A Few Notes on Simple Harmonic Oscillators

David Meyer

dmm613@gmail.com

Last update: December 15, 2024

1 Introduction

This story begins somewhere around 1658 when Robert Hooke [8] began experimenting with springs and masses. By 1660 Hooke had made two significant steps, namely the use of a balance controlled by a spiral spring and an improved escapement called the anchor escapement. In 1660 he discovered an instance of what we now call Hooke's Law while working on designs for the balance springs of clocks. However he only announced the general law of elasticity in his lecture Of Spring given in 1678 [7].

Interestingly, it was the addition of the balance spring to the balance wheel around 1658 by Robert Hooke (and Christiaan Huygens [19]) that greatly increased the accuracy of portable timepieces, transforming early pocket watches from expensive novelties to useful timekeepers.

These notes are organized (I hope) as follows: Section 2 looks at Hooke's Law in some detail. Section 4 considers some examples of simple harmonic oscillators, including Section 4.1 which looks at the simple pendulum, and Section 4.2 which considers LC circuits. Section 5 looks at the Quantum Harmonic Oscillator. Section 6 outlines how quantum fields fit into all of this. Finally, Section 7 offers a few observations and conclusions.

2 Hooke's Law

Hooke's law [22] is a law of physics that states that the force needed to extend or compress a spring by some distance scales linearly with respect to that distance. More specifically: $F = -kx$, where k is a constant characteristic of the spring (i.e., its stiffness) and x is the displacement from the equilibrium point (see Figure 1). The law is named after 17th-century British physicist Robert Hooke [8]. Hooke first stated the law in 1676 as a Latin anagram, and he published the solution of his anagram in 1678 as: ut tensio, sic vis ("as the extension, so the force" or "the extension is proportional to the force") [7].

2.1 The Mathematics of Hooke's Law

Hooke's Law describes a form of one-dimensional (scalar) simple harmonic motion, which as mentioned above occurs when the restoring force in a system (the force directed toward a stable equilibrium point) is proportional to the displacement from that equilibrium. Hooke's Law is most frequently stated as[1] $F \propto -x$. That is, the force F is proportional to the displacement x (hence Hooke's Law describes simple harmonic motion aka simple harmonic oscillation). More frequently this relationship is converted into an equality by the use of a constant k so that we have

$$F = -kx \tag{1}$$

We also know that Newton's Second Law of Motion [25] tells us that

$$F = ma \tag{2}$$

[1] In physics it appears to be convention to use x when x is a scalar and $\mathbf{x}$ when it is a vector in force and energy equations.

Setting Newton's force (Equation (2)) equal to Hooke's force (Equation (1)) gives us

$$ma = -kx \tag{3}$$

This implies that

$$\begin{aligned} ma &= -kx && \text{\# Equation (3)} \\ &\Rightarrow a = -\frac{k}{m}x && \text{\# solve for } a\text{, noting that } a \text{ and } x \text{ are functions of time} \\ &\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x && \text{\# } a = \frac{d}{dt}[v] = \frac{d}{dt}\left[\frac{dx}{dt}\right] = \frac{d^2x}{dt^2} \end{aligned}$$

This setup is shown modeling a spring and mass system[2] in Figure 1.

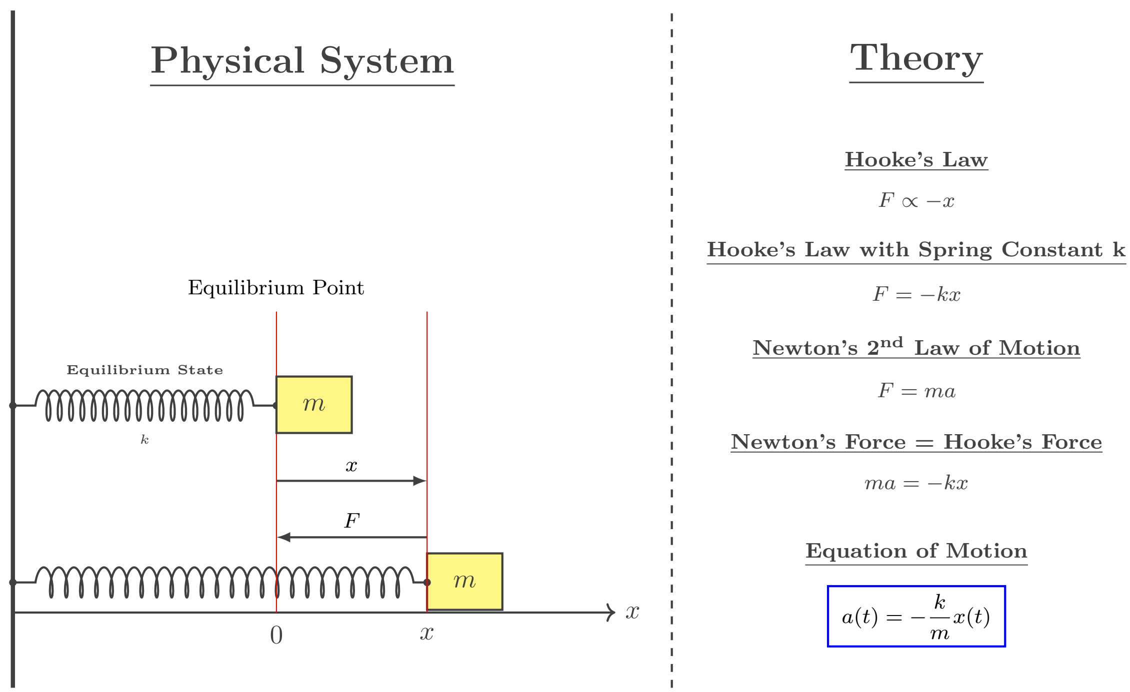

Figure 1: Spring and Mass System with Spring Constant k and Mass m

So now we know that

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{4}$$

The next question is how do we solve the second order ordinary differential equation (Equation (4)) for $x(t)$? Well, if we imagine the displacement ($x(t)$) plotted on the y axis against t (plotted on the x axis) we see the familiar pattern shown in Figure 2.

[2]Where the force is proportional to the distance x that the spring is stretched from its equilibrium position.

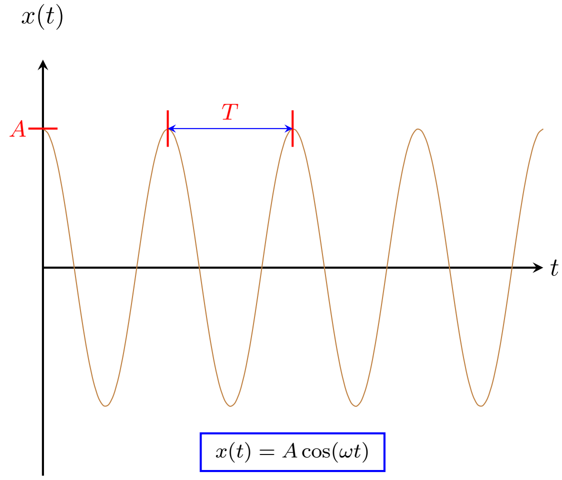

Figure 2: Simple Harmonic Oscillator Displacement Function

Here we can think of the spring being stretched to its maximum displacement (A) and then let go at time $t = 0$. The mass (m) will continue oscillating since in the Hooke's Law setup the only force acting on the mass is the force due to the spring, that is, $-kx$. This force is frequently referred to as the *restoring* force and is shown in Figure 1.

