

- (1) Let \mathcal{C} be a homotopical category, and let \mathcal{I} be any category. The category $\text{Fun}(\mathcal{I}, \mathcal{C})$ of functors from \mathcal{I} to \mathcal{C} becomes a homotopical category with weak equivalences defined object-wise. By choosing a homotopical category \mathcal{C} and a category \mathcal{I} , show that the limit functor

$$\lim: \text{Fun}(\mathcal{I}, \mathcal{C}) \rightarrow \mathcal{C}, \quad F \mapsto \lim F$$

is *not* a homotopical functor.

- (2) Prove that if \mathcal{I} is a discrete category (i.e. the only morphisms are identities) then the limit and colimit functors $\lim, \text{colim}: \text{Fun}(\mathcal{I}, \mathcal{C}) \rightarrow \mathcal{C}$ are homotopical functors. Conclude that products in $\text{ho}(\mathcal{C})$ are homotopy products and coproducts in $\text{ho}(\mathcal{C})$ are homotopy coproducts.
- (3) A *coequalizer* is the colimit of a diagram of shape $\bullet \rightrightarrows \bullet$ in a category.

- (a) Prove that the data of the coequalizer of two parallel morphisms $A \xrightleftharpoons[g]{f} B$ is equivalent to the data of the pushout of the diagram

$$A \xleftarrow{\nabla} A \amalg A \xrightarrow{(f,g)} B,$$

where $\nabla: A \amalg A \rightarrow A$ is the fold map.

- (b) Use part (a) to describe the homotopy coequalizer of two maps in the category **Top** of (unpointed) topological spaces¹.
- (4) A pointed space X is *well-based* if the inclusion of the basepoint is a cofibration. Let $f: X \rightarrow Y$ be a pointed map of well-based spaces.
- (a) Let $\text{cof}(f)$ be the homotopy cofiber of f . Prove that the homotopy cofiber of $Y \rightarrow \text{cof}(f)$ is homotopy equivalent to ΣX .
- (b) Prove the dual statement: if $\text{fib}(f)$ is the homotopy fiber of f , then the homotopy fiber of $\text{fib}(f) \rightarrow X$ is homotopy equivalent to ΩY .

REFERENCES

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¹To be precise, we assume all spaces are compactly generated and weakly Hausdorff.