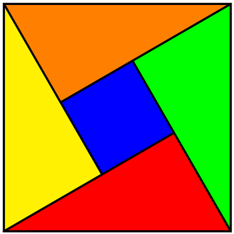
The Hopf Algebra Spectrum of Spherical Scissors Congruence

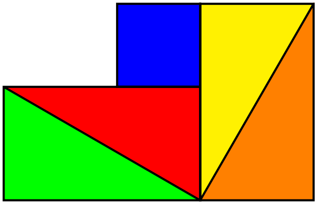
joint with Klang, Kuijper, Malkiewich, and Wittich

slides available at www.davidmehrle.com/ssc.pdf

Two polygons P and Q are *scissors congruent* if you can chop P into finitely many pieces and rearrange them to form Q.



P



Q

Two polygons are scissors congruent if and only if they have the same area.

HILBERT'S THIRD PROBLEM (1900)

Are any two polyhedra of equal volume scissors congruent?

Counterexample: any cube and tetrahedron of equal volume

Solved using Dehn invariant:

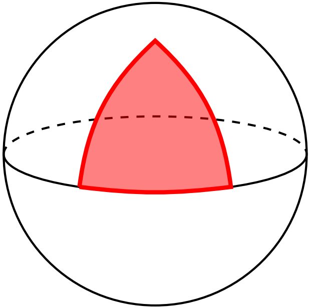
$$\begin{aligned} \text{Polyhedra} &\longrightarrow \mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\pi\mathbb{Z}) \\ P &\longmapsto \sum_{\substack{\text{edges } e \\ \text{in } P}} \text{length}(e) \otimes_{\mathbb{Z}} \text{angle}(e) \end{aligned}$$

THEOREM (Dehn, 1901) (Sydler, 1965)

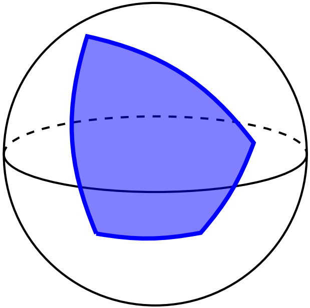
Dehn invariant and volume are complete scissors congruence invariants in dimensions 3 (Dehn) and 4 (Sydler).

GENERALIZED HILBERT'S THIRD PROBLEM

Are volume and Dehn invariant complete invariants in all dimensions? In spherical or hyperbolic geometry?



a spherical 2-simplex



a spherical polygon

Let $X^n = \mathbb{R}^n$ (Euclidean) or $X^n = S^n$ (Spherical).

DEFINITION

The *polytope group* $\mathcal{P}(X^n)$ is the abelian group with

- generators: polytopes $P \subseteq X^n$
- relations:

$$P = \sum_{i=1}^{m} P_i \quad \text{when} \quad P = \bigcup_{i=1}^{m} P_i, \quad \text{area}(P_i \cap P_j) = 0$$

$$P = \phi(P) \quad \text{for any isometry } \phi\colon X^n \to X^n$$

Polytopes P and Q are *scissors congruent* if $[P] = [Q]$ in $\mathcal{P}(X^n)$.

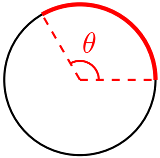
EXAMPLES

Area is a complete scissors congruence invariant in 2D:

$$\mathcal{P}(\mathbb{R}^2) \cong \mathbb{R}.$$

Angle is a complete SC invariant in S^1:

$$\mathcal{P}(S^1) \cong \mathbb{R}/2\pi\mathbb{Z}.$$



Dehn invariant, revisited:

$$\begin{aligned} \mathcal{P}(\mathbb{R}^3) &\longrightarrow \mathcal{P}(\mathbb{R}^1) \otimes_{\mathbb{Z}} \widetilde{\mathcal{P}}(S^1) \\ P &\longmapsto \sum_{\substack{\text{edges } e \\ \text{in } P}} \text{length}(e) \otimes_{\mathbb{Z}} \text{angle}(e) \end{aligned}$$

THEOREM (Sah, 1979)

The graded abelian group $\bigoplus_n \mathcal{P}(S^n)$ is a graded ring with join as multiplication. The quotient

$$\mathcal{S} := \bigoplus_{n \geq 0} \mathcal{P}(S^n)/([\mathrm{pt}])$$

is a commutative graded Hopf algebra.

Let $\widetilde{\mathcal{P}}(S^n)$ be the degree n piece of $\mathcal{S}$.

Coproduct is given by generalized Dehn invariants:

$$\mathcal{P}(X^n) \longrightarrow \mathcal{P}(X^{n-c}) \otimes_{\mathbb{Z}} \widetilde{\mathcal{P}}(S^{c-1}).$$

This also makes $\bigoplus_{n \geq 0} \mathcal{P}(\mathbb{R}^n)$ into an $\mathcal{S}$-comodule.

