Math 6490: Lie Algebras

but really this course is about Lie Groups

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Contents

Contents by Lecture

Administrative

Usually this course starts with the definition of Lie Algebras and ends with the classification of finite semisimple Lie Algebras. That course is boring, and most people wonder what it has to do with the Lie groups course. Our plan is to do a lecture or two about Lie groups and Lie algebras, but develop the rest of the prerequisites for the course along the way. We will deal with not only compact Lie groups, but also noncompact Lie groups. The course will be run like a seminar with students lecturing most of the time. Students should post notes or the abstract for their lecture.

The book for the course is *Lie Groups* by Daniel Bump.

1 Introduction and Examples

1.1 Lie groups

Definition 1.1. A **Lie group** is a group that also has the structure of a smooth manifold. All of the group operations (inverse, multiplication) are smooth.

A **Lie subgroup** is a closed submanifold of a Lie group.

Example 1.2.

- $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$, the invertible $n \times n$ real or complex matrices. These are manifolds because they're basically $\mathbb{R}^k$ minus a point, for some k.
- The **Borel group** B is the Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of upper triangular matrices. It is solvable as a group.
- $N = \left\{ \begin{pmatrix} 1 & x & \dots & x \\ 0 & 1 & & x \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{pmatrix} \right\}$. This is idempotent as a group.
- The **orthogonal group** $O(n) = \{g \in \mathrm{GL}(n, \mathbb{R}) \mid gg^T = I\}$. Another perspective is the set of matrices preserving the quadratic form
$$Q(\vec{x}, \vec{y}) = x_1 y_1 + \dots + x_n y_n.$$
That is,
$$O(n) = \{g \in \mathrm{GL}(n, \mathbb{R}) \mid Q(g\vec{x}) = Q(\vec{x}) \forall x \in \mathbb{R}^n\}.$$
This is a compact Lie group.
- The group preserving the quadratic form
$$Q_{p,q}(\vec{x}, \vec{y}) = x_1 y_1 + \dots + x_p y_p - x_{p+1} y_{p+1} - \dots - x_{p+q} y_{p+q}$$

is denoted $O_{p,q}$. This is another example of a Lie group, which may be noncompact if $q \neq 0$.

- The **symplectic group** is the group of matrices preserving the matrix $M = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ under conjugation. That is the symplectic group is

$$\mathrm{Sp}(n) = \{g \in \mathrm{GL}(n, \mathbb{R}) \mid gMg^{-1} = M\}$$

- The **unitary group** $U(n) \subseteq \mathrm{GL}(n, \mathbb{C})$ is the set of all unitary matrices, that is, matrices preserving the **Hermitian form**

$$H(\vec{x}, \vec{y}) = x_1\overline{y_1} + x_2\overline{y_2} + \ldots + x_n\overline{y_n}$$

  for $\vec{x} \in \mathbb{C}^n$. Counterintuitively, this is a real group because all of the matrices preserving this form are real. Hence, the underlying manifold of $U(n)$ is real.

  Complexifying this group gives back $\mathrm{GL}(n, \mathbb{C})$, so there is not a unique real version of every complex group, since $\mathrm{GL}(n, \mathbb{R})$ also complexifies to $\mathrm{GL}(n, \mathbb{C})$. We'll talk more about it later.

1.2 Lie Algebras

Definition 1.3. A **Lie algebra** $\mathcal{L}$ is an associative algebra over a field k with a bilinear map $[-,-]\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}$, called the a **Lie bracket**, that obeys the **Jacobi Identity**

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

for all $x, y, z \in \mathcal{L}$.

Example 1.4.

- $\mathfrak{gl}(n, k)$ is the Lie algebra of all $n \times n$ matrices over the field k with the bracket $[X, Y] = XY - YX$.
- $\mathfrak{sl}(n, k)$ is the subalgebra of $\mathfrak{gl}(n, k)$ consisting of all the traceless matrices.
- $\mathfrak{o}(n, k)$ is the subalgebra of $\mathfrak{gl}(n, k)$ consisting of all skew-symmetric matrices.
- $\mathfrak{sp}(n, k)$ is the subalgebra of $\mathfrak{gl}(n, k)$ consisting of all symmetric matrices.

The next question is how we go from the Lie Algebra to the Lie group and back. Already the notation makes it clear that they're related, but how?

Remark 1.5. Last time I stated that $U(n)$ isn't a complex Lie group, but a real one instead.

So why is $U(n)$ a real Lie group? Let's just start with the example of $U(1)$.

$$U(1) = \{z \mid z\bar{z} = 1\}.$$

Notice that complex conjugation isn't a complex analytic operation, so as soon as complex conjugation shows up we get a clue that it can't be a complex Lie group.

So now think of the Hermitian form

$$H(\vec{z}, \vec{w}) = \sum z_j \overline{w}_j.$$

Consider $\mathbb{C}^n \cong \mathbb{R}^{2n}$. We can write H as the sum of two forms,

$$H = H_{\mathrm{Re}} + iH_{\mathrm{Im}}$$

where H_{Re} is the orthogonal form and H_{Im} is the symplectic form. Hence, $\mathrm{O}(2n)$ and $\mathrm{Sp}(2n)$ both lie inside U_n.

