TRIANGULATED CATEGORIES NAME: SOLUTIONS

Due at the beginning of class on 25 March 2025

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Section 3.2] and [Wei94, Section 10.2]

(1) A spectrum X is *rational* if each of its homotopy groups is an $\mathbb{Q}$-vector space. Let $\mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}}$ be the full subcategory of $\mathrm{ho}(\mathcal{S}p)$ whose objects are rational spectra. Prove that $\mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}}$ is a triangulated subcategory of $\mathrm{ho}(\mathcal{S}p)$.

SOLUTION: Since the homotopy groups of a finite product/wedge of spectra are just the products of the homotopy groups of its summands, we know that $\mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}}$ is additive. Moreover, since Σ only affects the homotopy groups of a spectrum by shifting their degrees, we know that Σ restricts to an equivalence $\mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}} \to \mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}}$.

We are left with checking that the cofiber of a map of rational spectra is rational. Let $f : X \to Y$ be a map of rational spectra, and let $C = \mathrm{cof}(f)$. We have a long exact sequence in homotopy, from which we can extract a bunch of short exact sequences of the form:

$$0 \to \ker \delta_n \to \pi_n C \to \operatorname{im} \delta_n \to 0$$

where $\delta_n \colon \pi_n C \to \pi_{n-1} X$ is the connecting homomorphism. By exactness, we have

$$\ker \delta_n \cong \pi_n Y / \operatorname{im}(\pi_n X \to \pi_n Y),$$

which is a quotient of a $\mathbb{Q}$-vector space by a $\mathbb{Q}$-vector subspace, and thus rational. Similarly $\operatorname{im} \delta_n \subseteq \pi_{n-1} X$ and is thus a rational vector space. By choosing bases for these, we obtain isomorphisms

$$\pi_n C \cong \ker \delta_n \oplus \operatorname{im} \delta_n$$

for all n, showing that $\pi_n C$ is rational for all n and thus $C = \mathrm{cof}(f)$ is rational. Therefore $\mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}}$ is a triangulated subcategory of $\mathrm{ho}(\mathrm{Sp})$

(2) Define a triangulated functor $H \colon \mathcal{D}(\mathbb{Q}) \to \mathrm{ho}(\mathcal{S}p)$ such that $\pi_n H(V_\bullet) = H_n(V_\bullet)$ for all $n \in \mathbb{Z}$. *Hint: any chain complex of $\mathbb{Q}$-vector spaces is quasi-isomorphic to its homology.*

SOLUTION: Given a chain complex $V_\bullet$, define $H(V_\bullet) = \bigvee_{n \in \mathbb{Z}} \Sigma^n H(H_n(V_\bullet))$. That is, H takes a chain complex $V_\bullet$ to the wedge sum of shifts of the Eilenberg–MacLane spectra of its homology groups. Since quasi-isomorphisms induce isomorphisms on homology, this functor sends quasi-isos to weak equivalences. Therefore, it is a homotopical functor and descends to a functor $H \colon \mathcal{D}(\mathbb{Q}) \to \mathrm{ho}(\mathcal{S}p)$. Note that the image of this functor is contained in the full subcategory $\mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}}$ spanned by the rational spectra.

To show that this functor is triangulated, we must show that it is additive, that it commutes with translation, and it preserves triangles.

To see that this functor is additive, first note that H applied to the zero chain complex is just a point. Then:

$$\begin{aligned}
H(V_\bullet \oplus W_\bullet) &= \bigvee_{n\in\mathbb{Z}} \Sigma^n H(H_n(V_\bullet \oplus W_\bullet)) \\
&\cong \bigvee_{n\in\mathbb{Z}} \Sigma^n H(H_n(V_\bullet) \oplus H_n(W_\bullet)) && \text{homology distributes over products} \\
&\cong \bigvee_{n\in\mathbb{Z}} \Sigma^n (H(H_n(V_\bullet)) \vee H(H_n(W_\bullet))) && \text{Eilenberg–MacLane functor is additive (HW 5)} \\
&\cong \bigvee_{n\in\mathbb{Z}} \Sigma^n H(H_n(V_\bullet)) \vee \Sigma^n H(H_n(W_\bullet)) && \text{suspension commutes with coproducts} \\
&\cong \left(\bigvee_{n\in\mathbb{Z}} \Sigma^n H(H_n(V_\bullet))\right) \vee \left(\bigvee_{n\in\mathbb{Z}} \Sigma^n H(H_n(W_\bullet))\right) \\
&= H(V_\bullet) \vee H(W_\bullet)
\end{aligned}$$

So H is an additive functor.

To see that H commutes with translation, recall that translation in the derived category is shifting the chain complex by 1: $V_\bullet \mapsto V_{\bullet+1}$. Then

$$\begin{aligned}
H(V_{\bullet+1}) &= \bigvee_{k\in\mathbb{Z}} \Sigma^k H(H_k(V_{\bullet+1})) \\
&= \bigvee_{k\in\mathbb{Z}} \Sigma^k H(H_{k-1}(V_\bullet)) \\
&= \bigvee_{n\in\mathbb{Z}} \Sigma^{n+1} H(H_n(V_\bullet)) && \text{reindexing with } n = k-1 \\
&= \Sigma\left(\bigvee_{n\in\mathbb{Z}} \Sigma^n H(H_n(V_\bullet))\right) \\
&= \Sigma H(V_\bullet)
\end{aligned}$$

So H commutes with translation.

