LECTURE 06

§6.1 (Area Between Curves)
§6.2 (Setting up Integrals) 2 July 2018

1 PLAN

(1) Lecture on §6.1 (Area between curves)

(2) §6.1 Worksheet

(3) Short Break

(4) Lecture on §6.2 (Setting up integrals)

(5) §6.2 Worksheet

2 HANDOUTS

(1) §6.1 Worksheet (a few copies)

(2) §6.2 Worksheet (a few copies)

3 BEFORE CLASS

- Arrange desks into small groups of 4.
- Be early!
- Write this on the board:

Math 1910 Calculus II for Engineers

David Mehrle (dfm223@cornell.edu)

TA: John Whelan (jhw268@cornell.edu)

Today: §6.1 (Area Between Curves)
§6.2 (Setting up Integrals)

Tomorrow:

Don't forget to visit office hours if you haven't already! It's worth 5 points!

HW 1 due tomorrow!

Office hours: MT 3pm - 4pm, R 2pm - 4pm in Malott 256

4 ANNOUNCEMENTS

- There is an error on question 7 on the homework. It should be $N(d+1) - N(d)$ in the answer. Also, the first time $N(d)$ appears it should be $N'(d)$.
- Homework 1 due tomorrow.
- No class on Wednesday!

5 LECTURE ON §6.1

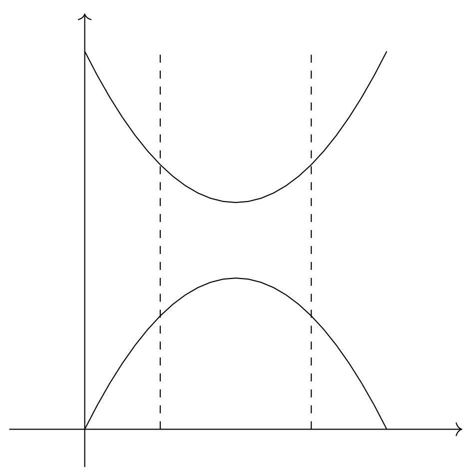
So far we've talked about the integral as the area under the curve, but what if we're asked to find a more complicated region, bounded by two curves?

Theorem 5.1. *Given two functions $f(x)$ and $g(x)$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$, the area of the region bounded between f and g is*

$$\int_a^b f(x) - g(x)\, dx$$

Example 5.2. Find the area of the region between $f(x) = x^2 - 4x + 10$ and $g(x) = 4x - x^2$ for $1 \leq x \leq 3$.

We want the bowtie region in the middle.



$$\int_1^3 f(x) - g(x)\, dx = \int_1^3 ((x^2 - 4x + 10) - (4x - x^2))\, dx = \int_1^3 (2x^2 - 8x + 10)\, dx = \left(\frac{2}{3}x^2 - 4x^2 + 10x\right)\Bigg|_1^3 = \frac{16}{3}$$

Okay, this works if both f and g are above the x-axis, but what if one or both are below it?

Question 5.3 (Think-pair-share)**.** Does it matter that f and g are both above the x-axis in the above example? The integral is signed area.

The answer is no, it doesn't matter. We can add a constant to both, and the area between them is just the difference. This isn't a signed area! So this is different from the normal integral, which is the signed area.

Definition 5.4. A region is **vertically simple** if any vertical line that intersects the region does so with f on top and g on bottom; they don't switch position over a vertically simple region.

However, if $f \geq g$ sometimes, and then $g \geq f$ sometimes, we have to correct for that.

Example 5.5. Find the area between the graphs of $f(x) = x - 12$ and $g(x) = x^2 - 5x - 7$ over $[-2, 5]$.

The first thought is that, as in the previous example, we would compute

$$\int_{-2}^{5} f(x) - g(x)\,dx,$$

but this doesn't work!

$$\int_{-2}^{5} f(x) - g(x)\,dx = \int_{-2}^{5} -x^2 + 6x - 5\,dx = -16\frac{1}{3}.$$

That's an area, but it's negative. Whoops.

