A Few Notes on Euler's Formula and Euler's Identity

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1 Why does a complex number z equal $re^{i\theta}$?

Our first question regards the nature of a complex number z, where the length of the line from the origin to the point z, $|z|$, equals r. In particular, why does a complex number $z = a + bi = re^{i\theta}$?

Well, consider the complex plane, shown in Figure 1:

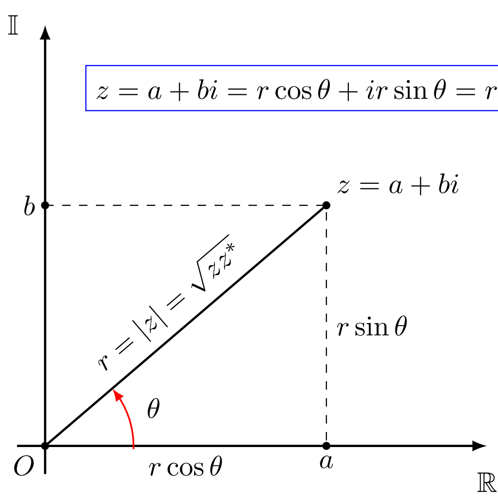

Figure 1: The Complex Plane

This picture makes it easy to see why $z = re^{i\theta}$. Specifically

$$
\begin{aligned}
z &= a + ib && \text{\# definition of a point } z \text{ in the complex plane} \\
&= r\cos\theta + ir\sin\theta && \text{\# switch to polar coordinates: } a = r\cos\theta \text{ and } b = r\sin\theta \\
&= r\left[\cos\theta + i\sin\theta\right] && \text{\# factor out } r \\
&= re^{i\theta} && \text{\# Euler's formula [1]: } e^{i\theta} = \cos\theta + i\sin\theta
\end{aligned}
$$

2 Euler's Formula

This is all good, but where does Euler's formula, depicted in Figure 2, come from?

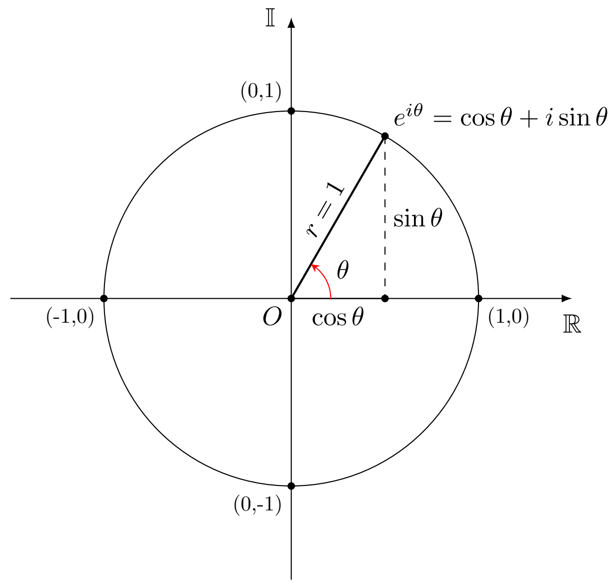

Figure 2: Euler's Formula, the Unit Circle, and the Complex Plane

There are several ways to derive Euler's formula. One method uses the Maclaurin series for $\cos\theta$, $\sin\theta$ and e^x [4]. To see how this works, notice that the Maclaurin series for $\cos\theta$ is

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \tag{1}$$

while the Maclaurin series for $\sin\theta$ is

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \tag{2}$$

Finally, the Maclaurin series for e^x is

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{3}$$

Now, if we set $x = i\theta$ in Equation (3) we see that

$$
\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots && \text{\# set } x = i\theta \text{ in Equation (3)} \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \cdots && \text{\# } i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ \ldots \\
&= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right] + \left[i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \cdots\right] && \text{\# group terms} \\
&= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right] && \text{\# factor out } i \text{ in the second term} \\
&= \cos\theta + i\sin\theta && \text{\# Equations (1) and (2)}
\end{aligned}
$$

So we wind up with

$$
\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta \tag{4}
$$

that is, Euler's formula.