We can check to see if our guess, $x(t) = A\cos(\omega t)$, is really a solution to Equation (4) as follows:

$$
\begin{aligned}
\frac{d^2x}{dt^2} &= -\frac{k}{m}x && \text{\# Equation (4)} \\
&\Rightarrow \frac{d^2}{dt^2}\Big[A\cos(\omega t)\Big] = -\frac{k}{m}x && \text{\# guess that } x(t) = A\cos(\omega t) \\
&\Rightarrow \frac{d}{dt}\Big[-\omega A\sin(\omega t)\Big] = -\frac{k}{m}x && \text{\# } \frac{d}{dt}\cos(u) = -\sin(u)\frac{du}{dt} \text{ with } u = \omega t \\
&\Rightarrow -\omega^2 A\cos(\omega t) = -\frac{k}{m}x && \text{\# } \frac{d}{dt}\sin(u) = \cos(u)\frac{du}{dt} \text{ with } u = \omega t \\
&\Rightarrow \omega^2 A\cos(\omega t) = \frac{k}{m}x && \text{\# cancel minus} \\
&\Rightarrow \omega^2 x = \frac{k}{m}x && \text{\# } x(t) = A\cos(\omega t) \\
&\Rightarrow \omega^2 = \frac{k}{m} && \text{\# cancel } x \\
&\Rightarrow \omega = \sqrt{\frac{k}{m}} && \text{\# take the square root of both sides}
\end{aligned}
$$

So $x(t) = A\cos(\omega t)$ is indeed a solution to Equation (4) in the case that

$$\omega = \sqrt{\frac{k}{m}} \tag{5}$$

Here A is the maximum displacement (or amplitude) and ω is the angular frequency (in radians per second). Angular frequency is defined to be

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{6}$$

where T is the period (time for one cycle in seconds) and f is the frequency in cycles/second [17]. Note that $f = \dfrac{1}{T}$ so f has units of seconds^{-1}.

Given this information we can write the displacement function $x(t)$ as in terms of the frequency f as follows:

$$x(t) = A\cos(2\pi f t)$$

We can also put Equations (5) and (6) together to find that

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

and

$$T = 2\pi\sqrt{\frac{m}{k}}$$

3 Total Energy of a Simple Harmonic Oscillator

We know that the total energy of an oscillator such as the spring and mass system we looked at above is

$$E_{\text{total}} = \text{KE} + \text{PE}$$

All good, but what is the kinetic energy (KE) and potential energy (PE) of such as system? Well, we know that $\text{KE} = \frac{1}{2}mv^2$ [13], but what about the potential energy of the system?

3.1 Potential Energy of a Simple Harmonic Oscillator

Potential energy is the energy a system has due to position, shape, or configuration. It is stored energy that is completely recoverable.

Definition 3.1. Conservative Force: A conservative force is one for which work done by or against the force depends only on the starting and ending points of a motion and not on the path taken [28].

Now we can define the potential energy (PE) for any conservative force. The work done against a conservative force to reach a final configuration depends on the configuration, not the path followed, and is the potential energy added. So in the case of a simple harmonic oscillator the work W done by a force F is equal to $\int_x F(x)\,dx$. That is, W equals the area under F [31]. Since work (and therefore potential energy) is a scalar quantity [6] we are only interested in the magnitude of F, kx, at each point x (the sign is the direction of the F vector in the one dimensional case). So if we set $F_1 = kx$ we get the curve shown in Figure 3, where we can see that the area under the curve is $\frac{1}{2}kx^2$. Thus the work done or potential energy stored is $\frac{1}{2}kx^2$.

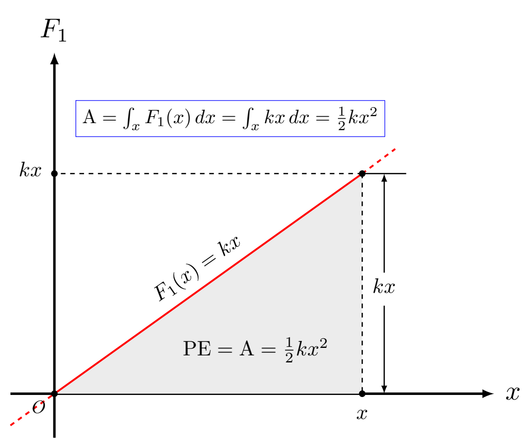

Figure 3: Potential Energy of a Simple Harmonic Oscillator

So now we know that $E_{\text{total}} = E(x, v) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$. This implies that

$$
\begin{aligned}
E(x,v) &= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 && \# \; E(x,v) = \mathrm{KE} + \mathrm{PE} \\
&= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 && \# \; v = \frac{dx}{dt} \\
&= \frac{1}{2}m\left(\frac{d}{dt}\Big[A\cos(\omega t)\Big]\right)^2 + \frac{1}{2}kx^2 && \# \; x(t) = A\cos(\omega t) \\
&= \frac{1}{2}m\big(-\omega A\sin(\omega t)\big)^2 + \frac{1}{2}kx^2 && \# \; \frac{d}{dt}\cos(u) = -\sin(u)\frac{du}{dt} \text{ with } u = \omega t \\
&= \frac{1}{2}m\omega^2A^2\sin^2(\omega t) + \frac{1}{2}kx^2 && \# \; \big(-\omega A\sin(\omega t)\big)^2 = \omega^2A^2\sin^2(\omega t) \\
&= \frac{1}{2}m\omega^2A^2\sin^2(\omega t) + \frac{1}{2}k\big(A\cos(\omega t)\big)^2 && \# \; x(t) = A\cos(\omega t) \\
&= \frac{1}{2}m\omega^2A^2\sin^2(\omega t) + \frac{1}{2}kA^2\cos^2(\omega t) && \# \; \big(A\cos(\omega t)\big)^2 = A^2\cos^2(\omega t) \\
&= \frac{1}{2}m\omega^2A^2\sin^2(\omega t) + \frac{1}{2}m\omega^2A^2\cos^2(\omega t) && \# \; \omega = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega^2 \text{ (Equation (5))} \\
&= \frac{1}{2}m\omega^2A^2\Big(\sin^2(\omega t) + \cos^2(\omega t)\Big) && \# \text{ factor out } \frac{1}{2}m\omega^2A^2 \\
&= \frac{1}{2}m\omega^2A^2 && \# \; \sin^2(\omega t) + \cos^2(\omega t) = 1 \\
&= \frac{1}{2}m(2\pi f)^2A^2 && \# \; \omega = 2\pi f \\
&= \frac{1}{2}4\pi^2f^2A^2m && \# \text{ expand } (2\pi f)^2 \text{ and rearrange} \\
&= 2\pi^2f^2A^2m && \# \; E_{\text{total}} = 2\pi^2f^2A^2m \\
&\Rightarrow E(x,v) \propto f^2A^2 && \# \; E(x,v) \text{ is proportional to } f^2A^2
\end{aligned}
$$

So we see that the oscillator has $E(x,v) = 2\pi^2f^2A^2m$, or said another way $E(x,v) \propto f^2A^2$. This result will become useful when we consider the quantum harmonic oscillator.

The next section drills down a bit on conservation of energy in the one-dimensional case.

