Localization Talk Outline

This outline is only a suggestion. There are many different directions that you could go with this talk. Use it as an excuse to learn about something you're curious about, and if would like to talk about something other than what's in the outline below, feel free to do so! The outline below may also be too much material. Feel free to focus on the important things and cut others if it's too much.

Read [vK13, Sections 3 and 4], or [Bou79, Sections 1 and 2]. I also found these slides of Aras Ergus helpful [Erg19].

Outline

(1) Motivation

- (a) So far, we've studied spectra using the notion of stable equivalence. But just as with spaces, it's often useful to study spectra with the notion of homology or cohomology instead of just homotopy.
- (b) Recall every spectrum E defines both a generalized cohomology theory E^*X via $E^nX = \pi_{-n}F(X, E)$ and a generalized homology theory E_*X by $E_nX = \pi_n(X \wedge E)$.
- (c) We can study spectra up to E-equivalence: a map $f\colon X \to Y$ of spectra is an E-equivalence if $E_*f\colon E_*X \to E_*Y$ is an isomorphism.
- (d) We will study spectra up to E-equivalence. This turns out to be an extremely fruitful pursuit – stable homotopy groups are often hard, but frequently some sort of E-homology is easier!

(2) Localization

- (a) Define E-acyclic and E-local spectra [vK13, Definition 3.2].
- (b) Prove that a map is an E-equivalence if and only if the fiber is E-acyclic.
- (c) Prove that a spectrum Z is E-local if and only if every E_*-equivalence $X \to Y$ induces an isomorphism $[X, Z] \cong [Y, Z]$.
- (d) Define an E-localization of a spectrum, and the E-localization functor [vK13, Definition 3.3].
- (e) State the theorem that a localization functor always exists [vK13, Theorem 3.10]. Don't prove it yet, we'll give an argument for the existence later.
- (f) State and prove the Whitehead theorem for E-local spectra: if $f\colon X \to Y$ is an E-equivalence of E-local spectra, then f is a stable equivalence [vK13, Proposition 3.4].
- (g) State [vK13, Propositions 3.6 and 3.7] and pick a few of the statements to prove. An important point is all of this is that if you have a (co)fiber sequence of spectra $X \to Y \to Z$, and any two of them are E-local, then so is the third.

(3) Examples

- (a) The first example is usually $E = H\mathbb{Q}$, the Eilenberg–MacLane spectrum of the rationals. Recall that $H\mathbb{Q} \simeq \mathbb{S}_{\mathbb{Q}}$, the rationalization of the sphere spectrum. Then an $H\mathbb{Q}$-acyclic spectrum is a spectrum X such that π_*X has only torsion (i.e. $\pi_*X \otimes \mathbb{Q} = 0$, or equivalently the $H\mathbb{Q}$-homology of X vanishes). Prove that an $H\mathbb{Q}$-local spectrum is a rational spectrum (i.e. π_*X is a rational vector space), and $H\mathbb{Q}$-localization is rationalization.
- (b) If E is a ring spectrum an X is an E-module, then X is E-local [vK13, Proposition 3.5]. Prove this.

(c) We say that a localization is *smashing* if $L_E X = E \wedge X$. Rationalization is a smashing equivalence, for example. If a localization is smashing, then $E = L_E(\mathbb{S})$ and $L_E(X) = L_E(\mathbb{S}) \wedge X$.

(d) Localization at $\mathbb{S}/p = \mathrm{hocofib}(\mathbb{S} \xrightarrow{p} \mathbb{S})$ can be constructed as $\mathrm{holim}_n\, X/p^n$, which looks a lot like the construction of the p-adics $\mathbb{Z}_p$. In fact, if the homotopy groups of X are finitely generated, then $\pi_n L_{\mathbb{S}/p} X \cong \pi_n X \otimes \mathbb{Z}_p$ [Bou79, Proposition 2.15]. Therefore, people often write $X_p^\wedge$ for $L_{\mathbb{S}/p}X$, and call it the p-completion of X.

