# A Few Notes on Maxwell's Equations

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Last Update: June 16, 2025
Initial Version: September 19, 2022

## 1 Introduction

Maxwell's equations are a set of four fundamental equations that describe the behavior of electric and magnetic fields in classical electromagnetism. They were formulated by the Scottish physicist James Clerk Maxwell in the 19th century and are crucial in understanding the behavior of electromagnetic waves, such as light (an idea previously articulated by Faraday [1]). Maxwell's equations are typically written in both differential and integral forms [18].

Maxwell's equations are fundamental to classical electromagnetism and have played a crucial role in the development of modern physics and technology, including the understanding of electromagnetic waves, the design of electrical circuits, and the development of technologies such as radio, television, and wireless communication, to name a few.

The rest of this note is organized as follows: Section 2 provides a few definitions needed to discuss Maxwell's equations. Section 3 discusses the differential form of Maxwell's equations, and Section 4 discusses the integral form of Maxwell's equations. Section 5 discusses a few of the properties of electromagnetic waves. Section 6 briefly discusses Oliver Heaviside's contributions to Maxwell's equations. Finally, Section 7 outlines some conclusions and final thoughts.

## 2 Definitions

**Definition 2.1.** A *field* is an algebraic structure[1] consisting of a non-empty set $\mathbb{K}$ equipped with two binary operations, $+$ (addition) and $\cdot$ (multiplication), satisfying the conditions[2]:

(A) $(\mathbb{K}, +)$ is an Abelian group with identity element 0 (called zero).

(M) $(\mathbb{K}\backslash\{0\}, \cdot)$ is an Abelian group with identity element 1.

(D) The distributive law $a(b + c) = ab + ac$ holds for all $a, b, c \in \mathbb{K}$.

Examples of important fields include

- $\mathbb{Q}$, the field of rational numbers
- $\mathbb{R}$, the field of real numbers
- $\mathbb{C}$, the field of complex numbers

[1]See Appendix A for a brief review of a few important algebraic structures.
[2]See Appendix B for more on groups and fields.

- $\mathbb{Z}_p$, the field of integers mod $p$ for prime $p$

Aside: Interestingly, every field is an integral domain, and integral domains have cancellation laws.

Ok, but why? Well, consider Lemma 2.1:

**Lemma 2.1.** $R$ has a cancellation law iff $R$ is an integral domain.

**Proof.** Consider the $\Rightarrow$ and $\Leftarrow$ cases:

($\Rightarrow$) Suppose $R$ has a cancellation law with $a, b \in R$, $a \neq 0$ and assume that $ab = 0$. Then

$$\begin{aligned} ab &= 0 && \text{\# by assumption} \\ &\Rightarrow a \cdot b = a \cdot 0 && \text{\# } 0 = a \cdot 0 \\ &\Rightarrow b = 0 && \text{\# (cancel } a \Rightarrow a \text{ is not a zero divisor)} \Rightarrow R \text{ is an integral domain} \end{aligned}$$

Thus $R$ has a cancellation law $\Rightarrow$ $R$ is an integral domain.

($\Leftarrow$) Suppose $R$ is an integral domain with $a, b, c \in R$, $a \neq 0$ and assume that $ab = ac$. Then

$$\begin{aligned} ab &= ac && \text{\# by assumption} \\ &\Rightarrow ab - ac = 0 && \text{\# subtract } ac \text{ from both sides} \\ &\Rightarrow a \cdot (b - c) = 0 && \text{\# factor out } a \\ &\Rightarrow b - c = 0 && \text{\# } R \text{ has no zero divisors and } a \neq 0 \Rightarrow b - c \text{ must equal zero} \\ &\Rightarrow b = c && \text{\# } (ab = ac \Rightarrow b = c) \Rightarrow R \text{ has a cancellation law} \end{aligned}$$

Thus $R$ is an integral domain $\Rightarrow$ $R$ has a cancellation law. ■

**Definition 2.2.** A vector space $V$ over a field $\mathbb{K}$ is an algebraic structure consisting of a non-empty set $V$ equipped with a binary operation $+$ (vector addition) and a scalar multiplication operation $(a, v) \in \mathbb{K} \times V \mapsto av \in V$ such that the following rules hold:

(VA) $(V, +)$ is an Abelian group with identity element $\mathbf{0}$ (the zero vector).

(VM) Rules for scalar multiplication:

(VM0) Closure: For any $a \in \mathbb{K}$ and $v \in V$ there is a unique element $av \in V$.

(VM1) Distributivity$_1$: For any $a \in \mathbb{K}$ and $u, v \in V$ we have $a(u + v) = au + av$.

