STABILITY NAME: SOLUTIONS

Due at the beginning of class on 11 March 2025

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Sections 2.4 and 2.5].

(1) Consider the commuting square of spectra

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

(a) Prove that this square is a homotopy pushout if and only if the induced map of homotopy cofibers $\mathrm{cof}(f) \to \mathrm{cof}(g)$ is a stable equivalence.

SOLUTION:

Thanks to Ea for their help with the backwards direction! :)

First we recall a few lemmas that will be useful.

Pasting Lemma. Consider the following commutative diagram in a category $\mathcal{C}$ with pushouts:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y \arrow[r] & Z \end{tikzcd}$$

such that the left square $\begin{smallmatrix} A \to B \\ \downarrow \ulcorner \downarrow \\ X \to Y \end{smallmatrix}$ is a pushout. Then the following are equivalent:

- The right square $\begin{smallmatrix} B \to C \\ \downarrow \quad \downarrow \\ Y \to Z \end{smallmatrix}$ is a pushout.
- The whole rectangle $\begin{smallmatrix} A \longrightarrow C \\ \downarrow \quad\quad \downarrow \\ X \longrightarrow Z \end{smallmatrix}$ is a pushout.

This lemma also holds for homotopy pushouts – a quick sketch of that argument can be found in the solutions for homework 3. There is also a dual notion for pullbacks and homotopy pullbacks: see [nLa].

Lemma. Consider the following square in $\mathcal{S}p$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h", "\simeq"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

where h is a stable equivalence. Then k is a stable equivalence if and only if the square is a homotopy pushout.

Now for the solution to the problem. Take homotopy cofibers of both f and g, realized as homotopy pushouts of $* \leftarrow A \xrightarrow{f} B$ and $* \leftarrow C \xrightarrow{g} D$. Assemble these diagrams into a cube as below:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dd] \arrow[dr] & & B \arrow[dd] \arrow[dr] & \\
& * \arrow[rr] \arrow[dd, equal] & & \mathrm{cof}(f) \arrow[dd, dashed] \\
C \arrow[rr, "g"] \arrow[dr] & & D \arrow[dr] & \\
& * \arrow[rr] & & \mathrm{cof}(g).
\end{tikzcd}$$

Note that the dashed map $\mathrm{cof}(f) \to \mathrm{cof}(g)$ is the map induced on pushouts by the maps between diagrams

$$\begin{tikzcd}
* \arrow[d, equal] & A \arrow[l] \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
* & C \arrow[l] \arrow[r, "g"] & D
\end{tikzcd}$$

Also note that by construction, the top and bottom faces are homotopy pushouts, and the front of the cube $\begin{tikzcd}[sep=small] * \arrow[r] \arrow[d, equal] & \mathrm{cof}(f) \arrow[d] \\ * \arrow[r] & \mathrm{cof}(g) \end{tikzcd}$ is a homotopy pushout if and only if the induced map $\mathrm{cof}(f) \to \mathrm{cof}(g)$ is a stable equivalence by the second lemma.

($\Longrightarrow$) Assume that the square $\begin{tikzcd}[sep=small] A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"] & D \end{tikzcd}$ is a homotopy pushout. Then all three of the top $\begin{tikzcd}[sep=small] A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ * \arrow[r] & \mathrm{cof}(f) \end{tikzcd}$, bottom $\begin{tikzcd}[sep=small] C \arrow[r, "g"] \arrow[d] & D \arrow[d] \\ * \arrow[r] & \mathrm{cof}(g) \end{tikzcd}$ and back $\begin{tikzcd}[sep=small] A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"] & D \end{tikzcd}$ of the cube are all homotopy pushouts. We want to show that the front $\begin{tikzcd}[sep=small] * \arrow[r] \arrow[d, equal] & \mathrm{cof}(f) \arrow[d] \\ * \arrow[r] & \mathrm{cof}(g) \end{tikzcd}$ is a homotopy pushout as well. First consider the back and bottom of the cube together

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
C \arrow[r, "g"] \arrow[d] & D \arrow[d] \\
* \arrow[r] & \mathrm{cof}(g)
\end{tikzcd}$$

Both of these are pushouts, so by the pasting lemma (rotated), so is the whole outer rectangle. This outer rectangle is the same outer rectangle as the front and top of the cube put together:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
* \arrow[r] \arrow[d] & \mathrm{cof}(f) \arrow[d] \\
* \arrow[r] & \mathrm{cof}(g)
\end{tikzcd}$$

Now in this rectangle, we know that the outer rectangle and the top square are pushouts. So again by the pasting lemma, the bottom square is a pushout. By the second lemma, this is equivalent to $\mathrm{cof}(f) \to \mathrm{cof}(g)$ being a stable equivalence.

