A Few Notes On The Riemann Zeta Function

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1 Introduction

TBD

1.1 Dirichlet Series

A Dirichlet Series [1] is an infinite series of the form

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1}$$

for $s \in \mathbb{C}$. Convention seems to be that it is unnecessary to note that $n \in \mathbb{N}^+$.

The complex variable s in Equation 1 is represented as $s = \alpha + it$. We use the notation $\Re(s)$ and $\Im(s)$ to indicate the real and imaginary parts of s respectively. That is, $\alpha = \Re(s)$ and $t = \Im(s)$.

Perhaps the most famous example of a Dirichlet series is the Riemann zeta function $\zeta(s)$, where we take $a_n = 1$ for all $n \in \mathbb{N}^+$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2}$$

2 Euler's Product Formula

Recall that the Riemann zeta function (Equation 2) is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $s \in \mathbb{C}$.

In 1737 Leonhard Euler [3] discovered the beautiful connection between the zeta function and the prime numbers by proving this identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \left(\frac{1}{1 - p^{-s}} \right) \tag{3}$$

The left side of Equation 3 is by definition $\zeta(s)$ (Equation 2). The infinite product on the right side of Equation 3 extends over all prime numbers p and is called a Euler Product [2]. A Euler Product is the expansion of a Dirichlet series (Equation 1) into an infinite product indexed by prime numbers:

$$\prod_{p \in \mathbb{P}} \left(\frac{1}{1 - p^{-s}} \right) = \frac{1}{1 - 2^{-s}} \cdot \frac{1}{1 - 3^{-s}} \cdot \frac{1}{1 - 5^{-s}} \cdot \frac{1}{1 - 7^{-s}} \cdot \frac{1}{1 - 11^{-s}} \cdots \frac{1}{1 - p^{-s}} \cdots$$

The Euler Product Formula (and therefore the zeta function) converges for $\Re(s) > 1$.

References

[1] John E. McCarthy. Dirichlet Series. `https://www.math.wustl.edu/~mccarthy/amaster-ds.pdf`, 2018. [Online; accessed 25-June-2021].

[2] Noam D. Elkies. Math 259: Introduction to Analytic Number Theory. `http://abel.math.harvard.edu/~elkies/M259.06/euler.pdf`, 1998. [Online; accessed 25-June-2021].

[3] Timothy Murphy. Euler's Product Formula. `https://www.maths.tcd.ie/pub/Maths/Courseware/428/Primes-II.pdf`, 2006. [Online; accessed 25-June-2021].