Homotopy Limits and Colimits

by
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References
- "Categorical homotopy theory" - E. Riehl
- "Homotopical categories: From model categories to $(\infty,1)$-categories" - E. Riehl
- "Spectra and stable homotopy theory" - C. Malkiewich
- "Model categories and their localizations" - P. Hirschhorn
- "Higher topos theory" - J. Lurie

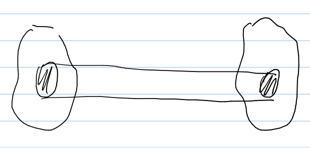
0. MOTIVATION

$\mathcal{C}$ homotopical cat. $\qquad \mathcal{M}$ model cat.

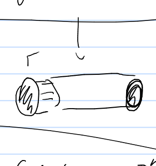
$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[d, hook] & D^n \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd} \qquad D^n \simeq *$$

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd} \qquad S^n \not\simeq *$$

colim: $Top^{D} \to Top$ is not homotopical.

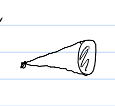


$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[d] & Cyl(S^{n-1} \hookrightarrow D^n) \arrow[d] \\ D^n \arrow[r] & \end{tikzcd}$$



$\simeq S^n$

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & Cyl(* \to D^n) \arrow[d] \\ * \arrow[r] & \cdot \end{tikzcd}$$



$\simeq S^n$

1 PRELIMINARIES

Def. A left deformation of $\mathcal{C}$ is an endofunctor

$$Q: \mathcal{C} \to \mathcal{C}$$

together w/ a nat. w.e.

$$q: Q \xRightarrow{\sim} id_{\mathcal{C}},$$

$\mathcal{C}_Q$ full subcat of $\mathcal{C}$ that contains image of Q

Def. A left def. of a functor $F: \mathcal{C} \to \mathcal{D}$ is a left def. $q: Q \xRightarrow{\sim} id_{\mathcal{C}}$ of $\mathcal{C}$ and a choice of $\mathcal{C}_Q$ s.t. F is homotopical on $\mathcal{C}_Q$.

ex. if $\mathcal{M}$ is a model cat., F is left Quillen, then the cofib. replacement is a left def. of F by Ken Brown

ex. $\mathcal{M} = CGWH$ $(=Top)$

Fact: CW complexes are cofibrant

CW approximation: if $X \in CGWH$ ex. CW Z & $f: Z \xrightarrow{\sim} X$.

Functorial cofib replacement:

$$Z = |Sing X_\bullet|$$

$$\Rightarrow \quad Q: CGWH \to CGWH$$
$$X \mapsto |Sing X_\bullet|$$

is a left def. of CGWH w/ $(CGWH)_Q = CW$

Thm. If $q: Q \Rightarrow id$ is a left def. of $F: \mathcal{C} \to \mathcal{D}$, then

$$\mathbb{L}F = F \circ Q$$

is a left derived functor of F.

(2.2.8)

2. HOMOTOPY COLIMITS

$\mathcal{J}$ small cat.

Def. The homotopy colimit functor (if it exists) is a left derived functor

$$hocolim := \mathbb{L}colim : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$$

of $colim: \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$.

To produce it, find a left def.

$$Q: \mathcal{C}^{\mathcal{J}} \to \mathcal{C}^{\mathcal{J}}$$

$q: Q \xRightarrow{\sim} id$ of colim.

pointwise
$\eta: F \Rightarrow G$ is w.e. iff $\eta_x: Fx \to Gx$ is for all $x \in \mathcal{J}$.

2.1 Simplicial model cat's

$F: \mathcal{J} \to \mathcal{M}$, $\quad G: \mathcal{J}^{op} \to sSet$

Def Bar constr. $B_\bullet(G, \mathcal{J}, F) \in s\mathcal{C}$

w/ $$B_n(G, \mathcal{J}, F) = \coprod_{\substack{j_0 \to \dots \to j_n \\ \text{in } \mathcal{J}}} G j_n \otimes F j_0$$

$\in sSet$

ex. classifying space of G

$$BG = |BG_\bullet| = |B_\bullet(*, G, *)|$$

$$\cdots \quad G \times G \times G \qquad G \times G \rightrightarrows G \rightleftarrows *$$

