THE SMASH PRODUCT NAME: ______________________

Due at the beginning of class on 9 April 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Sections 6.1 and 6.2].

(1) Let $F \dashv G$ be an adjoint pair of functors between monoidal categories.

(a) Show that if F is strong symmetric monoidal, then G is lax symmetric monoidal.

(b) If G is lax symmetric monoidal, can you say anything about F? Give a proof or counterexample.

(2) Every spectrum X is an $\mathbb{S}$-module. Describe the $\pi_*\mathbb{S}$-action on π_*X.

(3) Recall the cobordism spectrum MO from [Mal23, Example 2.1.21]. For a construction of MO as a symmetric/orthogonal spectrum, see [Sch07, Example 1.16].

(a) Prove that there is a pullback square of vector bundles

$$\begin{array}{ccc} \gamma_n \oplus \gamma_m & \longrightarrow & \gamma_{n+m} \\ \downarrow & & \downarrow \\ BO(n) \times BO(m) & \longrightarrow & BO(n+m), \end{array}$$

where $\gamma_k \to BO(k)$ is the tautological bundle.

(b) Use the pullback square to produce maps $MO(n) \wedge MO(m) \to MO(n+m)$ for all $n, m \geq 0$.

(c) Show that these maps make MO into a commutative ring orthogonal spectrum.

(4) Let R be a commutative ring spectrum.

(a) Show that the forgetful functor from the category of R-module spectra to symmetric (or orthogonal) spectra has a left adjoint.

(b) Let M be an R-module spectrum such that π_*M is free as a graded π_*R-module. Show that M is stably equivalent to a wedge sum of shifts of copies of R.

REFERENCES

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. `http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf`, October 2023.

[Sch07] Stefan Schwede. An untitled book project about symmetric spectra. `http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf`, 2007.