(VM2) Distributivity$_2$: For any $a, b \in \mathbb{K}$ and $v \in V$ we have $(a + b)v = av + bv$.

(VM3) Associativity: For any $a, b \in \mathbb{K}$ and $v \in V$ we have $(ab)v = a(bv)$.

(VM4) Identity: For any $v \in V$ we have $1v = v$ (where 1 is the identity element of $\mathbb{K}$).

Since vector spaces have two kinds of elements, namely elements of $\mathbb{K}$ and elements of $V$, we distinguish them by calling the elements of $\mathbb{K}$ scalars and the elements of $V$ vectors.

A vector space over the field $\mathbb{R}$ is often called a real vector space while a vector space over $\mathbb{C}$ is called a complex vector space.

## 2.1 Vectors

In this section we'll define the vector notation that we will use in these notes as well as a few important vector operations.

### 2.1.1 Notation

In these notes we will use boldface to represent a vector. Specifically, we will use $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ to represent a column or row vector in some $n$-dimensional space (usually $\mathbb{R}$ or $\mathbb{C}$). If $\mathbf{a}$ is a column vector then in matrix format

$$\mathbf{a} = \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{n\times 1}$$

If $\mathbf{a}$ is a row vector then in matrix format

$$\mathbf{a} = \underbrace{\begin{bmatrix} a_1 \; a_2 \; \ldots \; a_n \end{bmatrix}}_{1\times n}$$

The *transpose* of a vector $\mathbf{a}$, $\mathbf{a}^\intercal$, is defined as follows: If $\mathbf{a}$ is a column vector then

$$\mathbf{a}^\intercal = \begin{bmatrix} a_1 \; a_2 \; \ldots \; a_n \end{bmatrix}$$

Alternatively, if $\mathbf{a}$ is a row vector then

$$\mathbf{a}^\intercal = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Adding a "hat" to a vector denotes the unit vector. That is, for a vector $\mathbf{u}$, $\hat{\mathbf{u}}$ is defined to be

$$\hat{\mathbf{u}} := \frac{\mathbf{u}}{\|\mathbf{u}\|} \tag{1}$$

where $\|\mathbf{u}\|$ is the Euclidean Norm of the vector $\mathbf{u}$ (Definition 2.3). In words: $\hat{\mathbf{u}}$ is a vector of unit length in the $\mathbf{u}$ direction.

**Definition 2.3. Euclidean Norm: $\|\mathbf{x}\|$**

The Euclidean norm of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is defined to be

$$\|\mathbf{x}\| := \sqrt{\mathbf{x}\cdot\mathbf{x}} = \sqrt{\sum_{j=1}^{n} x_j^2}$$

### 2.1.2 Canonical Unit/Basis Vectors

$\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are defined to be the unit vectors in $\mathbb{R}^3$ in the $x$, $y$, and $z$ directions respectively. Note that $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are also the canonical basis vectors for $\mathbb{R}^3$ and have column vector format [4]:

$$\hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \hat{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

I have also seen $\mathbf{e}_i$ used to represent the $i^{th}$ basis vector in $\mathbb{R}^n$. So $\mathbf{e}_i$ is a vector with a one in the $i^{th}$ position and a zero in each of the other $n-1$ positions. In $\mathbb{R}^3$ this means that $\mathbf{e}_1 = \hat{\mathbf{i}}$, $\mathbf{e}_2 = \hat{\mathbf{j}}$, and $\mathbf{e}_3 = \hat{\mathbf{k}}$.

In general, the standard basis (which is sometimes called the computational basis) for the $n$-dimensional Euclidean space consists of the ordered set of $n$ distinct vectors

$$\{\mathbf{e}_i : 1 \leq i \leq n\}$$

where $\mathbf{e}_i$ is the $i^{\text{th}}$ basis vector, that is, it has a one in the $i^{\text{th}}$ coordinate (position) and zeros everywhere else[3]. The $\mathbf{e}_i$ have column vector format

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},\ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots,\ \mathbf{e}_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix},\ \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Using these definitions we can define the parametric form of some curve $C$ in $\mathbb{R}^3$, $\mathbf{r}(t)$, as follows:

$$\mathbf{r}(t) = g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}} + k(t)\,\hat{\mathbf{k}}.$$

where $t \in [a, b]$ and $g(t)$, $h(t)$, and $k(t)$ are scalar functions[4].