Thm [5.1.1] If $Q: \mathcal{M} \to \mathcal{M}$ cofib. replacement, then

$$B(id, \mathcal{J}, Q\circ -): \mathcal{M}^{\mathcal{J}} \to \mathcal{M}^{\mathcal{J}}$$

is a left def. of colim together w/

$$q: B(id, \mathcal{J}, Q\circ-) \Rightarrow Q \to id$$

In particular:
$$\boxed{\begin{aligned} hocolim &\simeq colim\, B(id_{\mathcal{J}}, \mathcal{J}, Q\circ-) \\ &\simeq B(*, \mathcal{J}, Q\circ-) \end{aligned}}$$

ex. homotopy coproducts

$\mathcal{J} = \bullet \quad \bullet$

$F: \mathcal{J} \to CGWH$ given by $A, B \in CGWH$

$$B_0(*, \mathcal{J}, Q\circ F) = * \times QA \amalg * \times QB = QA \amalg QB$$

$$B_1(*, \mathcal{J}, Q\circ F) = QA \amalg QB$$

$$\vdots$$

$$|B_\bullet(*, \mathcal{J}, Q\circ F)| = QA \amalg QB. \sim A \amalg B$$

2.2 Projective model structure

Def. If the assignment $\qquad \mathcal{M}$ model cat.
- W = ptwise w.e.
- Fib = ptwise fib.

constitutes a model cat str. on $\mathcal{M}^{\mathcal{J}}$, call it proj. model str.

Thm. If this model str. exists, then so does hocolim.
In prt., compute it by taking cofibr. replacement in $\mathcal{M}^{\mathcal{J}}$ & then computing colim.

Thm. If $\mathcal{C}$ cofib. generated, then proj. model ex. for all small $\mathcal{J}$.

ex. homotopy pushout.

Fact. $$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y & \end{tikzcd}$$ is cofibrant in proj. model str. if
- A, X, Y are cofibrant obj. &
- f, g are cofib's.

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd}$$

Remark: $ho(\mathcal{C}^{\mathcal{J}}) \to (ho\,\mathcal{C})^{\mathcal{J}}$ not usually an equivalence.

$\Rightarrow$ homotopy colimits are usually NOT colimits in $ho\,\mathcal{C}$.

3 EXAMPLES

3.1 Homotopy pushout

CGWH

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \amalg_{A \times \{0\}} A \times I \amalg_{A \times \{1\}} Y \end{tikzcd}$$

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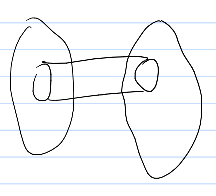

3.2 Cofiber & Suspension

mapping cone

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & cof \end{tikzcd}$$

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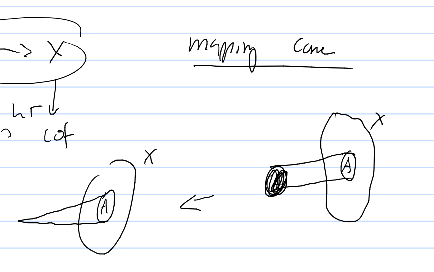

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & SX \end{tikzcd}$$

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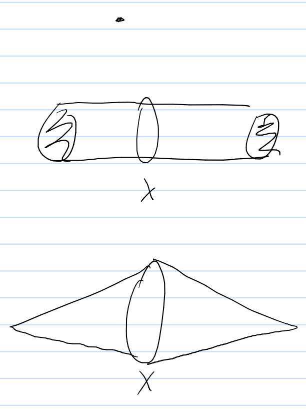
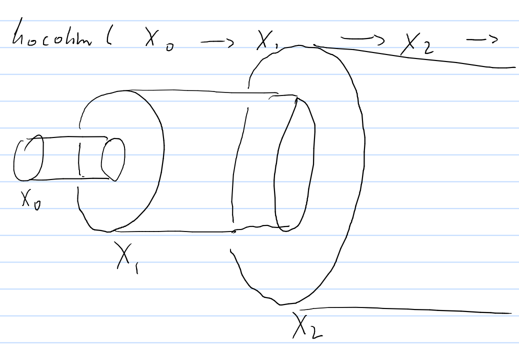


3.3. Mapping Telescope

= sequential homotopy colimit

$$hocolim(X_0 \to X_1 \to X_2 \to \dots)$$



ex. Telescopic localization

Y top. group $\qquad Y \xrightarrow{n} Y$ nth power multiplies homotopy classes by n

p prime, consider

$$hocolim(Y \xrightarrow{n} Y \xrightarrow{n} Y \to \dots) \simeq Y_p$$

$$\pi_i Y_p = (\pi_i Y)_p$$

For questions and/or further details, feel free to message me on Zulip or send me an email to sonjaf@nevada.unr.edu!