GENERALIZED COHOMOLOGY NAME: ____________________

Due at the beginning of class on 13 February 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Hat02, Sections 4.3 and 4.E].

(1) Let X be any space. Prove that $QX := \operatorname{colim}_n \Omega^n \Sigma^n X$ is an infinite loopspace.

(2) Let E be an infinite loopspace. What properties and/or structure must E have so that the associated generalized cohomology theory $E^*(X) := \bigoplus_i [X, E_i]$ has the structure of a graded commutative ring?

(3) Show that the infinite unitary group U is connected as a topological space. Use this to compute $\widetilde{K}^i(S^n)$ for all i and n.

(4) Let A be an abelian group. A *cohomology operation* is a natural transformation $\widetilde{H}^m(-;A) \to \widetilde{H}^n(-;A)$. The set of all cohomology operations forms a ring, called the *Steenrod algebra*, whose product is composition of operations.

  (a) For fixed m and n, prove that the set of all cohomology operations $\theta\colon \widetilde{H}^m(-;A) \Rightarrow \widetilde{H}^n(-;A)$ is in bijection with $H^n(K(A,m);A)$.

  (b) Prove that there are no nontrivial cohomology operations that decrease degree.

  (c) Prove that the set of cohomology operations which preserve degree are in bijection with the abelian group $\operatorname{Hom}(A,A)$.

REFERENCES

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge University Press, 2002.