typical Witt vectors of $\mathbb{F}_{p^n}$ is isomorphic to $\mathbb{Z}_p[\zeta]$, where ζ is a primitive root of unity of order $p^n - 1$. For reference, the minimal polynomial of ζ is

$$f(x) = \frac{x^{p^n-1} - 1}{x^{p-1} - 1}.$$

The Frobenius endomorphism is determined by $\zeta \mapsto \zeta^p$ and the Verschiebung endomorphism is the composite of multiplication by p with $\zeta \mapsto \zeta^{1/p}$ (ζ has a p-th root because p is coprime to $p^n - 1$).

3 Cyclotomic spectra and topological cyclic homology

Definition 3.1 ([Sch15, Definition 2.1]). An **orthogonal** G**-spectrum** consists of the following data:

(a) pointed spaces X_n with $O(n) \times G$-action preserving basepoints,

(b) maps $\sigma_n \colon X_n \wedge S^1 \to X_{n+1}$ that are G-equivariant with respect to G-actions on X_n, X_{n+1} and the trivial G-action on S^1,

such that

$$\sigma^m \colon X_n \wedge S^m \to X_{n+m}$$

is $O(n) \times O(m) \subseteq O(n+m)$-equivariant, and G-equivariant with respect to G-actions on X_n, X_{n+m} and trivial G-action on S^1.

A **map of orthogonal** G**-spectra** $f \colon X \to Y$ is a sequence of maps $f_n \colon X_n \to Y_n$ that are $O(n) \times G$-equivariant and the following diagrams commute for all n and m

$$\begin{tikzcd} X_n \wedge S^m \arrow[r, "f_n \wedge \mathrm{id}"] \arrow[d, "\sigma^m"] & Y_n \wedge S^m \arrow[d, "\sigma^m"] \\ X_{n+m} \arrow[r, "f_{n+m}"] & Y_{n+m} \end{tikzcd}$$

Remark 3.2. This isn't quite the definition given in [HM97], but it is equivalent. See [Sch15, Remark 2.7].

Orthogonal G-spectra have three different types of fixed points:

(1) **geometric fixed points** $\Phi^G X$, which is an orthogonal spectrum. The functor $\Phi^G \colon G\,\mathcal{S}p^O \to \mathcal{S}p^O$ is lax symmetric monoidal.

(2) **set-theoretic fixed points** X^G, given by $(X^G)_n = (X_n)^G$. This is again an orthogonal spectrum. It is also lax symmetric monoidal.

(3) **homotopy fixed points** $X^{hG} = \mathrm{hocolim}(X \colon NG \to \mathcal{S}p^O)$. This is an orthogonal spectrum whose homotopy groups can be computed by the homotopy fixed points spectral sequence

$$E^2_{i,j} = H^{-i}(G; \pi_j X) \implies \pi_{i+j}(X^{hG}).$$

We should also discuss

(4) **homotopy orbits** $X_{hG} = \mathrm{holim}(X \colon NG \to \mathcal{S}p^O)$. This is an orthogonal spectrum whose homotopy groups can be computed by the homotopy fixed points spectral sequence

$$E^2_{i,j} = H_i(G; \pi_j X) \implies \pi_{i+j}(X^{hG}).$$

These constructions are related in the following ways:

- There is a natural transformation $(-)^H \to \Phi^H$ for all subgroups $H \leq G$.

- By [HM97, Proposition 1.1], there is a fiber sequence for any C_{p^k}-spectrum X
$$X_{hC_{p^k}} \to X^{C_{p^k}} \to (\Phi^{C_p} X)^{C_{p^{k-1}}}$$
- There is a **norm map** $X_{hG} \to X^{hG}$. This roughly corresponds to the map $A_G \to A^G$, $a \mapsto \sum_{g \in G} a$ for a finite group G and a G-module A. The cofiber of this norm map is the **Tate construction** X^{tG}, whose homotopy groups are computed by the Tate spectral sequence
$$E^2_{i,j} = \widehat{H}^{-i}(G; \pi_j X) \implies \pi_{i+j}(X^{tG}),$$
Here, $\widehat{H}(G; A)$ denotes Tate cohomology, which splices together group homology and group cohomology to get a new cohomology theory.

Definition 3.3. Let $\mathcal{F}$ be a class of subgroups of G. A map $f\colon X \to Y$ of orthogonal G-spectra is an $\mathcal{F}$**-equivalence** if and only if $\Phi^H X \to \Phi^H Y$ is a stable equivalence of orthogonal spectra for all $H \in \mathcal{F}$.

Definition 3.4. A p**-cyclotomic spectrum** X is an orthogonal S^1-spectrum together with an $\mathcal{F}_p$-equivalence $\Phi^{C_p} X \to X$, where $\mathcal{F}_p$ is the class of finite p-subgroups of S^1 – the cyclic groups C_{p^n}.

