CELLULAR SPECTRA NAME: SOLUTIONS

Due at the beginning of class on 18 March 2025

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Section 2.6].

(1) Recall that the *free spectrum functor* $F_n \colon \mathcal{T}op_* \to \mathcal{S}p$ can be described by $F_n K \simeq \Sigma^{-n}\Sigma^\infty K$ for any $n \in \mathbb{Z}$.

(a) When $n \geq 0$, prove that F_n is left adjoint to evaluation $\mathrm{ev}_n \colon \mathcal{S}p \to \mathcal{T}op_*$, where ev_n is the functor that takes the n-th space of a spectrum: $\mathrm{ev}_n X = X_n$.

SOLUTION: The levels of the spectrum $F_n K$ are

$$(F_n K)_k = \begin{cases} \Sigma_{k-n} & k \geq n \\ * & k < n \end{cases}.$$

A map $f \colon F_n K \to X$ is determined by the level-wise maps $f_k \colon (F_n K)_k \to X_k$ compatible with the bonding maps. As $*$ is the zero object of pointed spaces, the data and coherence for the maps f_k for $k < n$ is automatic. Compatibility with the bonding maps for $k \geq n$ guarantees that

$$\begin{tikzcd}
\Sigma^{k-n+1}K \arrow[r, equal] \arrow[d, "\Sigma f_k"] & \Sigma^{k-n+1}K \arrow[d, "f_{k+1}"] \\
X_k \arrow[r, "\xi_k"] & X_{k+1}
\end{tikzcd}$$

and thus that the f_k are uniquely determined by f_n (and the bonding maps of X). Thus, it follows that maps $F_n K \to X$ are naturally bijective to maps $K \to X_n$, which completes the proof.

(b) Does F_n have a right adjoint when $n < 0$?

SOLUTION: For $n \geq 0$, $F_{-n} \simeq \Sigma^n \Sigma^\infty K$. As both Σ and Σ^∞ are left adjoints, it follows that F_{-n} is a composite of left adjoints and thus has a right adjoint.

(2) Consider the homotopy pushout/pullback square of spectra:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f'"] & B \arrow[d, "g'"] \\
A \arrow[r, "g"] & Y.
\end{tikzcd}$$

Prove that there is a Mayer–Vietoris-type long exact sequence of spectra:

$$\cdots \to \pi_{n+1} Y \to \pi_n X \to \pi_n A \oplus \pi_n B \to \pi_n Y \to \pi_{n-1} X \to \cdots$$

SOLUTION: We may assume Y is the homotopy push out $A \bigcup_X^h B$. There is a natural map $i : A \vee B \to Y$ which is a closed inclusion $i_n : A_n \vee B_n \to Y_n = A_n \cup_{X_n}^h B_n$ at each level, and therefore the cofiber at each level is homotopy-equivalent to the quotient of Y_n by the image of i_n:

$$\mathrm{cof}(i)_n = \mathrm{cof}(i_n) \simeq Y_n/(A_n \vee B_n) \cong \Sigma X_n$$

The last isomorphism can be seen by definition of the homotopy pushout:

$$Y_n/(A_n \vee B_n) = ((A_n \vee (X_n \wedge I_+) \vee B_n)/\sim)/(A_n \vee B_n) \cong (X_n \wedge I_+)/(X_n \wedge \{0\} \cup X_n \wedge \{1\}) \cong \Sigma X_n$$

Thus we have a cofiber/fiber sequence of spectra $A \vee B \to Y \to \Sigma X$. Recall that we can extend this fiber sequence to the left by taking loop spaces; in particular, we have a fiber sequence

$$\Omega\Sigma X \to A \vee B \to Y$$

Recall that $\Omega \simeq \Sigma^{-1}$. Hence the above fiber sequence yields

$$X \to A \vee B \to Y$$

Moreover, $A \vee B$ and $A \times B$ are stably equivalent, so we could alternatively write this as

$$X \to A \times B \to Y$$

Then since homotopy groups commute with products, and homotopy groups of spectra are always abelian, we have $\pi_*(A \vee B) \cong \pi_*(A) \times \pi_*(B) \cong \pi_*(A) \oplus \pi_*(B)$. Hence the induced long exact sequence associated to this fiber sequence is:

$$\ldots \pi_{n+1}(Y) \to \pi_n(X) \to \pi_n(A) \oplus \pi_n(B) \to \pi_n(Y) \to \pi_{n-1}(X) \to \ldots$$

(3) A spectrum X is called n-connect**ed** if $\pi_i X = 0$ for $i \leq n$, or n-connect**ive** if $\pi_i X = 0$ for $i < n$. Let $X \to Y \to Z$ be a cofiber/fiber sequence of spectra.