1.3 From Lie groups to Lie Algebras

Let G be a Lie group. Let $h \in G$. We get an automorphism of G by conjugation, $\Psi_h \colon g \mapsto hgh^{-1}$. We can think of $h \mapsto \Psi_h$ as a homomorphism $G \to \mathrm{Aut}(G)$.

By differentiating, we get a map Ad of G to $T_e(G)$, where $T_e(G)$ is the tangent space at the identity of G. Each $\mathrm{Ad}(g)$ is in $\mathrm{End}(T_e(G))$, with the map $X \mapsto hXh^{-1}$.

Note that if $\rho \colon G \to H$ is a diffeomorphism, then the diagram

$$\begin{array}{ccc} T_e(G) & \xrightarrow{d\rho|_e} & T_e(H) \\ \big\downarrow{\scriptstyle \mathrm{Ad}(g)} & & \big\downarrow{\scriptstyle \mathrm{Ad}(\rho(g))} \\ T_e(G) & \xrightarrow{d\rho|_e} & T_e(H) \end{array}$$

commutes.

If we differentiate again, we get a map which is usually called $\mathrm{ad} \colon T_e(G) \to T_e(G)$. If $X, Y \in T_e(G)$, then $\mathrm{ad}\, X(Y) \in T_e(G)$. The following then commutes

$$\begin{array}{ccc} T_e(G) & \xrightarrow{d\rho|_e} & T_e(H) \\ \big\downarrow{\scriptstyle \mathrm{ad}(X)} & & \big\downarrow{\scriptstyle \mathrm{ad}(d\rho(X))} \\ T_e(G) & \xrightarrow{d\rho|_e} & T_e(H) \end{array}$$

and we get $d\rho([X, Y]) = [d\rho(X), d\rho(Y)]$ as a consequence.

Definition 1.6. We define a Lie bracket on $T_e(G)$ by $[X, Y] := \mathrm{ad}(X)(Y)$. This makes $T_e(G)$ into a Lie algebra.

Example 1.7. Let $G = \mathrm{GL}(n, \mathbb{R})$. $T_e(G) = \mathrm{End}(\mathbb{R}^n) = M_{n\times n}(\mathbb{R})$. To see that the new bracket we defined by $\mathrm{ad}(X)(Y)$ agrees with the commutator bracket $[X, Y] = XY - YX$ on G, we need to do some work.

Consider

$$\exp(tX) := I + tX + \tfrac{1}{2}(tX)^2 + \dots$$

for X an $n \times n$ matrix. We see easily that $\exp(0) = I$ and

$$\frac{d}{dt}\exp(tX)\big|_{t=0} = X.$$

So we have a curve $\gamma_X(t) = \exp(tX)$ through e with tangent vector X. Now we can compute.

$$\begin{aligned}
\mathrm{ad}(X)(Y) &= \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\gamma_X(t)}(Y)) \\
&= \frac{d}{dt}\bigg|_{t=0} \exp(tX)Y\exp(tX)^{-1} \\
&= \gamma_X'(0)Y\gamma_X(0) + \gamma_X(0)Y(-\gamma_X(0)\gamma_X'(0)\gamma_X(0)^{-1}) \\
&= XY(I) - IY(IXI^{-1}) \\
&= XY - YX
\end{aligned}$$

Now, we can look at all the nice closed subgroups of $\mathrm{GL}(n, \mathbb{R})$, and see that the bracket on them is also the commutator bracket. This also gives a nice explanation of the crazy Jacobi identity. Namely, it says that

$$\mathrm{ad}\, X([Y, Z]) = [\mathrm{ad}(X)(Y), Z] + [Y, \mathrm{ad}(X)(Z)],$$

or in other words, ad is a derivation.

Remark 1.8. As a general principle, if G and H are Lie groups with G connected and simply connected, then a linear map $T_e(G) \to T_e(H)$ is the differential of a homomorphism $G \to H$ if and only if it preserves the bracket.

This is useful, but simply connected is a very strong assumption.

1.4 From Lie Algebras to Lie Groups

Given an algebra with a bracket, how does it interact with the multiplication on G? There are two maps

$$\exp\colon \mathfrak{g} \to G \qquad \text{and} \qquad \log\colon U \subset G \to \mathfrak{g}.$$

What is $\log(\exp X \exp Y) \in \mathfrak{g}$? Is there a formula for this in terms of the Lie bracket?

Yes, this is the **Campbell-Hausdorff formula.** This says that once we know the bracket on the Lie algebra, we actually know the multiplication on G.

Theorem 1.9 (Ado's Theorem)**.** *Every finite-dimensional Lie algebra $\mathfrak{g}$ has a faithful representation over a finite-dimensional vector space V. Thus, every finite-dimensional Lie algebra $\mathfrak{g}$ can be realized as a subalgebra of $GL(V)$ for some V.*

Left multiplication $L_g\colon G \to G$ is a smooth map, $L_g(h) = gh$. Then if U is any neighborhood of the identity, $L_g(U) = gU$ is a left translation of this neighborhood. This defines $(L_g)_*\colon T_e(G) \to T_g(G)$. For a tangent vector $X \in T_e(G)$, we get $L_{g*}X \in T_g(G)$. This defines a vector field, called a **left-invariant vector field**.

