HOMOTOPICAL CATEGORIES NAME: _____

Due at the beginning of class on 4 February 2025

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: Read §2.1 and §2.2 in [Rie14] or §B.1 in [HHR16].

(0) Introduce yourself on Zulip. Include:

(a) Your name and pronouns

(b) Two pictures of yourself: a serious one and a silly one

(c) Your university

(d) Your mathematical interests

(e) A nonmathematical interest – be specific!

- "I like reading fantasy books" :(
- "I just finished reading book 5 of the Stormlight Archive by Brandon Sanderson" :)

(1) A class of morphisms $\mathcal{W}$ in a category $\mathcal{C}$ satisfies the *two-out-of-three property* if given any two composable morphisms f and g, if any two of f, g, and gf are in $\mathcal{W}$, then so is the third.

(a) Prove that the class of weak equivalences $\mathcal{W}$ in a homotopical category $\mathcal{C}$ obeys the two-out-of-three property.

(b) Is the two-out-of-three property equivalent to the two-out-of-six property?

(2) Let $\mathcal{C}$ be any category equipped with a collection of morphisms $\mathcal{W}$. We say that $\mathcal{W}$ is *saturated* if every morphism f in $\mathcal{C}$ which becomes an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$ is in $\mathcal{W}$. We say that a homotopical category is *saturated* if the class of weak equivalences is saturated (i.e. if f becomes an isomorphism in $\mathrm{ho}(\mathcal{C})$, then $f \in \mathcal{W}$).

(a) Prove that if $\mathcal{W}$ is saturated, then $\mathcal{W}$ has the two-out-of-six property.

(b) Give an example of a homotopical category that is *not* saturated.

REFERENCES

[HHR16] Michael Hill, Michael Hopkins, and Douglas Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.

[Qui67] Daniel G. Quillen. *Homotopical algebra*, volume No. 43 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1967.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.