Instead, we need to divide the integral into vertically simple regions. We have

$$f(x) - g(x) = (x - 12) - (x^2 - 5x - 7) = -x^2 + 6x - 5 = -(x - 1)(x - 5).$$

The graphs intersect at $x = 1$ and $x = 5$. We have $f(x) - g(x) \geq 0$ for $1 \leq x \leq 5$ and therefore:

- $f(x) \geq g(x)$ on $[1, 5]$
- $g(x) \geq f(x)$ on $[-2, 1]$

This tells us to subdivide our region into two vertically simple regions over $[-2, 1]$ and $[1, 5]$. So the area is a sum of two intervals:

$$\begin{aligned}
\int_{-2}^{1} g(x) - f(x)\,dx + \int_{1}^{5} f(x) - g(x)\,dx &= \int_{-2}^{1} ((x^2 - 5x - 7) - (x - 12))\,dx + \int_{1}^{5} ((x - 12) - (x^2 - 5x - 7))\,dx \\
&= \int_{-2}^{1} x^2 - 6x + 5\,dx + \int_{1}^{5} -x^2 + 6x - 5\,dx \\
&= \left(\frac{1}{3}x^3 - 3x^2 + 5x\right)\Bigg|_{-2}^{1} + \left(-\frac{1}{3}x^3 + 3x^2 - 5x\right)\Bigg|_{1}^{5} \\
&= \frac{113}{3}
\end{aligned}$$

So we need to revise the previous theorem:

Theorem 5.6. *The area between the graphs of* $y = f(x)$ *and* $y = g(x)$ *over the interval* $[a, b]$ *is*

$$\int_a^b y_{\text{top}} - y_{\text{bottom}}\, dx$$

Where

$$y_{\text{top}} = \begin{cases} f(x) & f(x) \geq g(x) \\ g(x) & g(x) \geq f(x) \end{cases}$$

and similarly for y_{bottom}.

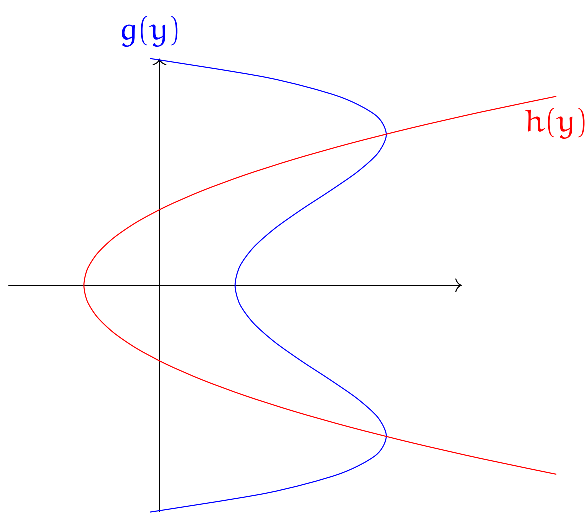
What if instead we want to integrate along the y-axis?

We may interpret $\int_a^b g(y)\, dy$ as the signed area between the graph of $x = g(y)$ and the y-axis. Areas to the right of the y-axis have positive area, to the left negative. This can be seen because $y = g(x)$ is the same function reflected across the line $x = y$.

Theorem 5.7. *The area between the graphs of* $y = f(x)$ *and* $y = g(x)$ *over the interval* $[c, d]$ *is*

$$\int_c^d x_{\text{right}} - x_{\text{left}}\, dy$$

Example 5.8. Calculate the area enclosed by the graphs of $h(y) = y^2 - 1$ and $g(y) = y^2 - \frac{1}{8}y^4 + 1$.



