Flag Varieties and Representations

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Contents

Lecture 1 June 30, 2015

Yesterday Tom Garrity said that "functions define the world" (and we all learn it as children). The better version is that **"categories define our universe"**).

Action of a group is always inside a category. We will have a categorical moment of zen for each category.

Definition 1. A **Category** is a collection of **objects** and a collection of **arrows / morphisms**. These must satisfy several properties:

1. Morphisms compose correctly. If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ then $A \xrightarrow{g \circ f} C$ is another morphism;
2. Composition is associative;
3. For each object X there is an identity arrow id_X such that for any $X \xrightarrow{f} Y$, $f \circ \mathrm{id}_X = f$ and for any $Z \xrightarrow{g} X$, $\mathrm{id}_X \circ g = g$.

Definition 2. In any category, $A \xrightarrow{f} B$ is an **isomorphism** if f has an inverse, i.e. there is a morphism $B \xrightarrow{g} A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. An **automorphism** is an isomorphism $X \to X$.

Left and right inverses are necessarily equal: if g_l and g_r are left/ right inverses, we have

$$g_l = g_l \circ \mathrm{id}_B = g_l \circ f \circ g_r = \mathrm{id}_A \circ g_r = g_r.$$

Definition 3. A **Monoid** is a category with one element.

Fun fact: a group can be defined as a monoid in which every morphism is an isomorphism. An example of a group is $\mathrm{Aut}(X) = \{\text{automorphisms of } X\}$ for any object X of any category.

We will talk about several categories, starting with the category of sets, and then vector spaces, and later the category of groups.

Category of Sets and Permutations

In this category, **Sets**, the objects are sets and the arrows are functions (but we'll call them mappings for now). We restrict attention to the category of finite sets, **finSets**, where the objects are finite sets. There is a sequence of finite sets:

$$\varnothing \subset [1] = \{1\} \subset [2] = \{1, 2\} \subset \ldots \subset [n] \subset \ldots.$$

These are all isomorphism classes in **finSets**.

Let's look at the endomorphisms of these objects in the category **finSets**. $\#\mathrm{End}([n]) = n^n$ and $\#\mathrm{Aut}([n]) = n!$. Note that endomorphisms don't commute, and automorphisms are just permutations. The symmetric group is $S_n = \mathrm{Aut}([n])$. (Which is $\mathrm{Perm}(n)$ in Aaron's notes but that's stupid).

Definition 4. Here's the **cycle notation** for permutations. For a permutation $1 \mapsto 2$, $2 \mapsto 3$, $4 \mapsto 5$, $5 \mapsto 4$, the notation is $(1\ 2\ 3)(4\ 5)$. Each parenthetical bit represents a cycle $1 \mapsto 2 \mapsto 3 \mapsto 1$ and $4 \mapsto 5 \mapsto 4$. The identity is $(1)(2)(3)(4)(5)$. To make our notation more efficient, we drop singletons.

Example 5.

$$S_3 = \{\mathrm{id}, (1\ 2), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$$

$$S_4 = \Big\{\mathrm{id}, (1\ 2), (1\ 3), (1\ 4), (2\ 3), (2\ 4), (3\ 4), (1\ 2\ 3), (1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 2),$$
$$(1\ 3\ 4), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\Big\}$$

Definition 6. The **sign** of a permutation is defined in fact for any $f\colon [n] \to [n]$ by

$$\mathrm{sgn}(f) \ := \ \prod_{i<j} \frac{f(j) - f(i)}{j - i}$$

Proposition 7. Let $f\colon [n] \to [n]$.

1. $\mathrm{sgn}(f) = 0$ unless f is invertible;
2. $\mathrm{sgn}(\sigma) = \pm 1$ if $\sigma \in S_n$;
3. $\mathrm{sgn}(\sigma \circ \tau) = \mathrm{sgn}(\sigma) \circ \mathrm{sgn}(\tau)$;
4. $\mathrm{sgn}(\mathrm{id}) = 1$;
5. $\mathrm{sgn}((i\ j)) = -1$.

Example 8. In S_3,

$$\mathrm{sgn}((1\ 2)) \ = \ \frac{1-2}{2-1} \cdot \frac{3-2}{3-1} \cdot \frac{3-1}{3-2} \ = \ -1.$$
$$\mathrm{sgn}((1\ 2\ 3)) \ = \ \frac{3-2}{2-1} \cdot \frac{1-2}{3-1} \cdot \frac{1-3}{3-2} \ = \ 1$$

Proof of Proposition 7.

1. If f is not a bijection, then it's not injective, so $f(j) = f(i)$ for some i, j. So there is a term $(f(j) - f(i))/(j - i) = 0$ in the product. Conversely, if f is a bijection then this cannot happen.

2. If f is a bijection, the sign must be ± 1 because the numerator and denominator have the same things, but in a different order. This means that $|f(i) - f(j)| = |k - \ell|$ for some $k, \ell \in [n]$ and $k - \ell$ or $\ell - k$ will appear in the denominator at some point cancelling the $f(i) - f(j)$. This happens for each i, j.
$$\frac{\prod_{i<j} |f(j) - f(i)|}{\prod_{i<j} |j - i|} = 1.$$

3.

$$\begin{aligned}\operatorname{sgn}(\sigma\circ\tau) &= \prod_{i<j}\frac{\sigma(\tau(i))-\sigma(\tau(j))}{j-i}\\ &= \prod_{i<j}\frac{\sigma(\tau(j))-\sigma(\tau(i))}{\tau(j)-\tau(i)}\frac{\tau(j)-\tau(i)}{j-i}\\ &= \prod_{\tau(i)<\tau(j)}\frac{\sigma(\tau(j))-\sigma(\tau(i))}{\tau(j)-\tau(i)}\prod_{i<j}\frac{\tau(j)-\tau(i)}{j-i} = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\end{aligned}$$

4. Clear.

5. Exercise.

□

Here's a fun fact. Note that

$$(a_1\ a_2\ a_3\ \dots\ a_n) = (a_1\ a_n)\dots(a_1\ a_3)(a_1\ a_2),$$

so we can conclude that any $\sigma \in S_n$ is a product of transpositions.

Another fun fact,

$$\sum_{\sigma\in\S_n}\operatorname{sgn}(\sigma) = 0$$

Lecture 2 — July 1, 2015

I'm going to start with a great line. Almost all algebraic geometry talks begin with this simple sentence: let k be a field.

Maybe $k = \mathbb{Q}$, or maybe $k = \mathbb{R}$, which is the completion of the first option, $\mathbb{Q}$, with respect to the ordinary norm $|\cdot|$. But we can also complete with respect to the p-adic norm $|\cdot|_p$ for p a prime (p is always prime, except on wikipedia where $p = 10$).

$$\mathbb{Q}\subset\mathbb{R}$$

Here's how the p-adic norm works. For n an integer, $|p^n|_p = \frac{1}{p^n}$, $|1|_p = 1$, and if $\frac{a}{b}\in\mathbb{Q}$ such that $p\nmid a$ and $p\nmid b$, then $|\frac{a}{b}|_p = 0$. Completing with respect to this norm gives the p-adic numbers $\mathbb{Q}_p$, also a field. And it too has an algebraic closure.

$$\mathbb{Q}\subset\mathbb{Q}_p$$

Of course, we also have the algebraic closure of $\mathbb{R}$, which is the familiar complex numbers $\mathbb{C}$. We can also take the algebraic closure of $\mathbb{Q}$ to get the algebraic numbers $\overline{\mathbb{Q}}$. Fields between $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ are **number fields** k, if they are finite extensions of $\mathbb{Q}$.

Problem 9. An open and very fundamental question in number theory. The **inverse galois problem**: does every finite group appear as the Galois group of some field extension $k/\mathbb{Q}$?

There are finite fields too, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and $\mathbb{F}_{p^n} \supset \mathbb{F}_p$. We can also take algebraic closures of $\mathbb{F}_p$, to get fields $\overline{\mathbb{F}_p}$, which are no longer finite.

There are fields even bigger than $\mathbb{C}$ as well, like the field of rational functions

$$\mathbb{C}(t) = \left\{ \frac{p(t)}{q(t)} \,\middle|\, p, q \in \mathbb{C}[t] \right\},$$

or it's algebraic closure. Or the field of Laurent series

$$\mathbb{C}((t)) = \left\{ \sum_{i=-n}^{\infty} a_i t^i \,\middle|\, a_i \in \mathbb{C} \right\}.$$

It's algebraic closure is the Puiseaux series $\overline{\mathbb{C}((t))}$. Maybe you have two variables, with a field $\mathbb{C}(t_1, t_2)$. Or n variables, such as $\mathbb{C}(t_1, \ldots, t_n)$. In some sense all of algebraic number theory lives inside the algebraic closure of one of these things. And then there are local fields, global fields, perfect fields, etc.

Okay enough about fields. When we say "let k be a field," we mean that $k = \mathbb{C}$ or $k = \mathbb{R}$.

Last time we talked about the category of sets, **Sets**. Let's define the product of two sets categorically.

Definition 10. The **product** of two sets S, T is a set U together with maps $p\colon U \to S$ and $q\colon U \to T$ such that for any other set Z with maps $f\colon Z \to S$ and $g\colon Z \to T$, there is a unique map $u\colon Z \to U$ such that $p \circ u = f$ and $q \circ u = g$.

$$\begin{tikzcd}
& Z \arrow[ddl, bend right, "f"'] \arrow[d, dashed, "u"] \arrow[ddr, bend left, "g"] & \\
& U \arrow[dl, "p"] \arrow[dr, "q"'] & \\
S & & T
\end{tikzcd}$$

Now let's talk about vector spaces over a fixed field k. We have a category called k**-Vect** of vector spaces over k, where the objects are k-vector spaces and the morphisms are linear maps. k**--Vect** is an **abelian category**, which in this case tells us that

$$\mathrm{Hom}(V, W) := \{\text{linear maps } f\colon V \to W\}$$

is also a k-vector space. A special case of this is the dual space $V^* := \mathrm{Hom}(V, k)$. After we choose a bases, we can think of $\mathrm{Hom}(k^m, k^n)$ as $n \times m$ matrices.

The hom-sets have more structure still! $\mathrm{End}(V) = \mathrm{Hom}(V, V)$ is a noncommutative ring of linear transformations (matrices after we choose a basis), and $\mathrm{GL}(V) = \mathrm{Aut}(V)$ is the invertible elements of $\mathrm{Hom}(V,V)$, which forms a group (it's the group of units of $\mathrm{End}(V)$).

The standard linear algebra thing to do next is to learn about determinants. The determinant is a map

$$\det\colon \mathrm{Hom}(V,V) \to k,$$

and it's well defined because it's independent of any choice of basis.

Definition 11. We'll define the **determinant** for a matrix $A = (a_{ij})$ as

$$\det(A) \;:=\; \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Next, some representation theory. We'll commit the cardinal sin of choosing a basis $e_1, \ldots, e_n$ for k^n.

Definition 12. The **permutation representation** of S_n on k^n corresponds to the action of S_n on vectors by permuting the coordinates. This is given by a map $S_n \to \mathrm{GL}(k^n)$, $\sigma \mapsto P_\sigma$, where P_σ is the permutation matrix with columns given by $P_\sigma e_i = e_{\sigma(i)}$.

Problem 13. Given a vector $\vec{v} = \sum_{i=1}^n v_i e_i \in k^n$, then what is $P_\sigma \vec{v}$?

Note that the determinant of a permutation matrix P_σ is the sign of its permutation: $\det(P_\sigma) = \mathrm{sgn}(\sigma)$.

Definition 14. A matrix A is **semisimple** if and only if there is an invertible matrix L such that LAL^{-1} is diagonal. This is exactly when A has a basis of eigenvectors.

Example 15. Let's find the eigenvalues of a rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

It has no real eigenvalues. We compute

$$\begin{aligned} \det\begin{bmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{bmatrix} \;&=\; (\lambda - \cos\theta)^2 + \sin^2\theta \\ &=\; \lambda^2 - 2\cos\theta\lambda + \cos^2\theta + \sin^2\theta \end{aligned}$$

$$\implies \lambda \;=\; \frac{2\cos\theta \pm \sqrt{4 - 4\cos^2\theta}}{2} = \cos\theta \pm i\sin\theta = e^{i\theta}$$

Let's find the eigenvectors too. We want the kernel of

$$\begin{bmatrix} e^{i\theta} - \cos\theta & \sin\theta \\ -\sin\theta & e^{i\theta} - \cos\theta \end{bmatrix} = \begin{bmatrix} i\sin\theta & \sin\theta \\ -\sin\theta & i\sin\theta \end{bmatrix},$$

which is spanned by

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ i \end{bmatrix},$$

corresponding to $e^{i\theta}$ and $e^{-i\theta}$, respectively.