3.2 Conservation of Energy: One Particle in One Dimension

The first thing to note here is that since we are in one-dimensional space (motion is in $\mathbb{R}^1$), quantities such as displacement, velocity, and acceleration, force, and energy are scalars. It seems to be convention to use, for example, $F = ma$ rather than $\mathbf{F} = m\mathbf{a}$, when working in $\mathbb{R}^1$.

Next, we will need the following definition:

Definition 3.2. Trajectory: A solution x(t) to the equation $F(x(t)) = m\ddot{x}(t)$, Newton's Second Law, is called a trajectory.

Now, consider the case of a general force function $F(x)$. Here we define the kinetic energy of a particle to be $\frac{1}{2}mv^2$. We also define the potential energy of a particle, $V(x)$, to be

$$V(x) = -\int F(x)\,dx \tag{7}$$

so that

$$\frac{d}{dx}V(x) = -F(x) \tag{8}$$

Then the total energy of a particle as a function of displacement and velocity, $E(x, v)$, is defined to be

$$E(x, v) = \frac{1}{2}mv^2 + V(x) \tag{9}$$

One of the main reasons this energy function is important is that it is *conserved*, meaning that its value along any trajectory is constant. Switching notation ($x = x(t)$ and $v = \dot{x}(t)$) and saying this in a slightly different way: An energy function is *conserved* if, for each trajectory $x(t)$ conforming to Newton's Second Law, a particle's total energy $E(x(t), \dot{x}(t))$ is independent of t. The conditions under which the total energy of a particle is conserved is the topic of Theorem 3.1.

Theorem 3.1. Suppose a particle's trajectory conforms to Newton's Second Law in the form $F(x(t)) = m\ddot{x}(t)$ and let V and E be as in Equations (7) and (9). In this case the total energy of the particle is conserved.

Proof. To show that the total energy of the particle is conserved we want to show that a particle's total energy does not change with time, that is, we want to show that $\frac{d}{dt}E(x(t), \dot{x}(t)) = 0$. As we will see, computing this derivative requires a bit of machinery from the chain and power rules [30, 32]:

$$\begin{aligned}
\frac{d}{dt}E(x(t), \dot{x}(t)) &= \frac{d}{dt}\left[\frac{1}{2}m(\dot{x}(t))^2 + V(x(t))\right] && \text{\# definition of } E(x(t), \dot{x}(t)) \text{ (Equation (9))} \\
&= \frac{d}{dt}\left[\frac{1}{2}m(\dot{x}(t))^2\right] + \frac{d}{dt}V(x(t)) && \text{\# since derivative is a linear operator [24]} \\
&= m\dot{x}(t)\ddot{x}(t) + \frac{d}{dt}V(x(t)) && \text{\# by the power \& chain rules: } \frac{d}{dt}\left[\frac{1}{2}m(\dot{x}(t))^2\right] = m\dot{x}(t)\ddot{x}(t) \\
&= m\dot{x}(t)\ddot{x}(t) + \left[\frac{d}{dx}V(x(t))\right]\dot{x}(t) && \text{\# } \underbrace{\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt}}_{\text{chain rule}} = \frac{dV}{dx}\dot{x} \Rightarrow \frac{d}{dt}V(x(t)) = \left[\frac{d}{dx}V(x(t))\right]\dot{x}(t) \\
&= \dot{x}(t)\cdot\left[m\ddot{x}(t) + \frac{d}{dx}V(x(t))\right] && \text{\# factor out } \dot{x}(t) \\
&= \dot{x}(t)\cdot\left[m\ddot{x}(t) - F(x(t))\right] && \text{\# since } \frac{d}{dx}V(x(t)) = -F(x(t)) \text{ (Equation (8))} \\
&= \dot{x}(t)\cdot 0 && \text{\# by Newton's 2}^{\text{nd}} \text{ law: } F = m\ddot{x} \Rightarrow m\ddot{x} - F = 0 \\
&= 0 && \text{\# } \frac{d}{dt}E(x(t), \dot{x}(t)) = 0
\end{aligned}$$

So we see that $\frac{d}{dt}E(x(t), \dot{x}(t)) = 0$ along any trajectory, which implies that $E(x(t), \dot{x}(t))$ is independent of t. Hence the total energy of the particle is conserved. ■

Energy is sometimes called a *conserved quantity* (or *constant of motion*), since a particle neither gains nor loses energy as it moves according to Newton's Second Law.

4 Other Simple Harmonic Oscillators

4.1 The Simple Pendulum

A pendulum is a weight (that is, a mass m in a gravitational field) suspended from a pivot so that it can swing freely [26]. When a pendulum is displaced sideways from its resting, equilibrium position, it is subject to a restoring force due to gravity that will accelerate it back toward the equilibrium position. When released, the restoring force acting on the pendulum's mass causes it to oscillate about the equilibrium position, swinging back and forth. The time for one complete cycle, a left swing and a right swing, is called the period (T). The period depends on the length of the pendulum and also to a slight degree on the amplitude (A), the width of the pendulum's swing. The free body diagram [20] for the simple pendulum is shown in Figure 4.

From the first scientific investigations of the pendulum around 1602 by Galileo Galilei, the regular motion of pendulums has been used for timekeeping. The pendulum clock invented by Christiaan Huygens in 1658 became the world's standard for timekeeping [19] and was used in homes and offices for 270 years until it was supplanted by the quartz clock in the 1930s [16]. Pendulums are also used in scientific instruments such as accelerometers and seismometers. Historically they were used as gravimeters to measure the acceleration of gravity in geo-physical surveys, and even as a standard of length. The word "pendulum" is new Latin, from the Latin *pendulus* meaning 'hanging' [33].

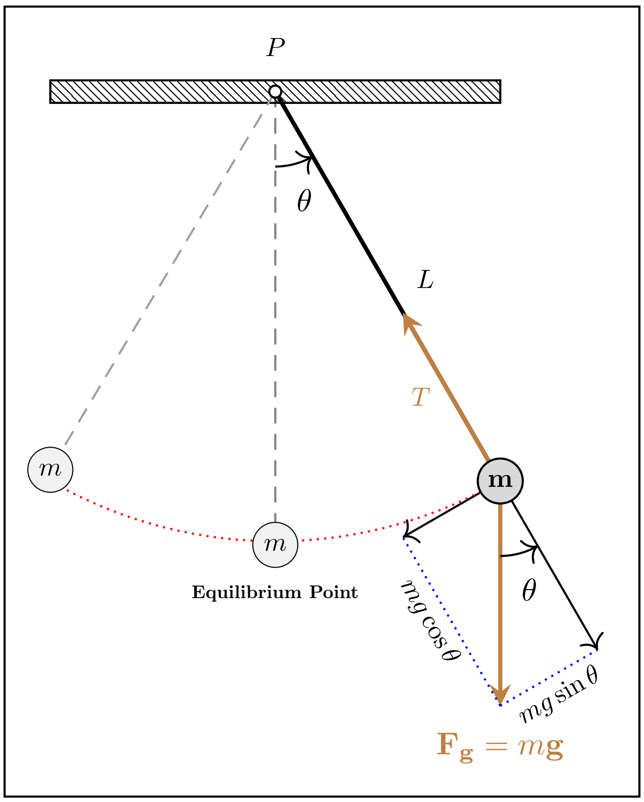

Figure 4: Free Body Diagram for a Simple Pendulum

The simple pendulum setup has a point mass m (the bob) on a string (or rod) of length L. Here the mass of the string is assumed to be negligible as compared to the mass of the bob. The pendulum pivots around a frictionless point P on the support. The only forces acting on the bob are the force of gravity (i.e., the weight of the bob) and tension from the string (T). The forces on the bob result in a net force of $-mg\sin\theta$ toward the equilibrium point. This force is called a *restoring force* since it points in the direction of the equilibrium point. The simple pendulum setup is shown in Figure 4.

