

Due at the beginning of class on 23 January 2024

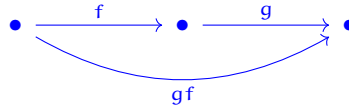
- Your answers should be neatly written and logically organized.
- You may collaborate on solving the problems, but the solutions you turn in should be your own.
- You may use any resource you find online (or elsewhere), but you must cite any resource you use.

Reading: Read §2.1 and §2.2 in [Rie14] or §B.1 in [HHR16].

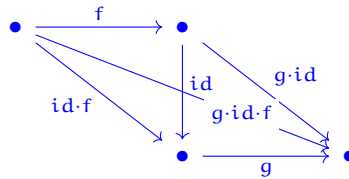
(1) A class of morphisms \mathcal{W} in a category \mathcal{C} satisfies the *two-out-of-three property* if given any two composable morphisms f and g , if any two of f , g , and gf are in \mathcal{W} , then so is the third.

(a) Prove that the class of weak equivalences \mathcal{W} in a homotopical category \mathcal{C} obeys the two-out-of-three property.

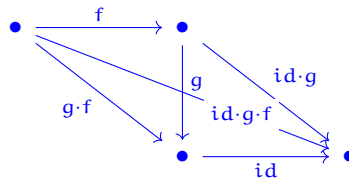
SOLUTION: Let \mathcal{C} be a homotopical category with weak equivalences the subcategory \mathcal{W} . We will show that \mathcal{W} has the two-out-of-three property by using its two-out-of-six property. Consider the morphisms f , g , and gf in the following diagram:



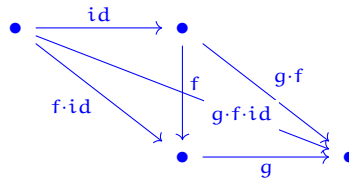
Suppose $f, g \in \mathcal{W}$. Then $gf \in \mathcal{W}$ by the two-out-of-six property applied to the diagram:



Suppose $g, gf \in \mathcal{W}$, then $f \in \mathcal{W}$ by the two-out-of-six property applied to the diagram:



Finally, suppose $f, gf \in \mathcal{W}$, then $g \in \mathcal{W}$ by the two-out-of-six property applied to the diagram:



- (b) Is the two-out-of-three property equivalent to the two-out-of-six property?

SOLUTION: No. It is weaker. For a counterexample, consider the category:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

Let $\mathcal{W} = \{gf, hg, \text{id}_A, \text{id}_B, \text{id}_C, \text{id}_D\}$. \mathcal{W} clearly satisfies the two-out-of-six property. It does not satisfy the two-out-of-three property: gf and hg are in \mathcal{W} , while hgf is not.

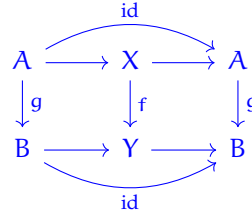
- (2) Let \mathcal{C} be any category equipped with a collection of morphisms \mathcal{W} . We say that \mathcal{W} is *saturated* if every morphism f in \mathcal{C} which becomes an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$ is in \mathcal{W} . We say that a homotopical category is *saturated* if the class of weak equivalences is saturated (i.e. if f becomes an isomorphism in $\text{ho}(\mathcal{C})$, then $f \in \mathcal{W}$).

- (a) Prove that if \mathcal{W} is saturated, then \mathcal{W} has the two-out-of-six property.

SOLUTION: Suppose f, g, h are composable in \mathcal{C} and $fg, gh \in \mathcal{W}$. Then, fg and gh are isomorphisms in $\text{ho } \mathcal{C}$. Isomorphisms satisfy 2-out-of-6, so f, g, h are isomorphisms in $\text{ho } \mathcal{C}$. Thus, $f, g, h \in \mathcal{W}$. This proves the claim. [Rie14, Lemma 2.1.10]

- (b) Give an example of a homotopical category that is *not* saturated.

SOLUTION: Consider the following category, which represents a morphism of retracts.



and let the weak equivalence be given by f and the identity morphisms. This satisfies 2-out-of-6 since the only way to obtain f via a composite of two morphisms is with f and an identity, and the only way to obtain an identity via a composite of two morphisms either uses identities or is one of the horizontal composites, but the maps $X \rightarrow A$ and $Y \rightarrow B$ cannot be applied before another morphism to either give f or an identity.

The weak equivalences are not closed under retract, since g is not a weak equivalence, but isomorphisms are closed under retract, so g must be an isomorphism in the homotopy category. Thus, this construction is a homotopical category which is not saturated.

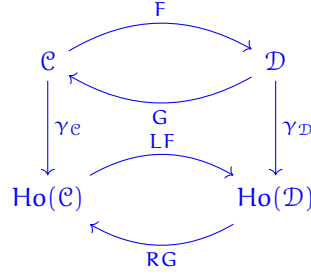
This example is nice because it is clear that it cannot have a model structure, as the weak equivalences in a model category are closed under retract by definition.