Lecture 3 July 2, 2015

First, a few comments about yesterday's lecture. At the very end, we said that in $\mathbb{C}^2$, $(1, i) \perp (1, -i)$. This is not true for the ordinary dot product. We need to use a **Hermitian inner product**, which caused some confusion.

We also forgot to talk about quotient spaces.

Definition 16. If V is a vector space and $U \leqslant V$ is a subspace, then we can form the **quotient vector space**

$$V/U := \{\vec{v} + U \mid v \in V\}.$$

Moreover, there is a natural, linear, surjective map $V \twoheadrightarrow V/U$.

This is interesting in representation theory, because one of the problems we'll be interested in is lifting a representation of V/U to a representation of V.

Okay, now let's move onto today's lecture. We're going to talk about group actions. Some examples of groups we're going to talk about include $\mathrm{Aut}(X)$ for X an object in some category **C**, or maybe $\mathrm{Aut}([n]) \cong S_n$, or maybe $\mathrm{GL}(V) = \mathrm{Aut}(V)$ for V a vector space.

Definition 17. The **category of groups**, which we call **Groups**, is the category in which the objects are groups and the morphisms are group homomorphisms.

Example 18. The exponential map is a group homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}_{>0}, \cdot)$, because $e^{x+y} = e^x e^y$. It has an inverse $\log$, but if we instead had the exponential $e\colon (\mathbb{C}, +) \to (\mathbb{C}^*, \cdot)$, it only has a local inverse.

Example 19. Another example we saw yesterday, which is the map $P\colon S_n \to \mathrm{GL}_n k$ that sends $\sigma \in S_n$ to the permutation matrix P_σ. There is an interesting commuting diagram of groups.

$$\begin{tikzcd} S_n \arrow[r, "\phi"] \arrow[d, "\mathrm{sgn}"] & \mathrm{GL}_n k \arrow[d, "\det"] \\ \{\pm 1\} \arrow[r] & k^* \end{tikzcd}$$

Definition 20. A group is **finitely presented** if it can be described with finitely many generators and finitely many relations, which we write

$$\left\langle x_1, \ldots, x_n \middle| w_1(x_1, \ldots, x_n) = w_2(x_1, \ldots, x_n) = \ldots = w_m(x_1, \ldots, x_n) = 1 \right\rangle.$$

Example 21. The **free group** on two generators is given by the finite presentation $\langle x, y \rangle$. We can draw it too! It's a nifty a fractal.

Each finitely presented group is a quotient of a free group by some normal subgroup that creates the relations.

Example 22. The **cyclic group** $C_n := \langle x \mid x^n = 1 \rangle$. The integers, $\mathbb{Z} \cong \langle x \rangle$ is the infinite cyclic group.

Example 23. The **dihedral group** $D_n := \langle x, y \mid x^n = y^2 = xyxy^{-1} = 1 \rangle$. It's not immediately clear that this group is finite, but we can list all of the elements as

$$\{1, x, x^2, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y\}$$

and then use the presentation to show that these are all of the elements. Here is another fun fact:

$$(xy)^2 = x(yx)y = x(x^{-1}y)y = 1.$$

Example 24. The symmetric group is generated by $x_i := (i\; i+1)$ for $1 \leqslant i \leqslant n$ according to the relations $x_i^2 = 1$, $x_i x_j = x_j x_i$ if $|i - j| \geqslant 2$, $(x_i x_{i+1})^3 = 1$. We can write this group as a Dynkin diagram by putting one dot per variable and a line between two dots if the corresponding variables satisfy $(x_i x_j)^2 = 1$.

$$\bullet \longrule \bullet \longrule \bullet \longrule \bullet \dashrightarrow \bullet$$

This is a Dynkin diagram of type A_n, which corresponds to the Coxeter group S_n.

Problem 25. The cyclic group C_{mn} is isomorphic to $C_m \times C_n$ if and only if $\gcd(m, n) = 1$.

Here's a really cool theorem about finitely generated abelian groups.

Theorem 26. Let A be a finitely generated abelian group. Then

$$A \cong \mathbb{Z}^r \times \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \times \cdots \times \mathbb{Z}/d_m\mathbb{Z}$$

uniquely, such that $d_i \mid d_{i+1}$ for $1 \leqslant i < n$.

We call the category of abelian groups **Ab**. For *any* subgroup B of an abelian group A, we have a quotient group A/B (for arbitrary groups, it only makes sense for normal subgroups).

Okay let's finally talk about group actions. An action of a group G on an object X in some category **C** can be defined in several ways, but the following is the cleanest.

Definition 27. A **group action** of a group G on $X \in$ **C**, where **C** is a category, is a homomorphism $\phi \colon G \to \mathrm{Aut}(X)$. We often write this $G \circlearrowright X$. If $X = V$ is a vector space, then ϕ is a **representation**.

Here's an equivalent and more familiar definition:

Definition 28. A **group action** of a group G on X is a map $G \times X \to X$ such that $1 \cdot x = x$ for all $x \in X$ and $g_1(g_2 x) = (g_1 g_2)x$.

Lecture 4 July 3, 2015

Today we're going to talk a little bit more about categories before we move on to G-modules.

Definition 29. Given two categories **C** and **D**, a **(covariant) functor** is a map $F\colon \mathbf{C} \to \mathbf{D}$ that assigns to each object $c \in \mathbf{C}$ an object $F(c) \in \mathbf{D}$, and to each arrow $f\colon c \to c'$ in **C**, an arrow $F(f)\colon F(c) \to F(c')$ in **D**, subject to the following conditions:

1. $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$;
2. $F(f \circ g) = F(f) \circ F(g)$.

A **contravariant functor** reverses the order of the arrows with composition, i.e. $F(f \circ g) = F(g) \circ F(f)$. The basic example of a functor is a **forgetful functor** that forgets the structure of some object and regards it as a more basic one. For example, you can forget that a vector space is a vector space and think of it as a set. In the diagram below, each functor is forgetful.

$$\begin{tikzcd} & \mathbf{Sets} & \\ k\text{-}\mathbf{Vect} \arrow[ur] \arrow[r] & \mathbf{Ab} \arrow[u] \arrow[r] & \mathbf{Groups} \arrow[ul] \end{tikzcd}$$

There are analogues to injective and surjective for functors too.

Definition 30. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is **faithful** if the map $F_*\colon \mathrm{Hom}_{\mathbf{C}}(X,Y) \to \mathrm{Hom}_{\mathbf{D}}(F(X), F(Y))$ is injective for each pair $X, Y \in \mathbf{C}$. It is **fully faithful** if F_* is bijective for each $X, Y \in \mathbf{C}$.

These are not synonyms for injective and surjective! A faithful functor need not be injective on objects or arrows. In the diagram above, the map **Ab** $\to$ **Groups** is fully faithful.

Okay, that was your moment of categorical zen. Now let's do some representation theory.

Example 31. Let's find the character table for $\mathbb{Z}/3\mathbb{Z}$. We first have the trivial representation, and then we have the rotations by $2\pi/3$ as another representation. What's the third one? It's an abelian group, so has three conjugacy classes and therefore three irreducible representations.

Try as hard as we can, we can't find the final representation over the real numbers. We have to move to the complex numbers. Let $\omega = e^{2\pi i/3}$. The character table for $\mathbb{Z}/3\mathbb{Z}$ is below.

$\mathbb{Z}/3\mathbb{Z}$	$\{0\}$	$\{1\}$	$\{2\}$
trivial	1	1	1
χ_1	1	ω	ω^2
χ_2	1	ω^2	ω

Wait... aren't the rows supposed to be orthogonal? They're not if we use the regular inner product. But they are if we use the **Hermitian inner product**

$$\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{n} x_i \overline{y_i}.$$

Caveat: If a character is thought of as a one-dimensional representation, then it is multiplicative, $\chi(gh) = \chi(g)\chi(h)$. But if $\chi = \operatorname{tr}(\rho)$ for a higher-dimensional representation, then it is *not* multiplicative!

Now we return to our regularly scheduled program of group actions. Let's pick a group action $\phi\colon G \to \operatorname{Aut}(S)$.

Definition 32. 1. The **orbit** of an element $s \in S$ is $\{\phi(g)(s) \mid g \in G\}$;

2. the **stabilizer** of an element $s \in S$ is $\{h \in G \mid \phi(h)(s) = s\}$;

3. the action is **transitive** if there is only one orbit.

We write H_s for the stabilizer of $s \in S$. Observe that $|H_s||S| = |G|$.

Example 33. The left multiplication action is $\phi\colon G \to \operatorname{Aut}_{\mathbf{Sets}}(G)$ given by $\phi(g)(h) = gh$. This is an example of a transitive action. Note that we think of $\operatorname{Aut}(G)$ as the automorphisms of G as a set, not as a group. Notice that $\phi(g)(h_1h_2) = gh_1h_2$ but $\phi(g)(h_1)\phi(g)(h_2) = gh_1gh_2$, so $\phi(g)$ is *not* an automorphism for each g.

Example 34. The conjugation action is $\psi\colon G \to \operatorname{Aut}_{\mathbf{Groups}}(G)$ given by $\psi(g)(h) = ghg^{-1}$. This differs from the previous example because here, $\psi(g)$ is an automorphism for each g, because

$$\psi(g)(x)\psi(g)(y) = gxg^{-1}gyg^{-1} = gxyg^{-1} = \psi(g)(xy).$$

The orbits of this action are the conjugacy classes of G.

Additionally, conjugation acts on the set of subgroups of G, by a new action $\psi'\colon G \to \operatorname{Aut}(\{H \mid H \leqslant G\})$ defined as $\psi'(g)(H) = gHg^{-1}$.

Definition 35. $H \leqslant G$ is **normal** if H is stabilized by G under the conjugation action. We write $H \lhd G$ if H is normal in G.

A normal subgroup is a union of conjugacy classes.

Example 36. There is a copy of the Klein four-group inside S_4 that is generated by $(1\ 2)(3\ 4)$, $(1\ 3)(2\ 4)$, $(1\ 4)(2\ 3)$, and it is normal. This is unique to the symmetric group S_4. For any other symmetric group S_n for $n \neq 4$, the only normal subgroups are A_n, S_n and $\{1\}$.

Proposition 37. $H \lhd G$ is normal if and only if G/H is a group. Conversely, given a map $\phi\colon G \to G$ is a homomorphism then $\ker \phi \leqslant G$ is normal.

Now we are equipped to talk about the lifting problem. Given V a vector space and U a subspace, we want to find a subspace W of V such that W is isomorphic to V/U.

$$\begin{tikzcd} V \arrow[r] & V/U \arrow[dl, bend left] \\ W \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

Lecture 5 July 6, 2015

Let G be a group.

Definition 38. A **G-module** is a vector space V together with a homomorphism $\phi\colon G \to \mathrm{GL}(V)$.

That is, a G-module is a representation of G. To write things concisely, we sometimes write $G \circlearrowright V$. Additionally, we write the action of G as $g \cdot \vec{v}$ rather that $\phi(g)(\vec{v})$.

Definition 39. In the **category of G-modules**, which we call **G-Mod**, the objects are G-modules $G \circlearrowright V$ and the morphisms are the intertwiners $f\colon V \to W$ such that $f(g\vec{v}) = gf(\vec{v})$. We call this property **G-linearity**.

A cute remark (taking what Tom did and putting it into weird notation to obscure it) is that there is a forgetful functor $F\colon$ **G-Mod** $\to$ **$\mathbb{C}$-Vect** which forgets the action of G and forgets that the morphisms are G-linear. Essentially, this is saying that there is a subset of the vector-space homomorphisms between $V, W \in$ **$\mathbb{C}$-Vect** that is G-linear.

$$\mathrm{Hom}_{G\text{-}\mathbf{Mod}}(V, W) \subset \mathrm{Hom}_{\mathbb{C}\text{-}\mathbf{Vect}}(V, W).$$

Lemma 40 (Schur's Lemma)**.** If V, W are irreducible G-modules, then

(a) $\mathrm{Hom}_{G\text{-}\mathbf{Mod}}(V, W) = 0$ if $V \not\cong W$;

(b) $\mathrm{Hom}_{G\text{-}\mathbf{Mod}}(V, W) \cong \mathbb{C}$ if $V \cong W$.

