Due at the beginning of class on 23 January 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: Read §2.1 and §2.2 in [Rie14] or §B.1 in [HHR16].

(1) A class of morphisms $\mathcal{W}$ in a category $\mathcal{C}$ satisfies the *two-out-of-three property* if given any two composable morphisms f and g, if any two of f, g, and gf are in $\mathcal{W}$, then so is the third.

(a) Prove that the class of weak equivalences $\mathcal{W}$ in a homotopical category $\mathcal{C}$ obeys the two-out-of-three property.

SOLUTION: Let $\mathcal{C}$ be a homotopical category with weak equivalences the subcategory $\mathcal{W}$. We will show that the $\mathcal{W}$ has the two-out-of-three property by using its two-out-of-six property. Consider the morphisms f, g, and gf in the following diagram:

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

Suppose $f, g \in \mathcal{W}$. Then $gf \in \mathcal{W}$ by the two-out-of-six property applied to the diagram:

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "\mathrm{id}\cdot f"'] \arrow[drr, "g\cdot\mathrm{id}\cdot f"] & \bullet \arrow[d, "\mathrm{id}"] \arrow[dr, "g\cdot\mathrm{id}"] & \\ & \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

Suppose $g, gf \in \mathcal{W}$, then $f \in \mathbb{W}$ by the two-out-of-six property applied to the diagram:

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "g\cdot f"'] \arrow[drr, "\mathrm{id}\cdot g\cdot f"] & \bullet \arrow[d, "g"] \arrow[dr, "\mathrm{id}\cdot g"] & \\ & \bullet \arrow[r, "\mathrm{id}"'] & \bullet \end{tikzcd}$$

Finally, suppose $f, gf \in \mathcal{W}$, then $g \in \mathcal{W}$ by the two-out-of-six property applied to the diagram:

$$\begin{tikzcd} \bullet \arrow[r, "\mathrm{id}"] \arrow[dr, "f\cdot\mathrm{id}"'] \arrow[drr, "g\cdot f\cdot\mathrm{id}"] & \bullet \arrow[d, "f"] \arrow[dr, "g\cdot f"] & \\ & \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

(b) Is the two-out-of-three property equivalent to the two-out-of-six property?

SOLUTION: No. It is weaker. For a counterexample, consider the category:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

Let $\mathcal{W} = \{gf, hg, \mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C, \mathrm{id}_D\}$. $\mathcal{W}$ clearly satisfies the two-out-of-six property. It does not satisfy the two-out-of-three property: gf and hg are in $\mathcal{W}$, while hgf is not.

(2) Let $\mathcal{C}$ be any category equipped with a collection of morphisms $\mathcal{W}$. We say that $\mathcal{W}$ is *saturated* if every morphism f in $\mathcal{C}$ which becomes an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$ is in $\mathcal{W}$. We say that a homotopical category is *saturated* if the class of weak equivalences is saturated (i.e. if f becomes an isomorphism in $\mathrm{ho}(\mathcal{C})$, then $f \in \mathcal{W}$).

(a) Prove that if $\mathcal{W}$ is saturated, then $\mathcal{W}$ has the two-out-of-six property.

SOLUTION: Suppose f, g, h are composable in $\mathcal{C}$ and $fg, gh \in \mathcal{W}$. Then, fg and gh are isomorphisms in $\mathrm{ho}\,\mathcal{C}$. Isomorphisms satisfy 2-out-of-6, so f, g, h are isomorphisms in $\mathrm{ho}\,\mathcal{C}$. Thus, $f, g, h \in \mathcal{W}$. This proves the claim. [Rie14, Lemma 2.1.10]

(b) Give an example of a homotopical category that is *not* saturated.

SOLUTION: Consider the following category, which represents a morphism of retracts.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"] \arrow[rr, bend left, "\mathrm{id}"] & X \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & Y \arrow[r] & B \end{tikzcd}$$

and let the weak equivalence be given by f and the identity morphisms. This satisfies 2-out-of-6 since the only way to obtain f via a composite of two morphisms is with f and an identity, and the only way to obtain an identity via a composite of two morphisms either uses identities or is oen of the horizontal composites, but the maps $X \to A$ and $Y \to B$ cannot be applied before another morphism to either give f or an identity.

The weak equivalences are not closed under retract, since g is not a weak equivalence, but isomorphisms are closed under retract, so g must be an isomorphism in the homotopy category. Thus, this construction is a homotopical category which is not saturated.

This example is nice because it is clear that it cannot have a model structure, as the weak equivalences in a model category are closed under retract by definition.

(c) (Optional) Show that the class of weak equivalences in a model category is saturated.

SOLUTION: [Rie14, Remark 2.1.9], which refers to [Qui67, Proposition 5.1]

(3) An *absolute left (or right) Kan extension* is a Kan extension that is preserved by any functor whatsoever. Let $F \colon \mathcal{C} \leftrightarrows \mathcal{D} \colon G$ be an adjunction between homotopical categories $\mathcal{C}$ and $\mathcal{D}$. Let $LF \colon \mathrm{ho}(\mathcal{C}) \to \mathrm{ho}(\mathcal{D})$ be the total left derived functor for F and $RG \colon \mathrm{ho}(\mathcal{D}) \to \mathrm{ho}(\mathcal{C})$ be the total right derived functor for G. Assume that LF and RG are absolute right/left Kan extensions. Prove that LF is left adjoint to RG.