Another common notation for vectors is $\vec{r}(t)$, where $\vec{r}(t)$ might be defined as follows:

$$\vec{r}(t) = g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}} + k(t)\,\hat{\mathbf{k}}$$

[3] This is called a "one-hot" encoding in machine learning, where the $i$'s might be the classes in a classification problem.

[4] A scalar function is a function $f$ such that $f : \mathbb{R}^n \to \mathbb{R}$.

## 2.2 A Few Definitions

| Symbol | What | Dimensions | Notes |
|---|---|---|---|
| $\mathbf{B}$ | Magnetic field | Newton/m/Ampere | $\mathrm{N} = \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$ |
| $\mathbf{E}$ | Electric field | Newton/Coulomb | $\mathrm{C} = \mathrm{A} \cdot \mathrm{s}$ |
| $\mathbf{J}$ | Current density | Ampere/m$^2$ | $\mathrm{A} = 6.241509074 \times 10^{18}\ \mathbf{e}/\mathrm{s}$ |
| $\rho$ | Charge density | Coulomb/m$^3$ | $\mathrm{C} = 6.241509074 \times 10^{18}\ \mathbf{e}$ |
| $c$ | Speed of light in a vacuum | m/s | $c = 2.99792458 \times 10^8$ m/s |
| $\epsilon_0$ | Permittivity of free space | Farad/m | $\epsilon_0 = 8.8541878128 \times 10^{-12}$ F/m |
| $\mu_0$ | Permeability of free space | Henry/m | $\mu_0 = 4\pi \times 10^{-7}$ H/m |

Table 1: A Few Definitions

### 2.2.1 Electric and Magnetic Constants in Maxwell's Equations

Maxwell's equations are usually defined using three constants: the speed of light ($c$), the electric permittivity of free space ($\epsilon_0$), and the magnetic permeability of free space ($\mu_0$). See Table 1.

To calculate the value of these constants, first note that the magnetic permeability of free space $\mu_0$ is taken to have the exact value $\mu_0 = 4\pi \times 10^{-7}$ H/m (see Table 1). So $\mu_0$ is given[5]. We also know that the speed of light $c$ is equal to $2.99792458 \times 10^8$ m/s so we also can consider $c$ as given.

Next, notice $\epsilon_0$ and $\mu_0$ are not independent but rather are related to the speed of light by the following equation [18]:

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{2}$$

Since we know the values of $\mu_0$ and $c$, we can find the value of $\epsilon_0$ by solving for it in Equation (2):

$$\epsilon_0 = \frac{1}{\mu_0 c^2} \tag{3}$$

Substituting the values of $\mu_0$ and $c$ into Equation (3) we get that $\epsilon_0 = 8.8541878128 \times 10^{-12}$ F/m.

Finally, note that in the presence of polarizable or magnetic media [20], the effective constants will have different values. In the case of a polarizable medium, called a dielectric, the comparison is stated as a relative permittivity or a dielectric constant. In the case of magnetic media, the relative permeability may also be stated.

[5] 1 Henry/m = 1 Newton/Ampere$^2$ so $\mu_0 = 4\pi \times 10^{-7}$ H/m $\approx 1.25663706143 \times 10^{-6}$ N/A$^2$.

### 2.3 Divergence and Curl

For some vector field $\mathbf{F}$, we call $\nabla \cdot \mathbf{F}$ the divergence of $\mathbf{F}$, while $\nabla \times \mathbf{F}$ is called the curl of $\mathbf{F}$ [5].

It is worth noting that the standard dot (or inner) product is defined as follows:

**Definition 2.4. Standard Inner Product: $\mathbf{a} \cdot \mathbf{b}$**

The inner product (aka dot product or scalar product) of two $n$-dimensional vectors $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$, usually denoted by either $\langle \mathbf{a}, \mathbf{b} \rangle$ or $\mathbf{a} \cdot \mathbf{b}$, is defined to be the scalar value

$$\mathbf{a} \cdot \mathbf{b} := \sum_{j=1}^{n} a_j b_j$$

Since $a_j$ and $b_j$ are scalars (and scalar multiplication commutes) the inner product commutes:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{j=1}^{n} a_j b_j = \sum_{j=1}^{n} b_j a_j = \mathbf{b} \cdot \mathbf{a} \tag{4}$$

This is all good, but here we are using the dot notation for the divergence and the divergence is missing a key property that the standard dot product has, namely, commutivity (Equation (4)). Specifically, the standard dot product commutes but the divergence does not:

$$\nabla \cdot \mathbf{F} \neq \mathbf{F} \cdot \nabla \tag{5}$$

One way to see this is to notice that the LHS of Equation (5) is a scalar function while the RHS is a differential operator.