4.1.1 Applying Newton's Laws to the Simple Pendulum

The pendulum is governed by Newton's laws applied to rotational motion, namely

$$\tau = I\alpha \tag{10}$$

where τ is the torque, I is the moment of inertia, and α is the angular acceleration [12].

Torque is interesting. It is a measure of rotational force, that is, it is a measure of twist. The moment of inertia of an object, also known as its rotational inertia, is a measure of how difficult it is to change the rotation of that object. It is an analogue of mass: mass is to force as rotational inertia is to torque. Angular acceleration is simply the change in velocity of the angle θ of the pendulum and is typically denoted[3] by either $\ddot{\theta}(t)$ or $\frac{d^2\theta}{dt^2}$.

A torque on a lever can be expressed as $\boldsymbol{\tau} = \mathbf{F} \times \mathbf{r}$, a force $\mathbf{F}$ multiplied by a distance from the pivot $\mathbf{r}$. In this case (Figure 4), $\mathbf{r} = L$ and $\mathbf{F} = m\mathbf{g}\sin\theta$, the force due to gravity that is perpendicular to the string. This force can be found by splitting the total gravitational force into its parallel and perpendicular parts through vector algebra (BTW, this is one of the reasons to like Lagrangian mechanics over Newtonian mechanics; Lagrangian mechanics uses (mostly) scalars whereas Newtonian mechanics uses vectors). This operation seems to be common and physics. In the case shown in Figure 4 we find that $\mathbf{F} = m\mathbf{g}\sin\theta$, the perpendicular force is the product of the total gravitational force $m\mathbf{g}$ and the scale factor $\sin\theta$.

Looking at the right hand side of Equation (10) we see we need to define I, the moment of inertia of the pendulum. In general the moment of inertia is found for any given object by $I = \sum mr^2$, where the sum is over each point in the object. Since the mass in our pendulum is concentrated in a single point at the bob, the sum is over a single term, the bob. Therefore

$$I = mL^2 \tag{11}$$

If we combine Equations (10) and (11) we see that

$$\begin{aligned}
\tau &= I\alpha && \text{\# definition of torque } \tau \text{ (Equation (10))} \\
&= mL^2\alpha && \text{\# definition of moment of inertia } I \text{ (Equation (11))} \\
&\Rightarrow -FL = mL^2\alpha && \text{\# } \tau = FL \\
&\Rightarrow -Lmg\sin\theta = mL^2\alpha && \text{\# } F = mg \sin\theta \\
&\Rightarrow -g\sin\theta = L\alpha && \text{\# cancel } L \\
&\Rightarrow \alpha = -\frac{g}{L}\sin\theta && \text{\# solve for } \alpha
\end{aligned}$$

Notice that the torque was made negative. This is by convention, since a negative torque indicates force in the clockwise direction, and the force of gravity perpendicular to the pendulum rod is clockwise at the initial state of the pendulum.

So now we know that

$$\alpha = -\frac{g}{L}\sin\theta \tag{12}$$

Now, if we let $\omega = \sqrt{\frac{g}{L}}$ and recall that $\alpha = \dfrac{d^2\theta}{dt^2}$ we see that

$$\begin{aligned}
\alpha &= -\frac{g}{L}\sin\theta && \text{\# Equation (12)} \\
&= -\omega^2\sin\theta && \text{\# } \omega = \sqrt{\frac{g}{L}} \\
\Rightarrow \frac{d^2\theta}{dt^2} &= -\omega^2 \sin\theta && \text{\# } \alpha = \frac{d^2\theta}{dt^2}
\end{aligned}$$

[3] Here again θ and $\theta(t)$ are used interchangeably.

Now we know that

$$\frac{d^2\theta}{dt^2} = -\omega^2 \sin\theta \tag{13}$$

All good, but there is an interesting problem with Equation (13): Unlike the equation of motion for the spring and mass problem (Equation (4)), we can't solve the second order differential equation of motion for the simple pendulum case (Equation (13)). However, we can solve a different but related problem which is called the "Small-Angle Pendulum Problem".

4.1.2 The Small-Angle Pendulum Problem

Interestingly, it turns out that for small θ (say, $\theta \leq 10°$) we see that $\sin\theta \approx \theta$. Said another way

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Using this constraint we can substitute θ for $\sin\theta$ in Equation (13) and write

$$\frac{d^2\theta}{dt^2} \approx -\omega^2\theta \tag{14}$$

This is a much simpler differential equation, one which we know how to solve[4]. In order to solve this differential equation, first notice that the second derivative of $\frac{d^2\theta}{dt^2}$ is the function itself (that is, θ) scaled by a constant ($-\omega^2$). This form is known as a second order linear homogeneous differential equation [15] and we can guess the function that exhibits this behavior. It's the exponential function:

$$\frac{d}{dx}e^x = e^x$$

If we set $\theta = e^{i\omega t}$ we see that

$$\begin{aligned}
\theta &= e^{i\omega t} && \text{\# definition of } \theta \\
\frac{d\theta}{dt} &= i\omega e^{i\omega t} && \text{\# chain rule: } \frac{d}{dt}e^u = e^u \cdot \frac{du}{dt} \text{ with } u = i\omega t \text{ and so } \frac{du}{dt} = i\omega \\
\frac{d^2\theta}{dt^2} &= \frac{d}{dt}\left[i\omega e^{i\omega t}\right] && \text{\# } \frac{d^2x}{dt^2} = \frac{d}{dt}\left[\frac{dx}{dt}\right] \\
&= i^2\omega^2 e^{i\omega t} && \text{\# chain rule: } \frac{d}{dt}\left[i\omega e^{i\omega t}\right] = (i\omega)\cdot\frac{d}{dt}\left[e^{i\omega t}\right] = (i\omega)(i\omega)e^{i\omega t} = i^2\omega^2 e^{i\omega t} \\
&= -\omega^2 e^{i\omega t} && \text{\# } i^2 = -1
\end{aligned}$$

So now we know that $\theta_1 = C_1 e^{i\omega t}$ is a solution to Equation (14), where $C_1 \in \mathbb{C}$. Notice also that $\theta_2 = C_2 e^{-i\omega t}$ is also a solution with $C_2 \in \mathbb{C}$. Here the (constant) coefficients C_1 and C_2 are preserved through differentiation and are essentially the integration constants. These constants represent the infinite number of solutions to the differential equation Equation (14), where are determined by the initial characteristics of the pendulum. The sum of these solutions is a more general solution, since either of the two constants be be set to θ to obtain the original solutions:

$$\theta = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \tag{15}$$

Taking the derivative of Equation (15) we get

[4]Notice that this equation has the same form as the equation of motion for the spring and mass problem, namely $\frac{d^2x}{dt^2} = -\omega^2 x$.

$$\frac{d\theta}{dt} = C_1 i\omega e^{i\omega t} - C_2 i\omega e^{-i\omega t} \tag{16}$$

