TRIANGULATED CATEGORIES NAME: ______________________

Due at the beginning of class on 12 March 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Section 3.2] and [Wei94, Section 10.2]

(1) (a) A spectrum X is *rational* if each of its homotopy groups is an $\mathbb{Q}$-vector space. Prove that the full subcategory of $\mathrm{ho}(\mathcal{S}p)$ consisting of the rational spectra is a triangulated subcategory.

(b) Define a triangulated functor $H\colon \mathcal{D}(\mathbb{Q}) \to \mathrm{ho}(\mathcal{S}p)$ such that $\pi_n H(V_\bullet) = H_n(V_\bullet)$ for all $n \in \mathbb{Z}$. *Hint: any chain complex of $\mathbb{Q}$-vector spaces is quasi-isomorphic to its homology.*

(2) Show that the class of stable equivalences is saturated in $\mathcal{S}p$.

(3) (a) By giving a counterexample, show that $\mathrm{ho}(\mathcal{S}p)$ is not an abelian category.

(b) Let X be a spectrum and $e\colon X \to X$ an *idempotent* map: $e \circ e = \mathrm{id}_X$. Construct a spectrum X_e such that for all $n \in \mathbb{Z}$,

$$\pi_n(X_e) = \mathrm{im}(\pi_n X \xrightarrow{e_*} \pi_n X).$$

Thus, idempotent maps have images in $\mathrm{ho}(\mathcal{S}p)$, even though it is not an abelian category.

(4) Let $\mathcal{C}$ be a triangulated category with shift functor Σ. Suppose that

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

is a commuting square in $\mathcal{C}$. Prove that there is a diagram as below, in which each row and each column is a triangle in $\mathcal{C}$, and the diagram commutes except for the bottom right square (marked with -1), which *anticommutes*: $fg = -gf$.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\ X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X' \arrow[d] \\ X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "-1"] & \Sigma X'' \arrow[d] \\ \Sigma X \arrow[r] & \Sigma Y \arrow[r] & \Sigma Z \arrow[r] & \Sigma^2 X \end{tikzcd}$$

REFERENCES

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.