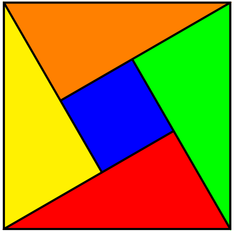
The Hopf Algebra Spectrum of Spherical Scissors Congruence

joint with Klang, Kuijper, Malkiewich, and Wittich

slides available at www.davidmehrle.com/ssc.pdf

Two polygons P and Q are *scissors congruent* if you can chop P into finitely many pieces and rearrange them to form Q.



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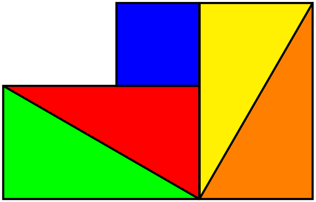

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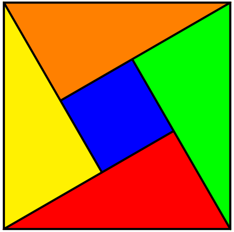

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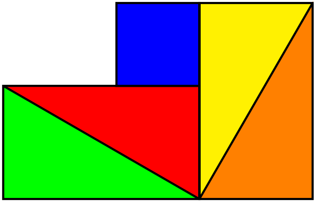

Q

THEOREM (Wallace 1807, Bolyai–Gerwein 1835)

Two polygons are scissors congruent if and only if they have the same area.

Hilbert's Third Problem (1900)

Are any two polyhedra of equal volume scissors congruent?

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Found using Dehn invariant:

$$\begin{aligned} \text{Polyhedra} &\longrightarrow \mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\pi\mathbb{Z}) \\ P &\longmapsto \sum_{\substack{\text{edges } e \\ \text{in } P}} \text{length}(e) \otimes_{\mathbb{Z}} \text{angle}(e) \end{aligned}$$

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THEOREM (Dehn, 1901) (Sydler, 1965)

Dehn invariant and volume are complete scissors congruence invariants in dimensions 3 (Dehn) and 4 (Sydler).

Generalized Hilbert's Third Problem

Are volume and Dehn invariant complete invariants in all dimensions? In spherical or hyperbolic geometry?

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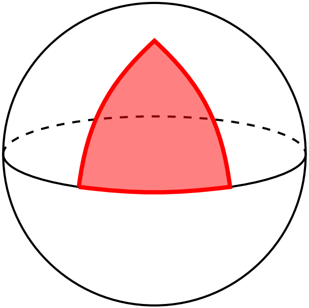

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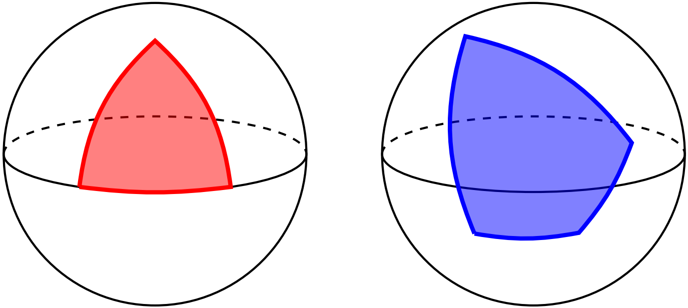

a spherical 2-simplex a spherical polygon

Let $X^n = \mathbb{R}^n$ (Euclidean) or $X^n = S^n$ (Spherical).

DEFINITION

The *polytope group* $\mathcal{P}(X^n)$ is the abelian group with

- generators: polytopes $P \subseteq X^n$
- relations:

$$P = \sum_{i=1}^{m} P_i \quad \text{when} \quad P = \bigcup_{i=1}^{m} P_i, \quad \text{area}(P_i \cap P_j) = 0$$

$$P = \phi(P) \quad \text{for any isometry } \phi\colon X^n \to X^n$$

Polytopes P and Q are *scissors congruent* if $[P] = [Q]$ in $\mathcal{P}(X^n)$.

