A Few Notes on the Fourier Series

David Meyer

dmm613@gmail.com

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1 Introduction

2 Vector Spaces: Linear Algebra vs. the Fourier Series

Here we will use the convention that the constant c denotes the constant function $f(x) = c$, for all x, when the context indicates that c is a function. On the other hand, the notation $c \cdot c$ denotes the scalar multiplication of the scalar value c with itself. Consequentially, I will use $\mathbf{1}$ to represent the constant function $f(x) = 1$ and use context to disambiguate the function $f(x)$ from the scalar value 1. For example, in Equation (1), $\langle \mathbf{1}, \mathbf{1} \rangle = \langle f(x), f(x) \rangle$, while the notation $1 \cdot 1$ represents the scalar multiplication of the scalar value 1 with itself.

What	**Linear Algebra**	**Fourier Series**
Vector Space	$\mathbb{R}^n$	Piecewise smooth 2π-periodic functions on $\mathbb{R}$
Inner Product	$\langle \mathbf{u}, \mathbf{v} \rangle = \sum\limits_{i=1}^{n} u_i v_i$	$\langle f(t), g(t) \rangle = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(t) g(t)\, dt$
Orthonormal Basis $\left(\mathbb{R}^3\right)$	$\{(1,0,0), (0,1,0), (0,0,1)\}$	$\{\mathbf{1}, \cos mt, \sin nt\},\ m, n \in \mathbb{N} \backslash \{0\}$
Representation of a Vector in the Basis	$\mathbf{x} = \sum\limits_{i=1}^{n} a_i \mathbf{e}_i$	$f(t) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nt + \sum\limits_{n=1}^{\infty} b_n \sin nt$
Coefficients are Projections	$a_i = \langle \mathbf{x}, \mathbf{e}_i \rangle$	$a_0 = \langle f(t), 1 \rangle$ $a_m = \langle f(t), \cos mt \rangle$ $b_m = \langle f(t), \sin mt \rangle$

Table 1: Vector Spaces: Linear Algebra vs. Fourier Series

2.1 Orthonormal Bases and the Fourier Series

Some people (specifically Rahul Narain (@narain@mathstodon.xyz)) feel that the orthonormal basis for the Fourier series should be $\{\mathbf{1}, \sqrt{2} \cos mt, \sqrt{2} \sin nt\},\ m, n \in \mathbb{N} \backslash \{0\}$.

Ok, but why is Narain arguing this? My guess is as follows: First, we know that the inner product of a vector $\mathbf{u}$ with itself equals 1 ($\langle \mathbf{u}, \mathbf{u} \rangle = 1$). However, for the Fourier series (where $\mathbf{1}$ represents the constant function $f(t) = 1$) we have $\langle \mathbf{1}, \mathbf{1} \rangle = 2$. Why? We can see this in Equation (1) and in a bit more detail in Figure 1.

$$\langle \mathbf{1}, \mathbf{1} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \cdot 1 \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} dt = \frac{1}{\pi} t \Big|_{-\pi}^{\pi} = \frac{1}{\pi} \big(\pi - (-\pi)\big) = \frac{1}{\pi} 2\pi = 2 \tag{1}$$

$$\begin{aligned}
\langle \mathbf{1}, \mathbf{1} \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) f(t) \, dt && \text{\# since } \mathbf{1} \text{ represents } f(t) = 1 \text{ and the definition of the inner product (Table 1)} \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \cdot 1 \, dt && \text{\# since in } f(t) = 1 \text{ for all } t \in \mathbb{R} \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} dt && \text{\# since } 1 \cdot 1 = 1 \\
&= \frac{1}{\pi} t \Big|_{-\pi}^{\pi} && \text{\# by the FToC [2]} \\
&= \frac{1}{\pi} \big(\pi - (-\pi)\big) && \text{\# evaluate at the end points} \\
&= \frac{1}{\pi} 2\pi && \text{\# simplify} \\
&= 2 && \text{\# } \langle 1, 1 \rangle = 2 \text{ [1]}
\end{aligned}$$

Figure 1: Value of $\langle 1, 1 \rangle$ for Fourier Series

On the other hand, $\langle \cos nt, \cos nt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 nt \, dt = \frac{1}{\pi} \pi = 1$. In the same way, $\langle \sin nt, \sin nt \rangle = 1$.

We can also see that the vectors in the basis $\{\mathbf{1}, \cos mt, \sin nt\}, m, n \in \mathbb{N} \backslash \{0\}$ are orthogonal:

$$\langle \mathbf{1}, \sin nt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \sin nt \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin nt \, dt = 0$$

Similarly, $\langle \mathbf{1}, \cos nt \rangle = \langle \cos mt, \sin nt \rangle = 0$.

Note that some authors set $\langle \mathbf{1}, \mathbf{1} \rangle = \int_{0}^{2\pi} dt = 2\pi$ [1].

3 Conclusions

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LaTeX Source

`https://www.overleaf.com/read/mtpfwbpmwcpg#b65e5d`

References

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