2.1 Aside: Why does $e^{2\pi ik} = 1$ for $k \in \mathbb{Z}$?

Consider the following:

- **Euler's formula relates the exponential function e^{ix} to the sine and cosine functions**

  The exponential function e^{ix} is connected to $\cos x$ and $\sin x$ by Euler's formula (Equation (4)).

- **Sine and cosine are periodic functions**

  Both the cosine and sine functions are periodic with a period of 2π. This means that $\cos(x+2\pi k) = \cos x$ for $k \in \mathbb{Z}$, and $\sin(x + 2\pi k) = \sin x$, again for $k \in \mathbb{Z}$.

- **$e^{2\pi ik}$ is related to the natural log of 1:** $e^z = 1 \Rightarrow z = \ln 1$

  The exponential function $e^z = 1$ has solutions when $z = 2\pi ik$ for $k \in \mathbb{Z}$, since

$$
\begin{aligned}
e^z &= e^{2\pi ik} && \text{\# set } z = 2\pi ik \text{ for } k \in \mathbb{Z} \\
&= \cos(2\pi k) + i\sin(2\pi k) && \text{\# by Euler's formula (Equation (4))} \\
&= 1 + i \cdot 0 && \text{\# since } \cos(2\pi k) = 1 \text{ and } \sin(2\pi k) = 0 \text{ for } k \in \mathbb{Z} \\
&= 1 && \text{\# since } i \cdot 0 = 0 \text{ and } 1 + 0 = 1 \\
&\Rightarrow e^{2\pi ik} = 1 && \text{\# by the previous lines} \\
&\Rightarrow e^z = 1 && \text{\# } z = 2\pi ik \text{ for } k \in \mathbb{Z}
\end{aligned}
$$

So interestingly, the expression $e^{2\pi ik} = 1$ reflects the periodic nature of the exponential function in the complex plane. This periodicity is shown in Figure 2.

2.2 Deriving Euler's Formula Using Polar Coordinates

There is also a clever way to derive Euler's formula using polar coordinates. This approach seems to require the fewest assumptions of the non-Maclaurin series derivations described in the this and the following sections. To see how this works, first notice that all non-zero complex numbers can be expressed in polar coordinates in a unique way. In particular, any number of the form e^{ix} (with $x \in \mathbb{R}$) which is non-zero can be expressed as[1]

$$e^{ix} = r\,(\cos\theta + i\sin\theta) \tag{5}$$

This is shown in Figure 1. In this case θ is the principal angle from the positive real axis (with say, $0 \leq \theta \leq 2\pi$) and r is its radius ($r > 0$). One of the features of this proof is that AFAICT it makes no assumption about the values of r and θ, except for the fact that they are functions of x (we can think of r as shorthand for $r(x)$; likewise θ is shorthand for $\theta(x)$). An interesting problem that has to be solved in this proof is what the relationships between x, r and θ are. These relationships (and the values of r and θ) will be determined in the course of the proof.

So to start, what do we know? Well, we know that when $x = 0$ the left-hand side of Equation (5) equals 1, which implies that r and θ, being functions of x, must satisfy the initial conditions that $r(0) = 1$ and $\theta(0) = 0$ (since $1 = 1 \cdot \big[\cos\theta(0) + i\sin\theta(0)\big] = 1 \cdot [\cos 0 + i\sin 0] = 1 \cdot [1 + i \cdot 0] = 1 \cdot [1 + 0] = 1$).

Now, if we differentiate both sides of Equation (5) with respect to x we see that