Example 41. $S_3 \circlearrowright \mathbb{C}^3$ by permuting the basis vectors, and $S_3 \circlearrowright \mathbb{C}$ by the trivial representation. Let $f\colon \mathbb{C}^3 \to \mathbb{C}$ be the function

$$f(v_1e_1 + v_2e_2 + v_3e_3) = v_1 + v_2 + v_3.$$

This is a $\mathbb{C}$-linear map, and in addition it's S_3-linear. Take a look at the kernel of f.

$$\ker f = \{\vec{v} \in \mathbb{C}^3 \mid v_1 + v_2 + v_3 = 0\}.$$

A basis for $\ker f$ is $\langle e_1 - e_2, e_2 - e_3 \rangle$. This space inherits an action of S_3 by way of its action on $\mathbb{C}^3$, and in fact this representation is irreducible.

$$\begin{tikzcd} & S_3 \circlearrowright \mathbb{C}^3 \arrow[r, "f"] & \mathbb{C} \circlearrowleft S_3 \\ \ker f \arrow[ur, hook] & & \end{tikzcd}$$

What's the matrix for the permutation $(1\ 2)$? Check it's action on the basis vectors.

$$(1\ 2) \cdot (e_1 - e_2) = e_2 - e_1 = -(e_1 - e_2)$$
$$(1\ 2) \cdot (e_2 - e_3) = e_1 - e_3 = (e_1 - e_2) + (e_2 - e_3)$$

So the matrix is given by

$$(1\ 2) \ = \ \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

Additionally, we can lift this representation to a representation on $\mathbb{C}^3$.

Problem 42. Find all the matrices for the action of S_3 on $\ker f$. Show that this representation is isomorphic to the other realization of the 2-dimensional irreducible representation of S_3 we found earlier.

Example 43. Another representation is $(\mathbb{C}, +) \to \mathrm{GL}_2\mathbb{C}$ given by

$$z \mapsto \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}.$$

This is a very important example! Similar to the Heisenberg group. The *only* stable line is the line $y = x$.

Theorem 44 (Maschke's Theorem)**.** Every representation of a finite group over $\mathbb{R}$ or $\mathbb{C}$ is a direct sum of irreducible subrepresentations. Each representation decomposes into irreducibles.

A more sophisticated version says that kG is semisimple so long as the characteristic of k and order of G are relatively prime.

Proof. Given $U \subset V$ invariant under G, we need to find a complimentary G-invariant subspace $W \subset V$ such that $U \oplus W = V$. That is, we need to solve the lifting problem of finding a subspace of V that is isomorphic to V/U. This will be an orthogonal compliment, but not with respect to the usual Hermitian inner product on $\mathbb{C}^n$.

Let's first talk about what inner product we're going to use here. The Hermitian inner product is

$$\vec{v} \cdot \vec{w} \ = \ \sum_{i=1}^{n} v_i \overline{w_i}.$$

This inner product isn't necessarily G-invariant, but we can fix that by averaging over all elements of the group G, as so

$$\langle \vec{v}, \vec{w} \rangle \;=\; \frac{1}{|G|} \sum_{g \in G} (g\vec{v}) \cdot (g\vec{w}).$$

Now we have a genuine **G-invariant bilinear form** on V.

Then we take $W = U^{\perp}$, the orthogonal compliment of U inside V with respect to our G-invariant bilinear form. We know that $\langle \vec{w}, \vec{u} \rangle = 0$ for all $\vec{w} \in W, \vec{u} \in U$. Claim that W is also G-invariant. Then, for any $h \in G$,

$$0 = \langle \vec{u}, \vec{w} \rangle = \langle h\vec{u}, h\vec{w} \rangle,$$

and we know that $h\vec{u} \in U$ since U is G-invariant, so now we know that $h\vec{w} \in W$, since it is perpendicular to anything in U. Hence, W is G-invariant as well.

This is really cool! It tells us that we can start with a representation V, pull out an irreducible component U, and then write V as $V = U \oplus U^{\perp}$. Now do it again with $U^{\perp}$, and eventually we will get down to a decomposition of V as

$$V = U_1 \oplus U_2 \oplus \ldots \oplus U_k$$

into irreducible representations. $\square$

What happens when we take the direct sum of all of the irreducible representations of a group? We get something called the regular representation.

Definition 45. The **regular representation** of a group G is the vector space with basis $\langle e_g \mid g \in G \rangle$, with an action of G given by $h \cdot e_g = e_{hg}$. This is also the **group-algebra** $\mathbb{C}[G]$, and it has dimension equal to the order of G, $\dim \mathbb{C}[G] = |G|$. We can extend the action $G \circlearrowright \mathbb{C}[G]$ linearly to an action $\mathbb{C}[G] \circlearrowright \mathbb{C}[G]$.

Here's a cool construction that shows that each irreducible representation appears at least once in $\mathbb{C}[G]$. Given a decomposition into irreducibles,

$$\mathbb{C}[G] \;=\; U_1 \oplus U_2 \oplus \ldots \oplus U_n,$$

and an irreducible representation $G \circlearrowright U.$, choose $\vec{u} \in U$. Construct the function $f_{\vec{u}} \colon \mathbb{C}[G] \to U$ defined by $f_{\vec{u}}(e_{\mathrm{id}}) = \vec{u}$. Then for any $g \in G$, $f_{\vec{u}}(e_g) = f_{\vec{u}}(ge_{\mathrm{id}}) = gf_{\vec{u}}(e_{\mathrm{id}}) = g \cdot \vec{u}$.

Now we have a G-linear map $f_{\vec{u}} \colon \mathbb{C}[G] \to U$. This decomposes as a bunch of maps on each irreducible component $f_i \colon U_i \to U$. By Schur's Lemma, each one of these maps is either an isomorphism or zero. But we cleverly constructed $f_{\vec{u}}$ so that it was nonzero, so it must be the case that $U \cong U_i$ for some i.

What about the converse? Now we want to know how many times each irreducible representation appears. In particular, how often is $U \cong U_i$ for each i? The answer is that a irreducible representation U appears $\dim U$-many times.

The proof of this is as follows. Let U be an irreducible representation. Consider $\mathrm{Hom}_G(\mathbb{C}[G], U)$. This space isomorphic to U as a vector space, with an isomorphism given by $f \mapsto f(e_{\mathrm{id}})$ and its inverse $\vec{u} \mapsto f_{\vec{u}}$. Then,

$$\begin{aligned} U &\cong \mathrm{Hom}(\mathbb{C}[G], U) \\ &\cong \mathrm{Hom}\left(\bigoplus_{i=1}^{n} U_i \ , \ U\right) \\ &\cong \bigoplus_{i=1}^{n} \mathrm{Hom}(U_i, U) \end{aligned}$$

Now by Schur's lemma, $\mathrm{Hom}(U_i, U) \cong \mathbb{C}$ if $U_i \cong U$, or zero otherwise. And these are isomorphic, so there must be $\dim U$-many nonzero items on the right. This comes from a rather silly observation, that

$$U \cong \bigoplus_{i=1}^{n} \mathrm{Hom}(U_i, U) \cong \bigoplus_{i=1}^{\dim U} \mathbb{C}.$$

Lecture 6 — July 7, 2015

Today, our groups are no longer finite (unless you maybe like to work over finite fields). They're going to be groups of matrices, **LieGroups**. Today we're going to talk about examples, and then for the next few days we'll talk about what it really means to be a Lie group. We'll talk about differential geometry and eventually, algebraic geometry, because the Lie groups we care about are algebraic.

For now, all I have to do is convince you that Lie groups are interesting. For now, let's let this *ad-hoc* definition suffice:

Definition 46 (Temporary)**.** A **Lie group** is a group of matrices defined by a polynomial condition on $\mathrm{GL}_n k$, where k is either $\mathbb{R}$ or $\mathbb{C}$.

Example 47. The **general linear group** is $\mathrm{GL}_n\mathbb{C} = \{$ invertible $n \times n$ matrices $\}$. This is contained in the set of all $n \times n$ matrices, $\mathfrak{gl}_n\mathbb{C}$, and $\mathrm{GL}_n\mathbb{C}$ is the compliment of the locus where the determinant vanishes.

$$\mathrm{GL}_n\mathbb{C} = \mathfrak{gl}_n\mathbb{C} \backslash \{X \in \mathfrak{gl}_n\mathbb{C} \mid \det X = 0\}$$

The same thing works over $\mathbb{R}$ instead of $\mathbb{C}$.

We have a multiplication map $m \colon \mathrm{GL}_n\mathbb{C} \times \mathrm{GL}_n\mathbb{C} \to \mathrm{GL}_n\mathbb{C}$ given by

$$(a_{ij}) \ , \ (b_{ij}) \ \mapsto \ \left(\sum_{k=1}^{n} a_{ik} b_{kj}\right)$$

And there is an inversion map $i \colon \mathrm{GL}_n\mathbb{C} \to \mathrm{GL}_n\mathbb{C}$ given by Kramer's rule. Both m and i are polynomial maps in the entries of matrices A and B.

Example 48.

$$\mathrm{GL}_2\mathbb{R} \;=\; \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; ad - bc = 0 \right\}$$

If you think about this really hard, you will see that this is like a cone. Because if $ad - bc = 0$, then so does $\lambda(ad - bc) = 0$ as well for any λ, and it's singular at the origin $a = b = c = d = 0$. We can think of it as a subset of $\mathbb{R}^4$ as well.

Example 49. The **special linear group** is a smooth submanifold of $\mathrm{GL}_n\mathbb{C}$.

$$\mathrm{SL}_n\mathbb{C} \;=\; \{A \in \mathrm{GL}_n\mathbb{C} \mid \det(A) = 1\}$$

Example 50. The length-preserving transformations of $\mathbb{R}^n$ are the **orthogonal matrices**.

$$O_n \;=\; \{A \in \mathrm{GL}_n\mathbb{R} \mid A\vec{v} \cdot A\vec{w} = \vec{v} \cdot \vec{w} \text{ for all } \vec{v}, \vec{w} \in \mathbb{R}^n\}$$

We can reinterpret the length-preserving condition as $A^T A = I$, because

$$\vec{v}^T \vec{w} = \vec{v} \cdot \vec{w} = A\vec{v} \cdot A\vec{w} = (A\vec{v})^T A\vec{w} = \vec{v}^T (A^T A)\vec{w}$$

Choosing $\vec{v} = \vec{e}_i$ and $\vec{w} = \vec{e}_j$, we see that $(A^T A)_{ij} = \delta_{ij}$ for all i.

The rows and columns of orthogonal matrices are of unit length and orthogonal.

Example 51. The length-preserving transformations of $\mathbb{C}^n$ are the **unitary matrices**.

$$U_n \;=\; \{A \in \mathrm{GL}_n\mathbb{C} \mid \langle A\vec{v}, A\vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle \text{ for all } \vec{v}, \vec{w} \in \mathbb{C}^n\}.$$

By the same argument as in the previous example, we see that unitary matrices must satisfy $A^* A = I$.

The rows and columns of unitary matrices are of unit length and orthogonal with respect to the Hermitian inner product.

Fun fact: U_n is not algebraic because of the complex conjugates.

Here are some interesting inclusions of Lie groups.

$$\begin{aligned} \mathrm{GL}_n\mathbb{C} &\supset U_n \\ \mathrm{SL}_n\mathbb{C} &\supset \mathrm{SU}_n \\ \mathrm{SO}_n\mathbb{C} &\supset \mathrm{SO}_n\mathbb{R} \end{aligned}$$

Along with the symplectic group, $\mathrm{Sp}_{2n}\mathbb{C}$, these are all of the Lie groups with only five exceptions. Let's do some examples.

Example 52. In dimension 1,

$$\begin{aligned} \mathrm{GL}(1, \mathbb{C}) &= \mathbb{C}^* \supset U(1) = \text{ unit circle} \\ \mathrm{SL}(1, \mathbb{C}) &= 1 \\ O(1, \mathbb{C}) &= \pm 1 \end{aligned}$$

In dimension n, we have a map $\det\colon U(n) \to U(1)$ that takes a unitary matrix to some value on the unit circle.

In dimension 2, we have

$$\mathrm{SO}(2,\mathbb{R}) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\}$$
$$\mathrm{SO}(2,\mathbb{C}) = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \middle| \, a^2+b^2=1, a,b\in\mathbb{C} \right\} = \left\{ \begin{bmatrix} z+z^{-1}/2 & -(z-z^{-1}/2i) \\ -(z-z^{-1}/2i) & z+z^{-1}/2 \end{bmatrix} \right\} \cong \mathbb{C}^*$$

We should talk about SU(2) at the same time we talk about SO$(3,\mathbb{R})$. As the rotations in $\mathbb{R}^3$, SO$(3,\mathbb{R})$ is the rotational symmetries of the two-sphere S^2.

