These problems are not due and will not be graded.

Reading: Read [Hat17] for an introduction to spectral sequences. Read [BC18] for information on the Adams spectral sequence.

(1) In this problem, we prove the following fact:

$$H^*(K(\mathbb{Z},n);\mathbb{Q}) = \begin{cases} \mathbb{Q}[x_n] & (n \text{ even}) \\ E_{\mathbb{Q}}(x_n) & (n \text{ odd}), \end{cases}$$

where $|x_n| = n$ and $E_{\mathbb{Q}}(x_n) \cong \mathbb{Q}[x_n]/(x_n^2)$ is an exterior $\mathbb{Q}$-algebra on a single generator in degree n.

(a) Compute $H^k(K(\mathbb{Z},n);\mathbb{Q})$ for $k \leq n$ without spectral sequences.
SOLUTION: By Hurewicz and the Universal Coefficient Theorem, it's zero in degree $0 < k < n$, and $\mathbb{Q}$ in degrees 0 and n.

(b) Use induction and the Serre spectral sequence for the fiber sequence

$$K(\mathbb{Z}, n-1) \to PK(\mathbb{Z},n) \to K(\mathbb{Z},n)$$

to verify the formula given.
SOLUTION: The spectral sequence in question has signature

$$E_2^{p,q} = H^p(K(\mathbb{Z},n); H^q(K(\mathbb{Z},n-1);\mathbb{Q})) \implies H^{p+q}(PK(\mathbb{Z},n);\mathbb{Q}).$$

However, the path space is contractible, so it converges to just a single $\mathbb{Q}$ in degree 0. We can use this to work backwards to figure out what the spectral sequence must look like. Knowing $H^*(K(\mathbb{Z},n-1);\mathbb{Q})$ will tell us the answer we seek in the inductive step.

For a base case, consider $K(\mathbb{Z},0)$ is just a collection of points, so it has no positive cohomology. By part (a), the cohomology is just $\mathbb{Q}$ in degree zero and zero otherwise.
We also know that $K(\mathbb{Z},1) = S^1$, so $H^*(K(\mathbb{Z},1);\mathbb{Q}) = \mathbb{Q}[x_1]/(x_1^2)$.

For the inductive step, assume that n is even. We have a spectral sequence with E_2-page

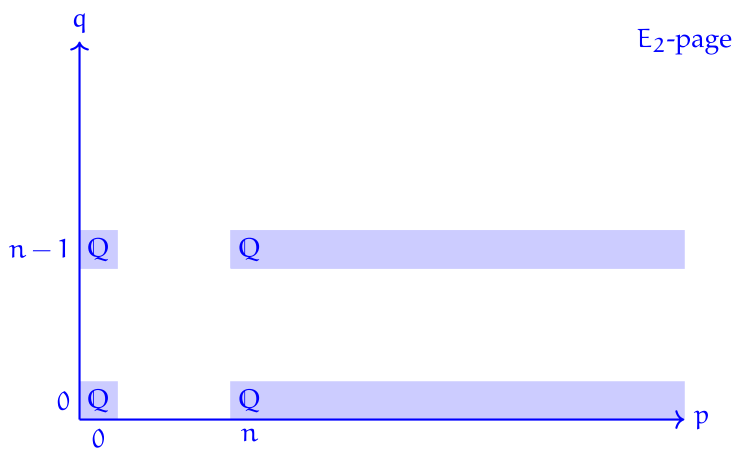
$$E_2^{p,q} = H^p(K(\mathbb{Z},n); H^q(K(\mathbb{Z},n-1);\mathbb{Q})).$$

Because $H^*(K(\mathbb{Z},n-1);\mathbb{Q})$ is exterior on a generator in degree $n-1$, we have

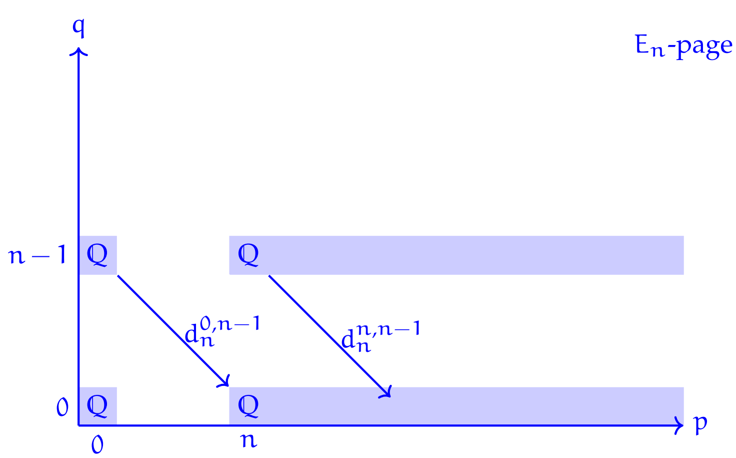
$$E_2^{p,q} = \begin{cases} H^p(K(\mathbb{Z},n);\mathbb{Q}) & (q = 0), \\ H^p(K(\mathbb{Z},n);\mathbb{Q}) & (q = n-1), \\ 0 & \text{otherwise}. \end{cases}$$

A picture of the possible nonzero terms in the E_2-page is below.

By part (a), $H^p(K(\mathbb{Z},n);\mathbb{Q})$ is $\mathbb{Q}$ for $p=0$ and $p=n$ and zero for $0<p<n$. So we can update our picture with this new information.