Scissors Congruence as K-Theory

Theorem (Zakharevich)

There is a K-theory spectrum $K(X^n)$ such that

$$\mathcal{P}(X^n) \cong \pi_0 K(X^n),$$

and a reduced K-theory spectrum $\widetilde{K}(X^n)$ such that

$$\widetilde{\mathcal{P}}(S^n) \cong \pi_0 \widetilde{K}(S^n).$$

Theorem (KKMMW)

The spectral Sah algebra

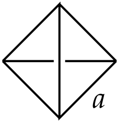
$$\mathbb{S} := \bigvee_{n \geq 0} \widetilde{K}(S^n)$$

is a Hopf algebra spectrum with $\pi_0 \mathbb{S} \cong \mathcal{S}$.

Obstacle

Trigonometry shows that $\mathcal{S} = \pi_0(\mathbb{S})$ is not cocommutative:

$\mathcal{P}(S^3) \ni$



$$\xmapsto{\text{Dehn}} a \otimes \theta, \qquad \cos(\theta) = \frac{\cos(a)}{1 + 2\cos(a)}$$

Theorem (Péroux–Shipley)

Coalgebras in a symmetric monoidal model category of spectra are necessarily cocommutative.

Consequences:

- Hopf algebra structure on $\mathbb{S}$ only exists in an ∞-category,
- or we must use operadic coalgebras.

WORKAROUND

THEOREM (Malkiewich)

Spherical scissors congruence K-theory is a Thom spectrum:

$$\widetilde{K}(S^n) \simeq \left(\Sigma^{-\mathbb{R}^n}\Sigma^\infty \Sigma \mathrm{S}\mathrm{T}(\mathbb{R}^n)\right)_{hO(n)^\delta},$$

where

- $T(\mathbb{R}^n) = \left|\text{poset of subspaces } U \text{ with } 0 \subsetneq U \subsetneq \mathbb{R}^n\right|$
- Σ and S are reduced and unreduced suspension
- Σ^∞ is suspension spectrum
- $\Sigma^{-\mathbb{R}^n}$ is desuspension with $O(n) \circlearrowright \mathbb{R}^n$
- $O(n)^\delta$ is orthogonal group with discrete topology

Workaround

Theorem (Malkiewich)

Spherical scissors congruence K-theory is a Thom spectrum:

$$\widetilde{K}(S^n) \simeq \left(\Sigma^{-\mathbb{R}^n}\Sigma^\infty \Sigma \mathrm{ST}(\mathbb{R}^n)\right)_{hO(n)^\delta}.$$

Let Dip be the category of finite-dimensional inner-product spaces and isometries. Define:

$$\begin{aligned} \mathfrak{S}\colon \mathrm{Dip} &\longrightarrow \mathrm{Sp}^O \\ V &\longmapsto \Sigma^{-V}\Sigma^\infty\Sigma\mathrm{ST}(V) \\ 0 &\longmapsto \mathbb{S} \end{aligned}$$

There is an equivalence of spectra $\mathcal{S} \simeq \operatorname{colim} \mathfrak{S}$.

Reductions

$\mathrm{Fun}(\mathrm{Dip}, \mathrm{Sp}^O)$ is symmetric monoidal via Day convolution,

$$\begin{tikzcd} \mathrm{Dip} \times \mathrm{Dip} \arrow[r, "F\times G"] \arrow[d, "\oplus"'] & \mathrm{Sp}^O \times \mathrm{Sp}^O \arrow[r, "\wedge"] & \mathrm{Sp}^O \\ \mathrm{Dip} \arrow[urr, "\boxtimes = \mathrm{Lan}_{\oplus}(F\wedge G)"'] & & \end{tikzcd}$$

and $\mathrm{colim}\colon \mathrm{Fun}(\mathrm{Dip}, \mathrm{Sp}^O) \to \mathrm{Sp}^O$ is strong symmetric monoidal

Reduction 1

To show $\mathcal{S} = \mathrm{colim}\, \mathfrak{S}$ is a Hopf algebra spectrum,
it suffices to show $\mathfrak{S}$ is a Hopf algebra in $\mathrm{Fun}(\mathrm{Dip}, \mathrm{Sp}^O)$.

Reductions

Reduction 1

It suffices to show $\mathfrak{S}$ is a Hopf algebra in $\mathrm{Fun}(\mathrm{Dip}, \mathrm{Sp}^O)$.

The functor

$$\begin{aligned} \mathrm{Fun}(\mathrm{Dip}, \mathrm{Top}_*) &\longrightarrow \mathrm{Fun}(\mathrm{Dip}, \mathrm{Sp}^O) \\ F &\longmapsto \left(V \mapsto \Sigma^{-V}\Sigma^{\infty}F(V) \right) \end{aligned}$$

is strong symmetric monoidal.

