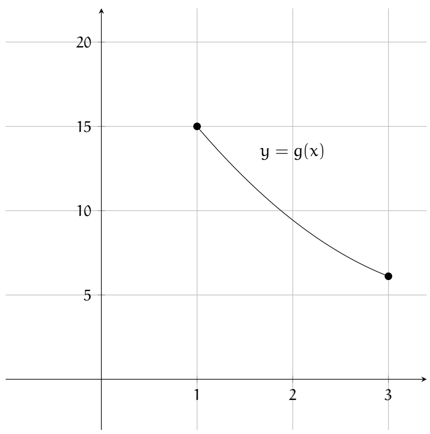
THE MEAN VALUE THEOREM

NAME: SOLUTIONS

March 31, 2017

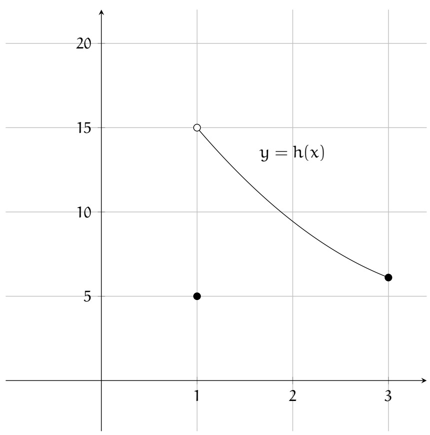
(1) Consider the graphs below. Can we guarantee that there is a point $(c, f(c))$, where the slope of the tangent line is the same as the slope of the secant line from a to b?

(a) $g(x)$ on the interval $[1, 3]$



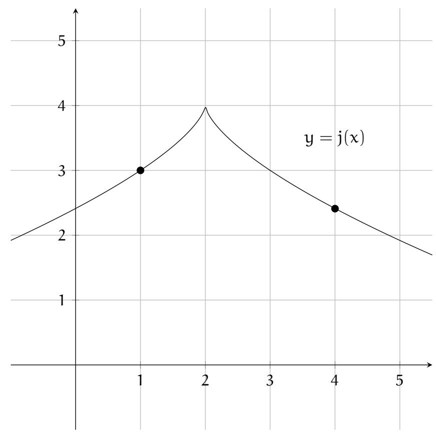
SOLUTION: Yes. The function g is continuous on $[1, 3]$ and differentiable on $(1, 3)$, so the mean value theorem applies.

(b) $h(x)$ on the interval $[1, 3]$



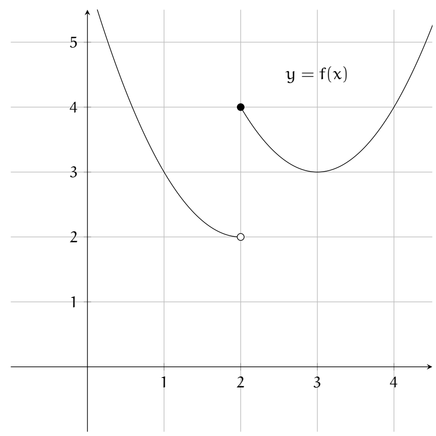
SOLUTION: No. $h(x)$ is not continuous at $x = 1$, so the mean value theorem doesn't apply.

(c) $j(x)$ on the interval $[1, 4]$



SOLUTION: No. $j(x)$ is not differentiable at $x = 2$, so the mean value theorem doesn't apply.

(d) $f(x)$ on the interval $[1, 3]$



SOLUTION: No. $f(x)$ is not continuous at $x = 2$, so the mean value theorem doesn't apply.

(2) I made some hot chocolate last night. It was 185°F. I let it cool while I played videogames. Twenty minutes later, the temperature of my not-quite-so-hot chocolate was 120°F. What does the Mean Value Theorem say about this situation? (Be specific to this case.)

SOLUTION: Over the course of 20 minutes, the average rate of cooling was $(185°\text{F} - 120°\text{F})(20 \text{ mins}) = 3.25 \frac{°\text{F}}{\text{min}}$. The mean value theorem tells us that there was a point in time during those 20 minutes when the instantaneous rate of cooling was exactly $3.24 \frac{°\text{F}}{min}$.

This material is based upon original Active Calculus materials produced by the University of Nebraska at Omaha.

(3) An elevator starts at ground level at time $t = 0$ seconds. At $t = 20$ seconds, the elevator has risen 100 feet. What does the Mean Value Theorem tell you about this situation?

SOLUTION: The average rate of rising was $(100 - 0)/20 = 5$ feet per second. The mean value theorem tells us that there was a point when the rate of rising was exactly 5 feet per second.

(4) Let $g(x) = |x^2 - 1|$.

(a) Do the hypotheses of the MVT hold on $[0, 3]$? Explain.

SOLUTION: No. The function $g(x)$ isn't differentiable at $x = 1$. (There are many ways to determine that, but the easiest is to draw a picture).

(b) Do the conclusions of the MVT hold on $[1, 3]$? Explain.

SOLUTION: Yes. Even though $g(x)$ isn't differentiable at $x = 1$, it is still continuous there. So the hypotheses of the mean value theorem are satsified.

(5) Does the MVT apply to $g(x) = x^{\frac{1}{3}}$ on $[0, 8]$? Why or why not? If so, find all values of c that satisfy the theorem.

SOLUTION: Yes. The function is not differentiable at $x = 0$, (the tangent line is vertical) but it is continuous there. So the MVT applies. To find all values c that satisfy the theorem, consider

$$\frac{g(8) - g(0)}{8 - 0} = g'(c)$$

Then solve for c. The left hand side simplifies to

$$\frac{2 - 0}{8 - 0} = \frac{1}{4}.$$

The right hand side simplifies to

$$g'(c) = \frac{1}{3c^{2/3}}.$$

We can set these equal to each other and solve for c.

$$\frac{1}{3c^{2/3}} = \frac{1}{4} \implies c^{2/3} = \frac{4}{3} \implies \boxed{c = \frac{8}{3^{3/2}}}$$

(6) Find all values of c which satisfy the MVT for $h(x) = x^3 + 6x + 2$ on $[-1, 3]$.

SOLUTION: The mean value theorem tells us that there is at least one c in $[-1, 3]$ such that

$$h'(c) = \frac{h(3) - h(-1)}{3 - (-1)}.$$

So we need to solve for c. We can substitute $h(x) = x^3 + 6x + 2$ in the above to get

$$3c^2 + 6 = \frac{47 - (-5)}{4} = \frac{52}{4} = 13$$

Then solve for c:

$$3c^2 + 6 = 13 \implies c^2 = \frac{7}{3} \implies \boxed{c^2 = \pm\sqrt{\frac{7}{3}}}$$

(7) A car travels 110 miles in 2 hours. What does the MVT tell you?

SOLUTION: At some point in time, the car was going 55 mph.

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