## 3 Maxwell's Equations (differential form)

The differential form of Maxwell's equations assumes the absence of magnetic or polarizable media since these media can change the values of the permittivity and the permeability (see Section (2.2.1)). The differential form of Maxwell's equations is:

1. **Gauss's Law for Electricity**

   $$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

   In words: a stationary charge ($\rho$) creates an electric field ($\mathbf{E}$) [13].

2. **Gauss's Law for Magnetism**

   $$\nabla \cdot \mathbf{B} = 0$$

   In words: magnetic monopoles can not exist [9].

3. **Faraday's Law of Electromagnetic Induction**

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

In words: a time-varying magnetic field ($\mathbf{B}$) creates an electric field ($\mathbf{E}$) [10].

4. **Ampere's Law with Maxwell's addition**

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

In words: an electric current ($\mathbf{J}$) or a time-varying electric field ($\mathbf{E}$) creates a magnetic field ($\mathbf{B}$) [14]. Maxwell's addition to Ampere's law is the term $\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (time-varying electric fields produce magnetic fields).

# 4 Maxwell's Equations (integral form)

1. **Gauss's Law for Electricity**

$$\oint_C \mathbf{E}\, d\mathbf{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

where $Q_{\text{enclosed}}$ is the enclosed charge. This equation implies that the electric field lines do not form a continuous closed path. In words: isolated positive or negative charges can exist.

2. **Gauss's Law for Magnetism**

$$\oint_C \mathbf{B}\, d\mathbf{A} = 0$$

This equation says that magnetic lines of force form a continuous closed path. In words: no isolated magnetic monopole can exist.

3. **Faraday's Law of Electromagnetic Induction**

$$\oint_C \mathbf{E}\, d\mathbf{l} = -\frac{d}{dt} \mathit{\Phi}_{\mathbf{B}}$$

where $\mathit{\Phi}_{\mathbf{B}}$ is the magnetic flux through a closed path. This equation says that the line integral of the electric field around any closed path is equal to the rate of change of magnetic flux through the closed path bounded by the surface.

4. **Ampere's Law with Maxwell's addition**

$$\oint_C \mathbf{B}\, d\mathbf{l} = \mu_0 I_{\text{enclosed}} + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \mathbf{E}\, d\mathbf{A}$$

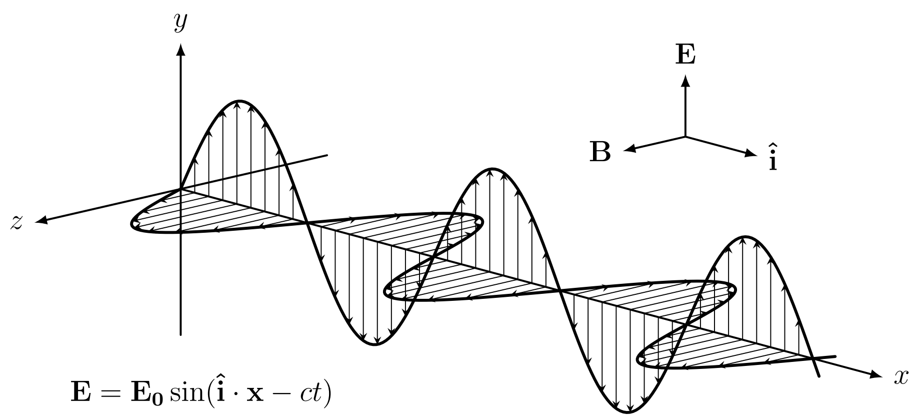
where $I_{\text{enclosed}}$ is the enclosed current. This equation says that both conduction and displacement current can produce a magnetic field.

## 5 Electromagnetic Waves

Classical electromagnetic waves, often referred to simply as electromagnetic waves, are a fundamental concept in classical physics that describes the behavior of electromagnetic radiation, including light, without considering the quantum mechanical nature of light (as described by quantum electrodynamics). These waves are described by classical electromagnetism, primarily through the equations developed by Maxwell in the $19^{\text{th}}$ century [18].