SOLUTION: Consider the diagram:

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[d, "\gamma_{\mathcal{C}}"'] & \mathcal{D} \arrow[l, bend left, "G"] \arrow[d, "\gamma_{\mathcal{D}}"] \\
\mathrm{Ho}(\mathcal{C}) \arrow[r, bend left, "LF"] & \mathrm{Ho}(\mathcal{D}) \arrow[l, bend left, "RG"]
\end{tikzcd}
$$

where F and G are adjoint functors with unit $\eta : id_{\mathcal{C}} \Rightarrow G \circ F$ and counit $\epsilon : F \circ G \Rightarrow id_{\mathcal{D}}$, (LF, α) is a total left derived functor where $\alpha : LF \circ \gamma_{\mathcal{C}} \Rightarrow \gamma_{\mathcal{D}} \circ F$, and (RG, β) is a total right derived functor where $\beta : \gamma_{\mathcal{C}} \circ G \Rightarrow RG \circ \gamma_{\mathcal{D}}$. Our aim is to show that LF and RG are adjoint. We will first produce the unit and counit of a potential adjunction. Since LF is an absolute right Kan extension, $RG \circ LF \cong Lan_{\lambda}(RG \circ \gamma_{\mathcal{D}} \circ F)$ with natural transformation $RG\alpha$ (the whiskered composite):

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{C}}"'] & \mathcal{D} \arrow[r, "\gamma_{\mathcal{D}}"] & \mathrm{Ho}(\mathcal{D}) \arrow[r, "RG"] & \mathrm{Ho}(\mathcal{C}) \\
\mathrm{Ho}(\mathcal{C}) \arrow[urr, "LF"] \arrow[urrr, dashed, "Lan_{\lambda}(RG \circ \gamma_{\mathcal{D}} \circ F)"'] & & &
\end{tikzcd}
$$

We would like to apply the universal property of the pair $(RG \circ LF, RG\alpha)$ to the functor $Id_{Ho(\mathcal{C})}$. To do so requires coming up with a natural transformation from $Id_{Ho(\mathcal{C})}$ to $RG \circ \gamma_{\mathcal{D}} \circ F$. Such a natural transformation is given by:

$$\beta F \circ \gamma_{\mathcal{C}}\eta : Id_{Ho(\mathcal{C})} \circ \gamma_{\mathcal{C}} = \gamma_{\mathcal{C}} \circ Id_{\mathcal{C}} \Rightarrow \gamma_{\mathcal{C}} \circ G \circ F \Rightarrow RG \circ \gamma_{\mathcal{D}} \circ F$$

Hence, we get a unique natural transformation $\overline{\eta} : Id_{Ho(\mathcal{C})} \Rightarrow RG \circ LF$. A similar argument produces a natural transformation $\overline{\epsilon} : RG \circ LF \Rightarrow Id_{Ho(\mathcal{D})}$ coming from the existence of the natural transformation $\gamma_{\mathcal{D}}\epsilon \circ \alpha G : LF \circ \gamma_{\mathcal{C}} \circ G \Rightarrow \gamma_{\mathcal{D}} \circ Id_{\mathcal{D}}$. A convenient way to determine whether $\overline{\eta}$ and $\overline{\epsilon}$ for an adjunction is to check the *triangular identities* (see page 85 of [**?**]):

$$\overline{\epsilon}LF \circ LF\overline{\eta} = Id_{LF},$$
$$RG\overline{\epsilon} \circ \overline{\eta}RG = Id_{RG}$$

We will show the first one as the second follows a similar argument. Since the universal property ensures a unique factorization for derived functors, it suffices to show:

$$\alpha \circ (\overline{\epsilon}LF \circ LF\overline{\eta})\gamma_{\mathcal{C}} = \alpha$$

