What does the series $S = \sum_{n=1}^{\infty} \left(\frac{a}{b}\right)^n$ converge to?

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1 First up: does this series converge?

Here we'll use the ratio test for convergence [2] and so we want to think of S as

$$S = \sum_{n=1}^{\infty} a_n \tag{1}$$

where $a_n = \left(\frac{a}{b}\right)^n$.

The usual form of the ratio test uses the limit $L = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. The ratio test tells us that

1. If $L < 1$ then the series converges absolutely.
2. If $L > 1$ then the series diverges.
3. If $L = 1$ (or the limit doesn't exist) then the test is inconclusive.

To apply the ratio test we want to compute the following limit:

$$L = \lim_{n\to\infty} \left| \frac{\left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right)^n} \right|$$

Since $\lim\limits_{n\to\infty} c = c$ for constant c and since $\dfrac{a}{b}$ is a constant with respect to n we see that the limit L is

$$L = \lim_{n\to\infty} \left| \frac{\left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right)^n} \right| = \lim_{n\to\infty} \left|\frac{a}{b}\right| = \frac{a}{b}$$

If $a < b$ then $\dfrac{a}{b} < 1$ and so by clause 1 of the ratio test we know that S converges absolutely.

2 Ok, S converges. What does it converge to?

Since we know that S converges absolutely when $a < b$, here is one way to think about the question:

$$
\begin{aligned}
S &= \sum_{n=1}^{\infty} \left(\tfrac{a}{b}\right)^n && \text{\# definiton of } S \text{ (Equation (1))} \\
&= \left(\tfrac{a}{b}\right)^1 + \left(\tfrac{a}{b}\right)^2 + \left(\tfrac{a}{b}\right)^3 + \cdots && \text{\# expand } S \\
\Rightarrow &\ \left(\tfrac{b}{a}\right) \cdot S = \left(\tfrac{b}{a}\right) \cdot \left[\left(\tfrac{a}{b}\right)^1 + \left(\tfrac{a}{b}\right)^2 + \left(\tfrac{a}{b}\right)^3 + \ \cdots\right] && \text{\# multiply both sides by } \left(\tfrac{b}{a}\right) \\
\Rightarrow &\ \left(\tfrac{b}{a}\right) \cdot S = 1 + \left[\left(\tfrac{a}{b}\right)^1 + \left(\tfrac{a}{b}\right)^2 + \left(\tfrac{a}{b}\right)^3 + \ \cdots\right] && \text{\# multiply through on right side} \\
\Rightarrow &\ \left(\tfrac{b}{a}\right) \cdot S = 1 + S && \text{\# definition of } S \\
\Rightarrow &\ \left(\tfrac{b}{a}\right) \cdot S - S = 1 && \text{\# subtract } S \text{ from both sides} \\
\Rightarrow &\ \left(\tfrac{b}{a}\right) \cdot S - \left(\tfrac{a}{a}\right) \cdot S = 1 && \text{\# multiply } S \text{ by } 1 = \tfrac{a}{a} \\
\Rightarrow &\ S \cdot \left[\tfrac{b}{a} - \tfrac{a}{a}\right] = 1 && \text{\# factor out } S \\
\Rightarrow &\ S \cdot \left[\tfrac{b-a}{a}\right] = 1 && \text{\# simplify} \\
\Rightarrow &\ S = \tfrac{a}{b-a} && \text{\# multiply both sides by } \tfrac{a}{b-a}
\end{aligned}
$$

So $S = \sum_{n=1}^{\infty} \left(\frac{a}{b}\right)^n = \frac{a}{b-a}$, where $a, b \in \mathbb{N}$ and $a < b$.

For example, if we let $a = 1$ and $b = 2$ then $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2-1} = 1$. Similarly, if $a = 1$ and $b = 3$ then $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3-1} = \frac{1}{2}$.

3 S is a Geometric Series

As pointed out by John Carlos Baez (@johncarlosbaez@mathstodon.xyz), S does not depend on a or b, but rather only on x.

More specifically, we can see that $S = \sum_{n=1}^{\infty} x^n$ is a geometric series with a first term of x and a common ratio of x. The general form of a geometric series is given by [1]:

$$S = a + ar + ar^2 + ar^3 + \ldots$$

In this case, $a = x$ is the first term of the series, and $r = x$ is the common ratio.

The sum of an infinite geometric series is well known and can be calculated using the formula:

$$S = \frac{a}{1-r} \tag{2}$$

for $|r| < 1$. If we then substitute $a = x$ and $r = x$ into Equation (2) we get:

$$S = \frac{x}{1-x} \tag{3}$$

So we can see that for S to converge, we need $|x| < 1$. If $|x| \geq 1$ the series diverges and does not have a finite sum.

Summary: $S = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ for $|x| < 1$. Otherwise, as we saw above, the series does not converge.

Finally, if $x = \frac{a}{b}$ then for $|x| < 1$

$$\begin{aligned}
S &= \frac{x}{1-x} && \text{\# Equation (3)} \\
&= \frac{\frac{a}{b}}{1-\frac{a}{b}} && \text{\# set } x = \tfrac{a}{b} \\
&= \frac{\frac{a}{b}}{\frac{b-a}{b}} && \text{\# get a common denominator} \\
&= \frac{a}{b-a} && \text{\# multiply by } 1 = \tfrac{\frac{b}{1}}{\frac{b}{1}}
\end{aligned}$$

So we see that if we set $x = \frac{a}{b}$ then $S = \frac{a}{b-a}$ when $a < b$. This is the result that we saw in Section 2.

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LaTeX Source

References

[1] Wikipedia Contributors. Geometric Series — Wikipedia, The Free Encyclopedia. `https://en.wikipedia.org/w/index.php?title=Geometric_series&oldid=1097138657`, 2022. [Online; accessed 11-July-2022].

[2] Wikipedia Contributors. Ratio Test — Wikipedia, The Free Encyclopedia. `https://en.wikipedia.org/w/index.php?title=Ratio_test&oldid=1075364794`, 2022. [Online; accessed 30-March-2022].