(e) An important theorem [vK13, Theorem 4.6]: if E and X are connective spectra, and $G = \pi_0 E$, then $L_E X \simeq L_{\mathbb{S}G} X$, where $\mathbb{S}G$ is the Moore spectrum of G. As a corollary, if X is bounded below, then $L_{H\mathbb{F}_p} X \simeq L_{\mathbb{S}/p} X = X_p^\wedge$.

(4) Constructing the Localization Functor

(a) Let $\mathcal{S}p_E$ denote the full subcategory of $\mathcal{S}p$ consisting of the E-local spectra. The inclusion functor $\mathcal{S}p_E \to \mathcal{S}p$ has a left adjoint, which we write $L_E : \mathcal{S}p \to \mathcal{S}p_E$ [Bou79]. This is E-localization.

(b) Let ${}_E\mathcal{S}p$ denote the full subcategory of $\mathcal{S}p$ consisting of the E-acyclic spectra. The inclusion functor ${}_E\mathcal{S}p \to \mathcal{S}p$ has a right adjoint, which we write ${}_E(-) : \mathcal{S}p \to {}_E\mathcal{S}p$ [Bou79].

(c) State but do not prove [Bou79, Theorem 1.1]: for any spectrum X, there is a fiber/cofiber sequence (natural in X)

$$_E(X) \xrightarrow{\theta_X} X \xrightarrow{\eta_X} L_E(X).$$

(d) By smashing this cofiber sequence with E, you can see that $\eta_X : X \to L_E(X)$ is an E-equivalence, so $L_E(X)$ is in fact an E-localization of X.

(e) Prove that L_E is an idempotent functor: $L_E(L_E(X)) \simeq L_E(X)$. (Use the Whitehead theorem for E-local spectra.)

(f) Prove that $X \to L_E X$ is initial among all maps from X to an E-local spectrum. Similarly, ${}_E X \to X$ is terminal among all maps from an E-acyclic spectrum to X.

(5) Bousfield Classes

(a) Two spectra E and E' are *Bousfield equivalent* if the following condition holds: X is E-acyclic if and only if X is E'-acyclic. Equivalently, a map of spectra is an E-equivalence if and only if it is an E'-equivalence. This is an equivalence relation on spectra.

(b) The equivalence class of a spectrum E with respect to this equivalence is relation is called the *Bousfield class* of E and denoted $\langle E \rangle$.

(c) There is a partial order on Bousfield classes given by $\langle E \rangle \leq \langle F \rangle$ if every F-acyclic spectrum is E-acyclic (the E-acyclics are contained in the F-acyclics).

(d) The Bousfield classes of spectra form a lattice with minimal element $\langle * \rangle$ and maximal element $\langle \mathbb{S} \rangle$.

(6) Localization as a computational tool.

(a) There is an *arithmetic fracture square*: a pullback/pushout in $\mathrm{ho}(\mathcal{S}p)$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_{p \text{ prime}} X_p^\wedge \arrow[d] \\ X_\mathbb{Q} \arrow[r] & \left(\prod_{p \text{ prime}} X_p^\wedge \right)_\mathbb{Q} \end{tikzcd}$$

This is an incredibly important tool for doing calculations – it enables us to compute the homotopy of X rationally and one prime at a time.

(b) The existence of this square follows from a more general fact: [vK13, Proposition 4.5]. State and prove this proposition, and deduce the fracture square as a corollary.

References

[Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[Erg19] Aras Ergus. The localization of spectra with respect to homology by A. K. Bousfield, eCHT Kan Seminar 2019. `https://www.aergus.net/academic/documents/assorted/bousfield-localization.pdf`, 2019.

[vK13] Paul van Koughnett. Spectra and localization. `https://people.math.harvard.edu/~hirolee/pretalbot2013/notes/2013-02-07-Paul-VanKoughnett-Bousfield_Localization.pdf`, 2013.