The key characteristics and features of classical electromagnetic waves include:

- **Wave Equation**: The behavior of classical electromagnetic waves is described by Maxwell's equations, which consist of four fundamental equations (see Sections 3 and 4):
  1. Gauss's Law for Electricity
  2. Gauss's Law for Magnetism
  3. Faraday's Law of Electromagnetic Induction
  4. Ampère's Law with Maxwell's addition
- **Wave Propagation**: Classical electromagnetic waves are solutions to Maxwell's equations and can propagate through space as self-sustaining waves of electric and magnetic fields. These waves travel at the speed of light in a vacuum, $c$, which is approximately $3.0 \times 10^8$ m/s.
- **Transverse Waves**: Electromagnetic waves are transverse waves, which means that the oscillations of the electric and magnetic fields are perpendicular to the direction of wave propagation. This is depicted in Figure 1. Contrast with longitudinal waves, where the oscillations are along the direction of propagation.
- **Continuous Spectrum**: Classical electromagnetic waves encompass a continuous spectrum of frequencies and wavelengths, from extremely low-frequency radio waves to extremely high-frequency gamma rays. Each portion of the spectrum corresponds to different types of electromagnetic radiation.
- **Wave Polarization**: Electromagnetic waves can be polarized, meaning that the electric field oscillates in a specific plane as the wave propagates. Polarization can be linear, circular, or elliptical, depending on the orientation of the electric field vector.
- **Superposition**: Like other waves, classical electromagnetic waves obey the principle of superposition, which means that multiple waves can combine to form a new wave through constructive or destructive interference [6].
- **Energy Transport**: Electromagnetic waves carry energy as they propagate through space. The intensity of the wave, which represents the energy per unit area per unit time, is related to the amplitude of the electric and magnetic fields.
- **Wave Reflection, Refraction, and Diffraction**: Classical electromagnetic waves exhibit behaviors such as reflection when they encounter boundaries, refraction when passing from one medium to another, and diffraction when encountering obstacles. These phenomena can be described by classical wave optics.




$$\mathbf{E} = \mathbf{E_0} \sin(\hat{\mathbf{i}} \cdot \mathbf{x} - ct)$$
$$\mathbf{B} = \mathbf{B_0} \sin(\hat{\mathbf{i}} \cdot \mathbf{x} - ct)$$
$$\mathbf{E} \cdot \hat{\mathbf{i}} = 0, \quad \mathbf{B} \cdot \hat{\mathbf{i}} = 0, \quad \mathbf{B} = \tfrac{1}{c}\hat{\mathbf{i}} \times \mathbf{E}$$

Figure 1: Propagation of an electromagnetic wave in the $\hat{\mathbf{i}}$ direction

It's important to note that while classical electromagnetic wave theory provides an accurate description of many electromagnetic phenomena, it does not account for the quantum nature of light, which is described by quantum electrodynamics (QED) [21]. In certain situations, particularly at very small scales or in the presence of intense electromagnetic fields, the quantum nature of light becomes significant, and classical wave theory may not fully explain the observed behavior.

# 6 Oliver Heaviside's Contributions to Maxwell's Equations

Oliver Heaviside made significant contributions to the understanding and reformulation of Maxwell's equations. His work helped simplify and clarify these equations, making them more accessible and useful for engineers and physicists.

A few of Heaviside's important contributions to Maxwell's equations include [12, 17]:

- **Vector Notation**: Heaviside introduced the modern vector notation for Maxwell's equations, which greatly simplified their mathematical expression. This notation represented electric and magnetic fields as vectors, making it easier to write and manipulate the equations. Heaviside's notation is still widely used today [5].
- **Telegrapher's Equations**: Heaviside derived what are now known as the Telegrapher's Equations, which describe the behavior of electrical signals in transmission lines, such as telegraph cables. These equations are a simplified form of Maxwell's equations and are crucial for understanding the propagation of electromagnetic waves in transmission lines [22].
- **Concept of Impedance**: Heaviside introduced the concept of electrical impedance, which is a measure of how much a medium resists the flow of alternating current. He showed how impedance relates to the propagation of electrical signals in transmission lines and how it affects the reflection and transmission of signals [11].
- **Elimination of Quaternions**: Heaviside eliminated the use of quaternions, a complex mathematical system that Maxwell had initially used to describe electromagnetic phenomena. He replaced quaternions with the more intuitive vector notation, making Maxwell's equations more accessible to a wider audience [8].

- **Improved Understanding of Displacement Current**: Heaviside clarified the concept of displacement current, which is a term introduced by Maxwell to account for changing electric fields. Heaviside's work helped physicists better understand this concept and its role in the equations [16].

- **Operational Calculus**: Heaviside is also known for his significant contributions to the field of electrical engineering and the development of operational calculus. Operational calculus is a mathematical technique used to solve differential equations, especially those that arise in the study of electrical circuits and systems. The operational calculus is also known as the Heaviside calculus in honor of his pioneering work in this area [19].

