Smash Product Talk Outline

Read [Mal23, Chapter 4] and the introduction to [Lew91].

(1) Symmetric monoidal categories

   (a) Define a monoidal category, and explain what it means for a monoidal category to be symmetric.
   (b) State the MacLane coherence theorem, and give its consequence – you can drop all mention of the coherence morphisms when working with a monoidal category.
   (c) Give several examples of monoidal categories.
   (d) Given
      - a symmetric monoidal category $(\mathcal{C}, \odot, 1)$
      - that is also a homotopical category, with class of weak equivalences $\mathcal{W} \subseteq \mathcal{C}$
      - and a left deformation $Q\colon \mathcal{C} \to \mathcal{C}$ with a full subcategory $\mathcal{C}_Q$ containing the image of Q,

      state the requirements for $\mathrm{ho}(\mathcal{C})$ to also become a symmetric monoidal category [Mal23, Lemma 4.1.7]. Describe the monoidal product and sketch the construction of the associativity and unit natural transformations. You should not verify that the coherence diagrams commute, we'll happily assume that they do.

(2) Adams' "handcrafted smash products"

   (a) Historically, the first attempt to define a smash product is Adams' handcrafted smash product.
   (b) Define the handcrafted smash product as in [Mal23, Definition 2.3.23]. Beware that there are some typos in the description of the bonding maps.
   (c) Explain the deficiencies of the smash product in this case: that it's not actually symmetric monoidal; we only have stable equivalences $\mathbb{S} \wedge X \simeq X$ and not isomorphisms. Illustrate this with a counterexample.
   (d) State [Lew91, Theorem 1.1] – smash products have to give up one of the things that we hope for in a category of spectra! This theorem shows that this is a hard problem.

(3) Smash product as a black box

   (a) Next week, we will modify our definition of spectra to make the smash product easier to define. But for now, we can give the smash product as a black box and work with it by properties. To be a little more clear, we can write $\widehat{\mathcal{S}p}$ for a hypothetical category of spectra with $\mathrm{ho}(\widehat{\mathcal{S}p}) \simeq \mathrm{ho}(\mathcal{S}p)$, and with all of the same functors and stuff that $\mathcal{S}p$ has, but $\widehat{\mathcal{S}p}$ is symmetric monoidal. Malkiewich doesn't change notation here, but we should to be less confusing.
   (b) Explain the properties that we will get out of the smash product [Mal23, Example 4.1.9].
   (c) Define monoids in a symmetric monoidal category and give several examples. Define ring spectra and their modules.
   (d) Define lax/strong symmetric monoidal functors, and explain that they send monoids to monoids. Give some examples.
   (e) Explain that the functors Σ^∞_+, H, and π_* will be symmetric monoidal after the black box [Mal23, Example 4.1.28].
   (f) Define a closed symmetric monoidal category, and give examples.

(g) State [Mal23, Lemma 4.1.33], but state it in the language of homotopical categories: Q is a left deformation and R is a right deformation, and $(Q, R)\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ becomes a right deformation for $\mathrm{Hom}(-,-)$. Sketch the proof.

(h) Explain the consequences of the above for our black box [Mal23, Example 4.1.35].

(i) Define dualizable objects [Mal23, Definition 4.2.1]. State the theorem/add to the black box that a spectrum X is dualizable if and only if it is a finite spectrum [Mal23, Example 4.2.11].

(4) Extra optional things to talk about if you have time or the inclination to do so

(a) Define the smash product on the Spanier–Whitehead category

(b) Give some consequences of duality: Poincaré Duality, Whitehead representability

(c) Discuss traces and fixed point formulas

REFERENCES

[Lew91] L. Gaunce Lewis, Jr. Is there a convenient category of spectra? *J. Pure Appl. Algebra*, 73(3):233–246, 1991.

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. `http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf`, October 2023.