A Note on Algebraic Structures

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1 A Few Algebraic Structures and Their Features

Structure	**ABO**[1]	**Identity**	**Inverse**	**Distributive**[2]	**Commutative**[3]	**Comments**
Semigroup	✓	no	no	N/A	no	$(S, \circ)$
Monoid	✓	✓	no	N/A	no	Semigroup with identity $\in S$
Group	✓	✓	✓	N/A	no	Monoid with inverses: $a \in S \backslash \{0\} \Rightarrow a^{-1} \in S$
Abelian Group	✓	✓	✓	N/A	✓$(\circ)$	Commutative group
Ring_+	✓	✓	✓	N/A	✓$(+)$	Abelian group under $+$
Ring_*	✓	yes/no	no	✓	no	Monoid under $*$
Division Ring	✓	✓$(+, *)$	✓$(+, *)$	✓	✓$(+)$	Ring with multiplicative inverses
Field	✓	✓$(+, *)$	✓$(+, *)$	✓	✓$(+, *)$	Division ring with commutative multiplication
Module	✓	✓$(+, *)$	✓$(+)$	✓	✓$(+)$	Abelian group under $+$, scalars $\in$ Ring
Vector Space	✓	✓$(+, *)$	✓$(+)$	✓	✓$(+)$	Abelian group under $+$, scalars $\in$ Field
Algebra over a Ring	yes/no	✓$(+, *)$	✓$(+)$	✓	✓$(+)$	Module with bilinear product[4]
Algebra over a Field	yes/no	✓$(+, *)$	✓$(+)$	✓	✓$(+)$	Vector space with bilinear product

Table 1: A Few Algebraic Structures and Their Features

1.1 Definitions

1. **ABO:** Associative Binary Operation
   - $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$
   - $x \circ y \in S$ for all $x, y \in S$ (S is closed under $\circ$)
2. **Distributive:** Distributive Property
   - Left Distributive Property: $x * (y + z) = (x * y) + (x * z)$ for all $x, y, z \in S$
   - Right Distributive Property: $(y + z) * x = (y * x) + (z * x)$ for all $x, y, z \in S$
   - $*$ is *distributive* over $+$ if $*$ is left and right distributive
3. **Commutative:** Commutative Property
   - $x \circ y = y \circ x$ for all $x, y \in S$

4. **Bilinear Map:** A bilinear map is a function combining elements of two vector spaces to yield an element of a third vector space, and is linear in each of its arguments [2]. Matrix multiplication is an example.

   More specifically, a bilinear map is a function $B : V \times W \to Z$ such that for all $v_1, v_2 \in V$, $w_1, w_2 \in W$, and scalars $\alpha \in \mathbb{F}$:

   $$B(\alpha v_1 + v_2, w) = \alpha B(v_1, w) + B(v_2, w)$$

   and

   $$B(v, \alpha w_1 + w_2) = \alpha B(v, w_1) + B(v, w_2)$$

   Note that it may be the case that $V = W = Z$.

2 Notes

- Table 1 implies that F $\subset$ R $\subset$ G $\subset$ M $\subset$ SG.
- Whether or not a ring has a multiplicative identity seems to depend on the field of study.

  In general the definition of a ring R doesn't require a multiplicative inverse in R ($a^{-1} \notin R$ for all $a \in R$) or that multiplication be commutative in R. Specifically: R is an Abelian group under $+$ but we don't require that multiplication be commutative (while $a + b = b + a$ for all $a, b \in R$, we don't require that $ab = ba$ for all $a, b \in R$). These are perhaps the main ways in which a ring differs from a field. In addition, as mentioned above in some cases R need not include a multiplicative identity ($1 \notin R$).
- VS $\subset$ Module since the scalars in a module come from a ring as opposed to a field like we find in vector spaces and F $\subset$ R [1].

LaTeX Source

https://www.overleaf.com/read/fcfcnyxmgzwv

References

[1] T. S. Blyth. *Module Theory: An Approach to Linear Algebra.* Oxford University Press, July 1977.

[2] Wikipedia contributors. Bilinear map — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Bilinear_map&oldid=1239420853, 2024. [Online; accessed 16-December-2024].