($\Longleftarrow$) Assume that $\mathrm{cof}(f) \to \mathrm{cof}(g)$ is a stable equivalence. The strategy for this direction is to take the homotopy pushout $P = C \cup_A^h B$ of $C \leftarrow A \xrightarrow{f} B$ and show that the map $P \to D$ is a stable equivalence.

To that end, let P be the homotopy pushout as below:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r, "g'"] \arrow[drr, bend right] \arrow[ur, phantom, "\ulcorner", very near end] & P \arrow[dr, "h"] & \\
 & & D
\end{tikzcd}$$

This fits into the cube as below, where $\mathrm{cof}(g')$ is the homotopy cofiber of $g' \colon C \to P$:

$$\begin{tikzcd}
A \arrow[rrrr, "f"] \arrow[dr] \arrow[ddd] & & & & B \arrow[ddl] \arrow[ddd] \arrow[dr] & \\
 & * \arrow[rrrr] \arrow[dddd] & & & & \mathrm{cof}(f) \arrow[dddl] \arrow[dddd] \\
 & & & P \arrow[dr, "h"] \arrow[ddr] & & \\
C \arrow[urrr, "g'"] \arrow[rrrr, "g"] \arrow[ddr] & & & & D \arrow[ddr] & \\
 & & & & \mathrm{cof}(g') \arrow[dr] & \\
 & * \arrow[urrr] \arrow[rrrr] & & & & \mathrm{cof}(g)
\end{tikzcd}$$

Within this outer cube, we have a smaller cube

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\
 & * \arrow[rr] \arrow[dd] & & \mathrm{cof}(f) \arrow[dd] \\
C \arrow[rr, "g'"] \arrow[dr] & & P \arrow[dr] & \\
 & * \arrow[rr] & & \mathrm{cof}(g').
\end{tikzcd}$$

whose back, bottom, and top are all homotopy pullback squares. By repeatedly applying the pasting lemma first to the back and bottom and then to the top and front, we learn that the front of the cube $\begin{tikzcd} * \arrow[r] \arrow[d] & \mathrm{cof}(f) \arrow[d] \\ * \arrow[r] & \mathrm{cof}(g') \end{tikzcd}$ is a homotopy pullback square. This means by the second lemma that $\mathrm{cof}(f) \simeq \mathrm{cof}(g')$.

By assumption, we also had $\mathrm{cof}(f) \simeq \mathrm{cof}(g)$. Hence, by the two-out-of-three property for stable equivalences, $\mathrm{cof}(g) \simeq \mathrm{cof}(g')$.

Now consider the map of cofiber sequences

$$\begin{tikzcd} C \arrow[r, "g'"] \arrow[d, "\mathrm{id}"] & P \arrow[r] \arrow[d, "h"] & \mathrm{cof}(g') \arrow[d, "\simeq"] \\ C \arrow[r, "g"] & D \arrow[r] & \mathrm{cof}(g) \end{tikzcd}$$

This induces a map of long exact sequences:

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{n+1}\mathrm{cof}(g') \arrow[r] \arrow[d, "\cong"] & \pi_n C \arrow[r, "g'_*"] \arrow[d, "="] & \pi_n P \arrow[r] \arrow[d, "h_*"] & \pi_n \mathrm{cof}(g') \arrow[r] \arrow[d, "\cong"] & \pi_{n-1} C \arrow[r] \arrow[d, "="] & \cdots \\ \cdots \arrow[r] & \pi_{n+1}\mathrm{cof}(g) \arrow[r] & \pi_n C \arrow[r, "g_*"] & \pi_n D \arrow[r] & \pi_n \mathrm{cof}(g) \arrow[r] & \pi_{n-1} C \arrow[r] & \cdots \end{tikzcd}$$

Then by the five-lemma, we must have that h_* is an isomorphism. Hence, $h\colon P \to D$ is a map which induces isomorphisms on homotopy, so it is by definition a stable equivalence.

Finally, this shows that D is stably equivalent to the homotopy pushout of $C \leftarrow A \xrightarrow{f} B$, so by definition $\begin{smallmatrix} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & D \end{smallmatrix}$ is a homotopy pushout square.

(b) Use this fact to prove that a commuting square of spectra is a homotopy pullback if and only if it is a homotopy pushout. You may use the fact that a sequence of spectra is a cofiber sequence if and only if it is a fiber sequence.

SOLUTION: A dual argument to the above demonstrates that a square is a homotopy pullback if and only if the induced map on fibers is a stable equivalence. However, we have seen that the cofiber is stably equivalent (functorially) to the suspension of the fiber, so the result follows from the fact that a map of spectra is a stable equivalence if and only if its suspension is.

