OPERADS TALK OUTLINE

This outline is only a suggestion. There are many different directions that you could go with this talk. Use it as an excuse to learn about something you're curious about, and if would like to talk about something other than what's in the outline below, feel free to do so!

Good introductions to operads include the following: [Bel17, Bra17, Sta04, Sar17]. Pick whichever reference that you find most accessible. The chapter [Man22] is a good overview of operads in stable homotopy theory, but assumes a little more background.

(1) Motivation

(a) In the study of ring spectra, we've seen general definitions of monoids and commutative monoids in any symmetric monoidal category. Recall these definitions.

(b) But things are often a lot more complicated in homotopy theory. We might have spectra that become (commutative) monoids in the stable homotopy category $\mathrm{ho}(\mathcal{S}\mathrm{p}) \simeq \mathrm{ho}(\mathcal{S}\mathrm{p}^{\Sigma}) \simeq \mathrm{ho}(\mathcal{S}\mathrm{p}^{O})$, but are not themselves (commutative) monoids in any choice of $\widehat{\mathcal{S}\mathrm{p}}$. This is one of the major difficulties in defining a symmetric monoidal category of spectra!

(c) Similarly, there are spaces that are not literally (commutative) monoids in $\mathcal{T}\mathrm{op}_*$, but still have many useful properties! For example, let X be a pointed space. The loopspace ΩX is not literally associative or unital, but close enough for many purposes!

(d) To take this further, we know that ΩX is associative and unital "up to homotopy" and $\Omega^2 X$ is commutative "up to homotopy," but what distinguishes $\Omega^3 X$ from $\Omega^2 X$? What extra structure does it have?

(e) Operads provide a way to encode this kind of structure and many other algebraic structures. I like to think of them as a vast generalization of monoids that work in many different categories. We'll mostly focus on operads in $\mathcal{T}\mathrm{op}_*$, and then at the end apply some of this to some choice of $\widehat{\mathcal{S}\mathrm{p}}$.

(2) Definition

(a) There are two kinds of operads, both of which are useful to talk about. The first is "(non-symmetric) operads," and the second is "symmetric operads." Depending on context, a symmetric operad is often just called an operad, and it's more common to specify that an operad is non-symmetric while assuming that all other operads are symmetric by default.

(b) Define a *(non-symmetric) operad* $\mathcal{O}$ in a symmetric monoidal category $(\mathcal{C}, \otimes, I)$. Through this talk, you can just use $(\mathcal{T}\mathrm{op}, \times, *)$ if you want. See [Bel17, Definition 1.1].

(c) Look up the conditions that are not included in [Bel17], but do not write them out in your talk. Instead give the pictorial intuition with trees as in [Bra17]. (Or [Sar17] or [Sta04].)

(d) If $\mathcal{O}$ is an operad in a category $(\mathcal{C}, \otimes, I)$ that admits a forgetful functor to $\mathcal{S}\mathrm{et}$, explain that you're supposed to think of elements of $\mathcal{O}(n)$ as "operations" with n inputs and one output. Again, you can take $\mathcal{C} = \mathcal{T}\mathrm{op}_*$ if you want.

(e) Define symmetric operads in a symmetric monoidal category $(\mathcal{C}, \otimes, I)$. If you want, you may assume again that this category is $(\mathcal{T}\mathrm{op}_*, \times, *)$. In a general symmetric monoidal category, you will have to make sense of a symmetric group action on an object; one way to do this is to define such a group action as a functor $\Sigma_n \to \mathcal{C}$, considering Σ_n as a one-object category.

(f) Define a morphism of operads. This should be pretty quick – it's just a sequence of morphisms $\mathcal{O} \to \mathcal{P}$ preserving all structure. You can be vague about exactly which diagrams commute.

(3) Examples of operads

(a) Given an object $X \in \mathcal{C}$, define the endomorphism operad on X. You can get as specific with this example as you'd like (i.e. pick a category $\mathcal{C}$ and pick an object X and talk about endomorphisms of X). This is really the prototypical example of an operad, and it explains why we think of elements of $\mathcal{O}(n)$ as operations on X.

(b) Given any operad $\mathcal{O}$ in $(\mathcal{C}, \otimes, I)$, define an $\mathcal{O}$-algebra in $\mathcal{C}$. This is just a morphism of operads $\mathcal{O} \to \mathrm{End}(X)$. (Again, perhaps $\mathcal{C} = \mathcal{T}op_*$).

