# SYMMETRIC/ORTHOGONAL SPECTRA TALK OUTLINE

Read [Mal23, Sections 6.1 and 6.2]. For more on symmetric spectra, see [Sch07]. For more on orthogonal spectra, see [Sch23] (take $G = e$). [Dug22, Page 4] has a good table comparing the advantages and disadvantages of the various categories of spectra.

## OUTLINE

(1) Symmetric and Orthogonal Spectra

  (a) Explain that the category of spectra $\mathcal{S}p$ that we have been working with so far is not exactly the right category of spectra. It doesn't have a smash product. We refer to $\mathcal{S}p$ as the category of *sequential spectra*.
  (b) Define symmetric spectra and the category $\mathcal{S}p^{\Sigma}$ [Mal23, Definition 6.1.2]
  (c) Define orthogonal spectra and the category $\mathcal{S}p^{O}$ [Mal23, Definition 6.1.3]
  (d) Explain that there are forgetful functors $\mathcal{S}p^{O} \to \mathcal{S}p^{\Sigma} \to \mathcal{S}p$.
  (e) Define the categories $\mathcal{S}$, $\mathcal{S}^{\Sigma}$, and $\mathcal{J} = \mathcal{S}^{O}$. Explain how we can view sequential spectra, symmetric spectra, and orthogonal spectra as functors out of these categories. See [Mal23, Definition 5.3.30, Lemma 5.3.32]. Introduce the term *diagram spectra* as in [Mal23, paragraph at the start of Section 6.1.4].
  (f) Define the free spectra $F^{\sigma}_n(K) := K \wedge \mathcal{S}^{\Sigma}(n, -)$ and $F^{O}_n(K) := K \wedge S^{O}(n, -)$, as in [Mal23, Definition 5.3.33].
  (g) Use these to describe the right adjoints to the forgetful functors.
  (h) Describe how to find (homotopy) limits and (homotopy) colimits in the categories of symmetric and orthogonal spectra [Mal23, Remark 6.1.16].
  (i) State [Mal23, Theorem 6.1.26] and its corollary, and pick a few items to prove. Don't prove all of them. The point is that sequential and symmetric spectra work basically the same as our sequential spectra from before.

(2) Equivalence of $\mathrm{ho}(\mathcal{S}p^{\Sigma})$ and $\mathrm{ho}(\mathcal{S}p)$ and the difficulty with weak equivalences in symmetric spectra

  (a) Define homotopy groups of symmetric and orthogonal spectra and define a $\pi_*$-isomorphism [Mal23, Definition 6.1.17].
  (b) Explain how a $\pi_*$-isomorphism in symmetric spectra doesn't capture the right notion of stable equivalence.
  (c) Define a stable equivalence of diagram spectra [Mal23, Definition 6.1.18].
  (d) Explain that stable equivalences and $\pi_*$-isomorphisms are equivalent in sequential spectra.
  (e) Prove that a $\pi_*$-isomorphism is a stable equivalence for all three types of spectra [Mal23, Proposition 6.1.24].
  (f) Define the *true homotopy groups* of a symmetric spectrum as the homotopy groups of $\pi_*(RX)$, where $R$ is the fibrant replacement functor [Mal23, Definition 6.1.35]. Explain that a map of symmetric spectra is a stable equivalence if it induces an isomorphism on the true homotopy groups.
  (g) State the theorem that $\pi_*$-isomorphisms and stable equivalences are the same for orthogonal spectra [Mal23, Proposition 6.1.32]. You don't have to prove it.

(h) State the theorem that with the stable equivalences, $\mathrm{ho}(\mathcal{S}p) \simeq \mathrm{ho}(\mathcal{S}p^{\Sigma}) \simeq \mathrm{ho}(\mathcal{S}p^{O})$ [Mal23, Proposition 6.1.34]. You don't have to prove it.

(3) Constructing the smash product

(a) Define the smash product of symmetric spectra [Mal23, Definition 6.2.1]. Indicate how it works for orthogonal spectra too.

(b) Give a few examples.

(c) Sketch the proof of [Mal23, Proposition 6.2.6], and state [Mal23, Theorem 6.2.8].

(d) State [Mal23, Theorem 6.2.22] and use it to prove [Mal23, Lemma 6.2.24]. Explain how to use this to prove that $\mathcal{S}p^{\Sigma}$ and $\mathcal{S}p^{O}$ are symmetric monoidal categories. Sketch the proof of Theorem 6.2.22 only if you think you'll have time.

(4) Ring spectra

(a) Explain how the smash product has a universal property similar to the universal property of the tensor product coming from bilinear homomorphisms. [Mal23, Definition 6.2.9].

(b) Define a ring spectrum [Mal23, Definition 6.2.26] and explain the remark [Mal23, Remark 6.2.27]: a ring spectrum structure is equivalent to a collection of maps $R_q \wedge R_p \to R_{p+q}$ that commute with bonding maps. See also [Sch07, Definition 1.3] for precisely which diagrams must commute.

(5) Extra things that you can do if you have the interest or time

(a) Describe the internal hom in $\mathcal{S}p^{\Sigma}$ according to [Mal23, Section 6.2.2], and show that the function spectra $F(X, -)$ are right adjoint to the smash product $X \wedge -$.

## References


[Dug22] Daniel Dugger. Stable categories and spectra via model categories. In *Stable categories and structured ring spectra*, pages 75–150. Cambridge: Cambridge University Press, 2022.

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. `http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf`, October 2023.

[Sch07] Stefan Schwede. An untitled book project about symmetric spectra. `http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf`, 2007.

[Sch23] Stefan Schwede. Lectures on equivariant stable homotopy theory. `http://www.math.uni-bonn.de/people/schwede/equivariant.pdf`, 2023.