  Operational calculus is primarily concerned with transforming differential equations into algebraic equations, which are often easier to manipulate and solve. Heaviside's work helped simplify the analysis of complex linear systems, making it more accessible to engineers and scientists.

  Some of the key concepts and techniques associated with Heaviside's operational calculus include:

  - **Laplace Transforms**: Heaviside introduced the use of Laplace transforms to solve linear differential equations with initial conditions. This transformation converts a differential equation into an algebraic equation, simplifying the solution process [7].

  - **Differential Operators**: Heaviside introduced the notion of differential operators, such as the D-operator, $D = \frac{d}{dt}$, which allows us to manipulate differential equations more efficiently [3, 15].

  - **Heaviside Step Function**: Heaviside introduced the unit step function (also known as the Heaviside step function) as a way to represent and model sudden changes in electrical circuits. The Heaviside step function is shown in Definition 6.1 and Figures 2, 3 and 4.

    **Definition 6.1. Heaviside Step Function:**

    $$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \\ \text{undefined or } 0.5 & t = 0 \text{ (depending on convention)} \end{cases}$$

    The Heaviside step function is shown in Figure 2.

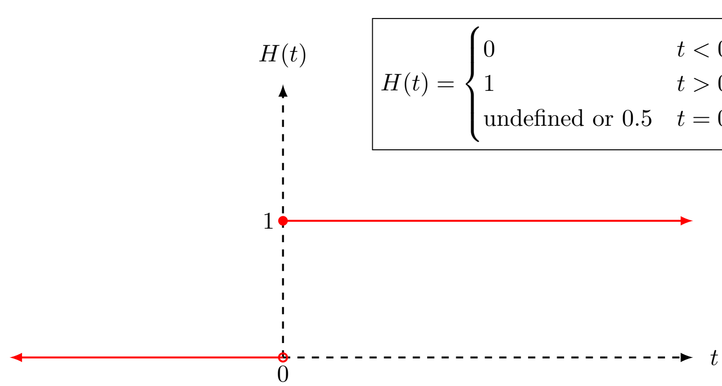



Figure 2: The Heaviside Function $H(t)$

Frequently we are interested in $H(t-a)$, which is sometimes written as $u_a(t)$ and is shown in Figure 3.

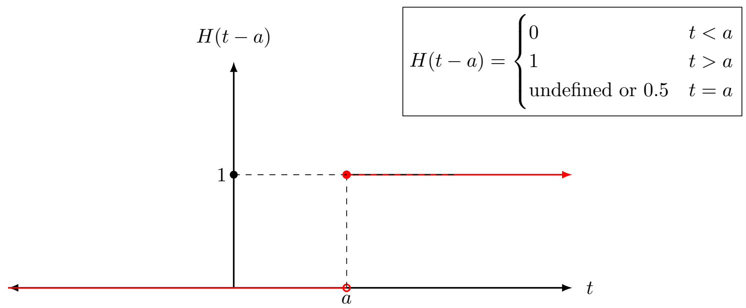



Figure 3: $H(t-a)$

Another useful version of the Heaviside step function, $H(a-t)$, is shown in Figure 4.

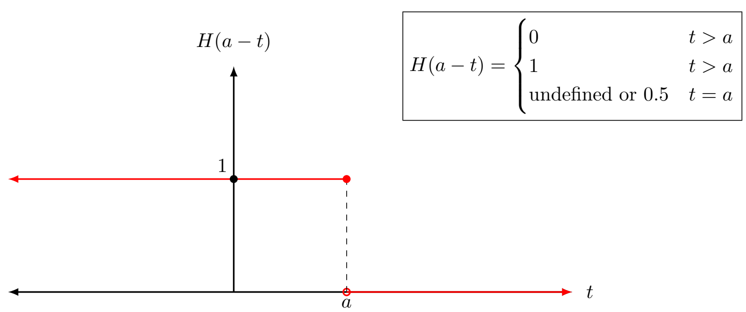



Figure 4: $H(a-t)$

Interestingly, the derivative of the Heaviside function $H(t)$ is the Dirac delta function $\delta(t)$. More specifically,

$$\frac{d}{dt}H(t) = \delta(t)$$

Heaviside's work in operational calculus and electrical engineering significantly influenced the development of modern control theory and signal processing. His contributions made it possible to analyze and design electrical and electronic systems more efficiently.

Operational calculus remains an important tool in engineering and physics, especially in the analysis of linear time-invariant systems. While the formalism of operational calculus has evolved since Heaviside's time, his pioneering work laid the foundation for these developments and continues to be influential in various engineering disciplines.