- (c) (Optional) Show that the class of weak equivalences in a model category is saturated.

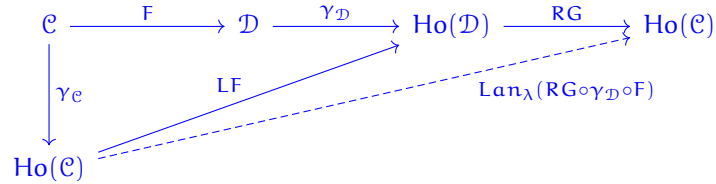
SOLUTION: [Rie14, Remark 2.1.9], which refers to [Qui67, Proposition 5.1]

- (3) An *absolute left (or right) Kan extension* is a Kan extension that is preserved by any functor whatsoever. Let $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ be an adjunction between homotopical categories \mathcal{C} and \mathcal{D} . Let $LF: \text{ho}(\mathcal{C}) \rightarrow \text{ho}(\mathcal{D})$ be the total left derived functor for F and $RG: \text{ho}(\mathcal{D}) \rightarrow \text{ho}(\mathcal{C})$ be the total right derived functor for G . Assume that LF and RG are absolute right/left Kan extensions. Prove that LF is left adjoint to RG .

SOLUTION: Consider the diagram:



where F and G are adjoint functors with unit $\eta : \text{id}_C \Rightarrow G \circ F$ and counit $\epsilon : F \circ G \Rightarrow \text{id}_D$, (LF, α) is a total left derived functor where $\alpha : LF \circ \gamma_C \Rightarrow \gamma_D \circ F$, and (RG, β) is a total right derived functor where $\beta : \gamma_C \circ G \Rightarrow RG \circ \gamma_D$. Our aim is to show that LF and RG are adjoint. We will first produce the unit and counit of a potential adjunction. Since LF is an absolute right Kan extension, $RG \circ LF \cong \text{Lan}_\lambda(RG \circ \gamma_D \circ F)$ with natural transformation $RG\alpha$ (the whiskered composite):



We would like to apply the universal property of the pair $(RG \circ LF, RG\alpha)$ to the functor $\text{Id}_{\text{Ho}(C)}$. To do so requires coming up with a natural transformation from $\text{Id}_{\text{Ho}(C)}$ to $RG \circ \gamma_D \circ F$. Such a natural transformation is given by:

$$\beta F \circ \gamma_C \eta : \text{Id}_{\text{Ho}(C)} \circ \gamma_C = \gamma_C \circ \text{Id}_C \Rightarrow \gamma_C \circ G \circ F \Rightarrow RG \circ \gamma_D \circ F$$

Hence, we get a unique natural transformation $\bar{\eta} : \text{Id}_{\text{Ho}(C)} \Rightarrow RG \circ LF$. A similar argument produces a natural transformation $\bar{\epsilon} : RG \circ LF \Rightarrow \text{Id}_{\text{Ho}(D)}$ coming from the existence of the natural transformation $\gamma_D \epsilon \circ \alpha G : LF \circ \gamma_C \circ G \Rightarrow \gamma_D \circ \text{Id}_D$. A convenient way to determine whether $\bar{\eta}$ and $\bar{\epsilon}$ form an adjunction is to check the *triangular identities* (see page 85 of [?]):

$$\bar{\epsilon} LF \circ LF \bar{\eta} = \text{Id}_{LF},$$

$$RG \bar{\epsilon} \circ \bar{\eta} RG = \text{Id}_{RG}$$

We will show the first one as the second follows a similar argument. Since the universal property ensures a unique factorization for derived functors, it suffices to show:

