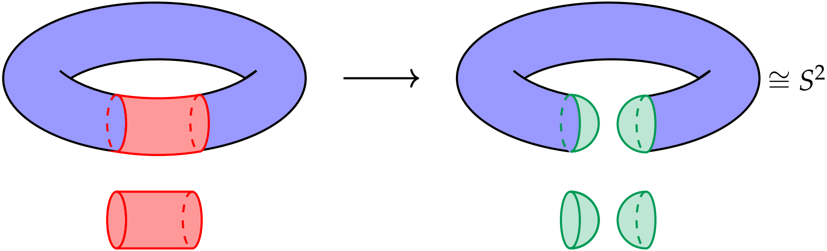
Combinatorial Tools in Equivariant Algebraic Topology

Part I
The Conjectures

Kervaire Problem (1960)

Can every smooth n-manifold become a sphere via surgery?



Surgery

Replace a submanifold by another with the same boundary

Kervaire Problem (1960)

Can every smooth n-manifold become a sphere via surgery?

(1963–1984) counterexamples with $n = 2, 6, 14, 30, 62$

(1969) Browder: counterexamples must have $n = 2^k - 2$

Theorem (Hill–Hopkins–Ravenel, 2016)

The only counterexamples have $n = 2, 6, 14, 30, 62$, or maybe 126

Key ingredient: equivariant homotopy theory

Theorem (Lin–Wang–Xu, December 2024)

There exist counterexample 126-manifolds (nonconstructive)

TELESCOPE CONJECTURE (Ravenel, 1984)

Two ways of computing $\pi_k S^n$ are the same

THEOREM (Burklund–Hahn–Levy–Schlank, 2023)

The Telescope Conjecture is false!

A key ingredient: equivariant homotopy theory

Will equivariant homotopy theory solve all our problems?

Not yet.

- Group actions are rigid; homotopy theory is floppy
- Computations are tough, even for small finite groups
- Needs new algebraic tools: Mackey/Tambara functors

The better we understand these tools, the more we can do!

Part II
Equivariant Algebraic Topology

Let G be a finite group.

Definition

A *G-space* is a topological space X with a G-action $G \times X \to X$:

$$1 \cdot x = x$$
$$g \cdot (h \cdot x) = (gh) \cdot x$$

such that $x \mapsto g \cdot x$ is continuous for all $g \in G$.

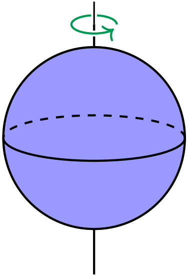
Definition

A *G-equivariant map* $f\colon X \to Y$ is a continuous map such that

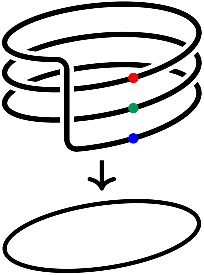
$$f(g \cdot x) = g \cdot f(x)$$

for all $g \in G$ and all $x \in X$.

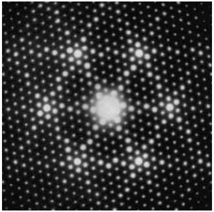
Examples



Rotation
$\mathbb{Z}/n$



Covering Spaces
$\mathbb{Z}/3$



$PbCr_3S_4$ crystals
Dihedral Group D_6

In algebraic topology, we study spaces using **invariants**

Definition

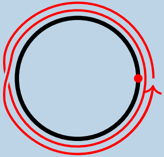
An invariant is data $F(X)$ built from a space X such that

$$X \simeq Y \implies F(X) \cong F(Y)$$

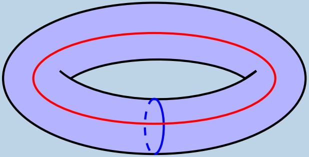
We mostly use the contrapositive: $F(X) \ncong F(Y) \implies X \not\simeq Y$

Example

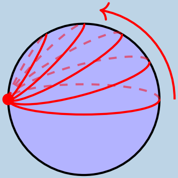
The fundamental group $\pi_1(X)$ is the group of homotopy classes of based loops in X.



$\pi_1(S^1) \cong \mathbb{Z}$



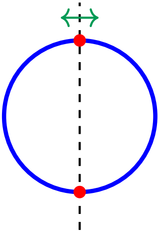
$\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$



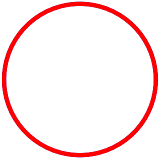
$\pi_1(S^2) = 0$

Question

How do you see group actions on spaces using invariants?