Fun fact: if $A \in$ SO$(3,\mathbb{R})$ then A has an eigenvector with eigenvalue 1. This is because the characteristic polynomial looks like

$$x^3 - \mathrm{tr}(A)x^2 + bx - \det(A),$$

and $\det(A) = 1$. This polynomial has three roots, and one of them must be real because all of the coefficients are real. These roots multiply to 1, so the real root must be ± 1.

Once we know that there is an eigenvector with eigenvalue 1, this corresponds to a direction that A leaves unchanged – the axis of rotation.

Now let's talk about SU(2). These matrices look like

$$\begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix}$$

such that $z\overline{z} + w\overline{w} = 1$. This is exactly the formula for the 3-sphere S^3.

We can rewrite the definition of an element of SU(2) by setting $z = a+bi$, $w = c+di$ with $a,b,c,d \in \mathbb{R}$. Then,

$$\begin{bmatrix} a+bi & -c+di \\ c+di & a-bi \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + c\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Call this $a\vec{1} + b\vec{i} + c\vec{j} + d\vec{k}$. Then the multiplication rules for these matrices are

$$\vec{i}^2 = \vec{j}^2 = \vec{k}^1 = \vec{i}\vec{j}\vec{k} = -\vec{1}$$

We got the unit quaternions!

$$\mathbb{H} = \{a + b\vec{i} + c\vec{j} + d\vec{k} \mid a^2+b^2+c^2+d^2 = 1\}$$

This is the equation for a sphere in $\mathbb{R}^4$, so $SU(2)$ is topologically the 3-sphere S^3.

Another fun fact. On the sphere S^3 there is an action of $SU(2)$ for which the latitudes of SU(2) are conjugacy classes and the longitudes are stabilizers!

$SU(2)$ is a double cover for $SO(3,\mathbb{R})$ as a manifold.

Lecture 7 July 8, 2015

A Lie group is a group in the category of smooth manifolds. That is the statement that we're trying to understand this week. So far, all we know is that they're sitting inside the Euclidean space $\mathbb{C}^{n^2}$ and are described by polynomial equations on the coordinates. For instance, SL$(n, \mathbb{C})$ is described by the equation $\det(A) = 1$.

Not Compact (Uncool)	Compact (Cool)
SL$(n, \mathbb{C})$	SU(n)
SO$(n, \mathbb{C})$	SO$(n, \mathbb{R})$

From here we're going to try and understand these things as manifolds. Today we're going to do some topology. Friday will be function orgy day.

Definition 53. A **topology** on a set X is a collection of subsets $U \subseteq X$ such that

(i) $\varnothing$, X are open;

(ii) $U_1, \ldots, U_n$ open implies that $U_1 \cap \ldots \cap U_n$ is open;

(iii) $\{U_i \mid i \in I\}$ each open implies $\bigcup_{i \in I} U_i$ is open.

Example 54. The Euclidean topology on $\mathbb{R}^n$ (or $\mathbb{C}^n \simeq \mathbb{R}^{2n}$) has open sets U that are unions of open balls $B_r(p) = \{x \in \mathbb{R}^n \mid \|x - p\| < r\}$. That is, every open set U is of the form

$$U = \bigcup_{i \in I} B_{r_i}(p_i).$$

Remark 55. I'm going to say "balls" a lot. I'll try not to giggle too much. Sorry about that.

Example 56. Let X be the real line with two origins, pictured below. This can be described as $\mathbb{R} \cup \{0'\}$, where $0'$ is a second origin. Open sets are open intervals in $\mathbb{R}$ and any open set containing 0 intersects every open set containing $0'$. It looks like this:

This space does not have a topology coming from a metric, and it is non-Hausdorff.

We want a category of topological spaces, so we should define maps between them.

Definition 57. A map $f \colon X \to Y$ of topological spaces is **continuous** if and only if $U \subset Y$ open implies $f^{-1}(U) \subset X$ is open.

Definition 58. A **homeomorphism** between topological spaces is a continuous function with a continuous inverse.

Example 59. Polynomials $P(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$ are continuous functions $P\colon \mathbb{R}^n \to \mathbb{R}$ with respect to the Euclidean topology.

If Y is a subset of X, and X is a topological space, we can define an induced topology on Y by stating that the open sets of Y are exactly $(U \cap Y)$ for U open in X. This has the following universal property. Let $i\colon Y \to X$ be the inclusion. The topology on Y is defined by stating that for any function $f\colon Z \to Y$ is continuous if and only if $i \circ f\colon Z \to X$ is continuous.

$$\begin{tikzcd} & X \\ Z \arrow[ur, dotted, "i\circ f"] \arrow[r, "f"'] & Y \arrow[u, hook, "i"'] \end{tikzcd}$$

This fact and the previous example allows us to conclude that all of our examples of Lie groups are topological spaces: they are subspaces of $\mathbb{C}^{n^2}$ defined by polynomial equations, which are continuous.

Note that $\mathbb{R}^n \not\simeq \mathbb{R}^m$ for $n \neq m$.

Definition 60. A topological space X is

- **Hausdorff** if and only if for all $x_1, x_2 \in X$, there are U_1, U_2 open such that $x_1 \in U_1$, $x_2 \in U_2$ yet $U_1 \cap U_2 = \varnothing$;
- **disconnected** if there are nonempty, distinct U_1, U_2 open in X such that $U_1 \cap U_2 = X$ yet $U_1 \cap U_2 = \varnothing$;
- **compact** if for all open covers $X = \bigcup_{\alpha \in A} U_\alpha$, there is a finite subcover $X = \bigcup_{i=1}^n U_{\alpha_i}$;
- **a manifold** if it is Hausdorff, locally homeomorphic to Euclidean space, and there is a countable cover of basic open sets.

We require that the intersections of charts on a manifold are compatible.

Theorem 61 ((Heine-Borel))**.** A subspace $X \subset \mathbb{R}^n$ with the induced Euclidean topology is compact if and only if X is closed and bounded.

Example 62. Let's think about SO$(n, \mathbb{R})$ as a subset of $\mathbb{R}^{n^2}$. Each matrix in SO$(n, \mathbb{R})$ is of the form

$$A = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \dots & \vec{u}_n \\ | & | & & | \end{bmatrix},$$

which is a vector in $\mathbb{R}^{n^2}$. Each of the columns is orthogonal to each other, so it looks like going out in some direction, and then taking a right angle turn in another direction, and so on. Since each vector is length one, then the furthest distance we travel from the origin is $\sqrt{n}$.

Additionally, $\mathrm{SO}(n, \mathbb{R})$ is closed because it is the inverse image of a closed set under bunch of polynomial equations:

$$\vec{u}_i \cdot \vec{u}_i = 1 \qquad \vec{u}_i \cdot \vec{u}_j = 0 \qquad \det(A) = 0.$$

So we conclude by Heine-Borel that $\mathrm{SO}(n, \mathbb{R})$ is compact.

Example 63. An example of non-compact Lie group is $\mathrm{SL}(2, \mathbb{R}) \subset \mathbb{R}^4$. It contains an unbounded line (well, excluding the origin)

$$t \mapsto \begin{bmatrix} t & 0 \\ 0 & \frac{1}{t} \end{bmatrix}.$$

Here's an example of gluing. Let $U_1 = \mathbb{R}^n$ and let $U_2 = \mathbb{R}^n$. Let $U_{12} = \mathbb{R}^n \backslash \{0\}$ and let $U_{21} = \mathbb{R}^n \backslash \{0\}$. Glue by the identity $\mathrm{id} = f_{12} \colon \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ to get an n-space with two origins. This is not Hausdorff. Every open set containing one origin intersects every open set containing the other.

If instead we glue along the map

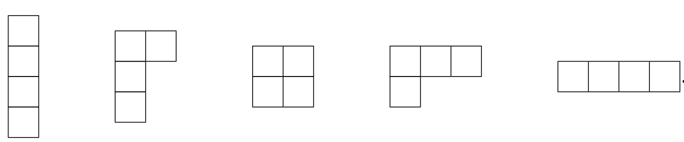
$$f_{12}(\vec{v}) = \frac{\vec{v}}{\|\vec{v}\|^2},$$

then we get the sphere. One origin becomes a point at ∞ and we get something homeomorphic to S^n in $\mathbb{R}^{n+1}$. This is the stereographic projection.

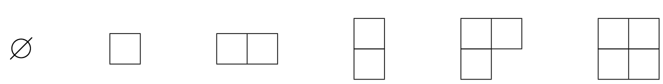
Lecture 8

July 9, 2015

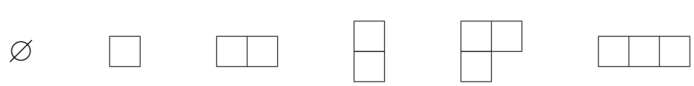
Today we're talking about Young Diagrams in a box. The conjugacy classes of the symmetric group are indexed by Young diagrams. For example, in S_4 we have

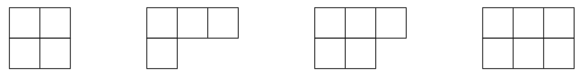


But this is too disorganized for me, so I want to put them in a box. For S_4, we're going to put them in a 2×2 box and call it $G(2, 4)$. The elements of $G(2, 4)$ are

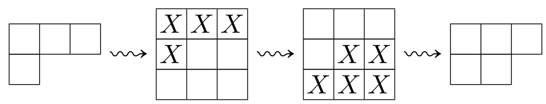


Or we could have $G(2, 5)$, in which we're trying to fit them in a box of height 2 and width $5 - 2 = 3$. How many Young diagrams can we fit in a box of this shape?

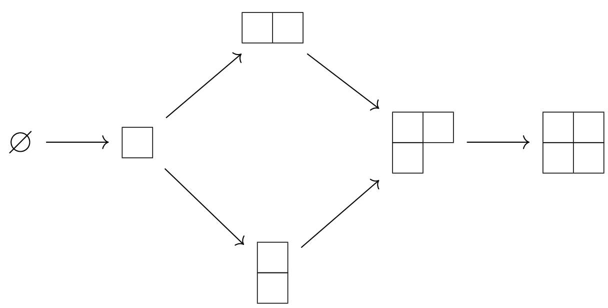




Notice that these have much more symmetry than the usual Young diagrams. There are dual diagrams for each diagram, which can be found by filling in the rest of the squares in the box and then rotating by π. For example, in a 3×3 box, we find the dual by



We also define a partial ordering on the Young diagrams by saying that one diagram is less than another if we can obtain the second by adding a single box. For example, in $G(2,4)$, we have



These diagrams are how we will describe some Grassmanians, but first we should talk about projective space.

Definition 64. **Projective space** $\mathbb{P}^n\mathbb{C}$ is the set of lines through the origin in $\mathbb{C}^{n+1}$. Namely,

$$\mathbb{P}^n\mathbb{C} \;=\; \{(x_0 \colon x_1 \colon \ldots \colon x_n) \in \mathbb{P}^n\mathbb{C} \mid x_i \neq 0 \text{ for some } i\},$$

where $(x_0 \colon x_1 \colon \ldots \colon x_n)$ means we consider points up to (nonzero) scalar multiplication.

We can obtain projective space by gluing two copies of $\mathbb{C}$. On the projective line $\mathbb{P}^1\mathbb{C}$, we have coordinates

$$(x_0 \colon x_1) \;=\; \begin{cases} (x_0/x_1, 1) & x_1 \neq 0 \\ (1, x_1/x_0) & x_0 \neq 0 \end{cases}$$

We can make the two cases into two copies of the complex numbers. Call them U_0 and U_1, where U_0 is the subset of $\mathbb{P}^1\mathbb{C}$ where $x_0 \neq 0$ and U_1 is the subset where $x_1 \neq 0$.

$$U_0 = \{(x_0 \colon x_1) \mid x_0 \neq 0\} = \{(1 \colon z) \mid z \in \mathbb{C}\}$$

$$U_1 = \{(x_0 \colon x_1) \mid x_1 \neq 0\} = \{(z \colon 1) \mid z \in \mathbb{C}\}$$

We glue along the function $f \colon U_0 \backslash \{(1 \colon 0)\} \to U_1$ given by

$$f(1, z) \;=\; (1/z, 1).$$

It is well-defined because we omit the only possible bad point $(1 \colon 0)$. Then, we identify a point in U_1 with its image under f. This gives us $\mathbb{CP}^1$ as two overlapping copies of $\mathbb{C}$, each missing a point that the other contains. This last point which is not in the domain of f is the point at infinity.

We can also do this in general, gluing copies of the affine plane to get projective space.

In $\mathbb{P}^2\mathbb{C}$, thinking about how to get all the lines through the origin in this way gives us a distinguished flag of subspaces

$$0 \subset \langle e_0 \rangle \subset \langle e_0, e_1 \rangle \subset \langle e_0, e_1, e_2 \rangle;$$

this is a point, and a line containing that point, and then a plane containing the line, and finally a 3-space.