$$\alpha \circ (\bar{\epsilon} LF \circ LF \bar{\eta}) \gamma_C = \alpha$$

$$\begin{aligned}
\alpha \circ (\bar{\epsilon}LF \circ LF\bar{\eta})\gamma_{\mathcal{C}} &= \alpha \circ \bar{\epsilon}(LF \circ \gamma_{\mathcal{C}}) \circ (LF)\bar{\eta}\gamma_{\mathcal{C}} \\
&= (\bar{\epsilon}(\gamma_{\mathcal{D}} \circ F)) \circ ((LF \circ RG)\alpha) \circ (LF)\bar{\eta}\gamma_{\mathcal{C}} \\
&= \bar{\epsilon}(\gamma_{\mathcal{D}} \circ F) \circ LF((RG)\alpha \circ \bar{\eta}\gamma_{\mathcal{C}}) \\
&= \bar{\epsilon}(\gamma_{\mathcal{D}} \circ F) \circ LF(\beta F \circ \gamma_{\mathcal{C}}\eta) \\
&= \bar{\epsilon}(\gamma_{\mathcal{D}} \circ F) \circ (LF)\beta F \circ (LF \circ \gamma_{\mathcal{C}})\eta \\
&= (\bar{\epsilon}\gamma_{\mathcal{D}} \circ (LF)\beta)F \circ (LF \circ \gamma_{\mathcal{C}})\eta \\
&= (\gamma_{\mathcal{D}}\epsilon \circ \alpha G)F \circ (LF \circ \gamma_{\mathcal{C}})\eta \\
&= \gamma_{\mathcal{D}}\epsilon F \circ \alpha(G \circ F) \circ (LF \circ \gamma_{\mathcal{C}})\eta \\
&= \gamma_{\mathcal{D}}\epsilon F \circ (\gamma_{\mathcal{D}} \circ F)\eta \circ \alpha \text{id}_{\mathcal{C}} \\
&= \gamma_{\mathcal{D}}(\epsilon F \circ F\eta) \circ \alpha \\
&= \gamma_{\mathcal{D}}\text{id}_F \circ \alpha \\
&= \text{id}_{\gamma_{\mathcal{D}} \circ F} \circ \alpha \\
&= \alpha
\end{aligned}$$

This is from [Rie14, Exercise 2.2.15], which cites [Mal07].

(4) Let \mathcal{C} be a homotopical category and let $L: \mathcal{C} \rightarrow \text{ho}(\mathcal{C})$ be the localization functor.

- (a) Let $c \in \mathcal{C}$. Prove that any natural transformation $\mathcal{C}(c, -) \Rightarrow F$ factors through $\text{ho}(\mathcal{C})(c, -)$, where $F: \mathcal{C} \rightarrow \mathbf{Set}$ is a homotopical functor.

SOLUTION: This is [HHR16, Proposition B.4]. By the Yoneda lemma,

$$\text{Nat}(\mathcal{C}(c, -), F) \cong F(c)$$

where $x \in F(c)$ corresponds to the natural transformation $\eta(x)$ with components

$$\eta(x)_d: \mathcal{C}(c, d) \rightarrow F(d): f \mapsto fx.$$

Since F is homotopical, and \mathbf{Set} is its own homotopy category, there is a diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathbf{Set} \\
\downarrow L & & \parallel \\
\text{ho } \mathcal{C} & \xrightarrow{\bar{F}} & \mathbf{Set}
\end{array}$$

witnessing that F factors through the localization L . Further, \bar{F} is the same as F on objects.

Thus $x \in F(c)$ also corresponds to the natural transformation $\bar{\eta}(x): \text{ho}(\mathcal{C})(c, -) \Rightarrow \bar{F}$ with components

$$\bar{\eta}(x)_d: \text{ho}(\mathcal{C})(c, d): f \mapsto fx.$$

This witnesses that an arbitrary natural transformation $\eta(x)$ is the composite of $L: \mathcal{C}(c, -) \Rightarrow \text{ho}(\mathcal{C})(c, -)$ and $\bar{\eta}(x) \circ L: \text{ho}(\mathcal{C})(c, -) \Rightarrow \bar{F} \circ L = F$ which proves the desired claim.

- (b) Let $c \in \mathcal{C}$ be an object such that $\mathcal{C}(c, -)$ is a homotopical functor. Prove that the natural transformation $\mathcal{C}(c, -) \rightarrow \text{ho}(\mathcal{C})(c, -)$ induced by L is a natural bijection.

SOLUTION: This is [HHR16, Corollary B.6]. By the previous problem, the identity natural transformation $\mathcal{C}(c, -)$ factors through $L: \mathcal{C}(c, -) \rightarrow \text{ho}(\mathcal{C})(c, -)$, and therefore L is injective.

Since $\mathcal{C}(c, -)$ is homotopical, if $f: d \rightarrow e$ is a weak equivalence, the composition map $f \circ -: \mathcal{C}(c, f): \mathcal{C}(c, d) \rightarrow \mathcal{C}(c, e)$ is an isomorphism. For $f: d \rightarrow c$ a weak equivalence, let $g \in \mathcal{C}(c, d)$ be the preimage of $1 \in \mathcal{C}(c, c)$ along the isomorphism $f \circ -: \mathcal{C}(c, d) \rightarrow \mathcal{C}(c, c)$. By construction, $f \circ g = 1$. This allows every zig-zag in $\text{ho}(\mathcal{C})(c, d)$ to be rewritten in terms of forward facing arrows, demonstrating that $L: \mathcal{C}(c, -) \rightarrow \text{ho}(\mathcal{C})(c, -)$ is surjective.

This completes the proof.

REFERENCES

- [HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.
- [Mal07] Georges Maltsiniotis. Le théorème de Quillen, d’adjonction des foncteurs dérivés, revisité. *C. R. Math. Acad. Sci. Paris*, 344(9):549–552, 2007.
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