$\mathbb{Z}/2$ acts by reflection
fixed points S^0



$\mathbb{Z}/2$ acts trivially
fixed points S^1

Answer

Consider the fixed point subspaces X^G as well!

$$X^G := \left\{ x \in X \,\middle|\, g \cdot x = x \text{ for all } g \in G \right\} \subseteq X$$

The Data Pipeline

$G \circlearrowright X$

$$\begin{tikzcd} & X^G \arrow[dl, hook] \arrow[dr, hook] & \\ X^H \arrow[dr, hook] & & X^K \arrow[dl, hook] \\ & X & \end{tikzcd}$$

$$\begin{tikzcd} & \pi_1^{\mathrm{st}}(X^G) \arrow[dl, shift left] \arrow[dr, shift left] & \\ \pi_1^{\mathrm{st}}(X^H) \arrow[ur, shift left] \arrow[dr, shift left] & & \pi_1^{\mathrm{st}}(X^K) \arrow[ul, shift left] \arrow[dl, shift left] \\ & \pi_1^{\mathrm{st}}(X) \arrow[ul, shift left] \arrow[ur, shift left] & \end{tikzcd}$$

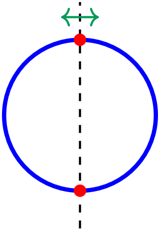
G-space $\longrightarrow$ Fixed Point Subspaces of X $\xrightarrow{\text{invariant}}$ Mackey Functors*

Theorem (Elmendorf, Piacenza, 1991)

For the purposes of homotopy theory, we may replace a G-space X by its collection of fixed point subspaces $\{X^H\}_{H \subseteq G}$

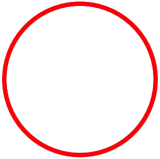
Example

reflection action



$$\begin{tikzcd} S^0 \arrow[d, hook] \\ S^1 \end{tikzcd} \xrightarrow{\pi_1^{st}} \begin{tikzcd} \mathbb{Z}/2 \arrow[d, shift left, "0"] \\ \mathbb{Z} \arrow[u, two heads, shift left] \end{tikzcd}$$

trivial action



$$\begin{tikzcd} S^1 \arrow[d, hook] \\ S^1 \end{tikzcd} \xrightarrow{\pi_1^{st}} \begin{tikzcd} \mathbb{Z} \arrow[d, shift left, "1"] \\ \mathbb{Z} \arrow[u, shift left, "2"] \end{tikzcd}$$

DEFINITION (Dress, 1971)

A Mackey Functor for G is the data:

- an abelian group $M(H)$ for each subgroup $H \subseteq G$
- restriction homomorphisms

$$\operatorname{res}^H_K \colon M(H) \to M(K)$$

- transfer homomorphisms

$$\operatorname{tr}^H_K \colon M(K) \to M(H)$$

- conjugation isomorphisms

$$c_g \colon M(K) \cong M(gKg^{-1})$$

with a "double coset formula" for $\operatorname{res} \circ \operatorname{tr}$, and other conditions

$$\begin{tikzcd}
 & M(G) \arrow[dl, shift left, "\operatorname{res}^G_H"] \arrow[dd, shift left, "\operatorname{res}^G_K"] \arrow[ddr, bend left, "\operatorname{res}"] & \\
M(H) \arrow[ur, shift left, "\operatorname{tr}^G_H"] \arrow[ddr, bend right, "\operatorname{res}^H_1"] & & \\
 & M(K) \arrow[uu, shift left, "\operatorname{tr}^G_K"] \arrow[d, shift left, "\operatorname{res}^K_{\mathbb{1}}"] \arrow[r, phantom, "\underset{c_g}{\cong}"] & M(gKg^{-1}) \arrow[uul, bend right, "\operatorname{tr}"'] \arrow[dl, bend left, "\operatorname{res}"] \\
 & M(\mathbb{1}) \arrow[loop below, "G"] \arrow[uul, bend left, "\operatorname{tr}^H_{\mathbb{1}}"] \arrow[u, shift left, "\operatorname{tr}^K_{\mathbb{1}}"] \arrow[ur, bend right, "\operatorname{tr}"'] &
\end{tikzcd}$$

Small Groups

A Mackey functor for $G = \mathbb{1}$ is just an abelian group

A Mackey functor M for $G = C_p$ (cyclic, order p) is the data:

$$\begin{tikzcd}
M(C_p) \arrow[d, "\mathrm{res}_{\mathbb{1}}^{C_p}"] \\
M(\mathbb{1}) \arrow[u, bend left, "\mathrm{tr}_{\mathbb{1}}^{C_p}"] \arrow[loop below, "C_p"]
\end{tikzcd}$$

$$\mathrm{res}_1^{C_p} \circ \mathrm{tr}_1^{C_p}(m) = \sum_{g \in C_p} g \cdot m$$

Question

Where do the transfers come from?