The two graphs intersect where $g(y) = h(y)$, or where

$$y^2 - \frac{1}{8}y^4 + 1 = y^2 - 1 \implies \frac{1}{8}y^4 - 2 = 0 \implies y = \pm 2.$$

The enclosed area is

$$\begin{aligned}\int_{-2}^{2} x_{\text{right}} - x_{\text{left}}\, dy &= \int_{-2}^{2} (2 - \frac{1}{8}y^4)\, dy \\ &= \left(2y - \frac{1}{40}y^5\right)\Big|_{-2}^{2} \\ &= \frac{32}{5}\end{aligned}$$

Remark 5.9 (General strategy for solving these kinds of problems)**.**

(1) Draw a picture!

(2) Find points of intersection.

(3) Divide into vertically simple regions.

(4) Write down an integral.

6 §6.1 WORKSHEET

- Divide students into groups of 4 (with one group of 5). Each group gets an area of the board to work at and one question to work on. They should draw the graph, shade the area, and set up (but don't evaluate) the integral.

7 SHORT BREAK

8 §6.2 LECTURE

Review similar triangles.

Definition 8.1. Two triangles ΔABC and $\Delta A'B'C'$ are **similar** if they have the same angles.

Theorem 8.2. *Given two similar triangles* ΔABC *and* $\Delta A'B'C'$*, the ratios of the side lengths are the same:*

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

This comes up *all the time!*

So far we've been using integrals to compute areas, but they are infinitely more practical than this. Any thing that can be thought of as the total amount of a bunch of small things can be written as an integral: mass, volume, population, work, etc. We'll get to all of these in time, but for now, we are using integrals to compute volumes.

Question 8.3 (Think-pair-share)**.** Which one has more volume: a straight cylinder, or a slanted cylinder?

The answer, of course, is that neither one does! This is an illustration of **Cavilieri's Principle**

Theorem 8.4 (Cavilieri's Principle)**.** *Solids with equal cross-sectional areas have the same volume.*

Think of this like a stack of coins: if you move every other coin off to the side, it's the same volume as the stack when it's straightened out.

This is a general strategy for finding volumes of shapes:

Theorem 8.5. *Let* $A(y)$ *be the area of a horizontal cross-section at height* y *of a solid body extending from* $y = a$ *to* $y = b$*. Then its volume is*

$$V = \int_a^b A(y)\, dy.$$

Example 8.6 (Volume of a pyramid)**.** Find the volume of a pyramid of height 12m whose base is a square of side 4m.

$$A(y) = s^2 = \frac{1}{9}(12 - y)^2.$$

$$V = \int_0^{12} A(y)\, dy = 64 \text{ m}^3.$$

Example 8.7 (Volume of a sphere)**.** Find the volume of a sphere of radius R.

$$A(x) = \pi(R^2 - x^2)$$

$$V = \int_{-R}^{R} \pi(R^2 - x^2)\, dx = \frac{4}{3}\pi R^3.$$

Remark 8.8 (General strategy for solving these problems)**.**

(1) Draw a picture! Indicate cross-sections.

(2) Find a formula for the cross-sectional area as a function of height.

(3) Write an integral of the cross-sectional areas.

(4) Evaluate.

We can also use these ideas to find mass: if an object has a linear mass density $\rho(x)$, then it's total mass is

$$M = \int_a^b \rho(x)\, dx.$$

Likewise, if we know that the population density of a city is given by a radial density function $\rho(r)$, where r is the distance from the city center, then the total population within r miles of the city center is

$$P = 2\pi \int_0^R r\rho(r)\, dr.$$

9 §6.2 Worksheet

Students work in groups at their table. One worksheet per group.

10 Debrief

- Stumbled through the examples for section 6.1. In particular, the one graph I didn't draw in the notes I failed to graph by hand.
- Some of the questions in the activity were too easy, and took way less time than the others. I didn't manage the groups who finished early very well.
- It would be better to put graphing $x = f(y)$ on this reading assignment, and save the cap of a sphere and similar triangles stuff for the next reading assignment.
- I need to look at the photo roster again to learn names.
- Spend less time at the board! Spend more time on activities!
- Students seemed bored.