We get this from first setting $z = 1$, and from this we get a bunch of lines but miss all those in the xy-plane because they don't pass through the plane $z = 1$ in $\mathbb{C}^3$. This corresponds to the subspace $\langle e_0, e_1, e_2 \rangle$; this is lines in $\langle e_0, e_1, e_2 \rangle$ yet not in $\langle e_0, e_1 \rangle$. Then by setting $y = 1$, we add all of the lines through the origin except the line $x = z = 0$. This corresponds to the portion of the flag $\langle e_0, e_1 \rangle$; this is lines in $\langle e_0, e_1 \rangle \backslash \langle e_0 \rangle$. Finally, we set $x = 1$ and get the final subspace of lines in $\langle e_0 \rangle$.

The generic behavior is

Lines	Subspace	Coordinates
$\ell \notin \langle e_0, e_1 \rangle$	$\mathbb{C}^2$	$z = 1$
$\ell \in \langle e_0, e_1 \rangle;\;\; \ell \notin \langle e_0 \rangle$	$\mathbb{C}$	$y = 1$
$\ell \in \langle e_0 \rangle$	$\mathbb{C}$	$x = 1$

Definition 65. A **flag** is a sequence of vector spaces increasing in dimension

$$V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n,$$

and a **full flag** for a vector space V is a sequence of spaces that increase in dimension by one each time, and end up filling the whole space. For instance, if V has dimension n, then a full flag is

$$0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = V,$$

and V_i has dimension i.

A generalization of flags are the Grassmanians.

Definition 66. The **Grassmanian** $G(m,n)$ is the set of subspaces of $\mathbb{C}^n$ of dimension m.

$$G(m,n) \;=\; \{W \subseteq \mathbb{C}^n \mid \dim W = m\}.$$

Much like projective space, we want to give coordinates to points in the Grassmanian. We will do this by describing the points with m coordinates. If the coordinates of $\mathbb{C}^n$ are $x_1, \dots, x_n$, then a point is described by

$$(x_{i_1}, \dots, x_{i_m})$$

for an index set $I = (1 \leqslant i_1 < i_2 < \dots < i_m \leqslant n)$, each $1 \leqslant i_j \leqslant n$. In projective space we set one coordinate to 1, but in a Grassmanian we set m coordinates to 1.

There is a much more sane way to describe these coordinates, much like the ratios $(x_0 \colon \dots \colon x_n)$ in $\mathbb{P}^n\mathbb{C}$.

Definition 67. The **Grassmann coordinates** for a point in $G(m,n)$ is an $m \times n$ matrix

$$X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots x_{mn} & \end{bmatrix}$$

where the basis of a space W is the rows of this matrix. We consider such matrices as equivalent up left-multiplication by $\mathrm{GL}(m,\mathbb{C})$, which only changes the basis but doesn't leave the space W.

Definition 68. From the Grassmann coordinates, we can define the **Plücker Coordinates** as points in $P^{\binom{n}{m}-1}$ by taking determinants of minors

$$\left(\det(M_{I_1}) \colon \det(M_{I_2}) \colon \dots \colon \det(M_{I_{\binom{n}{m}}})\right),$$

where $I_i \subset \{1, \dots, n\}$ are all the distinct subsets of $\{1, \dots, n\}$ and M_I is the minor of the Grassmann coordinate matrix X indexed by I.

This also gives us the **Plücker embedding** into $\mathbb{P}^{\binom{n}{m}-1}$.

Example 69. For $G(2,4)$, the Plücker

$$\begin{bmatrix} x_{11} & x_{12} & 1 & 0 \\ x_{21} & x_{22} & 0 & 1 \end{bmatrix} \xrightarrow{\text{Plücker}} (x_{11}x_{22} - x_{12}x_{21} \colon -x_{21} \colon x_{11} \colon x_{12} \colon 1)$$

Because all of the Grassmann coordinates appear, this map is injective!

Lecture 9 — July 10, 2015

The Grassmannian – Done Right

Recall that

$$\mathrm{Gr}(m,n) \;=\; \{W \subset \mathbb{C}^n \mid \dim(W) = n\}.$$

We also have a Plücker embedding

$$\mathrm{Gr}(m,n) \hookrightarrow \mathbb{P}^{\binom{n}{m}-1}\mathbb{C}.$$

Today we're going to learn about the Schubert cells which stratify the Grassmannian. The Schubert cells are indexed by Young diagrams in a box.

Definition 70. A **Young diagram in a box** is

$$\lambda = \{n - m \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m \geqslant 0\}$$

For one of these Young diagrams in a box we have m rows and $n-m$ columns, and the diagrams look like

$$m \text{ rows} \left\{ \underbrace{\begin{array}{|c|c|c|c|} \hline X & X & X & \\ \hline X & X & X & \\ \hline X & X & & \\ \hline & & & \\ \hline \end{array}}_{n-m \text{ columns}} \right.$$

Problem 71. There are $\binom{n}{m}$ such λ.

We also say that

$$|\lambda| = \lambda_1 + \ldots + \lambda_m \leqslant m(n-m),$$

and have a compliment of a Young diagram in a box given by

$$\lambda^c = \{n - m \geqslant (n-m) - \lambda_m \geqslant \ldots \geqslant (n-m) \geqslant \lambda_1 \geqslant 0\}$$

We have a partial order on these Young diagrams in boxes given by $\lambda \leqslant \mu$ if and only if $\mu = \lambda + \text{ some squares }$.

To begin with these Schubert cells, we fix a full flag

$$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \langle e_1, \ldots, e_{n-1} \rangle \subset \mathbb{C}^n$$

For ease of notation, call it

$$0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = \mathbb{C}^n$$

What is the generic behavior of a $W \subset \mathbb{C}^n$ of dimension m with respect to this flag? We might have

$$\dim(W \cap V_1) = 0, \ldots, \dim(W \cap V_{n-m}) = 0, \dim(W \cap V_{n-m}+1) = 1, \ldots \dim(W \cap V_n) = m.$$

Generically, we get something like

$$\dim(W \cap V_{n-m+i}) = i,$$

and the **dimension sequence** of $W \cap V_i$ looks like

$$(\underbrace{0, 0, \ldots, 0}_{n-m}, 1, 2, 3, \ldots, m).$$

We can also look at the **jump sequence** of successive differences in dimension. For the above, this is

$$(\underbrace{0, 0, \ldots, 0}_{n-m}, 1, 1, 1, \ldots, 1).$$

By some linear algebra shenanigans, observe that the vectors of V which lie in $W \cap V_{n-m+1}$ look like

$$(\underbrace{*, *, \ldots, *}_{n-m}, 1, 0, 0 \ldots, 0)$$

with respect to the basis that comes from our flag. And likewise, we can force the vectors of V which lie in $W \cap V_{n-m+2}$ to look like

$$(\underbrace{*, *, \ldots, *}_{n-m}, 0, 1, 0 \ldots, 0).$$

Putting these vectors into a matrix, we get the Grassmann coordinates! A point in W looks like

$$\begin{bmatrix} * & \ldots & * & 1 & 0 & \ldots & 0 \\ * & \ldots & * & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & \ldots & * & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

Now what about a non-generic subspace W? This means that the sequence of jumps looks like

$$(0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, \ldots).$$

If W has dimension m, there are m-many ones in the above sequence – the dimension of intersections increases by 1 a total of m times.

Example 72. What about Gr$(2, 4)$? We calculate with respect to the standard flag

$$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset \mathbb{C}^4.$$

The jump sequence $(0, 0, 1, 1)$ is the most generic, and corresponds to the matrix

$$\begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix}.$$

The jump sequence $(0, 1, 0, 1)$ is not quite so generic, and corresponds to the matrix

$$\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix}.$$

The jump sequences are themselves indexed by Young diagrams, and there is a partial order to them

$$\begin{tikzcd}
 & & 1001 \arrow[dr] & & \\
0011 \arrow[r] & 0101 \arrow[ur] \arrow[dr] & & 1010 \arrow[r] & 1100 \\
 & & 0110 \arrow[ur] & &
\end{tikzcd}$$

Look familiar? The jump sequences correspond to Young diagrams as follows. The most generic sequence corresponds to the empty Young diagram. Then for any other jump sequence, observe how many positions the left 1 had to move to get to it's position, and that is the first row of a Young diagram. The number of positions that the rightmost 1 moved to it's new position is the next row, and so on. The biggest Young diagrams correspond to the smallest subspaces in the stratification of the Grassmannian. So for example,

Example 73. The jump sequence $(0, 0, 1, 1)$ corresponds to an empty Young diagram. To get the jump sequence $(0, 1, 0, 1)$, the leftmost 1 moved one position, so we get

$$(0, 1, 0, 1) \rightsquigarrow \ydiagram{1}$$

To get the jump sequence $(1, 0, 0, 1)$, we moved the leftmost 1 by two positions, so

$$(1, 0, 0, 1) \rightsquigarrow \ydiagram{2}$$

The rest of them are as follows

$$(0, 1, 1, 0) \rightsquigarrow \ydiagram{1,1}$$

$$(1, 0, 1, 0) \rightsquigarrow \ydiagram{2,1}$$

$$(1, 1, 0, 0) \rightsquigarrow \ydiagram{2,2}$$

Another example. The jump sequence $(1, 0, 1, 0)$ corresponds to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix}.$$

Definition 74. Given a full flag $\mathcal{F}$ and a Young diagram in a box λ, the **Schubert cell** is

$$S_\lambda^{\mathcal{F}} = \left\{ \begin{matrix} s \text{ subspaces } W \subset \mathbb{C}^n \text{ corresponding} \\ \text{to } \lambda \text{ with respect to the flag } \mathcal{F} \end{matrix} \right\}$$

We won't prove this following result, although we will give a plausibility argument. The closure of S_λ in the Grassmannian is

$$\overline{S_\lambda} = \bigcup_{\lambda \leqslant \mu} S_\mu$$

What happens if we reverse the flag? If we take the flag

$$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \langle e_1, \ldots, e_{n-1} \rangle \subset \mathbb{C}^n$$

and substitute the flag

$$0 \subset \langle e_n \rangle \subset \langle e_n, e_{n-1} \rangle \subset \ldots \subset \langle e_n, \ldots, e_1 \rangle \subset \mathbb{C}^n,$$

what happens to the Schubert cells? You switch the rows and reverse them. Let's illustrate by example:

Example 75. In $\mathrm{Gr}(2,4)$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

These complimentary Schubert cells intersect in a point! This is Poincaré duality.

Lecture 10 — July 13, 2015

This week, we'll talk about some algebraic geometry related to the Lie groups that were introduced last week. The Lie groups we care about are

$$\mathrm{SL}(n,\mathbb{C}) \supset \mathrm{SU}(n)$$

$$\mathrm{SO}(n,\mathbb{C}) \supset \mathrm{SO}(n,\mathbb{R})$$

Of these, $\mathrm{SL}(n,\mathbb{C})$ is "algebraic," $\mathrm{SU}(n), \mathrm{SO}(n,\mathbb{C})$ and $\mathrm{SO}(n,\mathbb{R})$ are compact. We also care the spaces

$$\mathbb{P}^n\mathbb{C}, \;\; \mathrm{Gr}(m,n), \;\; \mathrm{Fl}(n),$$

which are both algebraic and compact. The rightmost one above is the flag variety.

As Tom Garrity said in his first lecture, "functions describe the world." This is very true of algebraic geometry, where everything eventually gets replaced with functions rather than objects.

For instance, we can realize this slogan in the group algebra $\mathbb{C}[G]$. All representations of a finite group G are contained inside $\mathbb{C}[G]$, which we can think of as functions on G, with multiplication convolution.

$$\mathbb{C}[G] = \{f \colon G \to \mathbb{C}\} = \left\{ \sum_{g \in G} f(g) g \right\}.$$

Let M be a manifold. Because M is a topological space, we know what it means for function $f: M \to \mathbb{R}$ to be continuous. But what does it mean for f to be differentiable? We can define this in terms of the functions on M.

Remark 76. From an action $G \circlearrowright M$ of a group on a manifold, we get an action of G on the functions $f: M \to \mathbb{R}$ defined by

$$g \cdot f(x) = f(g^{-1}x).$$

Hierarchy of Functions

There is a hierarchy of functions $f: \mathbb{R} \to \mathbb{R}$

(1) **continuous** functions, for which $\lim_{h \to 0} |f(x_0 + h) - f(x_0)| = 0$;

(2) **differentiable** functions, for which

$$\lim_{h \to 0} \frac{|f(x_0 + h) - y_1 h - f(x_0)|}{|h|} = 0,$$

for some y_1, which we call $f'(x_0)$;

(3) **smooth** (infinitely differentiable) functions, for which $f'(x)$, $f''(x)$, etc. are differentiable as functions of x.