Reduction 2

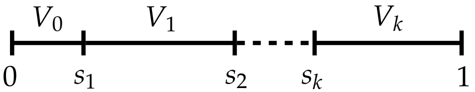
To show that $\mathfrak{S}$ is a Hopf algebra in $\mathrm{Fun}(\mathrm{Dip}, \mathrm{Sp}^O)$,
it suffices to show that $V \mapsto \Sigma ST(V)$ is Hopf in $\mathrm{Fun}(\mathrm{Dip}, \mathrm{Top}_*)$.

Reduction 2

It suffices to show that $V \mapsto \Sigma ST(V)$ is Hopf in $\mathrm{Fun}(\mathrm{Dip}, \mathrm{Top}_*)$.

A model for $\Sigma ST(V)$

$$\Sigma ST(V) \cong \left\{ \begin{array}{c} \text{order-preserving} \\ f\colon [0,1] \to \mathrm{Sub}(V) \end{array} \right\} \Big/ \begin{array}{c} f \sim g \text{ if } f \text{ and } g \text{ differ} \\ \text{only finitely often} \end{array}$$

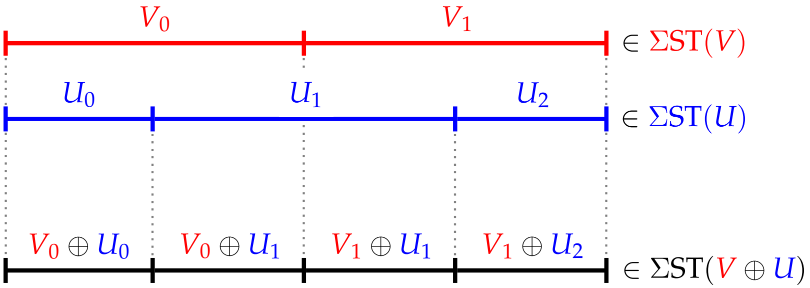


$\in \Sigma\mathrm{ST}(V)$

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_k$$

Product

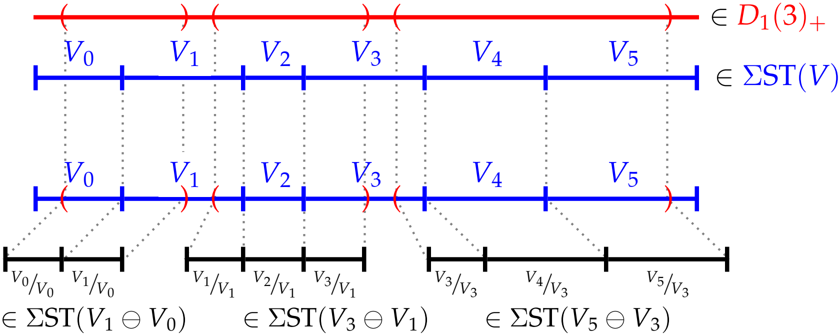
$$\big(\Sigma\mathrm{ST} \boxwedge \Sigma\mathrm{ST}\big)(W) = \bigvee_{V \oplus U = W} \Sigma\mathrm{ST}(V) \wedge \Sigma\mathrm{ST}(U) \longrightarrow \Sigma\mathrm{ST}(W)$$



Coproduct

$\Sigma\mathrm{ST}$ is a coalgebra for the little intervals operad D_1:

$$D_1(n)_+ \wedge \Sigma\mathrm{ST}(V) \to \Sigma\mathrm{ST}^{\boxed{\wedge} n}(V) = \bigvee_{V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n} \bigwedge_{i=1}^{n} \Sigma\mathrm{ST}(V_i \ominus V_{i-1})$$



Antipode

Lemma

A bialgebra B in a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ is Hopf if and only if the shear map is an isomorphism.

$$\text{shear}\colon B \otimes B \xrightarrow{\delta \otimes \mathrm{id}} B \otimes B \otimes B \xrightarrow{\mathrm{id} \otimes \mu} B \otimes B$$

Check the shear map of ΣST is iso. Solomon-Tits theorem gives

$$\Sigma ST(V) \simeq \bigvee_{\alpha} S^{\dim(V)-1},$$

so suffices to check the shear map is an isomorphism on H_*.

Follows from Sah's theorem that $\mathcal{S}$ is a Hopf algebra.

Theorem

$V \mapsto \Sigma\mathrm{ST}(V)$ is an (E_∞, E_1)-Hopf algebra in $\mathrm{Fun}(\mathrm{Dip}, \mathrm{Top}_*)$

THEOREM

$V \mapsto \Sigma\mathrm{ST}(V)$ is an (E_∞, E_1)-Hopf algebra in $\mathrm{Fun}(\mathrm{Dip}, \mathrm{Top}_*)$

COROLLARIES

- $\mathfrak{S}$ is an (E_∞, E_1)-Hopf algebra in $\mathrm{Fun}(\mathrm{Dip}, \mathrm{Sp}^O)$.
- $\mathcal{S}$ is an (E_∞, E_1)-Hopf algebra in Sp^O.