Examples

Area is a complete scissors congruence invariant in 2D:

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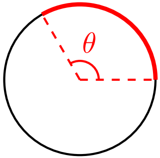

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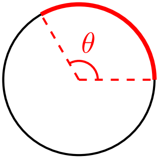

Dehn invariant, revisited:

$$\begin{array}{ccc} \mathcal{P}(\mathbb{R}^3) & \longrightarrow & \mathcal{P}(\mathbb{R}^1) \otimes_{\mathbb{Z}} \widetilde{\mathcal{P}}(S^1) \\ P & \longmapsto & \displaystyle\sum_{\substack{\text{edges } e \\ \text{in } P}} \text{length}(e) \otimes_{\mathbb{Z}} \text{angle}(e) \end{array}$$

THEOREM (Sah, 1979)

The graded abelian group $\bigoplus_n \mathcal{P}(S^n)$ is a graded ring with join as multiplication. The quotient

$$\mathcal{S} := \bigoplus_{n\geq 0} \mathcal{P}(S^n)/([\mathrm{pt}])$$

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This also makes $\bigoplus_{n\geq 0} \mathcal{P}(\mathbb{R}^n)$ into an $\mathcal{S}$-comodule.

Scissors Congruence as K-Theory

Theorem (Zakharevich)

There is a K-theory spectrum $K(X^n)$ such that

$$\mathcal{P}(X^n) \cong \pi_0 K(X^n),$$

and a reduced K-theory spectrum $\widetilde{K}(X^n)$ such that

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Theorem (KKMMW)

The spectral Sah algebra

$$\mathbb{S} := \bigvee_{n \geq 0} \widetilde{K}(S^n)$$

is a Hopf algebra spectrum with $\pi_0 \mathbb{S} \cong \mathcal{S}$.

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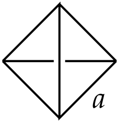

$$\xmapsto{\text{Dehn}} a \otimes \theta, \qquad \cos(\theta) = \frac{\cos(a)}{1 + 2\cos(a)}$$

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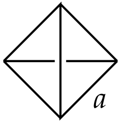

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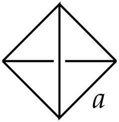
$\mathcal{P}(S^3) \ni$ 

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Coalgebras in a symmetric monoidal model category of spectra are necessarily cocommutative.

Consequences:

- Hopf algebra structure on $\mathbb{S}$ only exists in an ∞-category,
- or we must use operadic coalgebras.

WORKAROUND

THEOREM (Malkiewich)

Spherical scissors congruence K-theory is a Thom spectrum:

$$\widetilde{K}(S^n) \simeq \left(\Sigma^{-\mathbb{R}^n}\Sigma^\infty \Sigma S \mathrm{T}(\mathbb{R}^n)\right)_{hO(n)^\delta},$$

where

- $T(\mathbb{R}^n) = \left|\text{poset of subspaces } U \text{ with } 0 \subsetneq U \subsetneq \mathbb{R}^n\right|$
- Σ and S are reduced and unreduced suspension
- Σ^∞ is suspension spectrum
- $\Sigma^{-\mathbb{R}^n}$ is desuspension with $O(n) \circlearrowright \mathbb{R}^n$
- $O(n)^\delta$ is orthogonal group with discrete topology

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Let Dip be the category of finite-dimensional inner-product spaces and isometries. Define:

$$\begin{aligned} \mathfrak{S} \colon \mathrm{Dip} &\longrightarrow \mathrm{Sp}^O \\ V &\longmapsto \Sigma^{-V}\Sigma^\infty \Sigma \mathrm{ST}(V) \\ 0 &\longmapsto \mathbb{S} \end{aligned}$$

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There is an equivalence of orthogonal spectra $\mathcal{S} \simeq \operatorname{colim} \mathfrak{S}$.

Reductions

$\mathrm{Fun}(\mathrm{Dip}, \mathrm{Sp}^O)$ is symmetric monoidal via Day convolution,

$$\begin{tikzcd}
\mathrm{Dip} \times \mathrm{Dip} \arrow[r, "F \times G"] \arrow[d, "\oplus"'] & \mathrm{Sp}^O \times \mathrm{Sp}^O \arrow[r, "\wedge"] & \mathrm{Sp}^O \\
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Reduction 1

To show $\mathcal{S} = \mathrm{colim}\, \mathfrak{S}$ is a Hopf algebra spectrum,
it suffices to show $\mathfrak{S}$ is a Hopf algebra in $\mathrm{Fun}(\mathrm{Dip}, \mathrm{Sp}^O)$.