Smooth functions have Taylor series,

$$f(x_0 + h) \approx y_0 + y_1 h + y_2 \frac{h^2}{2} + \ldots$$

where $y_i = f^{(i)}(x_0)$. Of course, there are functions like $f(x) = e^{-1/x^2}$ that don't agree with their Taylor series, so we add a few new class to our hierarchy.

(4) **Analytic functions**, which agree with their taylor series everywhere;

(5) **rational functions**, which are ratios of polynomials $f(x) = p(x)/q(x)$ for $p, q \in \mathbb{R}[x]$.

Over the complex numbers, the hierarchy goes like this.

(1) **Continuous** functions;

(2) **holomorphic** functions (which are complex-differentiable and smooth and analytic)

$$\lim_{h \to 0} \frac{|f(z_0 + h) - y_1 z_0 - y_0|}{|h|} = 0,$$

where we set $f'(z_0) = y_1$ and set $y_0 = f(z_0)$.

So over the complex numbers, analytic, differentiable and smooth are the same notions. Holomorphic functions satisfy the **maximum principle**.

Proposition 77 (Maximum principle)**.** If $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic in a connected open set $U \subseteq \mathbb{C}$ and $z_0 \in U$ such that $|f(z_0)| \geqslant |f(z)|$ for all z in a neighborhood of z_0, then f is constant on D.

We can also talk about the hierarchy for functions in several variables $f\colon \mathbb{R}^n \to \mathbb{R}$.

Definition 78. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at a point $\vec{x}_0 \in \mathbb{R}^n$ if

$$\lim_{\vec{h}\to 0} \frac{|f(\vec{x}_0 + \vec{h}) - \vec{y}_1 \cdot \vec{h} - \vec{y}_0|}{|\vec{h}|} = 0.$$

The derivative is the gradient

$$\nabla f(\vec{x}_0) = \vec{y}_1 = \left(\frac{\partial f}{\partial x_1}(\vec{x}_0), \ldots, \frac{\partial f}{\partial x_n}(\vec{x}_0) \right).$$

We can also talk about infinitely differentiable and analytic functions.

What about multivariable functions over $\mathbb{C}$? A function $f\colon \mathbb{C}^n \to \mathbb{C}$ is holomorphic if and only if it is holomorphic in each variable separately. This fact is **Hartog's theorem**.

The **Implicit function theorem** tells us that if we have a vector-valued function $\vec{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ and write $\vec{f} = (f_1, \ldots, f_m)$, we can rewrite some portion of the image of $\vec{f}$ as the zero set of the functions $f_1, \ldots, f_m$. We can reinterpret this theorem as follows.

Theorem 79 (Implicit Function Theorem)**.** If $\vec{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ or $\vec{f}\colon \mathbb{C}^n \to \mathbb{C}^m$, then

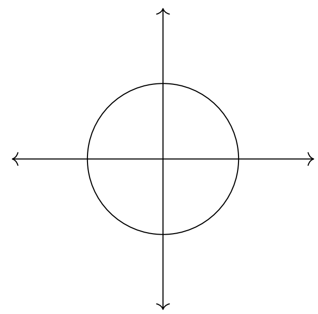
$$\mathbb{V}(\vec{f}) = \{\vec{x} \in \mathbb{C}^m \mid \vec{f}(\vec{x}) = 0\}$$

is an analytic manifold near any point $\vec{x}_0$ for which

$$\lim_{\vec{h}\to 0} \frac{|f(\vec{x}_0 + \vec{h}) - \mathrm{Jac}(\vec{f})(\vec{x}_0)\vec{h} - \vec{f}(\vec{x}_0)|}{|\vec{h}|} = 0,$$

and the Jacobian $\mathrm{Jac}(\vec{f})$ has full rank near $\vec{x}_0$.

Example 80. The circle $f(x,y) = x^2 + y^2 - 1$ has gradient $\nabla f = (2x, 2y)$, which has rank 1 at the point $(0,1)$. So we can rewrite the y-coordinate in terms of x near $(0,1)$, as $y = \sqrt{1-x^2}$.



Example 81. We can use this to analyze the Lie groups from before. Is SL$(n,\mathbb{C})$ a manifold? Well, elements X of SL$(n,\mathbb{C})$ are zeros of the polynomial $f(X) = \det(X) - 1$. So yes, it is a manifold. The partial derivatives of this equation can be found by expanding the determinant along minors M_{ij} corresponding to removing the i-th row and j-th column.

$$\frac{\partial f}{\partial x_{ij}} = (-1)^{i+j}\det(M_{ij})$$

Example 82. What about $O(n,\mathbb{C})$? This seems like a difficult Lie group to analyze, because the polynomial equation is a bit more complicated here. Orthogonal matrices A are zeros of $f(A) = A^TA - I$. We can expand this around the identity for $A = I + \varepsilon B$, where ε is small. On one hand,

$$f(I+\varepsilon B) = f(I) + \text{Jac}(f)\varepsilon B + O(\varepsilon^2)$$

but on the other hand,

$$f(I+\varepsilon B) = (I+\varepsilon B)^T(I+\varepsilon B) - I = 0 + \varepsilon(B^T + B) + \varepsilon^2 B^TB + O(\varepsilon^3),$$

so we conclude that Jac(f) is the map $B \mapsto B + B^T$.

This shows that $O(n,\mathbb{C})$ is a manifold, but we also know now how to compute the tangent space at the identity: it's the kernel of the Jacobian operator.

$$\ker(\text{Jac}(f)) = \{B \mid B + B^T = 0\} = \{\text{Skew-symmetric matrices}\}.$$

Example 83. We can analyze SL$(n,\mathbb{C})$ in the same way. Our polynomial is $f(A) = \det(A) - 1$, and near the identity

$$f(I+\varepsilon B) \approx \det(I+\varepsilon B) - 1 + \varepsilon\,\mathbf{tr}(B) + O(\varepsilon^2),$$

so Jac(f) = tr, and the tangent space is

$$\mathfrak{sl}(n,\mathbb{C}) = \{\text{ traceless } n\times n \text{ matrices }\}$$

Lecture 11

July 14, 2015

Tensor Madness

Between Tom Garrity and Sean McAfee, we've already covered most of what I want to say about tensors. So we'll fly through it really quickly.

Definition 84. Let V be a finite dimensional vector space over $\mathbb{C}$. The **dual space** V^* is

$$V^* = \{ \text{ linear } f \colon V \to \mathbb{C}\}.$$

If $G \circlearrowright V$, then we also get an action $G \circlearrowright V^*$ by $g \cdot f(x) = f(g^{-1} \cdot x)$.

Definition 85. The **tensor algebra** of a finite-dimensional vector space V is the space

$$TV \ = \ \bigoplus_{n=0}^{\infty} V^{\otimes n} \ = \ \mathbb{C} \oplus V \oplus (V \otimes V) \oplus \dots$$

If $G \circlearrowright V$, then we also have an action $G \circlearrowright TV$ by acting on the components of a tensor individually. For the d-th tensor power $V^{\otimes d}$, we have a right-action of S_d by permuting the coordinates. The action of S_d commutes with the action of G.

Joke 86. "I refuse to do it any other way, even if Serre tells me I have to. I know Serre. I don't know if he knows me, but ..." - Aaron Bertram.

We can break up $V^{\otimes d}$ into irreducible components under the action of S_d. One component is the symmetric tensors, $\text{Sym}^d V$, and another component is $\bigwedge^d V$, the antisymmetric tensors. We have a decomposition

$$V^{\otimes d} = S^d V \oplus \bigwedge^d V \oplus (\text{ other stuff }).$$

Example 87. For example, $\text{Sym}^2 V$ is the tensors that look like $x \otimes x$, $y \otimes y$, and $x \otimes y + y \otimes x$, and $\bigwedge^2 V$ is the tensors that looks like $x \otimes y - y \otimes x$.

Definition 88. The **symmetric algebra** over a finite-dimensional vector space V is

$$\text{Sym}(V) = \bigoplus_{n=0}^{\infty} \text{Sym}^n V = \mathbb{C} \oplus V \oplus \text{Sym}^2 V \oplus \dots$$

This is just the polynomial ring $\mathbb{C}[x_1, \dots, x_{\dim V}]$.

Definition 89. The **exterior algebra** over a finite-dimensional vector space V is

$$\bigwedge V = \bigoplus_{n=0}^{\infty} \bigwedge^n V = \mathbb{C} \oplus V \oplus \bigwedge^2 V \oplus \dots$$

Notice that for $n \geqslant \dim V$, $\bigwedge^n V = 0$. Hence, the exterior algebra remains finite dimensional even though the symmetric algebra is not finite dimensional. Additionally, $\det \in \bigwedge^n V^*$.

Definition 90. The **super polynomial algebra** over V is

$$\bigoplus_{d \geqslant 2}^{\infty} \left(\mathrm{Sym}^d V \oplus \bigwedge^d V \right)$$

The super-polynomial algebra is useful in physics.

We can also relate this to invariant theory once we choose a basis $e_1, \ldots, e_n$ for V, through the symmetric algebra.

$$\begin{aligned} \mathrm{Sym}(V) &= \mathbb{C}[x_1, \ldots, x_n] \\ &= \mathrm{Sym} \oplus \text{Anti-Sym} \oplus \ldots \\ &= \mathbb{C}[e_1, \ldots, e_n] \oplus \Delta \cdot \mathbb{C}[e_1, \ldots, e_n] \oplus \ldots, \end{aligned}$$

where Δ is the polynomial that's invariant under the action of A_n,

$$\Delta = \prod_{i<j} (x_i - x_j).$$

Definition 91. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, let

$$P_\lambda = x_1^{\lambda_1 + (n-1)} \cdots x_{n-1}^{\lambda_{n-1}+1} x_n^{\lambda_n} + \ldots + \mathrm{sgn}(\sigma) x_{\sigma(1)}^{\lambda_1 + (n-1)} \cdots x_{\sigma(n-1)}^{\lambda_{n-1}+1} x_{\sigma(n)}^{\lambda_n},$$

where the sum is over all permutations $\sigma \in S_n$. Then the **Schur Polynomial** is P_λ / P_0, and $P_0 = \Delta$.

Algebraic Geometry

To motivate this, let's think about the continuous characters of $U(1)$. These are maps $\chi\colon U(1) \to \mathbb{C}^\times$, and $\chi(e^{i\theta} e^{i\psi}) = \chi(e^{i\theta}) \chi(e^{i\psi})$. This tells us that $\chi(e^{i\theta})$ must lie on the unit circle. Moreover, χ maps roots of unity to roots of unity. From this, we can conclude that the only continuous characters of $U(1)$ are maps $\chi_k(g) = g^k$ for $k \in \mathbb{Z}$.

So all of the representations of $U(1)$ lie inside

$$\bigoplus_{k \in \mathbb{Z}} \mathbb{C} e^{ik\theta}$$

We can account for these characters in another way, by considering $U(1) \subset \mathbb{C}^\times = \mathrm{GL}(1, \mathbb{C})$. What are the algebraic functions on $\mathbb{C}^\times = \mathrm{GL}(1, \mathbb{C})$? These are the Laurent polynomials, because we don't have to worry about plugging in zero in the denominator.

$$\mathbb{C}[z, z^{-1}] = \bigoplus_{k \in \mathbb{Z}} \mathbb{C} z^k$$

This looks a lot like what we found for $U(1)$!

From this toy example, it makes sense to think about the algebraic functions on other Lie Groups that are more complicated, like GL$(n, \mathbb{C})$, Sp$(2n, \mathbb{C})$, SL$(n, \mathbb{C})$, or SO$(n, \mathbb{C})$. Given an algebraic group G, we want to write the coordinate ring of functions on the variety G as a sum of the irreps of the group G. This is what we did for $U(1)$ above.

We'll be able to forget the group entirely, and focus on just the functions on the group.

Example 92. For the variety SL$(n, \mathbb{C})$, the coordinate ring is

$$\mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}] \big/ \langle \det(x_{ij}) - 1 \rangle$$

When $n = 2$, we get

$$\mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}] \big/ \langle x_{11}x_{22} - x_{21}x_{12} - 1 \rangle.$$

The coordinate ring of GL$(n, \mathbb{C})$ is given by

$$\mathbb{C}[\{x_{ij}\}, y] \big/ \langle y \det(x_{ij}) - 1 \rangle = \mathbb{C}\big[\{x_{ij}\}, \det^{-1}\big]$$

We can't write down a polynomial equation for GL$(n, \mathbb{C})$ in n^2 dimensions, because it's the compliment of the zeros of a polynomial equation. Namely, it's the compliment of $\det(x_{ij}) = 0$. But we can do it in $n^2 + 1$ dimensions as above.