THEOREM (KKMMW)

Under mild cofibrancy assumptions, (E_∞, E_1)-bialgebras transfer to commutative bialgebras in the underlying ∞-category.

$\Longrightarrow \mathcal{S}$ is a Hopf algebra in $\mathbf{Sp} = N(\mathrm{Sp}^O)$.

Application

Primitive elements x in a Hopf algebra satisfy

$$\delta(x) = 1 \otimes x + x \otimes 1.$$

Lemma

Let H be a rational Hopf algebra.
Let $V \subseteq H$ be a sub-vector space of primitive elements.
The free graded-commutative algebra $\Lambda(V)$ is a subalgebra of H.

$$H = \pi_*(\mathcal{S}_{\mathbb{Q}}) \cong \bigoplus_{k,n} \pi_k \widetilde{K}(S^{n-1}) \otimes \mathbb{Q}.$$

$$V = \bigoplus_{k} \pi_k \widetilde{K}(S^{1}) \otimes \mathbb{Q} \quad (n = 2)$$

$\pi_*(\mathcal{S}_{\mathbb{Q}})$ is bigraded: $n =$ dimension, $k =$ homotopy degree

APPLICATION

THEOREM (Malkiewich)

$$\pi_k \widetilde{K}(S^1) \otimes \mathbb{Q} \cong \begin{cases} \Lambda^{k+1}\left({}^{\mathbb{R}}/_{\mathbb{Q}}\right) & (k \text{ even}) \\ 0 & (k \text{ odd}) \end{cases}$$

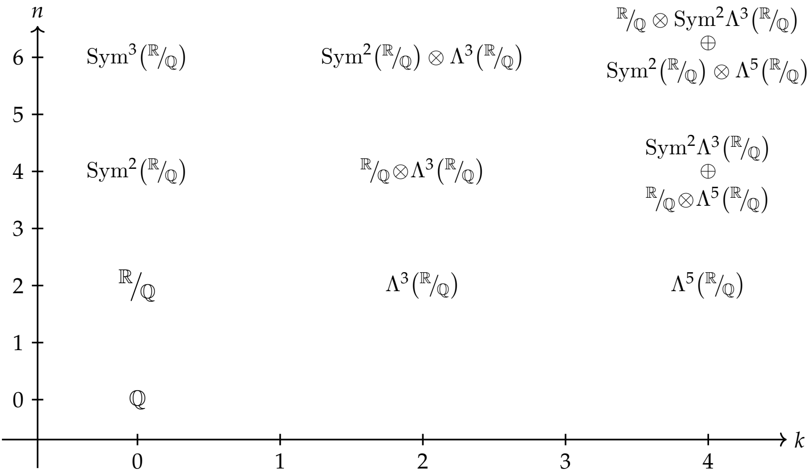
$$V = \bigoplus_k \pi_k \widetilde{K}(S^1) \otimes \mathbb{Q} \cong \bigoplus_{k \text{ even}} \Lambda^{k+1}\left({}^{\mathbb{R}}/_{\mathbb{Q}}\right)$$

THEOREM (KKMMW)

$\pi_*(\mathcal{S}_{\mathbb{Q}})$ has a Hopf subalgebra: the free commutative algebra on

$$\bigoplus_{k \text{ even}} \Lambda^{k+1}\left({}^{\mathbb{R}}/_{\mathbb{Q}}\right),$$

taken in dimension $n = 2$.



A large nonzero subalgebra of $\pi_*(\mathcal{S}_{\mathbb{Q}}) \cong \bigoplus_{k,n} \pi_k \widetilde{K}(S^{n-1}) \otimes \mathbb{Q}$

Thanks for listening!

BONUS: ABSTRACT NONSENSE

- $M_\bullet$ a symmetric monoidal simplicial model category
- $M_\bullet^\flat \subseteq M_\bullet$ a $\otimes$-closed full subcategory with all cofibrants
- $\mathbf{M}$ underlying ∞-category
- $\mathcal{O}_\bullet$ fibrant simplicial operad

THEOREM (KKMMW)

There is a canonical map of simplicial sets

$$N^s(\mathrm{Alg}_{\mathcal{O}_\bullet}(M_\bullet)) \longrightarrow \mathrm{Alg}_{N^s(\mathcal{O}_\bullet)}(\mathbf{M}).$$

There is a map of ∞-categories

$$N^s(\mathrm{BiAlg}_{E_\infty, E_1}(M_\bullet^\flat)) \longrightarrow \mathrm{CBiAlg}(\mathbf{M})$$

that sends Hopf algebras to Hopf algebras.