Any p-cyclotomic spectrum has maps
$$F\colon X^{C_{p^n}} \to X^{C_{p^{n-1}}}$$
$$V\colon X^{C_{p^{n-1}}} \to X^{C_{p^n}}$$
$$R\colon X^{C_{p^n}} \to X^{C_{p^{n-1}}}$$
defined as follows.

(1) F is the inclusion of fixed points.

(2) V is determined by the Mackey functor $S^1/C_{p^n} \mapsto \pi_* X^{C_{p^n}}$; see [HM97, Section 2.2].

(3) R is the composite $X^{C_{p^n}} \to (\Phi^{C_p} X)^{C_{p^{n-1}}} \xrightarrow{\sim} X^{C_{p^{n-1}}}$

Definition 3.5. Let X be a p-cyclotomic spectrum. Define the **topological restriction homology**
$$\mathrm{TR}(X, p) := \underleftarrow{\mathrm{holim}}\, X^{C_{p^n}} = \underleftarrow{\mathrm{holim}} \left(X \xleftarrow{R} X^{C_p} \xleftarrow{R} X^{C_{p^2}} \xleftarrow{R} \cdots \right)$$
The maps $F\colon X^{C_{p^n}} \to X^{C_{p^{n-1}}}$ induce a map $F\colon \mathrm{TR}(X, p) \to \mathrm{TR}(X, p)$. The **topological cyclic homology of** X is
$$\mathrm{TC}(X, p) := \mathrm{hofib}\left(\mathrm{TR}(X, p) \xrightarrow{F-1} \mathrm{TR}(X, p) \right).$$

Proposition 3.6 ([HM97, Proposition 1.5]). *For any ring spectrum A, there is a cyclotomic spectrum* $\mathrm{THH}(A)$.

In particular, $\mathrm{THH}(A)$ is a p-cyclotomic spectrum for each p by comparing [HM97, Definition 1.2] with [HM97, Definition 1.3].

Lemma 3.7 ([HM97, Lemma 2.3.1]). *For any commutative $\mathbb{F}_p$-algebra A, the following relations hold in $\pi_* \operatorname{THH}(A)^{C_{p^n}}$:*

$$\begin{aligned} V_* F_* &= p \\ F_* V_* &= p \\ x \cdot V_*(y) &= V_*(F_*(x) \cdot y) \\ R_* V_* &= V_* R_* \\ F_* V_* &= V_* F_* \end{aligned}$$

and moreover, F is a ring homomorphism.

Theorem 3.8 ([HM97, Theorem 2.3]). *Let A be a commutative ring. There is a natural isomorphism of rings*

$$\pi_0 \operatorname{THH}(A)^{C_{p^n}} \cong W_{n+1}(A)$$

that commutes with the Frobenius, verschiebung, and restriction maps on both sides.

Remark 3.9. If you're interested in the K-theoretic version of this theorem, there is a nice conceptual explanation due to Jonathan Campbell [Cam19]. Basically, there is a filtration on $K_0(\operatorname{End}(A))$, and the completion with respect to that filtration is $W(A)$. Speculatively, this filtration extends to the whole spectrum $K(\operatorname{End}(A))$ and the completion with respect to that filtration is then $\operatorname{TR}(A)$ – see Theorem 4.5 below. Then $\operatorname{THH}(A)$ and all of its C_{p^n} fixed points show up as the filtration quotients, and the trace map is the map from K to these filtration quotients.

4 Topological Cyclic Homology of perfect fields

The first step to compute topological cyclic homology is to compute topological Hochschild homology. In the case of $\mathbb{F}_p$, this has been kindly done for us by Bökstedt.

Theorem 4.1 ([HM97, Theorem 4.2]). $\pi_* \operatorname{THH}(\mathbb{F}_p) \cong \mathbb{F}_p[\sigma]$ *with* $|\sigma| = 2$.

In fact, something similar holds for any perfect field of characteristic p.

Theorem 4.2. *Let k be a perfect field of characteristic p. Then $\pi_* \operatorname{THH}(k) \cong k[\sigma]$ with $|\sigma| = 2$.*

Proof sketch. The Bökstedt spectral sequence for $\operatorname{THH}(k)$ is given by

$$E^2_{i,j}(k) = \operatorname{HH}_i^{\mathbb{F}_p}(\mathcal{A}_k)_j \implies \pi_{i+j}(H\mathbb{F}_p \wedge \operatorname{THH}(k)).$$

Where $\mathcal{A}_k = \pi_*(H\mathbb{F}_p \wedge Hk)$. In this case, $\mathcal{A}_k \cong \mathcal{A} \otimes k$, where $\mathcal{A} = \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$ is the dual Steenrod algebra. Hence, $E^2(k) \cong k \otimes E^2(\mathbb{F}_p)$. Therefore, $E^\infty(k) \cong k \otimes E^\infty(\mathbb{F}_p)$ as well since it is a spectral sequence of k-modules. $\square$