Lecture 12 July 15, 2015

Today we're going to talk about the category of **affine varieties**. In particular, we're going to focus on the ones that are **finite-type** over $\mathbb{C}$.

Let's start with an analogy to differential geometry. Let M be a topological manifold. We have a **sheaf** $\mathcal{C}_M$ of continuous functions on M. For any open set $U \subseteq M$,

$$\mathcal{C}_M(U) = \{ \text{ continuous } f\colon U \to \mathbb{R}\}.$$

There is another example of a sheaf on M which we call $\mathbb{Z}_M$. This is the sheaf of integer-valued continuous functions on any open set,

$$\mathbb{Z}_M(U) = \{ \text{ continuous } f\colon U \to \mathbb{Z}\}.$$

Although this looks rather boring, (these $f\colon U \to Z$ are constant on connected components of U) it contains a lot of information. From it, we can find all of the **Betti Numbers** of the manifold.

If M is instead a smooth manifold, there is another sheaf contained within $\mathcal{C}_M$ called $\mathcal{C}^\infty_M$. This is defined by

$$\mathcal{C}^\infty_M(U) = \{ \text{ smooth } f\colon U \to \mathbb{R}\}.$$

And there's even a finer sheaf, which we call $\mathcal{O}_M$,

$$\mathcal{O}_M(U) = \{ \text{ analytic } f\colon U \to \mathbb{R}\}.$$

To summarize, we have the following inclusions

$$\mathcal{C}_M \supset \mathcal{C}_M^\infty \supset \mathcal{O}_M \supset \mathbb{Z}_M.$$

In all of these cases, we have a topological space M with a sheaf of functions on it. This is what the picture will look like in algebraic geometry, but the topological space will not be Hausdorff (among other failings).

Definition 93. A **finitely-generated $\mathbb{C}$-algebra** is a $\mathbb{C}$-vector space A equipped with a multiplication operation of vectors such that there are $a_1, \ldots, a_n \in A$ so that the map $\mathbb{C}[x_1, \ldots, x_n] \to A$ defined by $x_i \mapsto a_i$ is surjective.

In analogy to the differential geometry setting, we will have a topological space Spec(A) and a sheaf of **regular functions** $\mathcal{O}_X(U)$.

There are two theorems of Hilbert that are critical to get us going in algebraic geometry.

Theorem 94 (Hilbert's Basis Theorem)**.** Any ideal $I \lhd \mathbb{C}[x_1, \ldots, x_n]$ is finitely generated.

Theorem 95 (Nullstellensatz)**.** The maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$ are the kernels of evaluation maps at points $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$, i.e. the kernels of maps $\text{ev}_{\vec{a}}\colon \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}$ defined by $\text{ev}_{\vec{a}}(f) = f(\vec{a})$.

$$\ker(\text{ev}_{\vec{a}}) = \langle x_1 - a_1, \ldots, x_n - a_n\rangle$$

Definition 96. The **max-spectrum** of a ring A is the set of all maximal ideals of A.

$$\text{maxSpec}(R) = \{ \text{ maximal ideals of } A\}$$

Remark 97. Let A be a $\mathbb{C}$-algebra with generators $a_1, \ldots, a_n$ and map $\phi\colon \mathbb{C}[x_1, \ldots, x_n] \to A$. Then if $\ker\phi = \langle f_1, \ldots, f_m\rangle$, we know that

$$A \cong {}^{C[x_1,\ldots,x_n]}\big/_{\ker\phi} = {}^{\mathbb{C}[x_1,\ldots,x_n]}\big/_{\langle f_1,\ldots,f_m\rangle},$$

so ideals of A are in bijection with ideals of $\mathbb{C}[x_1, \ldots, x_n]$ containing $f_1, \ldots, f_m$. We identify $\mathbb{C}[x_1, \ldots, x_n]/\ker\phi$ with the variety $\mathbb{V}(\ker\phi)$ corresponding to the ideal $\ker\phi$. In particular, the set of these f_i can be thought of as polynomials defining the variety $\mathbb{V}(\ker\phi)$, which has coordinate ring A. This gives a bijection

$$\text{maxSpec}(A) \longleftrightarrow \mathbb{V}(f_1, \ldots, f_m).$$

What's the topology on our space?

Definition 98. The **Zariski topology** on maxSpec(A) is the topology with the closed sets

$$Z(I) = \{\mathfrak{m} \mid I \subset \mathfrak{m}\} \subset \text{maxSpec}(A).$$

Joke 99. The Zariski topology is like Texas. There is nothing small in the Zariski topology. Every open set is dense! All open sets meet each other.

Problem 100. Show that the Zariski topology forms a topology on maxSpec(A).

In the Zariski topology, we have a basis of open sets given by

$$U_f = \text{maxSpec}(A) \backslash \mathbb{Z}(\langle f \rangle) = \{\mathfrak{m} \mid f \notin \mathfrak{m}\}$$

For an ideal $I = \langle f_1, \ldots, f_m \rangle$, we have an open set

$$\bigcup_{i=1}^{m} U_{f_i} = \text{maxSpec}(A) \backslash \mathbb{Z}(I).$$

An application of the basis theorem says that if we have an increasing chain of ideals,

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_n \subseteq \ldots$$

it must eventually stabilize, that is, there is some n such that for all $m \geqslant n$, $I_m = I_n$. This tells us that we cannot have infinitely decreasing chains of closed sets. They must stabilize

$$Z(I_1) \supseteq Z(I_2) \supseteq \ldots \supseteq Z(I_n) = Z(I_{n+1}) = \ldots = \bigcap_i Z(I_i).$$

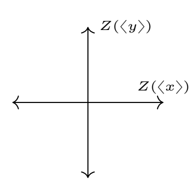
Definition 101. (a) A topological space in which every nested sequence of closed sets stabilizes is called **Noetherian**;

(b) a Noetherian topological space X is **irreducible** if for any closed $Z_1, Z_2 \subseteq X$ such that $X = Z_1 \cup Z_2$, we must have $X = Z_1$ or $X = Z_2$.

Fact 102. The Zariski topology on an integral domain is necessarily irreducible.

Example 103. Consider $A = \mathbb{C}[x, y]/\langle xy \rangle$. This is not a domain, because $x \neq 0$, $y \neq 0$, yet $xy = 0$. The maximal spectrum of A is the two coordinate axes:

$$Z(\langle xy \rangle) = Z(\langle x \rangle) \cup Z(\langle y \rangle).$$



The second thing that we needed in the differential geometry version of this was a sheaf of functions on the manifold. So here we're going to define the sheaf of regular functions on a domain A.

For any domain A, we can form the **field of fractions**

$$\mathbb{C}(A) = \left\{ \frac{a_1}{a_2} \mid a_1, a_2 \in A, a_2 \neq 0 \right\}$$

Regard elements $a \in A$ as functions on maxSpec(A), defined by

$$a(\mathfrak{m}) = a(\mathrm{mod}\ \mathfrak{m}) \in \mathbb{C}.$$

The reason that these elements live in $\mathbb{C}$ is because A is the coordinate ring of some variety, and so the elements of A are functions on that variety. Hence, we can imagine taking a polynomial f modulo a maximal ideal $\mathfrak{m}$ as evaluation of f at the point corresponding to $\mathfrak{m}$.

We call elements $a \in A$ **regular functions** on maxSpec(A). Any element $\phi = \frac{a}{b} \in \mathbb{C}(A)$ also defines a function on maxSpec(A) as

$$\phi(\mathfrak{m}) = \frac{a}{b}(\mathfrak{m}) = \frac{a(\mathrm{mod}\ \mathfrak{m})}{b(\mathrm{mod}\ \mathfrak{m})},$$

wherever $b \pmod{\mathfrak{m}} \neq 0$. For each $\phi \in \mathbb{C}(A)$, we write the **domain of of definition** as those points where $b \pmod{\mathfrak{m}}$ is nonzero.

$$U_\phi = \left\{ \mathfrak{m} \in X \mid \phi = \frac{a}{b} \text{ with } b(\mathfrak{m}) \neq 0 \right\}.$$

Definition 104. The **sheaf of regular functions** on $X = \mathrm{maxSpec}(A)$ is the sheaf $\mathcal{O}_X$ defined by

$$\mathcal{O}_X(U) = \{ \phi \in \mathbb{C}(X) \mid U \subseteq U_\phi \}.$$

Fact 105. $\mathcal{O}_X(X) = A$.

A consequence of this fact is also that $\mathcal{O}_X(U_f) = A[f^{-1}]$.

We now have all the information we need to define the category of affine varieties! For any two $\mathbb{C}$-algebras A, B and a homomorphism $h \colon A \to B$ we have a continuous map $f \colon Y = \mathrm{maxSpec}(B) \to X = \mathrm{maxSpec}(A)$ that comes from h. This also gives a map

$$f^* \colon \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$$

for each $U \subset X = \mathrm{maxSpec}(A)$ open. These data give a morphism of affine varieties.

So this category is indeed quite familiar! It's the opposite of the category of finitely-generated $\mathbb{C}$-algebras.

Lecture 13 July 16, 2015

Yesterday we talked about affine varieties. I want to talk about projective varieties today, without getting too bogged down in the details. It's going to be hard to do, but ideally I will give you an idea of what projective algebraic geometry while flying overhead.

Euclidean Geometry

Local		Global
Open Balls	$\xrightarrow{\text{gluing}}$	topological space
		zero of systems of equations
		spaces of orbits for $G \circlearrowright M$

In Algebraic Geometry, we do something a bit strange. We take the zero sets of systems of equations, which is a global object in Euclidean space, and make these our local notions. We have no implicit function theorem here, so assigning global coordinates is more difficult. So instead larger objects are generally made by gluing local patches.

Algebraic Geometry

Local		Global
Affine Variety		Projective Varieties
Spec(A)	$\xrightarrow[\text{regular fns}]{\text{gluing via}}$	Proj(A)
Zariski Topology		Spaces of Orbits
$\mathcal{O}_X$		$\mathcal{O}_X$

As usual, we won't talk about $\mathbb{C}^\times$ acting on affine varieties but instead $\mathbb{C}^\times$ acting on the functions on an affine variety. We assume that every algebra is a domain.

Definition 106. A $\mathbb{C}^\times$**-linearlized** $\mathbb{C}$**-algebra** A is an action of $\mathbb{C}^\times$ on A (written $\lambda(x)$ for $\lambda \in \mathbb{C}^\times$) such that

(a) the $\mathbb{C}^\times$-invariants $\{x \in A \mid \lambda(x) = x\}$ in A are just $\mathbb{C}$;

(b) the standard character space $A_1 = \{x \in A \mid \lambda(x) = \lambda x\}$ (the space on which $\mathbb{C}^\times$ acts) generates A as a $\mathbb{C}$-algebra, and in addition A_1 is finite dimensional.

Since every representation of $\mathbb{C}^\times$ is multiplication by λ^k for some k, then we break up A as the spaces invariant under each representation as

$$A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \dots$$

where $A_i = \{x \in A \mid \lambda(x) = \lambda^k x\}$.

Example 107. Let $A = \mathbb{C}[x_1, \ldots, x_n]$, and note that A decomposes as

$$\mathbb{C}[x_1, \ldots, x_n] = \bigoplus_{m=0}^{\infty} \mathbb{C}[x_1, \ldots, x_n]_{\deg = m}$$

Then the action of $\mathbb{C}^\times$ on the generators $\lambda(x_i) = \lambda x_i$ is ordinary multiplication. The space $A_1 = \mathbb{C}[x_1, \ldots, x_n]_{\deg=1}$ is the space corresponding to $k = 1$, and generates A (because it includes all x_i).

Definition 108. The **irrelevant ideal** of a graded ring

$$A = \bigoplus_{m=0}^{\infty} A_i$$

is the ideal

$$\mathfrak{m} = \bigoplus_{m>0}^{\infty} A_i.$$

This is the maximal $\mathbb{C}^\times$ invariant ideal.

Definition 109.

$$\text{maxProj}(A) = \{ \text{ maximal } \mathbb{C}^\times\text{-invariant homogeneous ideals} \mathfrak{m}_x \subsetneq \mathfrak{m}\}$$

Similarly, we can put a Zariski topology on maxProj(A) by declaring the closed sets to be

$$Z(I) = \{\mathfrak{m}_X \mid I \subset \mathfrak{m}_x\}$$

for $\mathbb{C}^\times$ invariant ideals I.

