$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}} = ?$$

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1 Introduction

One way to think about this question is the following:

$$
\begin{aligned}
S \quad &= \quad \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}} && \text{\# define } S \\
&\Rightarrow \quad S^2 = 2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}} && \text{\# square both sides} \\
&\Rightarrow \quad S^2 = 2 + S && \text{\# } S = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}} \\
&\Rightarrow \quad S^2 - S - 2 = 0 && \text{\# collect terms} \\
&\Rightarrow \quad (S - 2)(S + 1) = 0 && \text{\# factor} \\
&\Rightarrow \quad S \in \{2, -1\} && \text{\# solve for } S \\
&\Rightarrow \quad S = 2 && \text{\# positive root of } S \text{ (what about the negative root?)}
\end{aligned}
$$

So apparently $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}} = 2$.