(c) Give the example of the associative (non-symmetric) operad $\mathcal{A}ssoc$.

(d) Show that an $\mathcal{A}ssoc$-algebra in $(\mathcal{C}, \otimes, I)$ is an associative monoid in $\mathcal{C}$. Your proof need not be absolutely irrefutably rigorous – focus on giving intuition for this statement. [Sar17, Example 9].

(e) Give the example of the commutative (symmetric) operad $\mathcal{C}omm$.

(f) Show that a $\mathcal{C}omm$-algebra in $(\mathcal{C}, \otimes, I)$ is a commutative monoid. Again, prioritize intuition over rigor. [Sar17, Example 10].

(4) Operads in spaces

(a) Thus far, we haven't seen many examples of operads that aren't somehow trivial. We'll have to specialize to a particular category to get more interesting examples. From here on out, let's take $(\mathcal{C}, \otimes, I) = (\mathcal{T}op_*, \times, *)$.

(b) Give the example of the little n-cubes (symmetric) operad $\mathcal{C}_n$ [Bel17, Definition 3.1]. A special case is the little intervals operad $\mathcal{C}_1$. Draw some pictures.

(c) Explain how an n-fold loopspace $\Omega^n X$ is an algebra for the little n-cubes operad $\mathcal{C}_n$ [Bel17, right after Definition 3.1].

(d) Turn the category $\mathcal{O}p = \mathcal{O}p(\mathcal{T}op_*)$ of (symmetric) operads in $\mathcal{T}op_*$ into a homotopical category by defining a homotopy equivalence of operads.

(e) Give the following definition: an E_n-operad in $\mathcal{O}p$ is any operad that is weakly equivalent to the little n-cubes operad $\mathcal{C}_n$, in the homotopical category $\mathcal{O}p$. An E_∞-operad is any operad weakly equivalent to $\mathcal{C}omm$.

(f) Give the hierarchy of commutativity [Bel17, bottom of page 5]. In algebra, there's nothing between non-commutative and commutative. But in topology, we have an infinite hierarchy of types of commutativity between associative (an A_∞- or E_1-algebra) and commutative (an E_∞-algebra).

(g) Explain that there is a similar hierarchy of associativity [Bel17, page 4].

(h) State the recognition principle.

(5) Operads and Spectra (extra, if you have time). References for these facts can be found in [Man22].

(a) If X is E_∞-ring spectrum, then X is a commutative monoid in $\mathrm{ho}(\mathcal{S}p)$

(b) If X is E_1-ring spectrum, then X is a monoid in $\mathrm{ho}(\mathcal{S}p)$

(c) **Rectification Theorem:** Let $\widehat{\mathcal{S}p}$ be a symmetric monoidal category of spectra. If X is an E_1-algebra (= A_∞-algebra) in $\widehat{\mathcal{S}p}$, then there is a monoid $Y \in \widehat{\mathcal{S}p}$ such that $X \simeq Y$. Similarly, if X is an E_∞-algebra in $\widehat{\mathcal{S}p}$, then there is a commutative monoid $Y \in \widehat{\mathcal{S}p}$ such that $X \simeq Y$.

(d) If X is E_k for $k \geq 2$, then $\pi_* X$ is a commutative ring

(e) If X is E_1, then $\mathcal{B}i\mathcal{M}od(R)$ is a monoidal category

(f) If X is E_∞, then $\mathcal{LM}od(R)$ is a symmetric monoidal category

References

[Bel17] Eva Belmont. A quick introduction to operads. https://mathweb.ucsd.edu/~ebelmont/operads-talk.pdf, 2017.

[Bra17] Tai-Danae Bradley. What is an operad? https://www.math3ma.com/blog/what-is-an-operad-part-1, 2017.

[Man22] Michael A. Mandell. Operads and operadic algebras in homotopy theory. In *Stable categories and structured ring spectra*, volume 69 of *Math. Sci. Res. Inst. Publ.*, pages 183–247. Cambridge Univ. Press, Cambridge, 2022.

[Sar17] Maru Sarazola. Loop spaces and operads. https://sites.google.com/view/msarazola/notes-and-other-resources, 2017.

[Sta04] Jim Stasheff. What is ... an operad? *Notices Amer. Math. Soc.*, 51(6):630–631, 2004.