Example 110. If $A = \mathbb{C}[x_1, \ldots, x_n]$, we know that Spec$(A)$ is geometrically $\mathbb{C}^n$. The points in the space maxProj(A) correspond to the lines through the origin in $\mathbb{C}^n$, that is, $\mathfrak{m}_x = I(\ell)$ for some line ℓ through the origin. For a point $\vec{a} = (a_1, \ldots, a_n)$, the ideal correpsonding to the line through the origin and $\vec{a}$ is

$$\mathfrak{m}_{[\vec{a}]} = \langle x_i a_j - a_i x_j \mid i, j = 1, \ldots, n\rangle.$$

The irrelevant ideal corresponds to the origin, which doesn't define a line in projective space. We throw it out much like we throw out the origin when defining projective space.

So now we need to know the functions on this space. We define the rational functions on maxProj(A) by ratios of homogeneous polynomials of the same degree.

$$\mathbb{C}(A)_0 = \left\{ \frac{F}{G} \,\middle|\, F, G \in A_m \text{ for some } m \right\}.$$

These indeed define functions on the space, but I leave that as an exercise.

Problem 111. Show that $\phi(\mathfrak{m}_{[\vec{p}]}) = \phi(\text{mod } \mathfrak{m}_{[\vec{p}]})$ makes sense as a complex number.

As before, let U_ϕ be the domain of ϕ. Define a sheaf on $X = \text{maxProj}(A)$ by

$$\mathcal{O}_X(U) = \{\phi \in \mathbb{C}(A)_0 \mid U \subseteq U_\phi\}.$$

We can also get at these spaces by gluing affine varieties together. How do the affine spaces sit inside this projective space? Let $a \in A_1$. Then the domain we want for the corresponding variety is

$$A[a^{-1}]_0 = \left\{ \frac{F}{a^d} \;\middle|\; F \in A_d \right\} \subset \mathbb{C}(A)_0.$$

Hartshorne calls this $A_{(a)}$. Note that $\mathbb{C}(A)_0$ is the same as the field of fractions of $\mathbb{C}(A[a^{-1}]_0)$.

The affine variety corresponding to this algebra is maxSpec$(A[a^{-1}]_0)$, which is in bijection with the open set maxProj$(A)\backslash Z(a)$. The bijection is given by

$$\langle a_i/a - p_i \rangle \longleftrightarrow \langle a - a_i p_i \rangle,$$

where a_i generate A_1 as a $\mathbb{C}$-vector space.

Given a $\mathbb{C}^\times$-linearized $\mathbb{C}$-algebra with A_1 generated by $a_1, \ldots, a_n$ as a $\mathbb{C}$-vector space, we have a surjective map $\mathbb{C}[A_1] \twoheadrightarrow A$. We can think of $\mathbb{C}[A_1]$ as lines through the origin in the dual vector-space of A_1, which we call $\mathbb{P}^1(A_1)$. This contains maxProj(A).

So what have we done? Given a $\mathbb{C}^\times$-lienarized $\mathbb{C}$-algebra A, we have produced a projective variety. This comes with a sheaf of functions on it that is locally affine. In affine space, it's easy to go back from the variety to the algebra A, but in projective space, this is not the case. Given a projective variety, there are infinitely many ways to make it from a $\mathbb{C}^\times$-linearized $\mathbb{C}$-algebra. So it's a difficult question to determine when projective varieties are isomorphic.

Example 112. Let d be a fixed integer, and let A be a $\mathbb{C}^\times$-linearized $\mathbb{C}$-algebra,

$$A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \ldots \oplus A_d \oplus \ldots$$

What happens when we take only multiples of d?

$$\mathbb{C} \oplus A_d \oplus A_{2d} \oplus \ldots$$

This carries an action of $\mathbb{C}^\times$. What's maxProj of this space? It's just the **Veronese embedding**. We have to rescale by setting $\mu = \lambda^d$.

If A and B are both $\mathbb{C}^\times$-linearized $\mathbb{C}$-algebras, what space does $A \otimes_{\mathbb{C}} B$ determine? It's the product of maxProj(A) and maxProj(B).

$$\mathbb{C} \oplus A_1 \otimes B_1 \oplus A_2 \otimes B_2 \oplus \ldots$$

We can get the Grassmannian Gr(m, n) in this way too. If $A = \mathbb{C}[\{x_{ij}\}]$ where $X = \{x_{ij}\}$ are the Grassmann coordinates, then we can think about the ring

$$\mathbb{C} \oplus \langle \det(X_{I_1}) \rangle \oplus \langle \det(X_{I_1}) \rangle \oplus \dots$$

where I_i are all size m subsets of $\{1, 2, \dots, n\}$.

Lecture 14 July 17, 2015

Today I want to talk about flag manifolds and (hopefully) introduce quiver varieties so you can see what the big kids in the other room (graduate student seminar) are talking about.

Let's talk about the characteristic polynomial. Tom already talked about these things, but I'm going to talk about them again. This feeds right into the representation theory and Borel-Weil.

The characteristic polynomial is an invariant of SL$(n, \mathbb{C})$ acting on $n \times n$ matrices $A = (a_{ij})$ by conjugation.

$$\det(xI - A) = x^n - \text{tr}(A)x^{n-1} + \dots + (-1)^n \det(A),$$

and the intermediate terms come from sums of principal minors of the matrix. If you've never worked it out before, you should do so. We can also write this as

$$\det(xI - A) = x^n - p_1(A)x^{n-1} + \dots + (-1)^n p_n(A),$$

for some polynomials p_i in the coordinates of A. This is the ring of invariants of this action.

$$\begin{array}{rcl} \mathbb{C}[y_1, \dots, y_n] & = & \mathbb{C}[\{a_{ij}\}]^{\mathrm{SL}(n,\mathbb{C})} \\ y_k & \mapsto & p_k(a_{ij}) \end{array}$$

Using our algebraic geometry technology, we get a map on the spectra of the rings

$$\begin{array}{rcl} q\colon \text{maxSpec}(\mathbb{C}[\{a_{ij}\}]) & \to & \text{maxSpec}(\mathbb{C}[y_1, \dots, y_n]) \\ \langle x_{ij} - a_{ij} \rangle & \mapsto & \langle y_k - p_k(a_{ij}\rangle \end{array}$$

But we can think of this as sending a point (maximal ideal) associated to the matrix A to a point in $\mathbb{C}^n$ given by

$$A \quad \mapsto \quad (\text{tr}(A) = p_1(A), \dots, p_n(A) = \det(A))$$

Obeserve that q is onto: given a vector in $\mathbb{C}^n$ we can always cook up a matrix in SL$(n, \mathbb{C})$ whose coefficients of the characteristic polynomial are given by these coordinates.

Definition 113. The set of matrices which are upper triangular with 1′s on the diagonal is

$$\mathcal{N} = \left\{ \text{upper triangular matrices with } 1 \text{ on diagonals} \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}$$

Each of these is has the identity matrix in the closure of its orbit under conjugation by $\mathrm{SL}(n, \mathbb{C})$.

Example 114.

$$\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & a \end{bmatrix} \begin{bmatrix} t^{-1} & 0 \\ 0 & t \end{bmatrix} = \begin{bmatrix} 1 & at^2 \\ 0 & 1 \end{bmatrix} \xrightarrow{t \to 0} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let's try to tie all of this back to representation theory. Given a finite dimensional representationn of $\mathrm{SL}(n, \mathbb{C})$,

$$\rho \colon \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(V),$$

the character only depends on the conjugacy class of an element of $g \in \mathrm{SL}(n, \mathbb{C})$. If $g_t \in \mathcal{N}$, then

$$\mathrm{tr}(\rho(g_t)) = \mathrm{tr}(\rho(\mathrm{id})).$$

Since characters distinguish representations, and we can't tell the difference between conjugation by $\mathcal{N}$ and conjugation by the identity, then all of the irreducible representations of $\mathrm{SL}(n, \mathbb{C})$ lie inside the invariant ring of the regular representation.

$$\mathbb{C}[\mathrm{SL}(n, \mathbb{C})]^{\mathcal{N}}$$

Now let's consider $\mathrm{SL}(n, \mathbb{C}) \circlearrowright \mathbb{C}[\mathrm{SL}(n, \mathbb{C})]^{\mathcal{N}}$. This is related closely to the flag varieties we talked about earlier. Recall that the flag variety $\mathrm{Fl}(n)$ is the set of flags in $\mathbb{C}^n$.

$$\mathrm{Fl}(n) = \{V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset \mathbb{C}^n\}$$

Now fix the standard flag

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \langle e_1, \ldots, e_n \rangle.$$

$\mathrm{SL}(n, \mathbb{C})$ acts on $\mathrm{Fl}(n)$ by multiplying the basis vectors. What is the stabilizer of the standard flag? It has to fix subspaces $\langle e_1, \ldots, e_k \rangle$, so the matrix needs to be upper triangular.

Definition 115. The **Borel subgroup** of a Lie group is the stabilizer of a flag. In our case, this is upper triangular matrices

$$\mathcal{B} = \left\{ \begin{bmatrix} * & * & \ldots & * \\ 0 & * & \ldots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & * \end{bmatrix} \right\}$$

So we can think of the flag variety as $\mathrm{Fl}(n) \cong \mathrm{SL}(n, \mathbb{C})/\mathcal{B}$. Now let

$$\mathcal{T} = \mathcal{B}/\mathcal{N} = \{(t_1, \ldots, t_n) \mid t_1 t_2 \cdots t_n = 1\}.$$

This is often called the **torus** of this Lie group.

We want to understand $A = \mathbb{C}[\mathrm{SL}(n, \mathbb{C})]^{\mathcal{N}}$. This is a $\mathbb{C}$-algebra, as last lecture, and we have an action of $\mathcal{T}$ on A.

Example 116. For $n = 2$, $\mathcal{T} = \mathbb{C}^{\times}$, and

$$A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \ldots$$

We also have that

$$\mathrm{maxProj}(A) = \mathrm{Fl}(2) = \mathbb{P}^1\mathbb{C}.$$

This makes sense, because these things are just flags in $\mathbb{C}^2$ (so lines through the origin)! The **Borel-Weil** theorem says that this is all of the representations of $\mathrm{SL}(2, \mathbb{C})$.

Example 117. In general, we have an action of $\mathcal{T} = (\mathbb{C}^{\times})^{n-1}$ on $\mathbb{C}[\mathrm{SL}(n, \mathbb{C})]^{\mathcal{N}}$.

Remark 118.

$$\mathrm{maxSpec}\left(\mathbb{C}[\mathrm{SL}(n, \mathbb{C})]^{\mathcal{N}}\right) \cong {}^{\mathrm{SL}(n,\mathbb{C})}\!\big/_{\mathcal{N}} \xrightarrow{\mathcal{T}} \mathrm{Fl}(n)$$

This whole discussion is a machine for generating representations of $\mathrm{SL}(n, \mathbb{C})$. The procedure is as follows

(1) take a character χ of the torus $\mathcal{T}$;

(2) we get a $\mathbb{C}$-algebra $\mathbb{C}[\mathrm{SL}(n, \mathbb{C})]^{\mathcal{N}_\chi}$ with an action of $\mathbb{C}^{\times}$;

(3) take maxProj of this space to get the representation on the flag variety $\mathrm{Fl}(n)$.

In our last ten minutes, let's talk briefly about quiver varieties.

Definition 119. A **quiver** is a directed graph.

Example 120. Here's a quiver that Aaron Bertram studied for quite some time. It's the quiver for the projective plane.

$$\bullet \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \bullet \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \bullet$$

Definition 121. A **quiver representation** is an assignment to each vertex of the quiver a vector space and to each arrow a directed graph. In addition, these arrows should commute.

Example 122. To the quiver from example 120, we might have a representation

$$V_1 \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; V_2 \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; V_3$$

with some maps $T_{1,2,3} \colon V_1 \to V_2$ and $S_{1,2,3} \colon V_2 \to V_3$.

Let Q be a quiver with representation V. Choosing a basis for each of these vector spaces in a quiver representation, the maps become matrices. We can act on these spaces by change of basis via

$$\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \cdots \times \mathrm{GL}(V_n)/\mathbb{C}^\times$$

The **quiver variety** is the spectrum of the invariant ring of the **quiver algebra**, that is, the quiver variety is

$$\mathrm{Spec}\left(\mathbb{C}[Q]^{\mathrm{SL}(n_1)\times\cdots\times\mathrm{SL}(n_k)}\right)$$

These are just some things that are big in algebraic geometry right now. Flag varieties, quiver varieties, Grassmannians, etc.