Lemma 4.3 ([HM97, Proposition 4.1, Addendum 4.3]). *For any p-cyclotomic spectrum X, there is a commutative diagram whose rows are cofiber sequences:*

$$\begin{tikzcd} X_{hC_{p^n}} \arrow[r] \arrow[d, equal] & X^{C_{p^n}} \arrow[r] \arrow[d, "\Gamma"] & X^{C_{p^{n-1}}} \arrow[d, "\widehat{\Gamma}"] \\ X_{hC_{p^n}} \arrow[r] & X^{hC_{p^n}} \arrow[r] & X^{tC_{p^n}} \end{tikzcd}$$

When $X = \mathrm{THH}(\mathbb{F}_p)$, the maps Γ and $\widehat{\Gamma}$ induce equivalences of connective covers.

Therefore, to compute the homotopy groups of $\mathrm{THH}(k)^{C_{p^n}}$, it suffices to compute the homotopy groups fo $\mathrm{THH}(k)^{tC_{p^n}}$ for all n. This is an improvement, because each of the three spectra along the bottom row has a spectral sequence to compute its homotopy groups. In particular, we have the Tate spectral sequence

$$\widehat{E}^2_* = \widehat{H}^*(C_{p^n}; \pi_* \mathrm{THH}(\mathbb{F}_p))$$

We can describe this $\widehat{E}^2$-page completely by the theorem of Bökstedt.

$$\widehat{E}^2_* \cong \Lambda_{\mathbb{F}_p}(u_n) \otimes \mathbb{F}_p[t, t^{-1}] \otimes \mathbb{F}_p[\sigma],$$

with $|u| = (-1, 0)$, $|t| = (-2, 0)$ and $|\sigma| = (0, 2)$.

We can compare this spectral sequence with the spectral sequence for the S^1 Tate construction on $\mathrm{THH}(\mathbb{F}_p)$ to learn that t and σ must be permanent cycles. Moreover, we have already computed $\pi_0 \mathrm{THH}(\mathbb{F}_p)^{C_{p^n}}$ in Theorem 3.8. This is enough to tell us all about the differentials in this spectral sequence. The possible differentials are

$$d^{2r+1} u_n = t^{r+1} \sigma^r$$

on the r-th page. $\widehat{\Gamma}$ tells us that we must have $r = n$, and the $2n+2$ page becomes

$$\widehat{E}^{2n+2}_* = \mathbb{F}_p[t, t^{-1}, \sigma] / t^{n+1} \sigma^n.$$

We may compare with the spectral sequence for the mod p homotopy groups $\pi_*(\mathrm{THH}(\mathbb{F}_p)^{tC_{p^n}}; \mathbb{F}_p)$ to learn that all extensions are maximally nontrivial, and therefore,

$$\pi_* \mathrm{THH}(\mathbb{F}_p)^{tC_{p^n}} \cong \mathbb{Z}/p^n[\widehat{\sigma}, \widehat{\sigma}^{-1}],$$

where $|\widehat{\sigma}| = 2$.

Theorem 4.4 ([HM97, Theorem 4.5]). *For all $n \geq 0$,*

$$\pi_* \mathrm{THH}(k)^{C_{p^n}} \cong W_{n+1}(k)[\sigma_{n+1}]$$

with $|\sigma_n| = 2$. Moreover,

$$\begin{aligned} F_*(\sigma_{n+1}) &= \sigma_n \\ V_*(\sigma_n) &= p\sigma_{n+1} \\ R_*(\sigma_{n+1}) &= p\sigma_n \end{aligned}$$

and F_*, V_*, R_* agree with the Witt vector Frobenius, Verschiebung, and restriction on $W_{n+1}(k)$.

Proof sketch. For $\mathbb{F}_p$, the computation of the homotopy groups follows immediately from the computation of $\pi_* \mathrm{THH}(k)^{tC_{p^{n+1}}}$ previously, and the fact that $\widehat{\Gamma}$ from Lemma 4.3 is an equivalence of connective covers.

From there, the theorem can be shown for a general perfect field k of characteristic p by induction on n. The base case $n = 0$ is Theorem 4.2. The induction step follows from the norm cofibration sequence for $\mathrm{THH}(k)$:

$$\mathrm{THH}(k)_{hC_{p^n}} \to \mathrm{THH}(k)^{C_{p^n}} \to \mathrm{THH}(k)^{C_{p^{n-1}}}. \qquad \square$$

Theorem 4.5. $\mathrm{TR}(k) \simeq HW(k)$.

Proof. To prove that $\mathrm{TR}(k)$ is an Eilenberg–MacLane spectrum, we need only show that it has a single homotopy group in degree zero.