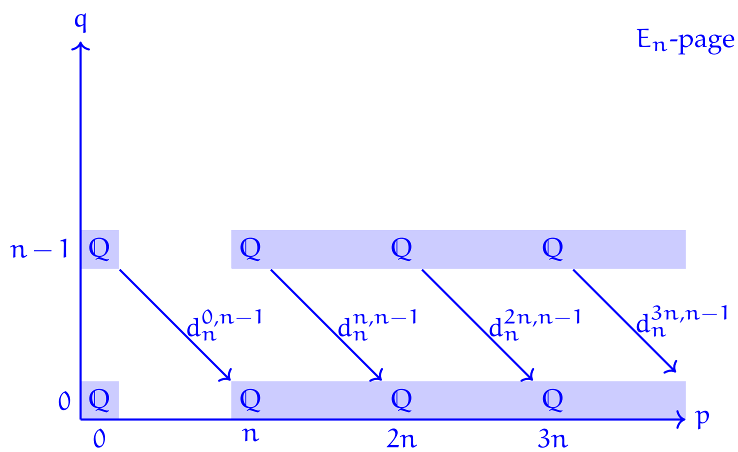
To learn more, we have to think about differentials. In this spectral sequence with this indexing, differentials d_r go down $r-1$ and right r. So there are no differentials until the E_n-page, at which point we might have some differentials. Let's draw them in to our picture.



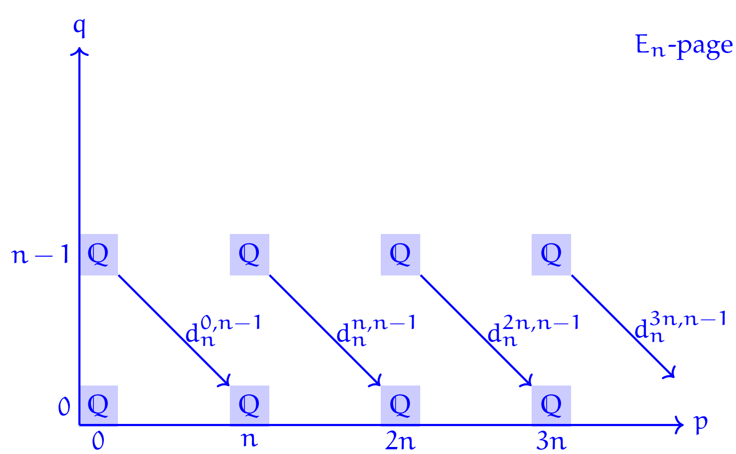
If these differentials were zero, the $\mathbb{Q}$s not at the origin would survive to the E_∞-page. But we know this cannot happen, since the cohomology of the path space is contractible, and the E_∞-page should just be a single $\mathbb{Q}$ at the origin.

Knowing what we know about the E_n-page, the only way that the $\mathbb{Q}$s in the row at $q = n-1$ vanish on the E_{n+1}-page is if a differential exiting from degree $(p, n-1)$ is injective. Similarly, the only way that a $\mathbb{Q}$ in the $q = 0$ row vanishes is if the differential entering degree $(p, 0)$ is surjective. Hence, $d_n^{0,n-1}$ is an isomorphism. Similarly, $d_n^{n,n-1}$ is injective, meaning that $E_n^{2n,0}$ contains $\mathbb{Q}$ as a summand. But if it is larger than just $\mathbb{Q}$, it the differential entering would not be surjective, and this term would survive to the E_∞-page. Therefore, we get a $\mathbb{Q}$ in degree $(2n, 0)$.

Since $E_n^{p,0} = H^p(K(\mathbb{Z}, n); \mathbb{Q})$ and $E_n^{p,n-1} = H^p(K(\mathbb{Z}, n); \mathbb{Q})$ are isomorphic, this means that we must have a $\mathbb{Q}$ in degree $(2n, n-1)$ as well. By the same logic as the previous paragraph, these propogate and we get $\mathbb{Q}$ in degrees $(kn, 0)$ and $(kn, n-1)$ on the E_n-page for all $k \geq 0$.



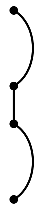
We can now eliminate terms in degrees $(p, 0)$ with $n < p < 2p$, because there is nothing to surject onto them. This means that there are no terms in degrees $(p, n-1)$ with $n < p < 2n$ as well, and then there is nothing to surject onto degrees $(p, 0)$ with $2n < p < 3n$, and so on. We end up with an E_n-page that looks like the following, with all differentials isomorphisms.

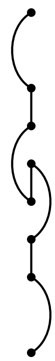


One can learn the multiplicative structure of the cohomology from this, but we'll save that for another day.

The case n odd is similar.

(2) Let $\mathcal{A}_1$ be the subalgebra of the Steenrod algebra $\mathcal{A}$ generated by Sq^1 and Sq^2. A depiction of $\mathcal{A}_1$ as a module over itself appears in [BC18, Figure 3]. For each of the following $\mathcal{A}_1$-modules M, draw the first few stages of a projective resolution of M and write an Adams chart for $\mathrm{Ext}_{\mathcal{A}}^{s,t}(M, \mathbb{Z}/2)$.

(a) $M_0 =$  SOLUTION: [BC18, Example 4.4.2, Figure 15]

(b) $M_1 =$  SOLUTION: [BC18, Example 4.4.2, Figure 18]

(c) $\Sigma^{-2}\widetilde{H}^*(\mathbb{C}P^\infty; \mathbb{Z}/2) =$  SOLUTION: [BC18, Example 4.5.6]

REFERENCES

[BC18] Agnès Beaudry and Jonathan A. Campbell. A guide for computing stable homotopy groups. In *Topology and quantum theory in interaction*, volume 718 of *Contemp. Math.*, pages 89–136. Amer. Math. Soc., Providence, RI, 2018.

[Hat17] Allen Hatcher. Spectral Sequences. `https://pi.math.cornell.edu/~hatcher/AT/SSpage.html`, 2017.