STABLE PHENOMENA IN SPACES NAME: ______________________

Due at the beginning of class on 6 February 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Sto22, Chapter 2].

(1) A theorem of Serre shows that $\pi_i(S^n)$ for $i > 2$ is a finite abelian group, except for two classes of exceptions: $\pi_n(S^n) \cong \mathbb{Z}$ and $\pi_{4j-1}(S^{2j}) \cong \mathbb{Z} \oplus M$, where M is a torsion $\mathbb{Z}$-module. Use this to prove that the stable homotopy groups $\pi_i^s(S^0)$ are finite abelian for $i > 0$.

(2) Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category. A *monoid* in $\mathcal{C}$ is an object M together with morphisms $m\colon M \otimes M \to M$ and $i\colon 1 \to M$ such that the following diagrams commute:

$$\begin{tikzcd} M \otimes M \otimes M \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes m"'] & M \otimes M \arrow[d, "m"] \\ M \otimes M \arrow[r, "m"] & M \end{tikzcd} \qquad \begin{tikzcd} 1 \otimes M \arrow[r, "i\otimes id"] \arrow[dr, "\cong"'] & M \otimes M \arrow[d, "m"] & M \otimes 1 \arrow[l, "id\otimes i"'] \arrow[dl, "\cong"] \\ & M & \end{tikzcd}$$

A *morphism of monoids* $f\colon M \to N$ is one that commutes with the structure morphisms:

$$\begin{tikzcd} M \otimes M \arrow[r, "f\otimes f"] \arrow[d, "m"'] & N \otimes N \arrow[d, "m"] \\ M \arrow[r, "f"] & N \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "i"] \arrow[dr, "i"'] & M \arrow[d, "f"] \\ & N \end{tikzcd}$$

(M, m, i) is a *commutative monoid* if $f = f \circ s$, where $s\colon M \otimes M \to M \otimes M$ is the symmetry in $\mathcal{C}$.

(a) Let M and C be objects in $\mathcal{C}$. Prove that if M is a monoid and C is a comonoid, then $\mathcal{C}(C, M)$ is a monoid in the ordinary sense: a set with an associative and unital operation.

(b) Let M in $\mathcal{C}$ be a monoid in two different ways: (M, m, i) and (M, n, j). Further assume that m and n are morphisms of monoids. Prove that M is a commutative monoid and the two structures are the same.

(c) For any spaces X and Y, prove that $[X, \Omega^2 Y]$ and $[\Sigma X, \Omega Y]$ are abelian groups.

(3) Let $f\colon X \to Y$ be a map between simply connected spaces such that $f_*\colon H_i(X) \to H_i(Y)$ is an isomorphism for $i \leq n$. We will show that f is an n-connected map.

(a) Let C be the homotopy cofiber of f, and let F be the homotopy fiber of $Y \to C$. Use the Hurewicz theorem to show that C is n-connected and $F \to Y$ is an n-connected map.

(b) Use the Blakers–Massey theorem to show that $X \to F$ is at least 2-connected.

(c) Show that f is at least 2-connected. Iterate your argument from part (b) to show that f is n-connected.

(4) Let $X_0 \to X_1 \to X_2 \to \cdots$ be a sequence of spaces. Prove that $\Omega \operatorname{hocolim}_i X_i \simeq \operatorname{hocolim}_i \Omega X_i$. Use this to show that homotopy groups commute with sequential homotopy colimits.

References

[Sto22] Bruno Stonek. Introduction to stable homotopy theory. https://bruno.stonek.com/stable-homotopy-2022/stable-online.pdf, July 2022.