Taking the derivative one more time we see that

$$\begin{aligned}
\frac{d^2\theta}{dt^2} &= \frac{d}{dt}\Big[C_1 i\omega e^{i\omega t} - C_2 i\omega e^{-i\omega t}\Big] && \text{\# Equation (16)} \\
&= \frac{d}{dt}\Big[C_1 i\omega e^{i\omega t}\Big] - \frac{d}{dt}\Big[C_2 i\omega e^{-i\omega t}\Big] && \text{\# derivative is a linear operator} \\
&= C_1 i\omega \frac{d}{dt}\Big[e^{i\omega t}\Big] - C_2 i\omega \frac{d}{dt}\Big[e^{-i\omega t}\Big] && \text{\# factor out } C_1, C_2, i \text{ and } \omega \text{ (not functions of } t\text{)} \\
&= C_1 i\omega e^{i\omega t}(i\omega) - C_2 i\omega e^{-i\omega t}(-i\omega) && \text{\# } \frac{d}{dt}e^u = e^u \frac{du}{dt} \text{ with } u = i\omega t \text{ and } i^2 = -1 \\
&= i^2 C_1 \omega^2 e^{i\omega t} + i^2 C_2 \omega^2 e^{-i\omega t} && \text{\# } (-1) * (-1) = 1\text{, collect terms, rearrange} \\
&= -C_1 \omega^2 e^{i\omega t} - C_2 \omega^2 e^{-i\omega t} && \text{\# } i^2 = -1 \\
&= -\omega^2 (C_1 e^{i\omega t} + C_2 e^{-i\omega t}) && \text{\# factor out } -\omega^2 \\
&= -\omega^2 \theta && \text{\# } \theta = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \text{ (Equation (15))} \\
&\Rightarrow \frac{d^2\theta}{dt^2} = -\omega^2\theta && \text{\# Equation (14)}
\end{aligned}$$

This result tells us that Equation (14) encompasses all of the possible solutions to the small-angle pendulum problem. All good, but one question we might ask is what happens when we apply Euler's formula [5] to Equation (15)? Recall that Euler's Formula states that $e^{ix} = \cos x + i \sin x$ so we see that

$$\begin{aligned}
\theta &= C_1 e^{i\omega t} + C_2 e^{-i\omega t} && \text{\# Equation (15)} \\
&= C_1 \cos(\omega t) + C_1 i \sin(\omega t) + C_2 e^{-i\omega t} && \text{\# } C_1 e^{i\omega t} = C_1 \cos(\omega t) + C_1 i \sin(\omega t) \\
&= C_1 \cos(\omega t) + C_1 i \sin(\omega t) + C_2 \cos(-\omega t) + C_2 i \sin(-\omega t) && \text{\# } C_2 e^{-i\omega t} = C_2 \cos(-\omega t) + C_2 i \sin(-\omega t) \\
&= C_1 \cos(\omega t) + C_1 i \sin(\omega t) + C_2 \cos(\omega t) - C_2 i \sin(\omega t) && \text{\# } \sin(-\theta) = -\sin\theta \text{ and } \cos(-\theta) = \cos\theta \\
&= (C_1 + C_2)\cos(\omega t) + i(C_1 - C_2)\sin(\omega t) && \text{\# group terms, rearrange} \\
&= A\cos(\omega t) + B\sin(\omega t) && \text{\# } A = C_1 + C_2 \text{ and } B = i(C_1 - C_2)
\end{aligned}$$

So now we can write $\theta = A\cos(\omega t) + B\sin(\omega t)$ where $A, B \in \mathbb{R}$ are constants defined by the initial conditions of the pendulum.

Since θ is a linear combination of periodic functions it is also periodic [4]. We can also note here that adding two periodic functions with the same period results in another periodic function with that same period [4]. Since the periods of both $\cos(\omega t)$ and $\sin(\omega t)$ are $\frac{2\pi}{\omega}$ we know that the period of the pendulum, recalling that we defined $\omega = \sqrt{\frac{g}{L}}$ and applying the small-angle approximation, is

$$T \approx \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

An interesting outcome here is that this approximation of the period does not rely on the maximum angular displacement of the pendulum, which is in accordance with Galileo's observations as described in [1].

Summary: It turns out that the small-angle approximation for the pendulum period is fairly good and many papers study the error in this approximation. For example, when $\theta < \frac{\pi}{4}$ the relative (or approximation) error[5] R is less than 3.84%. However, even this small inaccuracy can add up: after about 13 full oscillations of a pendulum with $\omega = \pi$ and $\theta_{\max} = \frac{\pi}{4}$, the approximation is out of phase by half a period with the exact pendulum. This means that after 13 oscillations, when the approximate pendulum is at equilibrium, the exact pendulum is maximally displaced, and vice-versa [10].

Finally, pendulums with small angles do not appear to be uncommon. For example, many clocks have long, thin pendulums, essentially large L and small $\theta_{\max}$. In practice these two constants are used produce clocks with long and accurate periods [27].

4.2 The LC Circuit Oscillator

Amazingly, it turns out that the equation of motion for the series LC circuit, shown in Figure 5, has the same form as the equations of motion for both spring and mass system and the small-angle pendulum system [2, 11], namely

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

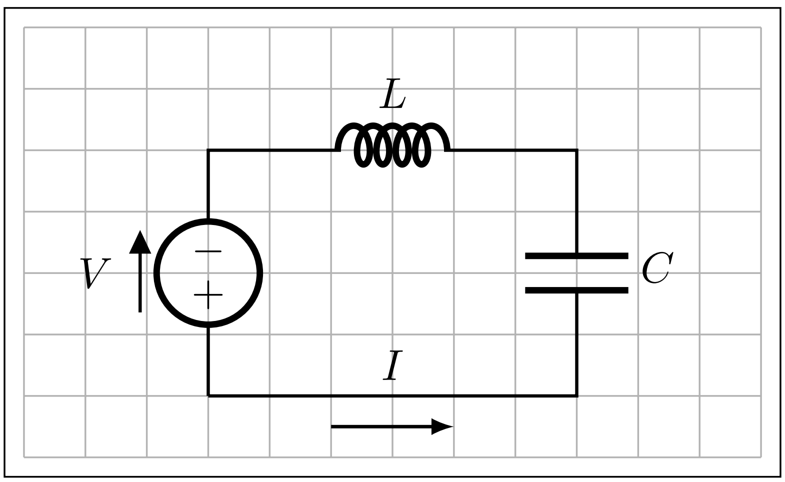

Figure 5: Series LC Circuit

Ok, but why? First, let

$$\begin{array}{lcl} V_L(t) & - & \text{denote the voltage drop across the inductor} \\ V_R(t) & - & \text{denote the voltage drop across the resistor} \\ V_C(t) & - & \text{denote the voltage drop across the capacitor} \\ V(t) & - & \text{denote the voltage increase across the power source} \end{array}$$

Then the voltage drop across the various components of the circuit[6] is given by [2]:

$$\begin{array}{ll} \text{Inductor:} & V_L = L\dot{I} \\ \text{Resistor:} & V_R = RI \\ \text{Capacitor:} & \dot{V}_C = \left(\frac{1}{C}\right) I \end{array}$$

[5]The Relative Error R is computed as follows: $R = 100 * \left(\frac{T_{\text{exact}} - T_{\text{approx}}}{T_{\text{exact}}}\right)$ [18].

