Sequential Spectra 2 Talk Outline

Read [Mal23, Section 2.1.4] and [Mal23, Sections 2.3 and 2.5].

Outline

(1) Examples

  (a) Suspension spectra, including the sphere spectrum and zero spectrum
  (b) Eilenberg–MacLane spectra
  (c) Complex K-theory KU
  (d) Real K-theory KO [Mal23, Example 2.5.28]. This is the infinite loopspace $\mathbb{Z} \times BO$, where O is the infinite orthogonal group. As a cohomology theory, this is like complex K-theory except for real vector bundles. There is also a Bott periodicity theorem for this spectrum, but it's 8-fold periodic, with homotopy groups (in order, starting at zero).

$$\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \ldots$$

  (Note: these should extend infinitely far into the negative degrees as well.)

  (e) In the last talk on sequential spectra, we learned that Σ and Ω are inverse equivalences on $\mathcal{S}p$. Give examples of the shift desuspension spectra from [Mal23, Example 2.1.8] and negative spheres in $\mathcal{S}p$.

(2) Thom Spectra

  (a) In this section, you should basically just go through [Mal23, Section 2.1.4].
  (b) Introduce Thom spaces and give a few examples.
  (c) Define Thom spectra, and give a few examples.
  (d) Construct the Thom spectrum MO, and explain the connection of MO to cobordism of manifolds [Mal23, Example 2.5.36].
  (e) Also mention the Thom spectrum MU [Mal23, Example 2.5.37].
  (f) Describe the homotopy of MO and MU [Mal23, Examples 2.5.36, 2.5.37], and how MO splits into a bunch of Eilenberg–MacLane spectra.

(3) Operations on Spectra [Mal23, Section 2.3]. You should divide this section into several smaller subsections as you feel are appropriate.

  (a) Product and coproduct (wedge). Examples of products and wedges of suspension spectra and Eilenberg–MacLane spectra. Last time, we learned that products and coproducts are the same, so we often denote the coproduct/product of two spectra X and Y by $X \oplus Y$, like with Abelian groups.
  (b) Explain that the homotopy category of spectra with the stable equivalences inverted is an additive category, which tells us a lot about how to work with spectra! We'll see more on this soon.
  (c) Define the shift operator sh_d [Mal23, Definition 2.3.4] and prove that $sh_d X$ is stably equivalent to $\Sigma^d X$.
  (d) Define smashing a space K with a spectrum X, and the function spectrum $F(K, X)$. Note that $K \wedge - \dashv F(K, -)$ as functors on spectra.

(e) Define the mapping spectrum $\mathrm{Map}(X, Y)$, also sometimes denoted $F(X, Y)$. Define Spanier–Whitehead duals of spaces and spectra.

(f) Describe how to compute limits, colimits, homotopy limits, and homotopy colimits in the category of spectra. Give the examples of homotopy pushouts and homotopy pullbacks, and homotopy fibers and cofibers. Last time, we learned that homotopy pushouts and homotopy pullbacks are the same in $\mathcal{S}p$, but you should issue the warning that in general, limits and colimits are not the same.

(g) Now that you know how to construct cofibers, you can define the spectrum $\mathbb{S}/p$ [Mal23, Example 2.5.29]. This is the cofiber of the map $\Sigma^{-1}\Sigma^\infty p \colon \mathbb{S} \to \mathbb{S}$, where $p \colon S^1 \to S^1$ is a degree p map. Explain how this is related to the universal coefficient sequence.

(h) Define the rationalization of the sphere spectrum $\mathbb{S}_\mathbb{Q}$ [Mal23, Example 2.5.30]. By a homework problem, we know that $\mathbb{S}_\mathbb{Q}$ is equivalent to $H\mathbb{Q}$. Prove this! (it should be short)

(i) Define p-localization of a spectrum, and p-completion [Mal23, Examples 2.5.31, 2.5.32]. This can be a quick aside after rationalization.

(j) Explain how to rationalize or take any spectrum mod-p [Mal23, Example 2.5.33].

(k) State the theorem that the rationalization of any spectrum is just a sum of shifted copies of $H\mathbb{Q}$, cf. [Mal23, Example 2.5.34]. Give the example for rationalized K-theory $KU_\mathbb{Q}$.

(4) If you find yourself with extra time and motivation

(a) Define the Atiyah–Hirzebruch spectral sequence and use it to do a sample computation. Ask me for a reference if you want to do this.

REFERENCES

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.