$$\begin{aligned}
\frac{d}{dx}e^{ix} &= \frac{d}{dx}\Big[r\,(\cos\theta + i\sin\theta)\Big] && \text{\# differentiate both sides with respect to } x\\
&\Rightarrow\quad ie^{ix} = \frac{d}{dx}\Big[r\,(\cos\theta + i\sin\theta)\Big] && \text{\# apply the chain rule to the LHS}\\
&\Rightarrow\quad ie^{ix} = \frac{dr}{dx}(\cos\theta + i\sin\theta) + r\frac{d}{dx}\Big[\cos\theta + i\sin\theta\Big] && \text{\# apply the product rule to the RHS}\\
&\Rightarrow\quad ie^{ix} = \frac{dr}{dx}(\cos\theta + i\sin\theta) + r\left[\frac{d}{dx}\cos\theta + i\frac{d}{dx}\sin\theta\right] && \text{\# derivative is a linear operator [3]}\\
&\Rightarrow\quad ie^{ix} = \frac{dr}{dx}(\cos\theta + i\sin\theta) + r\left[(-\sin\theta)\frac{d\theta}{dx} + (i\cos\theta)\frac{d\theta}{dx}\right] && \text{\#}\ \frac{d}{dx}\cos u = -\sin u\frac{du}{dx},\ \frac{d}{dx}\sin u = \cos u\frac{du}{dx}\\
&\Rightarrow\quad ie^{ix} = \frac{dr}{dx}(\cos\theta + i\sin\theta) + r(-\sin\theta + i\cos\theta)\frac{d\theta}{dx} && \text{\# factor out}\ \frac{d\theta}{dx}\\
&\Rightarrow\quad ie^{ix} = (\cos\theta + i\sin\theta)\frac{dr}{dx} + r(-\sin\theta + i\cos\theta)\frac{d\theta}{dx} && \text{\# put the RHS into a more convenient form}
\end{aligned}$$

Note that we are looking for an expression that is unique in terms of r and θ so we would like to get rid the e^{ix} term. One way to do this is to substitute $r(\cos\theta + i\sin\theta)$ for e^{ix} (Equation (5)) on the left hand side of the last equation above, which gives us

$$ir(\cos\theta + i\sin\theta) = (\cos\theta + i\sin\theta)\frac{dr}{dx} + r(-\sin\theta + i\cos\theta)\frac{d\theta}{dx}$$

Multiplying through by i on the left hand side gives

[1]Note to self: remember to observe the difference between Equation (5), where e is raised to the ix power, and Euler's formula, where e is raised to the $i\theta$ power.

$$r(i\cos\theta - \sin\theta) = (\cos\theta + i\sin\theta)\frac{dr}{dx} + r(-\sin\theta + i\cos\theta)\frac{d\theta}{dx}$$

and multiplying through on both sides side gives

$$ir\cos\theta - r\sin\theta = (\cos\theta)\frac{dr}{dx} + (i\sin\theta)\frac{dr}{dx} + (-r\sin\theta)\frac{d\theta}{dx} + (ir\cos\theta)\frac{d\theta}{dx} \tag{6}$$

Next, equating the imaginary and real parts on both sides of Equation (6) we see that

$$ir\cos\theta = i\sin\theta\frac{dr}{dx} + ir\cos\theta\frac{d\theta}{dx}$$

and

$$-r\sin\theta = \cos\theta\frac{dr}{dx} - r\sin\theta\frac{d\theta}{dx}$$

Now we have is a system of two equations in two unknowns: $\frac{dr}{dx}$ and $\frac{d\theta}{dx}$. To solve this system, first assign $\alpha = \frac{dr}{dx}$ and $\beta = \frac{d\theta}{dx}$ so that

$$r\cos\theta = (\sin\theta)\alpha + (r\cos\theta)\beta \tag{7}$$

and

$$-r\sin\theta = (\cos\theta)\alpha - (r\sin\theta)\beta \tag{8}$$

Next, by multiplying Equation (7) by $\cos\theta$ and Equation (8) by $\sin\theta$ we get

$$r\cos^2\theta = (\sin\theta\cos\theta)\alpha + (r\cos^2\theta)\beta \tag{9}$$

and

$$-r\sin^2\theta = (\sin\theta\cos\theta)\alpha - (r\sin^2\theta)\beta \tag{10}$$

We can eliminate α by subtracting Equation (10) from Equation (9) to get

$$r(\cos^2\theta + \sin^2\theta) = r(\cos^2\theta + \sin^2\theta)\beta \tag{11}$$

and since $\cos^2\theta + \sin^2\theta = 1$ Equation (11) simplifies to

$$r = r\beta \tag{12}$$

Since $r > 0$ for all x Equation (12) tells us that β, which we set equal to $\frac{d\theta}{dx}$, must equal 1. If we substitute $\frac{d\theta}{dx} = 1$ back into Equations (7) and (8) we see that

$$
\begin{aligned}
0 &= (\sin\theta)\alpha \\
0 &= (\cos\theta)\alpha
\end{aligned}
$$

which implies that $\alpha\sin\theta = \alpha\cos\theta$ which in turn implies that $\alpha = 0$. But why does $\alpha = 0$? One way to think about this is to consider that $\alpha\sin\theta = \alpha\cos\theta \Rightarrow \dfrac{\sin\theta}{\cos\theta} = \tan\theta = 1$. This occurs when $\theta = \dfrac{\pi}{4} + n\pi$ for $n \in \mathbb{Z}$. Since neither $\sin\theta$ nor $\cos\theta$ equals zero for these values of θ we can conclude that α must equal zero.