[6]In the following we use "dot" notation to represent the time derivative of x. Specifically $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$.

We also know from Kirchoff's Voltage Law (sometimes called the loop rule) [23] that

$$V = V_L + V_R + V_C \tag{17}$$

For an LC circuit Kirchoff's Voltage Law tells us that $V = V_L + V_C$ (Equation (17), minus V_R since there is no resistor in the LC circuit). Substituting the voltage values into this equation we get

$$V = L\dot{I} + V_C$$

Differentiating both sides and remembering that $\dot{V}_C = \left(\frac{1}{C}\right) I$ we get

$$\dot{V} = L\ddot{I} + \left(\frac{1}{C}\right) I \tag{18}$$

Since $\dot{V} = 0$ (it is a constant source, say a battery), we can see that

$$\begin{aligned}
0 &= L\ddot{I} + \left(\frac{1}{C}\right) I && \text{\# Equation (18) with } \dot{V} = 0 \\
&= \ddot{I} + \left(\frac{1}{LC}\right) I && \text{\# divide both sides by } L \\
\Rightarrow &\quad \ddot{I} = -\left(\frac{1}{LC}\right) I && \text{\# rearrange} \\
\Rightarrow &\quad \ddot{I} = -\omega_0^2 I && \text{\# set } \omega_0 = \sqrt{\frac{1}{LC}}
\end{aligned}$$

What we can notice is that $\ddot{I} = -\omega_0^2 I$ has the same general form as the equation of motion for both the spring and mass and the simple pendulum systems. We can write the general equation of motion for these systems as

$$\ddot{\psi} = -\omega_0^2 \psi \tag{19}$$

4.2.1 The RLC Circuit Oscillator

If we add dampening to the spring and mass system it is equivalent to adding a resistor (denoted with a R) to the LC circuit. The series RLC circuit is shown in Figure 6.

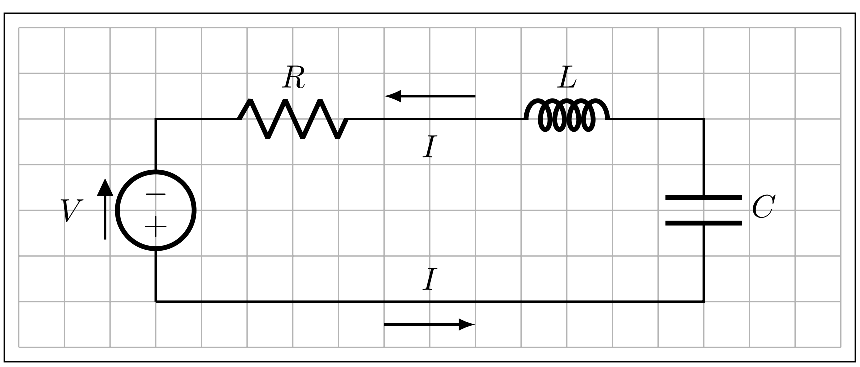

Figure 6: Series RLC Circuit

So the question is what exactly is the equivalence between a RLC circuit and a damped spring and mass system? The answer is that they are both damped simple harmonic oscillators. Ok, but why?

If we add a resistor to the series LC circuit shown in Figure 5 we get the RLC circuit shown in Figure 6. We can use Kirchoff's Voltage Law again to see that

$$V = V_L + V_R + V_C$$

Substituting the voltage drop values into this equation we get

$$V = L\dot{I} + RI + V_C$$

Taking the derivative of both sides and recalling that $\dot{V}_C = \left(\frac{1}{C}\right) I$ we get

$$\dot{V} = L\ddot{I} + R\dot{I} + \left(\frac{1}{C}\right) I \tag{20}$$

and so

$$\begin{aligned}
\dot{V} &= L\ddot{I} + R\dot{I} + \left(\frac{1}{C}\right) I && \text{\# Equation (20)} \\
&\Rightarrow\ 0 = L\ddot{I} + R\dot{I} + \left(\frac{1}{C}\right) I && \text{\# } V \text{ is a constant power source so } \dot{V} = 0 \\
&\Rightarrow\ L\ddot{I} + R\dot{I} + \left(\frac{1}{C}\right) I = 0 && \text{\# rearrange} \\
&\Rightarrow\ \ddot{I} + \left(\frac{R}{L}\right)\dot{I} + \left(\frac{1}{LC}\right) I = 0 && \text{\# divide through by } L
\end{aligned}$$

So the equation of motion for the series RLC circuit oscillator is

$$\ddot{I} + \left(\frac{R}{L}\right)\dot{I} + \left(\frac{1}{LC}\right) I = 0 \tag{21}$$

The general form of the equation of motion for a damped harmonic oscillator is

$$\ddot{\psi} + \beta\dot{\psi} + \omega_0^2 \psi = 0 \tag{22}$$

and we can see that Equation (21) is an instance of Equation (22) with $\psi = I$, $\beta = \frac{R}{L}$ and $\omega_0 = \sqrt{\frac{1}{LC}}$.

Next, consider the equation of motion for the damped spring and mass system. There we know that

$$\begin{aligned}
ma &= -kx - b\dot{x} && \text{\# Equation (3) minus the damping force } b\dot{x} \\
&= -b\dot{x} - kx && \text{\# rearrange} \\
&\Rightarrow\ ma + b\dot{x} + kx = 0 && \text{\# add } b\dot{x} + kx \text{ to both sides} \\
&\Rightarrow\ a + \left(\frac{b}{m}\right)\dot{x} + \left(\frac{k}{m}\right) x = 0 && \text{\# divide through by } m \\
&\Rightarrow\ \ddot{x} + \left(\frac{b}{m}\right)\dot{x} + \left(\frac{k}{m}\right) x = 0 && \text{\# } a = \frac{d^2x}{dt^2} = \ddot{x}
\end{aligned}$$

So we see that equation of motion for the damped spring and mass system is

$$\ddot{x} + \left(\frac{b}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0 \tag{23}$$

where $\left(\frac{b}{m}\right)\dot{x}$ is a new term that represents the damping force (e.g. resistance or friction). Here again the equation of motion, Equation(23), is of the same form as Equation (22). Finally, the displacement function for the damped harmonic oscillator is shown in Figure 7.

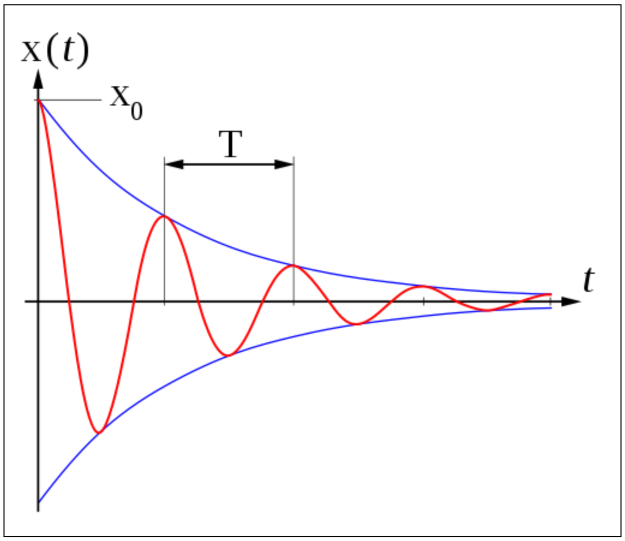

Figure 7: Damped Harmonic Oscillator Displacement Function [3]

For the damped small-angle pendulum case the equation of motion is

$$\ddot{\theta} + b\dot{\theta} + \left(\frac{g}{L}\right)\theta \approx 0 \tag{24}$$

which again has the same form as Equation (22).