Oliver Heaviside's contributions to Maxwell's equations were instrumental in simplifying and modernizing the mathematical description of electromagnetism. His work made it easier for engineers and scientists to apply these equations to practical problems and laid the foundation for many developments in electrical engineering and telecommunications.

# 7 Conclusions

# 8 Acknowledgements

Thanks to Larry Lang for suggesting that I add some text about Oliver Heaviside's important contributions to Maxwell's equations (see Section 6). Thanks also to rrogers (@rrogers@mathstodon.xyz) for pointing out Heaviside's contributions to the development of operational calculus.

# LaTeX Source

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# Appendix A

## A Brief Review of a Few Algebraic Structures

| Structure | ABO[1] | Identity | Inverse | Distributive[2] | Commutative[3] | Comments |
|---|---|---|---|---|---|---|
| Semigroup | ✓ | no | no | N/A | no | $(S, \circ)$ |
| Monoid | ✓ | ✓ | no | N/A | no | Semigroup plus identity $\in S$ |
| Group | ✓ | ✓ | ✓ | N/A | no | Monoid plus inverse $\in S$ |
| Abelian Group | ✓ | ✓ | ✓ | N/A | ✓$(\circ)$ | Commutative group |
| $\text{Ring}_+$ | ✓ | ✓ | ✓ | ✓ | ✓$(+)$ | Abelian group under $+$ |
| $\text{Ring}_*$ | ✓ | yes/no | no | ✓ | no | Monoid under $*$ |
| $\text{Field}_{(+,*)}$ | ✓ | ✓$(+,*)$ | ✓$(+,*)$ | ✓ | ✓$(+,*)$ | Abelian group under $+$ and $*$ |
| Vector Space | ✓ | ✓$(+,*)$ | ✓$(+)$ | ✓ | ✓$(+)$ | Abelian group under $+$, scalars $\in$ Field |
| Module | ✓ | ✓$(+,*)$ | ✓$(+)$ | ✓ | ✓$(+)$ | Abelian group under $+$, scalars $\in$ Ring |

Table 2: A Few Algebraic Structures and Their Features

## Abbreviations

1. **ABO:** Associative Binary Operation
   - $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$
   - $x \circ y \in S$ for all $x, y \in S$ ($S$ is closed under $\circ$)

2. **Distributive:** Distributive Property
   - Left Distributive Property: $x * (y + z) = (x * y) + (x * z)$ for all $x, y, z \in S$
   - Right Distributive Property: $(y + z) * x = (y * x) + (z * x)$ for all $x, y, z \in S$
   - $*$ is *distributive* over $+$ if $*$ is left and right distributive
3. **Commutative:** Commutative Property
   - $x \circ y = y \circ x$ for all $x, y \in S$

<u>**Notes**</u>

- Table 2 implies that F $\subset$ R $\subset$ G $\subset$ M $\subset$ SG.
- Whether or not a ring has a multiplicative identity seems to depend on the field of study.

  In general the definition of a ring $R$ doesn't require a multiplicative inverse in $R$ ($a^{-1} \notin R$ for all $a \in R$) or that multiplication be commutative in $R$. Specifically: $R$ is an Abelian group under $+$ but we don't require that multiplication be commutative (while $a + b = b + a$ for all $a, b \in R$, we don't require that $ab = ba$ for all $a, b \in R$). These are perhaps the main ways in which a ring differs from a field. In addition, as mentioned above in some cases $R$ need not include a multiplicative identity ($1 \notin R$).
- F $\subset$ VS since the field axioms require a multiplicative inverse ($a^{-1}$) while vector spaces do not. Fields are also commutative under $*$ and vector spaces are not.
- VS $\subset$ Module since the scalars in a module come from a ring as opposed to a field like we find in vector spaces and F $\subset$ R [2].

# Appendix B

# Fields and Vector Spaces

## <u>Fields</u>

A *field* is an algebraic structure $\mathbb{K}$ in which we can add and multiply elements such that the following laws hold:

### Addition Laws

(FA0) Closure: For any $a, b \in \mathbb{K}$ there is a unique element $a + b \in \mathbb{K}$.

(FA1) Associativity: For all $a, b, c \in \mathbb{K}$ we have $a + (b + c) = (a + b) + c$.

(FA2) Identity: There is an element $0 \in \mathbb{K}$ such that $a + 0 = 0 + a = a$ for all $a \in \mathbb{K}$.

(FA3) Inverse: For any $a \in \mathbb{K}$ there exists $-a \in \mathbb{K}$ such that $a + (-a) = (-a) + a = 0$.