Let E/F be a Galois extension with (finite) Galois group $\mathrm{Gal}(E/F)$

There is a $\mathrm{Gal}(E/F)$-Mackey functor M with $M(H) = (E^H, +)$

Restrictions are inclusions:

$$\mathrm{res}^H_K \colon E^H \hookrightarrow E^K$$

Transfers are sums over orbits (field-theoretic traces):

$$\begin{aligned} \mathrm{tr}^H_K \colon E^K &\longrightarrow E^H \\ a &\longmapsto \sum_{gK \in H/K} g \cdot a \end{aligned}$$

Slogan

Mackey Functors $\pi_1^{\mathrm{st}}(X^H)$ encode a "Galois theory" of G-spaces

Part III
Equivariant Algebra

The category of G-Mackey functors is:

- abelian $\Longrightarrow$ algebra!
- symmetric monoidal $\Longrightarrow$ commutative algebra!
- generated by finitely many projectives $\Longrightarrow$ homological algebra!

EXAMPLE

When $G = \mathbb{1}$, this is just abelian groups and commutative rings

QUESTION

What is a commutative ring for G-Mackey functors?

Let E/F be a Galois extension with (finite) Galois group $\mathrm{Gal}(E/F)$

There is a $\mathrm{Gal}(E/F)$-Mackey functor M with $M(H) = E^H$

Restrictions are inclusions:

$$\mathrm{res}^H_K \colon E^H \hookrightarrow E^K$$

Transfers are sums over orbits (field-theoretic traces):

$$\begin{aligned} \mathrm{tr}^H_K \colon E^K &\longrightarrow E^H \\ a &\longmapsto \sum_{\sigma \in \mathrm{Gal}(E^K/E^H)} \sigma(a) \end{aligned}$$

Also have field-theoretic norms:

$$a \longmapsto \prod_{\sigma \in \mathrm{Gal}(E^K/E^H)} \sigma(a)$$

Definition (Tambara, 1993)

A Tambara Functor for G is the data:

- a commutative ring $T(H)$ for each subgroup $H \subseteq G$
- restriction (ring) homomorphisms

$$\mathrm{res}^H_K \colon T(H) \to T(K)$$

- transfer homomorphisms (for $+$)

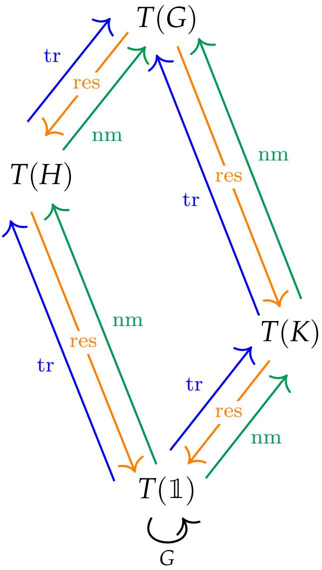
$$\mathrm{tr}^H_K \colon T(K) \to T(H)$$

- norm homomorphisms (for $\times$)

$$\mathrm{nm}^H_K \colon T(K) \to T(H)$$

- conjugation isomorphisms

with "double coset formulas" for $\mathrm{res} \circ \mathrm{tr}$ and $\mathrm{res} \circ \mathrm{nm}$, and …



Tambara Functors for Small Groups

A Tambara functor for $G = \mathbb{1}$ is a commutative ring

A Tambara functor T for $G = C_p$ is the data:

$$\begin{tikzcd} T(C_p) \arrow[d, "\mathrm{res}"] \\ T(\mathbb{1}) \arrow[u, bend left, "\mathrm{tr}"] \arrow[u, bend right, "\mathrm{nm}"'] \arrow[loop below, "C_p"] \end{tikzcd}$$

$$\mathrm{nm}(0) = 0$$

$$\mathrm{res} \circ \mathrm{tr}(x) = \sum_{g \in C_p} g \cdot x \qquad \mathrm{res} \circ \mathrm{nm}(x) = \prod_{g \in C_p} g \cdot x$$

(and other conditions)