Finally, a few of the analogies between mechanical and electrical systems are shown in Table 1.

Mechanical Quantity	**Symbol**	**Electrical Quantity**	**Symbol**
Mass	m	Inductance	L
Spring Constant	k	Capacitance	C
Force	F	Voltage	V
Velocity	v	Current	I
Damping Constant	b	Resistance	R

Table 1: Analogies Between Mechanical and Electrical Harmonic Oscillators

5 Quantum Harmonic Oscillators

Before diving into the structure and operation of the quantum harmonic oscillator, let's take a minute and look at the history of how we got here.

We pick up this story in 1851, when Gustav Kirchhoff met Robert Wilhelm Bunsen. Both men had been interested in why heated objects glowed with characteristic colors and intensities. Bunsen had just accepted a position at the University of Heidelberg, and Kirchhoff eventually moved to Heidelberg in 1854, beginning a fruitful collaboration with Bunsen that resulted in the establishment of the field of spectroscopy, which involves the analysis of the composition of chemical compounds through the spectra they produce [21, 29].

Intrigued by the different colors produced when various substances were heated in a flame, Bunsen wanted to use the colors the colors to identify chemical elements and compounds. Broadening the concept, Kirchhoff suggested that Bunsen not only pay attention to the immediately visible colors but also that he study the spectra of color components produced by passing the light produced by each substance through a prism. Thus was the field of spectroscopy born.

In 1859, Kirchhoff noted that dark lines found in the Sun's spectrum were further darkened when the sunlight passes through a sodium compound heated by a bunsen burner. From this, he concluded that the original dark lines, called Fraunhofer lines after the scientist who discovered them, result from sodium in the Sun's atmosphere. This opened up a new technique for analyzing the chemical composition of stars [14].

That same year Kirchhoff researched the manner in which radiation is emitted and absorbed by various substances, and formulated what is now known as Kirchoff's Law of Thermal Radiation [9]: In a state of thermal equilibrium the radiation emitted by a body is equal to the radiation absorbed by the body. By 1860, Bunsen and Kirchhoff were able to assign distinct spectral characteristics to a number of metals. Together they discovered cesium (1860) and rubidium (1861) while studying the chemical composition of the Sun via its spectral signature.

In 1862 Kirchoff introduced the concept of a "blackbody," a body that is both a perfect emitter and absorber of heat radiation. Later research into black body radiation was pivotal in the the development of the quantum theories that emerged at the beginning of the twentieth century. So how did the study of blackbodies and blackbody radiation lead to the development of quantum mechanics? To understand this we'll first take a closer look at exactly what blackbodies and blackbody radiation are.

5.1 So What Exactly are Blackbodies and Why are They Important?

To start, recall that light itself is an electromagnetic wave. In other words, light is an electric field that is oscillating. So, if you place a charged particle in a ray of light that particle will feel an oscillating force. The charge carried by a particle is also the source of an electric field. So if the charged particle is oscillating, the electric field it generates will also oscillate, that is, the charged particle will emit light. Ok, but what is a blackbody?

First, when light is incident on a body, three things can happen: the body can *reflect* some of all of the light, the body can *transmit* some or all of the light, or the body can *absorb* some of all of the light. A body that only absorbs and does not reflect or transmit is called a blackbody. Any light of any wavelength that is incident on the blackbody will not be reflected and will not be transmitted; rather, it will disappear inside the body.

So what does it imply to say that light is absorbed by a blackbody? Well, in both the wave and particle descriptions of light light is a carrier of energy. So when light is absorbed by a body, it means that the energy carried by that light is absorbed by the body and the internal energy of the body increases. When light interacts with the surface of the body, the protons and electrons begin to oscillate because of the charger they carry (sound familiar? see Section 4); that is, they gain kinetic energy. The bottom line is that the energy carried by the light is transferred to kinetic energy of the charged particles in the blackbody. Note that since the electron is much less massive than the proton it is the electrons that hold much of this (kinetic) energy.

We know from statistical mechanics that the temperature of a body T is proportional to the average kinetic energy of the particles in the body, that is

$$T_{\text{body}} \propto \overline{\text{KE}}_{\text{particles}}$$

So when the blackbody absorbs light, the kinetic energy of its electrons ($\overline{\text{KE}}_{\text{particles}}$) increases and so the temperature of the body (T_{body}) increases. Since the electrons of a blackbody which have temperature will oscillate their electric field will also oscillate and as such they will emit light. This light is called *black body radiation.*

So blackbodies can be heated by the light that is incident on their surfaces, but they can also be heated by other processes, such as in stars where fusion provides the energy (somewhat paradoxically the brightest objects (stars) are also almost perfect blackbodies). The effect on the electrons in the star, however, is the same: they gain kinetic energy, they move around, create oscillating electric fields, and light is emitted.

Next we can observe that the hotter the body is the larger an electron's kinetic energy, and therefore the larger the changes in their electric fields.

$$\begin{aligned}
T_{\text{body}} \uparrow \quad &\Rightarrow \quad \overline{\text{KE}}_{\text{particles}} \uparrow && \text{\# increase in } T_{\text{body}} \Rightarrow \text{increase in average kinetic energy } (\overline{\text{KE}}) \\
&\Rightarrow \quad \overline{v}_{\text{electrons}} \uparrow && \text{\# increase in } \overline{\text{KE}} \Rightarrow \text{increase in average frequency} \\
&\Rightarrow \quad \nabla \vec{E} \uparrow && \text{\# increase in } \overline{\text{KE}} \Rightarrow \text{increase in their energy field } (\nabla \vec{E}) \\
&\Rightarrow \quad P_{\text{light}} \uparrow && \text{\# increase in } \vec{E} \Rightarrow \text{increase in the power of the light emitted } (P_{\text{light}}) \\
&\Rightarrow \quad P_{\text{light}} = f(T) && \text{\# the power of the light emitted from a blackbody is a function of its temperature}
\end{aligned}$$

So what we have learned is there is a direct relationship between the temperature of the surface of the blackbody and the intensity of the light it emits. This relationship is called Stefan-Boltzmann's Law and is typically written as

$$P = \sigma A T^4 \tag{25}$$

In words, Equation (25) says that the power (P) emitted by a blackbody is proportional to its area (A) and temperature to the forth power (T^4). The proportionality constant σ is called Stefan-Boltzmann's constant and

$$\sigma = 5.67 \times 10^{-8} W m^{-2} K^{-4}$$

Now would be a good time to remember that a blackbody is just a model; there is no such thing as a perfect blackbody; in reality objects do not absorb one hundred percent of the incident light. Rather, in real bodies there is some reflection and transmission. Such a (real) body is called a *greybody.* There is a quantity, emissivity, which tells us how close a greybody is to a theoretical blackbody.

Definition 5.1. Emissivity: An object's emissivity, e, measures how a greybody's behavior compares to a theoretical blackbody.