Now, recall that $\alpha = \dfrac{dr}{dx}$ so now we know that $\dfrac{dr}{dx} = 0$ and $\dfrac{dr}{dx} = 0 \Rightarrow r(x) = a$ for some constant a. Similarly, since $\dfrac{d\theta}{dx} = 1$ we know that $d\theta = dx$ and integrating both sides we see that

$$
\int d\theta = \int dx \Rightarrow \theta(x) = x + C \tag{13}
$$

for some constant C.

Going back to the very beginning where we saw that $r(0) = 1$ and since we know that $r(x) = a$ for all x, r must equal 1 for all x. Similarly because $\theta(0) = 0$ we see that $\theta(0) = 0 + C$ (Equation (13)), so it must be that C equals zero. Said another way, now we know that $\theta(x) = x$ or more concisely, $\theta = x$.

So $r = 1$ and $\theta = x$. Substituting these values back into Equation (5) we see that

$$
\begin{aligned}
e^{ix} &= r \cdot [\cos\theta + i\sin\theta] && \text{\# Equation (5)} \\
&= 1 \cdot [\cos x + i\sin x] && \text{\# } r = 1 \text{ and } \theta = x \\
&= \cos x + i\sin x && \text{\# Euler's formula}
\end{aligned}
$$

2.3 Another Approach To Euler's Formula Without Maclaurin Series

A different way to derive Euler's formula without using Maclaurin series starts by defining

$$
f(\theta) = e^{-i\theta}\,(\cos\theta + i\sin\theta)
$$

for all real θ. Then

$$
\begin{aligned}
f(\theta) \quad = \quad & e^{-i\theta}\left(\cos\theta + i\sin\theta\right) && \text{\# definition of } f(\theta) \\
\Rightarrow \quad & f'(\theta) = \frac{d}{d\theta}\Big[e^{-i\theta}\left(\cos\theta + i\sin\theta\right)\Big] && \text{\# take derivative of both sides} \\
\Rightarrow \quad & f'(\theta) = e^{-i\theta}\cdot\frac{d}{d\theta}(\cos\theta + i\sin\theta) + \frac{d}{d\theta}e^{-i\theta}(\cos\theta + i\sin\theta) && \text{\# product rule} \\
\Rightarrow \quad & f'(\theta) = e^{-i\theta}(i\cos\theta - \sin\theta) + \frac{d}{d\theta}e^{-i\theta}(\cos\theta + i\sin\theta) && \text{\# } \frac{d}{d\theta}(\cos\theta + i\sin\theta) = i\cos\theta - \sin\theta \\
\Rightarrow \quad & f'(\theta) = e^{-i\theta}(i\cos\theta - \sin\theta) + ie^{-i\theta}(\cos\theta + i\sin\theta) && \text{\# chain rule} \\
\Rightarrow \quad & f'(\theta) = e^{-i\theta}\cdot\Big[(i\cos\theta - \sin\theta) - i(\cos\theta + i\sin\theta)\Big] && \text{\# factor out } e^{-i\theta} \\
\Rightarrow \quad & f'(\theta) = e^{-i\theta}\cdot\Big[(i\cos\theta - \sin\theta) - (i\cos\theta - \sin\theta)\Big] && \text{\# multiply through by } i \\
\Rightarrow \quad & f'(\theta) = e^{-i\theta}\cdot 0 && \text{\# } \Big[(i\cos\theta - \sin\theta) - (i\cos\theta - \sin\theta)\Big] = 0 \\
\Rightarrow \quad & f'(\theta) = 0 && \text{\# simplify}
\end{aligned}
$$

