Ring Spectra Talk Outline

This outline is only a suggestion. There are many different directions that you could go with this talk. Use it as an excuse to learn about something you're curious about, and if would like to talk about something other than what's in the outline below, feel free to do so!

Read [Mal23, Section 6.2.5] about symmetric and orthogonal ring spectra. For more on symmetric spectra, see [Sch07]. For more on orthogonal spectra, see [Sch23] (take $G = e$).

(1) Ring spectra

(a) Pick a symmetric monoidal category of spectra, such as $\mathcal{S}p^{\Sigma}$ or $\mathcal{S}p^{O}$. You can also be vague about it and just use the black box $\widehat{\mathcal{S}p}$ from [Mal23, Section 4.3].

(b) Define associative and commutative ring spectra as monoids/commutative monoids in a symmetric monoidal model category of spectra. Some terminology that doesn't appear in Malkiewich's book: associative ring spectra are called A_∞-ring spectra or E_1-ring spectra, and commutative ring spectra are E_∞-ring spectra. These are the names of operads: a commutative ring spectrum is an algebra in $\mathcal{S}p^{O}$ over an E_∞-operad, for example. There are also E_n-ring spectra, but they are significantly more complicated to define.

(c) Give the explicit definition of ring spectra in $\mathcal{S}p^{\Sigma}$ or $\mathcal{S}p^{O}$ as a collection of appropriately equivariant map $\mu_{m,n} \colon R_m \wedge R_n \to R_{m+n}$ with unit maps satisfying some diagrams [Sch07, Definition 1.3].

(d) Give examples of ring spectra: the sphere spectrum is a ring spectrum in a unique way, since it is the unit for the smash product. It is the initial commutative ring spectrum.

(e) The Eilenberg–MacLane spectrum functor is lax symmetric monoidal, so if R is a ring (which is a monoid in abelian groups for the tensor product), then HR is a ring spectrum. It is commutative if R is commutative. This is a consequence of the general fact that the image of a monoid under a lax symmetric monoidal functor is a monoid.

(f) Similarly, the suspension spectrum of any monoid in $(\mathcal{T}op_*, \wedge, S^0)$ is a ring spectrum, because Σ^∞ is strong symmetric monoidal. If we have a topological group G, then $\Sigma^\infty(G_+)$ is called the *spherical group ring* and sometimes written $\mathbb{S}[G]$.

(g) Define a map of (commutative) ring spectra (i.e. a ring homomorphism) and the category of algebras in spectra [Mal23, Definition 6.2.32]. We say that a two ring spectra are stably equivalent if their underlying spectra are stably equivalent.

(h) Another consequence of the general fact that lax symmetric monoidal functors preserve monoids: if R is a ring spectrum, then $\pi_* R$ is a graded commutative ring.

(i) For example, $\pi_*(HA)$ is just A in degree zero as a ring. $\pi_*\mathbb{S}$ is a very complicated ring in which every element of positive degree is nilpotent (this is a consequence of the nilpotence theorem of Devinatz–Hopkins–Smith due to Nishida).

(j) Another example, which you don't need to prove anything about. The complex K-theory spectrum KU is a ring spectrum, with $\pi_* KU \cong \mathbb{Z}[\beta^{\pm 1}]$, with $|\beta| = 2$.

(2) Module spectra

(a) Define left modules over a ring spectrum, both implicitly [Mal23, Definition 6.2.32] and explicitly [Sch07, Definition 1.5]. If R is commutative, the distinction between right and left modules is not important.

(b) Define homomorphisms of modules, and the category of left modules over a ring spectrum. A stable equivalence of R-modules is an R-module homomorphism that is a stable equivalence on the underlying spectra.

(c) As an example, any spectrum is a module over $\mathbb{S}$. This is a general fact: any object is a module over the unit in a symmetric monoidal category.

(d) A significantly harder example is that the homotopy category of HA modules for a ring A is equivalent to the derived category of A. See [Ric22, Section 3.1] for details and various versions of this theorem that have been proven over the years.

(e) For any commutative ring spectrum R and R-modules M and N, we can define a relative smash product (like a relative tensor product) by coequalizing the actions:

$$M \wedge R \wedge N \rightrightarrows M \wedge N \to M \wedge_R N.$$

This makes the category of R-modules into a symmetric monoidal category. Note that this only works when R is commutative.

(f) Similarly, we can define the *relative function spectrum* by equalizing the actions:

$$F_R(M, N) \to F(M, N) \rightrightarrows F(R \wedge M, N),$$

where the first map is induced by $R \wedge M \to M$ and the second comes from

$$F(M, N) \to F(R \wedge M, R \wedge N) \to F(R \wedge M, N).$$

This makes the category of R-modules into a closed symmetric monoidal category.

(g) Define commutative R-algebra spectra as commutative monoids in this symmetric monoidal category of R-modules. There is a category of these in which R is the initial R-algebra spectrum.

(h) Example: commutative $\mathbb{S}$-algebra spectra are commutative ring spectra.

(i) Example: commutative HA-algebra spectra are commutative differential graded A-modules (see [Ric22, Theorem 3.4] for a precise statement).

(j) State but do not prove the Schwede–Shipley Theorem [SS03, Theorem 3.1.1]: any stable model category with a compact generator (and some other mild hypotheses) is equivalent to the category of R-modules for some ring spectrum R. (Recall that earlier we saw that the homotopy category of any stable model category is triangulated. It's not so important exactly what a stable model category is, but this theorem is included for cultural knowledge.)

(3) Optional extra things you might want to talk about if you have time. Send me an email if you want references on any particular thing.

(a) Counterexamples to expected behavior of ring spectra. For example, the cofiber of $\mathbb{S} \xrightarrow{2} \mathbb{S}$ is not a ring spectrum (not even an associative one!). See the introduction to [Ric22].

(b) Define operads and algebras over them and use this to discuss E_n-ring spectra.

(c) Smith ideals of Ring spectra [Hov14]

(d) Galois theory of structured ring spectra

References

[Hov14] Mark Hovey. Smith ideals of structured ring spectra. https://arxiv.org/abs/1401.2850, 2014.

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