Now we can calculate the power emitted by a greybody as follows

$$P = e \sigma A T^4 \tag{26}$$

We can see that when the emissivity of an object is equal to one ($e = 1$), the object is an ideal blackbody. When the emissivity is equal to zero ($e = 0$) we call the object is ideal white body. Finally, if $0 < e < 1$ we call the object a greybody.

Interestingly, for humans

$$\begin{aligned}
e &= 0.97 && \text{\# humans are greybodies that are almost blackbodies} \\
T &= 37^\circ C && \text{\# average human temperature is about } 37^\circ C \\
P &= e\sigma A T^4 \approx 810 W && \text{\# human emit about 810 W of electromagnetic radiation}
\end{aligned}$$

So humans do emit light, we just can't see it (it's in the infrared part of the spectrum). Surprisingly, animals and plants turn out to be have high emissivity (are good blackbodies), while minerals such as aluminium or silver have low emissivity (high reflectivity). This is why aluminium ($e = 0.05$) is used in household mirrors and silver ($e = 0.02$) is used in high-grade mirrors; they are both nearly perfect reflectors.

6 Quantum Fields

7 Conclusions

Acknowledgements

Thanks to Chris Lonvick (lonvick@gmail.com) for his thoughtful suggestions on Figure 4 and to Dave Plonka for his helpful comments on Table 1. Thanks also to Bill Westfield for noticing an error in my description of the accuracy of Huygens' pendulum clock.

LaTeX Source

`https://www.overleaf.com/read/xjmyvksvtztb`

References

[1] Albert Van Helden et al. The Galileo Project. http://galileo.rice.edu/sci/instruments/pendulum.html, 1995. [Online; accessed 17-December-2021].

[2] Arthur Mattuck. RLC Circuits. https://ocw.mit.edu/courses/mathematics/18-03-differential-equations-spring-2010/readings/supp_notes/MIT18_03S10_chapter_8.pdf, 2010. [Online; accessed 19-December-2021].

[3] beltoforion.de. Damped Harmonic Oscillator. https://beltoforion.de/en/harmonic_oscillator/, 2015. [Online; accessed 24-December-2021].

[4] Bill Casselman. Periodic Functions. https://personal.math.ubc.ca/~cass/courses/m256-8b/pdf/periodic.pdf, 2005. [Online; accessed 17-December-2021].

[5] David Meyer. A Few Notes on Euler's Formula and Euler's Identity. https://davidmeyer.github.io/qc/euler.pdf, 2021. [See https://davidmeyer.github.io/qc/].

[6] Doug Davis. Elastic Potential Energy. https://www.ux1.eiu.edu/~cfadd/1150/06WrkEng/ElPE.html, 2002. [Online; accessed 26-September-2022].

[7] Robert Hooke. Lectures De potentia restitutiva: Or, Of spring explaining the power of springing bodies, 1678.

[8] J J O'Connor and E F Robertson. Robert Hooke. https://mathshistory.st-andrews.ac.uk/Biographies/Hooke, August 2002. [Online; accessed 9-December-2021].

[9] Nick Connor. What is Kirchhoff's Law of Thermal Radiation – Definition. https://www.thermal-engineering.org/what-is-kirchhoffs-law-of-thermal-radiation-definition, May 2019. [Online; accessed 30-December-2021].

[10] Nicolas Graber-Mitchell. Finding the Period of a Simple Pendulum. https://arxiv.org/pdf/1805.00002.pdf, May 2018.

[11] Paul Avery and Yasu Takano. RLC Circuits. http://www.phys.ufl.edu/courses/phy2049/f06/lectures/2049_ch31_avery.pdf, December 2006. https://www.phys.ufl.edu/courses/phy2049/f06/overview.shtml.

[12] R. Nave. Simple Harmonic Motion. http://hyperphysics.phy-astr.gsu.edu/hbase/shm.html, 2005. [Online; accessed 16-December-2021].

[13] R. Nave (HyperPhysics). Kinetic Energy. http://hyperphysics.phy-astr.gsu.edu/hbase/ke.html, 2022. [Online; accessed 25-September-2022].

[14] Science History Institute. Robert Bunsen and Gustav Kirchhoff. https://www.sciencehistory.org/historical-profile/robert-bunsen-and-gustav-kirchhoff, May 2021. [Online; accessed 30-December-2021].

[15] Tseng, Z. S. Second Order Linear Partial Differential Equations. http://www.personal.psu.edu/sxt104/class/Math251/Notes-PDE%20pt1.pdf, 2016. [Online; accessed 16-December-2021].

[16] Warren A. Marrison. The Evolution of the Quartz Crystal Clock. http://www.ieee-uffc.org/main/history.asp?file=marrison, 1948. [Online; accessed 15-December-2021].

[17] Wikipedia Contributors. Angular Frequency — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Angular_frequency&oldid=1058712937, 2021. [Online; accessed 10-December-2021].

[18] Wikipedia Contributors. Approximation Error — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Approximation_error&oldid=1056596112, 2021. [Online; accessed 18-December-2021].

[19] Wikipedia Contributors. Christiaan Huygens — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Christiaan_Huygens&oldid=1046113855, 2021. [Online; accessed 27-September-2021].

[20] Wikipedia Contributors. Free Body Diagram — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Free_body_diagram&oldid=1058537164, 2021. [Online; accessed 15-December-2021].

[21] Wikipedia Contributors. Gustav Kirchhoff — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Gustav_Kirchhoff&oldid=1059244105, 2021. [Online; accessed 30-December-2021].

[22] Wikipedia Contributors. Hooke's Law — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Hooke%27s_law&oldid=1045035875, 2021. [Online; accessed 21-September-2021].

[23] Wikipedia Contributors. Kirchhoff's Circuit Laws — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Kirchhoff%27s_circuit_laws&oldid=1059111176, 2021. [Online; accessed 20-December-2021].

[24] Wikipedia Contributors. Linearity of Differentiation — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Linearity_of_differentiation&oldid=1026177205, 2021. [Online; accessed 1-September-2021].

[25] Wikipedia Contributors. Newton's Laws of Motion — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Newton%27s_laws_of_motion&oldid=1037183516, 2021. [Online; accessed 25-August-2021].

[26] Wikipedia Contributors. Pendulum — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Pendulum&oldid=1058923597, 2021. [Online; accessed 15-December-2021].

[27] Wikipedia Contributors. Pendulum Clock — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Pendulum_clock&oldid=1060793234, 2021. [Online; accessed 18-December-2021].

[28] Wikipedia Contributors. Potential Energy — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Potential_energy&oldid=1028956270, 2021. [Online; accessed 25-August-2021].

[29] Wikipedia Contributors. Robert Bunsen — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Robert_Bunsen&oldid=1059421613, 2021. [Online; accessed 30-December-2021].

[30] Wikipedia Contributors. Chain Rule — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Chain_rule&oldid=1102134961, 2022. [Online; accessed 27-September-2022].

[31] Wikipedia Contributors. Conservative Force — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Conservative_force&oldid=1087071434, 2022. [Online; accessed 25-September-2022].

[32] Wikipedia Contributors. Power Rule — Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/w/index.php?title=Power_rule&oldid=1111058436, 2022. [Online; accessed 27-September-2022].

[33] William Morris. The American heritage dictionary of the English language. https://archive.org/details/americanheritag00morr/page/969/mode/2up, 1980. [Online; accessed 15-December-2021].