Example: Burnside Functor $\mathbb{A}$

For each G, there is a Burnside functor $\mathbb{A}$ with origin in topology. Each $\mathbb{A}(H)$ is built from finite G-sets.

$$\mathbb{A} \text{ for } C_p: \quad \begin{tikzcd} \mathbb{Z}[t]/(t^2 - pt) \arrow[d, "t \mapsto p"] \\ \mathbb{Z} \arrow[u, bend left, "x \mapsto tx"] \arrow[u, bend right, "x \mapsto x + \left(\frac{x^p - x}{p}\right)t"'] \arrow[loop below, "\text{trivial}"] \end{tikzcd}$$

This Tambara functor $\mathbb{A}$ plays the role of $\mathbb{Z}$.

- Every Mackey functor is an $\mathbb{A}$-module
- $\mathbb{A}$ is the initial Tambara functor
- $\mathbb{A}$ is the unit for the tensor product

Representation Theory

Definition

Let X be a G-space. The space X^n has actions of both G^n and S_n, which combine to an action of the wreath product $G \wr S_n$.

Theorem (Cornelius–Dominguez–Modi–Mehrle–Rose–Stapleton, 2024)

Connect $\mathbb{A}(G \wr S_n)$ to the representation theory of S_n; use this to find computationally effective formulas for nm in $\mathbb{A}$.

Work-in-Progress (Calle–Chan–Mehrle–Quigley–Spitz–Van Niel)

Every Tambara functor T has a character theory: a Tambara functor $\Gamma(T)$ with $T \hookrightarrow \Gamma(T)$, where $\Gamma(T)$ is easier to study.

Commutative Algebra

Theorem (Nakaoka, 2014)

There is a robust theory of prime ideals for Tambara functors.

Theorem (Chan–Mehrle–Quigley–Spitz–Van Niel, 2024)

Use $\Gamma(\mathbb{A})$ for $G = C_p$ to describe the Tambara affine line, and relate affine Tambara algebraic geometry to invariant theory.

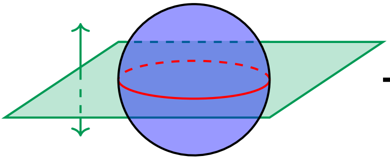
Long-Term Goal

Algebraic geometry and invariant theory of Tambara functors

Part IV
Homotopical Combinatorics

A Non-Example

$\mathbb{Z}/2$ acts on S^2 by reflection in equator



$$\longrightarrow \quad \begin{tikzcd} S^1 \arrow[d, hook] \\ S^2 \end{tikzcd} \quad \xrightarrow{\pi_1^{st}} \quad \begin{tikzcd} \mathbb{Z} \arrow[d, shift left] \\ 0 \arrow[u, shift left] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[d, "0"] \\ 0 \arrow[u, bend left, "0"] \arrow[u, bend right, "\mathrm{nm}?"'] \arrow[loop below, "C_p"] \end{tikzcd}$$

$$\begin{aligned} & 0 = 1 && \text{in the ring } 0 \\ \Longrightarrow \quad & \mathrm{nm}(0) = \mathrm{nm}(1) && \\ \Longrightarrow \quad & 0 = 1 && \text{in } \mathbb{Z} \end{aligned}$$

Contradiction! No map nm can exist

Want to allow Tambara functors with a subset of the norms

QUESTION

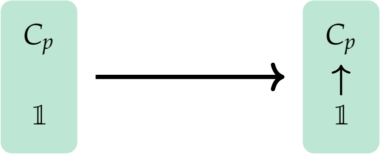
Which combinations of norms are allowable?

DEFINITION (Rubin, 2020)

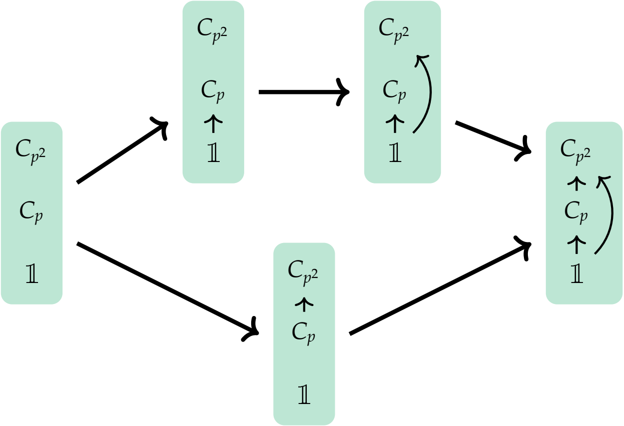
A G-transfer system is a partial order $\to$ on subgroups of G:

- (refinement) if $K \to H$, then $K \subseteq H$
- (conjugation) if $K \to H$, then $gKg^{-1} \to gHg^{-1}$ for all $g \in G$
- (restriction) if $K \to H$ and $L \subseteq H$, then $K \cap L \to L$

Transfer systems for $G = C_p$:



Transfer systems for $G = C_{p^2}$:



THEOREM (Balchin–Barnes–Roitzheim, 2021)

There are $\mathrm{Cat}(n+1) = \frac{1}{n+2}\binom{2n+2}{n+1}$ transfer systems for C_{p^n}.

Incomplete Tambara Functors

Definition

Let τ be a transfer system for G. A τ-Tambara functor is a Tambara functor with only those norms parameterized by τ.

$$\begin{tikzcd}
C_{p^2} & & T(C_{p^2}) \arrow[d, "\mathrm{res}"] \\
C_p \arrow[u] & & T(C_p) \arrow[u, bend left, "\mathrm{tr}"] \arrow[u, bend right, "\mathrm{nm}"'] \arrow[d, "\mathrm{res}"] \\
\mathbb{1} & & T(\mathbb{1}) \arrow[u, bend left, "\mathrm{tr}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C_{p^2} & & T(C_{p^2}) \arrow[d, "\mathrm{res}"] \\
C_p & & T(C_p) \arrow[u, bend left, "\mathrm{tr}"] \arrow[d, "\mathrm{res}"] \\
\mathbb{1} \arrow[u] \arrow[uu, bend right] & & T(\mathbb{1}) \arrow[u, bend left, "\mathrm{tr}"] \arrow[u, bend right, "\mathrm{nm}"'] \arrow[uu, bend right=60, "\mathrm{nm}"']
\end{tikzcd}$$

Free Algebras

$\mathbb{Z}[x]$ is a free algebra with one generator

The free τ-Tambara functor $\mathbb{A}[\tau, H]$ is determined by:

- a transfer system τ
- a subgroup $H \subseteq G$

$$\mathbb{A}[\tau, C_p] :$$

$$\begin{tikzcd}
C_{p^2} \\
C_p \arrow[u] \\
\mathbb{1}
\end{tikzcd}$$

τ

$$\begin{tikzcd}
\mathbb{A}(C_{p^2})[\mathrm{tr}(x_i), \mathrm{nm}(x_i)]/\sim \arrow[d, "\mathrm{res}"] \\
\mathbb{A}(C_p)[x_1, \ldots, x_{[G:\,H]}] \arrow[u, bend left, "\mathrm{tr}"] \arrow[u, bend right, "\mathrm{nm}"'] \arrow[d, "\mathrm{res}"] \\
\mathbb{A}(\mathbb{1})[\mathrm{res}(x_i)] \arrow[u, bend left, "\mathrm{tr}"]
\end{tikzcd}$$

$\mathbb{Z}[x]$ is a free $\mathbb{Z}$-algebra with one generator

$\mathbb{Z}[x]$ is a free $\mathbb{Z}$-module with basis $\{1, x, x^2, x^3, \ldots\}$

QUESTION

Is the free τ-Tambara functor $\mathbb{A}[\tau, H]$ free as a Mackey functor?

THEOREM (Hill–Mehrle–Quigley, 2023)

Let G be solvable. Then $\mathbb{A}[\tau, H]$ is free as a Mackey functor iff:

(a) $H \to G$ in τ,

(b) τ has no arrows below H.

THEOREM (Hill–Mehrle–Quigley, 2023)

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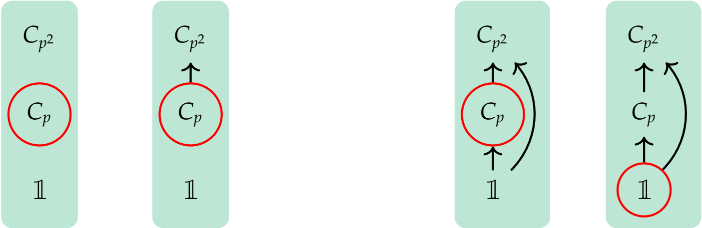

not free | free | not free | free

Combinatorial Question

How many $\mathbb{A}[\tau, H]$ are free?