Since $f'(\theta) = 0$ we know that $f(\theta)$ is a constant (that is, $f(x) = a$ for some constant a). We also know that

$$f(0) = e^{-i0}\cdot\big[\cos 0 + i\sin 0\big] = 1\cdot\big[1 + 0\big] = 1$$

Now, since $f(\theta)$ is a constant and $f(0) = 1$ we know that $f(\theta) = 1$ for all real θ. Dividing both sides by $e^{-i\theta}$ yields

$$\frac{f(\theta)}{e^{-i\theta}} = \frac{1}{e^{-i\theta}} = \frac{e^{-i\theta}\cdot(\cos\theta + i\sin\theta)}{e^{-i\theta}}$$

and we wind up with Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

2.4 One More Approach To Euler's Formula

Finally, while the approach in this section is similar to the approach proposed in Section 2.3, some people prefer this approach as it seems to require fewer assumptions (as I mentioned above, the approach taken in Section 2.2 seems to require the fewest assumptions). In any event, the proof goes as follows:

$$
\begin{aligned}
z \quad = \quad & \cos\theta + i\sin\theta && \text{\# definition of a complex number (Figure 1)} \\
\Rightarrow \quad & \frac{dz}{d\theta} = \frac{d}{d\theta}\left[\cos\theta + i\sin\theta\right] && \text{\# take the derivative of both sides} \\
\Rightarrow \quad & \frac{dz}{d\theta} = \frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}i\sin\theta && \text{\# derivative is a linear operator} \\
\Rightarrow \quad & \frac{dz}{d\theta} = -\sin\theta + i\cos\theta && \text{\# } \frac{d}{d\theta}\cos\theta = -\sin\theta \text{ and } \frac{d}{d\theta}i\sin\theta = i\cos\theta \\
\Rightarrow \quad & \frac{dz}{d\theta} = iz && \text{\# } iz = i\left[\cos\theta + i\sin\theta\right] = -\sin\theta + i\cos\theta
\end{aligned}
$$

So now we know that $\frac{dz}{d\theta} = iz$, which implies that $\frac{dz}{z} = i\,d\theta$. This means that

$$\begin{aligned}
\frac{dz}{z} &= i\,d\theta && \text{\# result from above} \\
&\Rightarrow \int \frac{dz}{z} = \int i\,d\theta && \text{\# integrate both sides} \\
&\Rightarrow \ln|z| = i\theta + C && \text{\# } \int \frac{dz}{z} = \ln|z| \text{ and } \int i\,d\theta = i\,\theta + C \\
&\Rightarrow z = e^{i\theta + C} && \text{\# exponentiate both sides: } e^{\ln|z|} = z \text{ and } e^{i\theta+C} = e^{i\theta+C}
\end{aligned}$$

Now, we know $z(0) = \cos 0 + i \sin 0 = 1 + i \cdot 0 = 1$. We also know that $z(0) = e^{0i+C} = e^C = 1$. Since $e^C = 1$ we know that C must equal 0. Putting all of this together we see that $z = e^{i\theta} = \cos\theta + i\sin\theta$, that is, Euler's formula.

3 Euler's Identity

Something interesting happens when we set $\theta = \pi$ in Euler's formula. We get Euler's identity [2].

$$\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta && \text{\# Euler's formula (Equation (4))} \\
&\Rightarrow e^{i\pi} = \cos\pi + i\sin\pi && \text{\# set } \theta = \pi \\
&\Rightarrow e^{i\pi} = -1 + i \cdot 0 && \text{\# } \cos\pi = -1 \text{ and } \sin\pi = 0 \\
&\Rightarrow e^{i\pi} = -1 + 0 && \text{\# } i \cdot 0 = 0 \\
&\Rightarrow e^{i\pi} = -1 && \text{\# simplify} \\
&\Rightarrow e^{i\pi} + 1 = 0 && \text{\# Euler's identity}
\end{aligned}$$

4 Conclusions

5 Acknowledgements

Thanks to Dave Neary for suggesting a non-Maclaurin series derivation of Euler's formula and encouraging my study of the non-Maclaurin series derivations of Euler's formula.

LaTeX Source

`https://www.overleaf.com/read/gczxmjgfrxxb`

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