(FA4) Commutativity: For any $a, b \in \mathbb{K}$ we have $a + b = b + a$.

### Multiplication laws

(FM0) Closure: For any $a, b \in \mathbb{K}$, there is a unique element $ab \in \mathbb{K}$.

(FM1) Associativity: For all $a, b, c \in \mathbb{K}$ we have $a(bc) = (ab)c$.

(FM2) Identity: There is an element $1 \in \mathbb{K}$, $1 \neq 0$, such that $a1 = 1a = a$ for all $a \in \mathbb{K}$.

(FM3) Inverse: For any $a \in \mathbb{K}$ with $a \neq 0$, there exists $a^{-1} \in \mathbb{K}$ such that $aa^{-1} = a^{-1}a = 1$.

(FM4) Commutativity: For any $a, b \in \mathbb{K}$ we have $ab = ba$.

**Distributive law**

(D) Distributivity: For all $a, b, c \in \mathbb{K}$, we have $a(b + c) = ab + ac$.

Note the similarity of the addition and multiplication laws. We say that $(\mathbb{K}, +)$ is an *Abelian* group if (FA0)-(FA4) hold. (FM0)-(FM4) say that $(\mathbb{K}\backslash\{0\}, \cdot)$ is also an Abelian group (we have to leave out 0 because as (FM3) says, 0 does not have a multiplicative inverse).

Examples of fields include $\mathbb{Q}$ (the rational numbers), $\mathbb{R}$ (the real numbers), $\mathbb{C}$ (the complex numbers), and $\mathbb{Z}_p$ (the integers mod $p$, for $p$ a prime number).

Associated with any field $\mathbb{K}$ is a non-negative integer called its characteristic, which is defined as follows: the characteristic of a field $\mathbb{K}$, often denoted $\mathrm{char}(\mathbb{K})$, is the smallest number of times one must use the field's (or ring's) multiplicative identity (1) in a sum to get the additive identity (0). If this sum never reaches the additive identity the field is said to have characteristic zero. That is,

$$\mathrm{char}(\mathbb{K}) = \begin{cases} n & n \text{ is the smallest positive number such that } \underbrace{1 + 1 + \cdots + 1}_{n} = 0 \\ 0 & \text{if the sum of ones never reaches } 0 \end{cases}$$

Important examples such as $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ have characteristic zero, while $\mathbb{Z}_p$ has characteristic $p$ (for prime $p$).

## Vector Spaces

Let $\mathbb{K}$ be a field. A vector space $V$ over $\mathbb{K}$ is an algebraic structure in which we can add two elements of $V$ and multiply an element of $V$ by an element of $\mathbb{K}$ (this is called *scalar multiplication*) such that the following rules hold:

**Addition Laws**

(VA0) Closure: For any $u, v \in V$ there is a unique element $u + v \in V$.

(VA1) Associativity: For all $u, v \in V$ we have $u + (v + w) = (u + v) + w$.

(VA2) Identity: There is an element $0 \in V$ such that $v + 0 = 0 + v = v$ for all $v \in V$.

(VA3) Inverse: For any $v \in V$, there exists $-v \in V$ such that $v + (-v) = (-v) + v = 0$.

(VA4) Commutativity: For any $u, v \in V$ we have $u + v = v + u$.

**Scalar multiplication laws**

(VM0) Closure: For any $a \in \mathbb{K}$, $v \in V$ there is a unique element $av \in V$.

(VM1) Distributivity$_1$: For any $a \in \mathbb{K}$, $u, v \in V$ we have $a(u + v) = au + av$.

(VM2) Distributivity$_2$: For any $a, b \in \mathbb{K}$, $v \in V$ we have $(a + b)v = av + bv$.

(VM3) Associativity: For any $a, b \in \mathbb{K}$, $v \in V$ we have $(ab)v = a(bv)$.

(VM4) Identity: For any $v \in V$ we have $1v = v$ (where 1 is the element given by (FM2)).

Again, we can summarize (VA0)–(VA4) by saying that $(V, +)$ is an Abelian group.

One of the most important examples of a vector space over a field $\mathbb{K}$ is the set $\mathbb{K}^n$ of all $n$-tuples with elements from $\mathbb{K}$ (should prove that $\mathbb{K}^n$ is a vector space). Addition and scalar multiplication in $\mathbb{K}^n$ are defined by the following rules:

$$
\begin{aligned}
(u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \\
a(v_1, v_2, \ldots, v_n) &= (av_1, av_2, \ldots, av_n)
\end{aligned}
$$

Note that one of the key features of a vector space is closure under *componentwise* addition, as shown above.