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Reduction 2

To show that $\mathfrak{S}$ is a Hopf algebra in $\mathrm{Fun}(\mathrm{Dip}, \mathrm{Sp}^O)$,
it suffices to show that $V \mapsto \Sigma ST(V)$ is Hopf in $\mathrm{Fun}(\mathrm{Dip}, \mathrm{Top}_*)$.

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A model for $\Sigma ST(V)$

$$\Sigma ST(V) \cong \left\{ \begin{array}{c} \text{order-preserving} \\ f\colon [0,1] \to \mathrm{Sub}(V) \end{array} \right\} \Big/ \begin{array}{c} f \sim g \text{ if } f \text{ and } g \text{ differ} \\ \text{only finitely often} \end{array}$$

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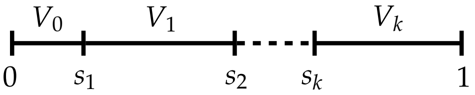
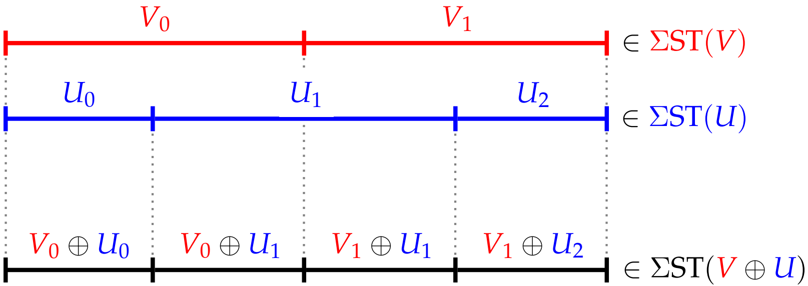


$\in \Sigma\mathrm{ST}(V)$

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_k$$

Product

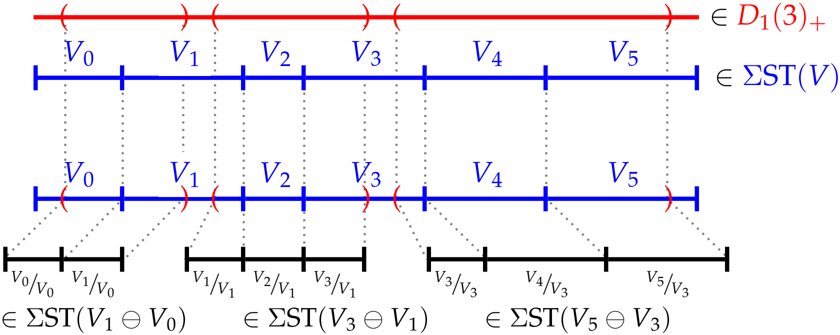
$$\big(\Sigma\mathrm{ST} \boxtimes \Sigma\mathrm{ST}\big)(W) = \bigvee_{V \oplus U = W} \Sigma\mathrm{ST}(V) \wedge \Sigma\mathrm{ST}(U) \longrightarrow \Sigma\mathrm{ST}(W)$$



Coproduct

$\Sigma\mathrm{ST}$ is a coalgebra for the little intervals operad D_1:

$$D_1(n)_+ \wedge \Sigma\mathrm{ST}(V) \to \Sigma\mathrm{ST}^{\boxslash n}(V) = \bigvee_{V_0\subseteq V_1\subseteq\ldots\subseteq V_n} \bigwedge_{i=1}^{n} \Sigma\mathrm{ST}(V_i \ominus V_{i-1})$$



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Lemma

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Theorem (KKMMW)

Under mild cofibrancy assumptions, (E_∞, E_1)-bialgebras transfer to commutative bialgebras in the underlying ∞-category.

$\Longrightarrow \mathcal{S}$ is a Hopf algebra in $\mathbf{Sp} = N(\mathrm{Sp}^O)$.

Application

Primitive elements x in a Hopf algebra satisfy

$$\delta(x) = 1 \otimes x + x \otimes 1.$$

Lemma

Let H be a rational Hopf algebra.

Let $V \subseteq H$ be a sub-vector space of primitive elements.