(2) An *semiadditive category* is a category $\mathcal{A}$ with a zero object 0 that admits all finite products and coproducts, such that the canonical morphism $X \amalg Y \to X \times Y$ is an isomorphism.

(a) Show that $\mathrm{ho}(\mathcal{S}p)$ is a semiadditive category.

SOLUTION: For this, we can combine previous results from class and from the homework. The zero object is the zero spectrum/point spectrum $*$. We know $\mathcal{S}p$ has all finite products and coproducts, which are also the homotopy product and the homotopy coproduct. So by an exercise on homework 2, products/coproducts in $\mathcal{S}p$ are also the products in $\mathrm{ho}(\mathcal{S}p)$. Finally, we saw in class that $X \vee Y \simeq X \times Y$, so products and coproducts in the homotopy category agree.

An *additive functor* between semiadditive categories is a functor that preserves the zero object and preserves finite products/coproducts.

(b) Is the Eilenberg–MacLane spectrum functor $H\colon \mathcal{A}b \to \mathrm{ho}(\mathcal{S}p)$ an additive functor?

SOLUTION: Yes. This question is asking if $H(A \oplus B) \simeq HA \vee HB$ and if H sends the zero object to the zero object.

The zero object in $\mathcal{A}b$ is the zero abelian group, and the zero object in $\mathcal{S}p$ is the point spectrum. $H0$ is the spectrum whose n-th space is $K(0, n) = *$, so it is the point spectrum.

Note that $HA \times HB \simeq HA \vee HB$, so it suffices to show that $H(A \oplus B) \simeq HA \times HB$. This is true because $\pi_n(X \times Y) \cong \pi_n(X) \oplus \pi_n(Y)$, so $K(A \oplus B, n) = K(A, n) \times K(B, n)$. The result for spectra follows.

(3) Let $k \geq 0$. Define the *shift functor* $\mathrm{sh}_k \colon \mathcal{S}p \to \mathcal{S}p$ by $\mathrm{sh}_k(X)_n = X_{k+n}$.

(a) Prove that there is a natural stable equivalence $\Sigma \simeq \mathrm{sh}_1$.

SOLUTION: There is a natural map $\sigma : \Sigma X \to \mathrm{sh}_1 X$ given by the structure map:

$$\sigma_n : (\Sigma X)_n = \Sigma X_n \to X_{n+1} = (\mathrm{sh}_1 X)_n$$

in degree n. We must show that σ induces an isomorphism on all homotopy groups. We have the following commutative diagrams for all k induced by σ:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{k+n}(\Sigma X_n) \arrow[r, "(\Sigma\sigma_n)_*\circ\Sigma"] \arrow[d, "(\sigma_n)_*"] & \pi_{k+n+1}(\Sigma X_{n+1}) \arrow[r] \arrow[d, "(\sigma_{n+1})_*"] & \cdots \\
\cdots \arrow[r] & \pi_{k+n}(X_{n+1}) \arrow[r, "(\sigma_{n+1})_*\circ\Sigma"'] & \pi_{k+n+1}(X_{n+2}) \arrow[r] & \cdots
\end{tikzcd}$$

The colimit of the top row as $n \to \infty$ is $\pi_k(\Sigma X)$ and the colimit of the bottom row is $\pi_k(\mathrm{sh}_1 X)$. Hence, $\sigma_* : \pi_k(\Sigma X) \to \pi_k(\mathrm{sh}_1 X)$ is the colimit of the maps $(\sigma_n)_* : \pi_{n+k}(\Sigma X_n) \to \pi_{n+k}(X_{n+1})$ as $n \to \infty$. This induced map is the isomorphism seen in class (see also [Mal23, Remark 2.4.2] and hence σ is a stable equivalence.

(b) Define functors sh_k for $k < 0$. Prove that $\mathrm{sh}_{-1} \simeq \Omega$.

SOLUTION: If $k < 0$, define the shift functor $\mathrm{sh}_k : \mathrm{Sp} \to \mathrm{Sp}$ by:

$$\mathrm{sh}_k(X)_n = \begin{cases} * & \text{if } k+n < 0 \\ X_{k+n} & \text{otherwise} \end{cases}$$

We have the natural map $\mathrm{sh}_{-1}X \to \Omega X$ given by inclusion of the basepoint $* \to \Omega X_0$ in degree zero, and the adjoints of X's structure maps $(\mathrm{sh}_{-1}X)_n = X_{n-1} \to \Omega X_n$ for $n \geq 1$. This map is a stable equivalence because it induces exactly the isomorphism $\pi_n(\Omega X) \to \pi_{m+1}(X) = \pi_m(\mathrm{sh}_{-1}X)$ seen in class.

REFERENCES

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[nLa] nLab. Pasting law. https://ncatlab.org/nlab/show/pasting+law+for+pullbacks.