A Few Notes On The Taylor Series

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1 Introduction

2 Taylor Series

3 Taylor Inequality

Suppose that the function $f(x)$ is infinitely differentiable (smooth) at $x = a$. Then as we saw in Section 2, the Taylor series for $f(x)$, centered at $x = a$, is

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{1}$$

We can divide $T(x)$ into a Taylor polynomial of degree n, denoted $T_n(x)$, and an infinite series $R_n(x)$, called the *Taylor remainder*, such that

$$T(x) = T_n(x) + R_n(x) \tag{2}$$

Here $T_n(x) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ and $R_n(x) = \sum\limits_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$.

Understanding how $R_n(x)$ relates to $T(x)$, that is, $R_n(x) = T(x) - T_n(x)$, will allow us to understand, among other things, how good of an approximation $T(x)$ is to $f(x)$, whether or not $T(x)$ converges, and to what.

BTW, a useful form of Equation 2 to think of $R_n(x)$ as the difference between $f(x)$ and the Taylor polynomial of degree n for $f(x)$ centered at a. That is

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \tfrac{f^{(k)}(a)}{k!}(x-a)^k && \text{\# Taylor series of } f(x) \\
&\Rightarrow \quad T_n(x) = \sum_{k=0}^{n} \tfrac{f^{(k)}(a)}{k!}(x-a)^k && \text{\# definition of the Taylor polynomial } T_n(x) \\
&\Rightarrow \quad R_n(x) = f(x) - T_n(x) && \text{\# definition of the Taylor remainder } R_n(x) \\
&\Rightarrow \quad R_n(x) = \sum_{k=0}^{\infty} \tfrac{f^{(k)}(a)}{k!}(x-a)^k - \sum_{k=0}^{n} \tfrac{f^{(k)}(a)}{k!}(x-a)^k && \text{\# expand } f(x) \text{ and } T_n(x) \\
&\Rightarrow \quad R_n(x) = \tfrac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{(n+1)} && \text{\# see comment below}
\end{aligned}
$$

The last line above, $R_n(x) = \sum_{k=n+1}^{\infty} \tfrac{f^{(k)}(a)}{k!}(x-a)^k = \tfrac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{(n+1)}$, follows because for $k > n+1$ $f^{(k)}(a) = 0$ ($f^{(k)}(x)$ for $k > n+1$ has a $(x-a)^k$ term which is zero at $x = a$).

So now we can see that the Taylor remainder $R_n(x)$ is a kind of error in our Taylor polynomial $T_n(x)$'s estimate of $f(x)$:

$$
R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{(n+1)} \tag{3}
$$

We can use Equations 2 and 3 to state the Taylor Remainder Theorem:

Theorem 3.1. Taylor Remainder Theorem: If $|f^{(n)}| \leq M$ for $|x-a| \leq d$ then the remainder $R_n(x)$ of the Taylor series $T(x)$ satisfies the inequality

$$
\left|R_n(x)\right| \leq \frac{M}{(n+1)!}|x-a|^{n+1}
$$

for $|x-a| \leq d$, where M and d are constants.

Sketch of Proof:

Acknowledgements