

Due April 30th.

- Your answers should be neatly written and logically organized.
- Do your best to solve these problems by yourself, but ask for help from others if you're stuck. Asking for help is usually a good move with research problems!
- The solutions you turn in should be your own.
- You may use any resource you find online (or elsewhere), but you must cite any resource you use.

ESSAY QUESTION

Using examples from class (or elsewhere), write 1-2 pages of \TeX^* to argue for or against the following claim:

Spectra are primarily important in algebraic topology as a tool for studying stable homotopy groups of spaces.

Cite any resources you use.

PROBLEMS

Answer 3 out of the following 4 problems.

- (1) Define a category $\mathcal{Z}\mathcal{P}$ of “zpectra” whose objects are \mathbb{Z} -indexed sequences of spaces $\dots, X_{-2}, X_{-1}, X_0, X_1, X_2, \dots$ together with structure maps $\sigma_i: \Sigma X_i \rightarrow X_{i+1}$ for all $i \in \mathbb{Z}$. A *morphism of zpectra* $f: X \rightarrow Y$ is a sequence of continuous maps $f_i: X_i \rightarrow Y_i$ that commute with the structure maps of X and Y .
The *stable homotopy groups* of a zpectrum X are defined by $\pi_k X := \text{colim}_{n \in \mathbb{Z}} \pi_{n+k} X_n$. A *stable equivalence of zpectra* is a map that induces isomorphisms on stable homotopy groups.
Prove that the homotopy category of the homotopical category of zpectra and stable equivalences is equivalent to $\text{ho}(\mathcal{S}\mathcal{P})$.
- (2) Let $X \xrightarrow{f} Y \rightarrow Z$ be a cofiber sequence such that f is zero in $\text{ho}(\mathcal{S}\mathcal{P})$. Show that $Z \simeq Y \vee \Sigma X$.
- (3) Let $\widehat{\mathcal{S}\mathcal{P}}$ be any symmetric monoidal category of spectra. Given a spectrum X , define $T(X) := \bigvee_{n \geq 0} X^{\wedge n}$.
(a) Prove that $T(X)$ is an associative ring spectrum.
(b) Prove that the functor $T: \widehat{\mathcal{S}\mathcal{P}} \rightarrow \text{Mon}(\widehat{\mathcal{S}\mathcal{P}})$ is left adjoint to the forgetful functor $U: \text{Mon}(\widehat{\mathcal{S}\mathcal{P}}) \rightarrow \widehat{\mathcal{S}\mathcal{P}}$, where $\text{Mon}(\widehat{\mathcal{S}\mathcal{P}})$ is the category of monoids in $\widehat{\mathcal{S}\mathcal{P}}$, i.e. associative ring spectra.
- (4) Let X be a symmetric spectrum. Recall that the symmetric spectrum $\text{sh}^1 X$ is the symmetric spectrum with $(\text{sh}^1 X)_n = X_{1+n}$, where Σ_n acts on X_{1+n} as the subgroup of Σ_{1+n} consisting of those permutations of $\{1, \dots, n+1\}$ leaving 1 fixed.
(a) Construct a symmetric spectrum $(\text{sh}^{-1} X)$ with n -th space $(\Sigma_n)_+ \wedge_{\Sigma_{n-1}} X_n$.
(b) Show that sh^{-1} is a functor on symmetric spectra which is left adjoint to sh^1 .

*Or 1-2 pages single spaced in a word processor, about 500-1000 words. I am not a stickler for essay length.