Group	# subgroups H	# transfer systems τ	# pairs (τ, H)	# free	% free
C_p	2	2	4	2	50
C_{p^2}	3	5	15	4	≈ 27
C_{p^3}	4	14	56	9	≈ 16
C_{p^4}	5	42	210	23	≈ 11
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
C_{p^n}	$n+1$	$\mathrm{Cat}(n+1)$	$P(n)$	$F(n)$	$\frac{F(n)}{P(n)}$

$$P(n) = (n+1)\,\mathrm{Cat}(n+1) \qquad F(n) = \sum_{i=0}^{n} \mathrm{Cat}(i)$$

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} F(i)}{\sum_{i=0}^{n} P(i)} = 0$$

THEOREM (Hill–Mehrle–Quigley, 2023)

Fix a bijection $\mathbb{G} \colon \mathbb{N} \to \{\text{isomorphism classes of finite groups}\}$.

Let $F(G)$ be the number of pairs (τ, H) such that $\mathbb{A}[\tau, H]$ is free.

Let $P(G)$ be the total number of pairs (τ, H) for G.

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} F(\mathbb{G}(i))}{\sum_{i=1}^{n} P(\mathbb{G}(i))} = 0$$

SLOGAN

"Free incomplete Tambara functors are almost never free."

WORK-IN-PROGRESS (Bingham-Franchere-Jones-Mehrle-Shoults-Yousef)

Computer code and recursive formulas to enumerate free transfer systems for C_{p^n}, C_{pq^n}, $C_{p^2q^n}$, $\ldots$

Theorem (Mehrle–Quigley–Stahlhauer, 2024)

Let G be a cyclic p-group for an odd prime p. If $\mathbb{A}[\tau, H]$ is free, we construct well-behaved Koszul resolutions.

Theorem (Mehrle–Quigley–Stahlhauer, 2024)

Let G be a cyclic p-group, any prime p. If $\mathbb{A}[\tau, H]$ is *not* free, then it is infinite dimensional: there is a module with no finite resolution.

Goal

A theory of minimal resolutions for Tambara functors

Takeaways

Will equivariant algebraic topology solve all our problems?

Not yet. But sooner rather than later!

- Renewed interest in the field
- We understand the tools much better than 10 years ago
- New computational aids, e.g. homotopical combinatorics

Work-in-Progress (Guillou–Keyes–Mehrle)

Apply what we've learned about Tambara functors to make new calculations in equivariant homotopy theory.

Thank you!

Bonus: Mackey Functors

$\mathcal{F}\mathrm{in}^G$ = category of finite G-sets and G-equivariant functions

$\mathcal{S}\mathrm{pan}(\mathcal{F}\mathrm{in}^G)$ = category of finite G-sets and spans of finite G-sets

$$U \xleftarrow{} X \xrightarrow{} V$$

Definition

A Mackey functor is a product-preserving functor

$$M\colon \mathcal{S}\mathrm{pan}(\mathcal{F}\mathrm{in}^G) \to \mathcal{A}\mathrm{b}$$

$$M(H) := M(G/H)$$

$$\mathrm{res}^H_K := M\left(G/H \twoheadleftarrow G/K \xrightarrow{\mathrm{id}} G/K\right)$$

$$\mathrm{tr}^H_K := M\left(G/K \xleftarrow{\mathrm{id}} G/K \twoheadrightarrow G/H\right)$$

Bonus: Tambara Functors

$\mathcal{B}\mathrm{ispan}(\mathcal{F}\mathrm{in}^G)$ = category of finite G-sets & bispans of finite G-sets

$$\begin{tikzcd} & X \arrow[r] \arrow[dl] & Y \arrow[dr] & \\ U & & & V \end{tikzcd}$$

Definition

A Tambara functor is a product-preserving functor

$$T\colon\ \mathcal{B}\mathrm{ispan}(\mathcal{F}\mathrm{in}^G) \to \mathcal{S}\mathrm{et}$$

such that each $T(U)$ is a commutative ring

$$\mathrm{res}^H_K := T\left(G/H \twoheadleftarrow G/K \xrightarrow{\mathrm{id}} G/K \xrightarrow{\mathrm{id}} G/K\right)$$

$$\mathrm{tr}^H_K := T\left(G/K \xleftarrow{\mathrm{id}} G/K \xrightarrow{\mathrm{id}} G/K \twoheadrightarrow G/H\right)$$

$$\mathrm{nm}^H_K := T\left(G/K \xleftarrow{\mathrm{id}} G/K \twoheadrightarrow G/H \xrightarrow{\mathrm{id}} G/H\right)$$