

# SYMMETRIC & ORTHOGONAL SPECTRA NAME: \_\_\_\_\_

**Due at the beginning of class on 15 April 2024**

- Your answers should be neatly written and logically organized.
- You may collaborate on solving the problems, but the solutions you turn in should be your own.
- You may use any resource you find online (or elsewhere), but you must cite any resource you use.

**Reading:** [Mal23, Sections 6.1 and 6.2] and [Sch23, Section 1].

- (1) Let  $X$  be a symmetric spectrum such that  $\Sigma_n$  acts trivially on  $X_n$  for all  $n$ .
  - (a) Prove that the orbit space  $(S^n)_{\Sigma_n}$  is contractible for all  $n \geq 2$ , with  $\Sigma_n$ -action on  $S^n$  by permutation of coordinates, viewing  $S^n$  as the one-point compactification of  $\mathbb{R}^n$ .
  - (b) Show that the (naive) homotopy groups of  $X$  are trivial.
- (2) A *symmetric ring spectrum* is a symmetric spectrum  $R$  together with  $\Sigma_n \times \Sigma_m$ -equivariant multiplication maps  $\mu_{n,m}: R_n \wedge R_m \rightarrow R_{n+m}$  and unit maps  $\iota_0: S^0 \rightarrow R_0$  and  $\iota_1: S^1 \rightarrow R_1$  satisfying associativity, unit, multiplicativity, and centrality conditions (see [Sch07, Definition 1.3]).  
Show that a symmetric ring spectrum in the sense above is a monoid in the monoidal category of symmetric spectra, with smash product as in [Mal23, Definition 6.2.1].
- (3) Cobordism of manifolds is captured by the spectrum  $MO$ . Read about this spectrum in [Mal23, Example 2.1.20] and [Sch07, Example 1.16].
  - (a) Prove that there is a pullback square of vector bundles

$$\begin{array}{ccc} \gamma_n \oplus \gamma_m & \longrightarrow & \gamma_{n+m} \\ \downarrow & & \downarrow \\ BO(n) \times BO(m) & \longrightarrow & BO(n+m), \end{array}$$

where  $\gamma_k \rightarrow BO(k)$  is the tautological bundle.

- (b) Use the pullback square to produce unit maps  $\mu_{n,m}: MO(n) \wedge MO(m) \rightarrow MO(n+m)$  for all  $n, m \geq 0$ .
- (c) Define unit maps  $\iota_0: S^0 \rightarrow MO(0)$  and  $\iota_1: S^1 \rightarrow MO(1)$ .
- (d) Show that these maps make  $MO$  into a commutative ring orthogonal spectrum.

## REFERENCES

- [Mal23] Cary Malkiewich. Spectra and stable homotopy theory. [http://people.math.binghamton.edu/malkiewich/spectra\\_book\\_draft.pdf](http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf), October 2023.
- [Sch07] Stefan Schwede. An untitled book project about symmetric spectra. <http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf>, 2007.
- [Sch23] Stefan Schwede. Lectures on equivariant stable homotopy theory. <http://www.math.uni-bonn.de/people/schwede/equivariant.pdf>, 2023.