(a) Prove that if X and Z are n-connected, then so is Y.

SOLUTION: Suppose X and Z are n-connected. The cofiber sequence induces a long exact sequence on homotopy groups:

$$\cdots \to \pi_{k+1}(Z) \to \pi_k(X) \to \pi_k(Y) \to \pi_k(Z) \to \pi_{k-1}(X) \to \ldots$$

Since X and Z are n-connected we have $\pi_k(X) = \pi_k(Z) = 0$ for all $k \leq n$. Thus we can extract sequences $0 \to \pi_k(Y) \to 0$ from above. By exactness, $\pi_k(Y) = 0$ for all $k \leq n$, i.e. Y is n-connected (or $(n+1)$-connective.

(b) What can you say about connectivity of Z if X and Y are n-connected? What can you say about connectivity of X if Y and Z are n-connected?

SOLUTION: Suppose X and Y are n-connected. As above, we can extract sequences $0 \to \pi_k(Z) \to 0$ for all $k \leq n$, showing that $\pi_k(Z) = 0$ for all $k \leq n$. Thus, Z is at least n-connected.
Now Suppose Y and Z are n-connected. We can extract sequences $0 \to \pi_k(X) \to 0$ for all $k \leq n-1$, showing that X is at least $(n-1)$-connected. At degree n, we have:

$$\cdots \to \pi_{n+1}(Y) \to \pi_{n+1}(Z) \to \pi_n(X) \to 0$$

So we cannot determine whether X is n-connected or not, only that it is $(n-1)$-connected (or n-connective).

(4) Prove that the following three conditions are equivalent:

(a) X is a *finite spectrum*, i.e. X is stably equivalent to a cellular spectrum with finitely many stable cells.

(b) X is stably equivalent to a free spectrum $F_k K \simeq \Sigma^{-k}\Sigma^\infty K$ for a finite cell complex K.

(c) X is bounded below and the direct sum of the homology groups $\bigoplus_k H_k(X;\mathbb{Z})$ is finitely generated as an abelian group.

SOLUTION: (a) $\implies$ (b): Let the stable cells of X be given by $c_1, \ldots, c_k$ so that c_i is of dimension m_i. By definition, the c_i are attached as $(m_i + n_i)$-cells in X_{n_i} for some n_i. Choose $n = \max n_i$ so that the c_i can be taken to be attached as $(m_i + n)$-cells in X_n. Then, build a finite cell complex K with these $(m_i + n)$-cells. This induces a map $F_n K \to X$, and this is a stable equivalence because these spectra have the same cells.

(b) $\implies$ (c): Write $H_*(X)$ for $\bigoplus_k H_k(X;\mathbb{Z})$. Suppose X is stably equivalent to $F_n K$. Then, $H_*(X) \cong H_*(F_n K) \cong H_{*+n}(K)$. Since K is a finite cell complex, its homology can be computed as the homology of its cellular chain complex. Since K has finitely many cells, the cellular chain complex is finitely generated (finitely many total generators, not just in each degree), and thus its homology is finitely generated (finitely many total generators) since quotients of finitely generated abelian groups are finitely generated. Thus, $H_*(X)$ is finitely generated.

(c) $\implies$ (a): Suppose $H_*(X)$ is finitely generated. In particular, $H_*(X)$ is bounded below. Wlog, we may suppose that $H_*(X)$ is non-negatively graded by using the fact that the shift functor is an equivalence. By algebra nonsense, $H_*(X)$ is also finitely presented. Perhaps even better, it has a 2-term resolution of finitely generated free abelian groups. Use this free resolution to make a cell complex K (of spaces) with cells given by the generators and attaching maps given by the degrees according to the map in the resolution. By construction $H_*(K) \cong H_*(X)$, which can be verified by using cellular homology to compute $H_*(K)$. Further, the fact that these cells correspond to homology classes in X gives a map $\Sigma^\infty K \to X$ witnessing this homology isomorphism. Then since $\Sigma^\infty K$ and X are bounded below, the Hurewicz theorem implies that this map is a stable equivalence. Finally, note $\Sigma^\infty K$ is a cellular spectrum with finitely many stable cells. This completes the proof.

REFERENCES

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.