

Due at the beginning of class on 29 April 2024

- Your answers should be neatly written and logically organized.
- You may collaborate on solving the problems, but the solutions you turn in should be your own.
- You may use any resource you find online (or elsewhere), but you must cite any resource you use.

Reading: Good introductions to operads include the following: [Bel17, Bra17, Sta04, Sar17]. Pick whichever reference that you find most accessible. The survey article [Man22] is a much more comprehensive overview of operads in stable homotopy theory, but it is also much more technical.

- (1) Let X be an E_k -algebra in \mathcal{Top}_* for some $k \geq 2$. Show that $\pi_0(X)$ and $\pi_1(X)$ are abelian groups.
- (2) Let $\widehat{\mathcal{S}p}_{\geq 0}$ be the full subcategory of the $\widehat{\mathcal{S}p}$ on the connective spectra.
 - (a) Show that there is an adjunction $\pi_0: \widehat{\mathcal{S}p}_{\geq 0} \rightleftarrows \mathcal{Ab}: H$.
 - (b) Let A be an abelian group. Show that the endomorphism operads $\text{End}_{\widehat{\mathcal{S}p}}(HA)$ and $\text{End}_{\mathcal{Ab}}(A)$ are weakly equivalent as operads in \mathcal{Top}_* .
 - (c) Show that the structure of an \mathcal{O} -algebra on HA is equivalent to the structure of an $\pi_0(\mathcal{O})$ -algebra on A , where $\pi_0(\mathcal{O})$ is the operad in $(\text{Set}, \times, \{*\})$ constructed from \mathcal{O} by taking $\pi_0(\mathcal{O})(n) = \pi_0(\mathcal{O}(n))$.
 - (d) If A is a commutative ring, show that HA is an E_∞ -algebra.
- (3) Let $(\mathcal{C}, \otimes, I)$ and (\mathcal{D}, \odot, J) be symmetric monoidal categories enriched in \mathcal{Top}_* , and let \mathcal{O} be an operad of pointed spaces. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a lax symmetric monoidal functor.
 - (a) If X is an \mathcal{O} -algebra in \mathcal{C} , show that $F(X)$ is an \mathcal{O} -algebra in \mathcal{D} .
 - (b) Show that the suspension spectrum of a topological monoid is an E_1 -ring spectrum.
 - (c) If A is a commutative ring, show that HA is an E_∞ -ring spectrum. Give a different proof than your solution to problem 2(d).

REFERENCES

- [Bel17] Eva Belmont. A quick introduction to operads. <https://mathweb.ucsd.edu/~ebelmont/operads-talk.pdf>, 2017.
- [Bra17] Tai-Danae Bradley. What is an operad? <https://www.math3ma.com/blog/what-is-an-operad-part-1>, 2017.
- [Man22] Michael A. Mandell. Operads and operadic algebras in homotopy theory. In *Stable categories and structured ring spectra*, volume 69 of *Math. Sci. Res. Inst. Publ.*, pages 183–247. Cambridge Univ. Press, Cambridge, 2022.
- [Sar17] Maru Sarazola. Loop spaces and operads. <https://sites.google.com/view/msarazola/notes-and-other-resources>, 2017.
- [Sta04] Jim Stasheff. What is ... an operad? *Notices Amer. Math. Soc.*, 51(6):630–631, 2004.