Finally, we must check that H preserves distinguished triangles. Consider a distinguished triangle in $\mathcal{D}(\mathbb{Q})$, which is quasi-isomorphic to one of the form

$$V_\bullet \xrightarrow{f_\bullet} W_\bullet \to \mathrm{cone}(f_\bullet) \to V_{\bullet+1}$$

We want to show that

$$H(V_\bullet) \xrightarrow{H(f_\bullet)} H(W_\bullet) \to H(\mathrm{cone}(f_\bullet))$$

is a cofiber sequence in $\mathrm{ho}(\mathcal{S}p)$. Consider the morphism g from the cofiber of $H(f_\bullet)$ to $H(\mathrm{cone}(f_\bullet))$:

$$\begin{tikzcd}
H(V_\bullet) \arrow[r, "H(f_\bullet)"] \arrow[d, equal] & H(W_\bullet) \arrow[r] \arrow[d, equal] & \mathrm{cof}(H(f_\bullet)) \arrow[d, dashed, "g"] \\
H(V_\bullet) \arrow[r, "H(f_\bullet)"] & H(W_\bullet) \arrow[r] & H(\mathrm{cone}(f_\bullet))
\end{tikzcd}$$

Taking homotopy, we get a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}\,\mathrm{cof}(H(f_\bullet)) \arrow[r] \arrow[d, "g_*"] & \pi_n H(V_\bullet) \arrow[r] \arrow[d, equal] & \pi_n H(W_\bullet) \arrow[r] \arrow[d, equal] & \pi_n\,\mathrm{cof}(H(f_\bullet)) \arrow[r] \arrow[d, "g_*"] & \pi_{n-1} H(V_\bullet) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \pi_{n+1} H(\mathrm{cone}(f_\bullet)) \arrow[r] & \pi_n H(V_\bullet) \arrow[r] & \pi_n H(W_\bullet) \arrow[r] & \pi_n H(\mathrm{cone}(f_\bullet)) \arrow[r] & \pi_{n-1} H(V_\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

The top row is exact because it's the LES of a cofiber sequence. The bottom row is exact because it's isomorphic to the LES in homology for the distinguished triangle $V_\bullet \xrightarrow{f_\bullet} W_\bullet \to \mathrm{cone}(f_\bullet)$ in $\mathcal{D}(\mathbb{Q})$. Then the five lemma shows that $\mathrm{cof}(H(f_\bullet))$ is stably equivalent to $H(\mathrm{cone}(f_\bullet))$. Therefore, H sends triangles to triangles.

We have shown that H is additive, commutes with shift functors, and sends triangles to triangles. So H is a triangulated functor.

(3) (a) Show that in a triangulated category $\mathcal{C}$, all monomorphisms are split monomorphisms.

SOLUTION: Given a monic $f\colon A \to B$, we may extend to an exact triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

and more powerfully to an exact sequence

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B \to \ldots.$$

It follows that $-\Sigma f \circ h = 0$ and thus that $f \circ \Sigma^{-1} h = 0$ and thus that $\Sigma^{-1} h = 0$ and $h = 0$ since f is monic. It follows that $B \cong A \oplus C$ so that f is split.

(b) Show that the triangulated category $\mathcal{D}(\mathbb{Z})$ is not an abelian category.

SOLUTION: Assume for sake of contradiction that $\mathcal{D}(\mathbb{Z})$ is abelian. Consider the quotient homomorphism $\mathbb{Z}/4 \to \mathbb{Z}/2$, considered as chain complexes concentrated in degree zero inside $\mathcal{D}(\mathbb{Z})$. Since we are assuming $\mathcal{D}(\mathbb{Z})$ is abelian, this has a kernel, which is a monomorphism $K \hookrightarrow \mathbb{Z}/4$. In the triangulated category $\mathcal{D}(\mathbb{Z})$, all monomorphisms are split. But that would imply that the short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

is split as well, which is false. Hence, $\mathbb{Z}/4 \to \mathbb{Z}/2$ has no kernel in $\mathcal{D}(\mathbb{Z})$, and therefore $\mathcal{D}(\mathbb{Z})$ is not an abelian category.

(c) Show that $\mathrm{ho}(\mathcal{S}p)$ is not an abelian category.

SOLUTION: The strategy here is to apply the Eilenberg–MacLane spectrum functor H to the previous example.

Assume for sake of contradiction that $\mathrm{ho}(\mathcal{S}p)$ is abelian. Consider the map $H(\mathbb{Z}/4) \to H(\mathbb{Z}/2)$ induced by the quotient homomorphism $\mathbb{Z}/4 \to \mathbb{Z}/2$. The kernel of this in $\mathrm{ho}(\mathcal{S}p)$ is a spectrum K with a monomorphism $f\colon K \hookrightarrow H(\mathbb{Z}/4)$. Since $\mathrm{ho}(\mathcal{S}p)$ is triangulated, this monomorphism is split: there is a left inverse $g\colon H(\mathbb{Z}/4) \to K$ such that $gf = \mathrm{id}_K$.

The exact triangle $K \to H(\mathbb{Z}/4) \to H(\mathbb{Z}/2)$ is a fiber sequence in spectra, and yields a long exact sequence

$$\cdots \to \pi_1 H(\mathbb{Z}/2) \to \pi_0 K \xrightarrow{\pi_0 f} \pi_0 H(\mathbb{Z}/4) \to \pi_0 H(\mathbb{Z}/2) \to \pi_{-1} K \to \cdots$$

which simplifies to the short exact sequence

$$0 \to \pi_0 K \xrightarrow{\pi_0 f} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

Hence, $\pi_0 K \cong \mathbb{Z}/2$. Since f has a left inverse g, $\pi_0 f$ has a left inverse $\pi_0 g$, which implies that the SES above splits. This is a contradiction, so $\mathrm{ho}(\mathcal{S}p)$ cannot be abelian.

References

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.