Given $X \in T_e(G)$, let V be the left-invariant vector field V. There exists an integral curve $c_X\colon [0,1] \to G$ such that ${}^{d}\!/_{dt}\big|_{t_0} c = V_{c(t_0)}$ and $c(0) = e$.

Now we claim that c is a homomorphism, that is, $c_X(t+s) = c_X(t)c_X(s)$ for $t, t+s \in [0,1]$. To verify this, we check that both are solutions to the same differential equation with the same initial conditions. Then we can extend c to a global solution to the equation, $c\colon \mathbb{R} \to G$. This is called a **one-parameter subgroup** of G.

Example 1.10.

$$\exp\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} = \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix} \qquad \exp\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\exp\begin{bmatrix} ta & 0 \\ 0 & tb \end{bmatrix} = \begin{bmatrix} e^{ta} & 0 \\ 0 & e^{tb} \end{bmatrix} \qquad \exp\begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$$

Proposition 1.11. *The exponential map $\exp\colon \mathfrak{g} \to G$ is the unique map with the differential $d\exp_0 = \mathrm{id}$ and whose restriction to the lines tX is a one-parameter subgroup $c_X(t)$.*

1.5 More about the Campbell-Hausdorff formula

Definition 1.12. The map $\log\colon G \to \mathfrak{g}$ is given by

$$\log g = (g-1) - \frac{(g-1)^2}{2} + \frac{(g-1)^3}{3} + \dots$$

Definition 1.13.

$$X * Y = \log(\exp X \exp Y)$$

Theorem 1.14.

$$X * Y = (X+Y) + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \dots$$

2 Representations

Definition 2.1. A representation of a Lie algebra $\mathfrak{g}$ is a homomorphism of the Lie algebra into $\mathfrak{gl}(V)$. V is usually a finite-dimensional complex vector space, and $\mathfrak{g}$ is a real Lie algebra.

Definition 2.2. We can complexify $\mathfrak{g}$ by taking $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. This is a complex Lie algebra denoted $\mathfrak{g}_{\mathbb{C}}$, with bracket extended from that on $\mathfrak{g}$.

Remark 2.3. If V and W are real vector spaces, then

$$\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Hom}(V, W) \cong \mathrm{Hom}_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}, W \otimes_{\mathbb{R}} \mathbb{C}).$$

Example 2.4. The complexification of $\mathfrak{gl}(n, \mathbb{R})$ is $\mathfrak{gl}(n, \mathbb{C})$.

Remark 2.5. Every complex matrix X can be written uniquely as $X_1 + iX_2$ with X_1 skew-symmetric and X_2 symmetric. This is done by setting $X_1 = \frac{1}{2}(X - X^T)$ and $X_2 = \frac{1}{2i}(X + X^T)$.

We can apply this remark to $U(n)$.

Example 2.6. Recall that $\mathfrak{u}(n)$ is the Lie algebra of the Lie group $U(n)$, and that this is a *real* Lie algebra. This is the Lie algebra of skew-hermitian matrices in $\mathfrak{gl}(n, \mathbb{C})$.

We can think of $\mathfrak{u}(n)$ as the set of all pairs (X_1, X_2) of skew symmetric real matrices X_1 and symmetric real matrices X_2, by taking the real and imaginary parts of a skew-hermitian matrix.

When we complexify, we get pairs (Y_1, Y_2) of complex matrices, with Y_1 skew-symmetric and Y_2 symmetric. Hence, by the remark above, we get all complex matrices.

Hence, the complexification of $\mathfrak{u}(n)$ is $\mathfrak{gl}(n, \mathbb{C})$.

The fact that both $\mathfrak{u}(n)$ and $\mathfrak{gl}(n, \mathbb{R})$ are real versions of $\mathfrak{gl}(n, \mathbb{C})$ gives us several ways to create representations of $\mathfrak{gl}(n, \mathbb{C})$. We can take representations of $\mathfrak{u}(n)$ and complexify to get a representation of $\mathfrak{gl}(n, \mathbb{C})$, or we can take representations of $\mathfrak{gl}(n, \mathbb{R})$ and complexify.

Definition 2.7. A $\mathfrak{g}$-representation is **irreducible** if 0 and the whole space V are the only $\mathfrak{g}$-invariant subspaces.

Example 2.8. Let

$$\mathfrak{g}_{\mathbb{C}} = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, b, c \in \mathbb{C} \right\}.$$

Then $V = \mathbb{C}^2$ is not irreducible as a representation, and it is also not a direct sum of irreducible representations.

Example 2.9. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, the Lie algebra of traceless 2×2 complex matrices. This has a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then the standard representation $\mathfrak{sl}(2, \mathbb{C}) \circlearrowright \mathbb{C}^2$ is irreducible.

We can construct more representations by taking tensor products of this standard 2-dimensional representations. But the tensor product is not usually irreducible, and it's a difficult problem in general to decompose the tensor product of two representations into irreducible representations.