SEQUENTIAL SPECTRA

NAME: _____

Due at the beginning of class on 27 February 2024

- *Your answers should be neatly written and logically organized.*

- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*

- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Sections 2.1, 2.2, 2.3, 2.4, and 2.5].

(1) Eilenberg-MacLane spectra are characterized by their homotopy groups: if any other spectrum X satisfies $\pi_i X = 0$ for $i \neq 0$, then $X \simeq H(\pi_0 X)$.

  (a) Prove that $H(\mathbb{Z}/p)$ is the cofiber of the map obtained by applying the functor H to $p \colon \mathbb{Z} \to \mathbb{Z}$.

  (b) Prove that the rationalization of S is stably equivalent to $H\mathbb{Q}$.

  (c) Prove that the functor $H \colon \mathcal{A}b \to \mathcal{S}p$ is a pointed functor in the sense that it sends the zero object in $\mathcal{A}b$ to the zero object in $\mathcal{S}p$.

  (d) Is $H \colon \mathcal{A}b \to ho(\mathcal{S}p)$ an additive functor?

(2) Let $ev_0 \colon \mathcal{S}p \to \mathcal{T}op_*$ be the functor that evaluates a spectrum at its zeroth space: $ev_0 X = X_0$.

  (a) Prove that Σ^∞ is left adjoint to ev_0.

  (b) For any spectrum X, let $\Omega^\infty X = ev_0 RX$, where R is the fibrant replacement functor [Mal23, Proposition 2.2.9]. Prove that there is an adjunction $\Sigma^\infty \colon ho(\mathcal{T}op_*) \leftrightarrows \colon ho(\mathcal{S}p) \colon \Omega^\infty$.

(3) Consider the commuting square of spectra

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{g} & D
\end{array}
$$

  (a) Prove that this square is a homotopy pushout if and only if the induced map of homotopy cofibers $cof(f) \to cof(g)$ is a stable equivalence.

  (b) Use this fact to prove that a commuting square of spectra is a homotopy pullback if and only if it is a homotopy pushout. You may use the fact that a sequence of spectra is a cofiber sequence if and only if it is a fiber sequence.

(4) Let $k \geq 0$. Define the *shift functor* $sh_k \colon \mathcal{S}p \to \mathcal{S}p$ by $sh_k(X)_n = X_{k+n}$.

  (a) Prove that there is a natural stable equivalence $\Sigma \simeq sh_1$.

  (b) Define functors sh_k for $k < 0$. Prove that $sh_{-1} \simeq \Omega$.

REFERENCES

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.