$$\begin{aligned}
\alpha \circ (\overline{\epsilon} LF \circ LF\overline{\eta})\gamma_{\mathcal{C}} &= \alpha \circ \overline{\epsilon}(LF \circ \gamma_{\mathcal{C}}) \circ (LF)\overline{\eta}\gamma_{\mathcal{C}} \\
&= (\overline{\epsilon}(\gamma_{\mathcal{D}} \circ F)) \circ ((LF \circ RG)\alpha) \circ (LF)\overline{\eta}\gamma_{\mathcal{C}} \\
&= \overline{\epsilon}(\gamma_{\mathcal{D}} \circ F) \circ LF((RG)\alpha \circ \overline{\eta}\gamma_{\mathcal{C}} \\
&= \overline{\epsilon}(\gamma_{\mathcal{D}} \circ F) \circ LF(\beta F \circ \gamma_{\mathcal{C}}\eta) \\
&= \overline{\epsilon}(\gamma_{\mathcal{D}} \circ F) \circ (LF)\beta F \circ (LF \circ \gamma_{\mathcal{C}})\eta \\
&= (\overline{\epsilon}\gamma_{\mathcal{D}} \circ (LF)\beta)F \circ (LF \circ \gamma_{\mathcal{C}})\eta \\
&= (\gamma_{\mathcal{D}}\epsilon \circ \alpha G)F \circ (LF \circ \gamma_{\mathcal{C}})\eta \\
&= \gamma_{\mathcal{D}}\epsilon F \circ \alpha(G \circ F) \circ (LF \circ \gamma_{\mathcal{C}})\eta \\
&= \gamma_{\mathcal{D}}\epsilon F \circ (\gamma_{\mathcal{D}} \circ F)\eta \circ \alpha \mathrm{id}_{\mathcal{C}} \\
&= \gamma_{\mathcal{D}}(\epsilon F \circ F\eta) \circ \alpha \\
&= \gamma_{\mathcal{D}}\mathrm{id}_F \circ \alpha \\
&= \mathrm{id}_{\gamma_{\mathcal{D}} \circ F} \circ \alpha \\
&= \alpha
\end{aligned}$$

This is from [Rie14, Exercise 2.2.15], which cites [Mal07].

(4) Let $\mathcal{C}$ be a homotopical category and let $L\colon \mathcal{C} \to \mathrm{ho}(\mathcal{C})$ be the localization functor.

(a) Let $c \in \mathcal{C}$. Prove that any natural transformation $\mathcal{C}(c,-) \Rightarrow F$ factors through $\mathrm{ho}(\mathcal{C})(c,-)$, where $F\colon \mathcal{C} \to \mathbf{Set}$ is a homotopical functor.

SOLUTION: This is [HHR16, Proposition B.4]. By the Yoneda lemma,

$$\mathrm{Nat}(\mathcal{C}(c,-), F) \cong F(c)$$

where $x \in F(c)$ corresponds to the natural transformation $\eta(x)$ with components

$$\eta(x)_d\colon \mathcal{C}(c,d) \to F(d)\colon f \mapsto fx.$$

Since F is homotopical, and **Set** is its own homotopy category, there is a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "L"'] & \mathbf{Set} \arrow[d, equal] \\
\mathrm{ho}\,\mathcal{C} \arrow[r, "\overline{F}"] & \mathbf{Set}
\end{tikzcd}$$

witnessing that F factors through the localization L. Further, $\overline{F}$ is the same as F on objects. Thus $x \in F(c)$ also corresponds to the natural transformation $\overline{\eta}(x)\colon \mathrm{ho}(\mathcal{C})(c,-) \Rightarrow \overline{F}$ with components

$$\overline{\eta}(x)_d\colon \mathrm{ho}(\mathcal{C})(c,d)\colon f \mapsto fx.$$

This witnesses that an arbitrary natural transformation $\eta(x)$ is the composite of $L\colon \mathcal{C}(c,-) \Rightarrow \mathrm{ho}(\mathcal{C})(c,-)$ and $\overline{\eta}(x) \circ L\colon \mathrm{ho}(\mathcal{C})(c, L-) \Rightarrow \overline{F} \circ L = F$ which proves the desired claim.

(b) Let $c \in \mathcal{C}$ be an object such that $\mathcal{C}(c,-)$ is a homotopical functor. Prove that the natural transformation $\mathcal{C}(c,-) \to \mathrm{ho}(\mathcal{C})(c,-)$ induced by L is a natural bijection.

SOLUTION: This is [HHR16, Corollary B.6]. By the previous problem, the identity natural transformation $\mathcal{C}(c,-)$ factors through $L\colon \mathcal{C}(c,-) \to \mathrm{ho}(\mathcal{C})(c,-)$, and therefore L is injective. Since $\mathcal{C}(c,-)$ is homotopical, if $f\colon d \to e$ is a weak equivalence, the composition map $f\circ\colon \mathcal{C}(c,f)\colon \mathcal{C}(c,d) \to \mathcal{C}(c,e)$ is an isomorphism. For $f\colon d \to c$ a weak equivalence, let $g \in \mathcal{C}(c,d)$ be the preimage of $1 \in \mathcal{C}(c,c)$ along the isomorphism $f\circ\colon \mathcal{C}(c,d) \to \mathcal{C}(c,c)$. By construction, $f \circ g = 1$. This allows every zig-zag in $\mathrm{ho}(\mathcal{C})(c,d)$ to be rewritten in terms of forward facing arrows, demonstrating that $L\colon \mathcal{C}(c,-) \to \mathrm{ho}(\mathcal{C})(c,-)$ is surjective.

This completes the proof.

References

[HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.

[Mal07] Georges Maltsiniotis. Le théorème de Quillen, d'adjonction des foncteurs dérivés, revisité. *C. R. Math. Acad. Sci. Paris*, 344(9):549–552, 2007.

[Qui67] Daniel G. Quillen. *Homotopical algebra*, volume No. 43 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1967.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.