The free graded-commutative algebra $\Lambda(V)$ is a subalgebra of H.

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$\pi_*(\mathcal{S}_{\mathbb{Q}})$ is bigraded: $n =$ dimension, $k =$ homotopy degree

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Theorem (Malkiewich)

$$\pi_k \widetilde{K}(S^1) \otimes \mathbb{Q} \cong \begin{cases} \Lambda^{k+1}\left(\mathbb{R}/\mathbb{Q}\right) & (k \text{ even}) \\ 0 & (k \text{ odd}) \end{cases}$$

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$$V = \bigoplus_k \pi_k \widetilde{K}(S^1) \otimes \mathbb{Q} \cong \bigoplus_{k \text{ even}} \Lambda^{k+1}\left(\mathbb{R}/\mathbb{Q}\right)$$

APPLICATION

THEOREM (Malkiewich)

$$\pi_k \widetilde{K}(S^1) \otimes \mathbb{Q} \cong \begin{cases} \Lambda^{k+1}\left({}^{\mathbb{R}}/_{\mathbb{Q}}\right) & (k \text{ even}) \\ 0 & (k \text{ odd}) \end{cases}$$

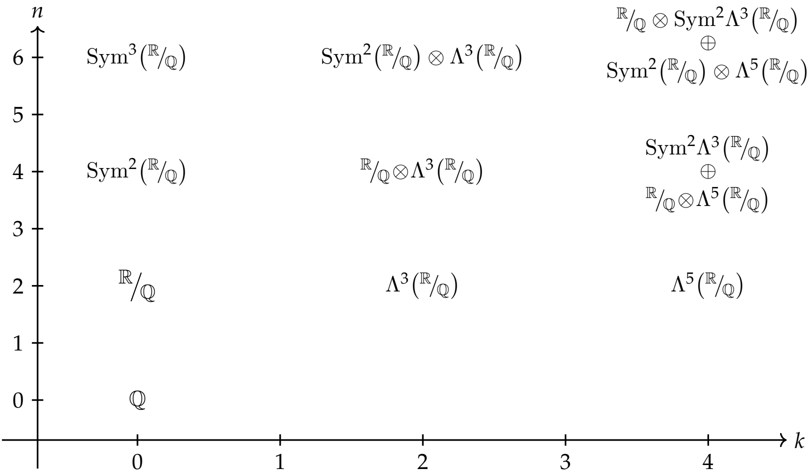
$$V = \bigoplus_k \pi_k \widetilde{K}(S^1) \otimes \mathbb{Q} \cong \bigoplus_{k \text{ even}} \Lambda^{k+1}\left({}^{\mathbb{R}}/_{\mathbb{Q}}\right)$$

THEOREM (KKMMW)

$\pi_*(\mathcal{S}_{\mathbb{Q}})$ has a Hopf subalgebra: the free commutative algebra on

$$\bigoplus_{k \text{ even}} \Lambda^{k+1}\left({}^{\mathbb{R}}/_{\mathbb{Q}}\right),$$

taken in dimension $n = 2$.



A large nonzero subalgebra of $\pi_*(\mathcal{S}_{\mathbb{Q}}) \cong \bigoplus_{k,n} \pi_k \widetilde{K}(S^{n-1}) \otimes \mathbb{Q}$

Thanks for listening!

BONUS: ABSTRACT NONSENSE

- $M_\bullet$ a symmetric monoidal simplicial model category
- $M_\bullet^\flat \subseteq M_\bullet$ a $\otimes$-closed full subcategory with all cofibrants
- $\mathbf{M}$ underlying ∞-category
- $\mathcal{O}_\bullet$ fibrant simplicial operad

THEOREM (KKMMW)

There is a canonical map of simplicial sets

$$N^s(\mathrm{Alg}_{\mathcal{O}_\bullet}(M_\bullet)) \longrightarrow \mathrm{Alg}_{N^s(\mathcal{O}_\bullet)}(\mathbf{M}).$$

There is a map of ∞-categories

$$N^s(\mathrm{BiAlg}_{E_\infty,E_1}(M_\bullet^\flat)) \longrightarrow \mathrm{CBiAlg}(\mathbf{M})$$

that sends Hopf algebras to Hopf algebras.