Recall that

$$\mathrm{TR}(k) = \underleftarrow{\mathrm{holim}} \left(\mathrm{THH}(k) \xleftarrow{R} \mathrm{THH}(k)^{C_p} \xleftarrow{R} \mathrm{THH}(k)^{C_{p^2}} \leftarrow \cdots \right)$$

Therefore, after taking homotopy, we find

$$\mathrm{TR}_*(k) = \underleftarrow{\lim} \left(W_1[\sigma_1] \xleftarrow{R} W_2[\sigma_2] \xleftarrow{R} W_3[\sigma_3] \leftarrow \cdots \right).$$

The fact that homotopy groups commute with this inverse limit requires some work. In particular, one must check that the Bousfield–Kan spectral sequence has a vanishing line of slope at least 1 on the first page.

Recall that $W(k) = \underleftarrow{\lim}\, W_n(k)$. So when we didn't have the variables σ_n, we obtain the Witt vectors. To prove that the inverse limit above is isomorphic to $\mathbb{Z}_p$, it suffices to show that the class of each σ_n is zero in the inverse limit.

To see this, note that we have $W_1(k) = W(k)/pW(k) \cong k$, and k has characteristic p. Note that $\sigma_n = p\sigma_{n-1}$. So in the inverse limit, there are equivalences

$$[\sigma_n] = [p\sigma_{n-1}] = [p^2\sigma_{n-2}] = \cdots = [p^{n-1}\sigma_1].$$

But $p^{n-1}\sigma_1 \in W_1(k) \cong k$, and is therefore zero. So no σ_n survives in the limit. Hence,

$$\mathrm{TR}_*(k) \cong \mathbb{Z}_p,$$

with $\mathbb{Z}_p$ in degree zero. Therefore, there is an equivalence of spectra

$$\mathrm{TR}(k) \simeq HW(k). \qquad \square$$

Lemma 4.6. *Let F be the fiber of the zero map $0\colon HA \to HB$ between two Eilenberg–MacLane spectra. Then $F \simeq HA \vee \Sigma^{-1}HB$.*

Proof. By the long exact sequence in homotopy, $\pi_0 F \cong A$ and $\pi_{-1} F \cong B$. To show that F splits as the sum of its homotopy, it suffices to show that the cofiber sequence

$$\Sigma^{-1}HB \to F \xrightarrow{f} HA$$

splits. To that end, apply the functor $\mathrm{Map}(HA, -)$ to the fiber sequence

$$F \xrightarrow{f} HA \xrightarrow{0} HB$$

to get

$$\mathrm{Map}(HA, F) \to \mathrm{Map}(HA, HA) \to \mathrm{Map}(HA, HB).$$

The map on the right sends everything to the zero morphism. In particular, the identity $HA \to HA$ goes to zero, and therefore comes from a map $g\colon HA \to F$ such that $fg = \mathrm{id}_{HA}$. Therefore, the sequence

$$\Sigma^{-1}HB \to F \xrightarrow{f} HA$$

splits and $F \simeq HA \vee \Sigma^{-1}HB$. □

Theorem 4.7 ([HM97, Theorem B])**.** $\mathrm{TC}(k) \simeq H\mathbb{Z}_p \vee \Sigma^{-1}H\,\mathrm{coker}(F-1)$

Proof. By definition, $\mathrm{TC}(k)$ is the homotopy fiber of $F - 1\colon \mathrm{TR}(k) \to \mathrm{TR}(k)$. By the previous theorem, $\mathrm{TR}(k) \simeq HW(k)$, and F corresponds to the Witt vector Frobenius. The long exact sequence in homotopy gives

$$0 \to \mathrm{TC}_0(k) \to W(k) \xrightarrow{F-1} W(k) \to \mathrm{TC}_{-1}(k) \to 0.$$

This shows that $\mathrm{TC}(k)$ has the same homotopy type as $H\ker(F-1) \vee \Sigma^{-1}H\,\mathrm{coker}(F-1)$. Note that $\ker(F-1)$ is isomorphic to the copy of $\mathbb{Z}_p$ inside of $W(k)$, so $H\ker(F-1) \simeq H\mathbb{Z}_p$.

For $k = \mathbb{F}_p$, the map $F - 1$ is zero, and therefore is a (generalized) Eilenberg–MacLane spectrum by Lemma 4.6. In general, $\mathrm{TC}(k)$ is a $\mathrm{TC}(\mathbb{F}_p)$-module spectrum, and therefore is itself a (generalized) Eilenberg–MacLane spectrum by a theorem of Shipley. The homotopy type determines that it must be:

$$\mathrm{TC}(k) \simeq H\mathbb{Z}_p \vee \Sigma^{-1}H\,\mathrm{coker}(F-1).$$ □

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