A Few Notes on Density Operators, Expectation Values and Matrix Shapes

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1 Introduction

These notes started life as an experiment in drawing matrices and their shapes (see Section 4). However, it has evolved into a more ad-hoc collection of notes covering a few topics in quantum mechanics. So its a WIP. We start with a review of Orthonormality, Completeness, and Projection...

2 Orthonormality, Completeness, and Projection

As we saw above, unitary matrices are matrices which satisfy

$$\mathbf{U}^{-1} = \mathbf{U}^{\dagger} \tag{1}$$

Unitary matrices are ubiquitous and important in quantum mechanics, in particular because they have the following unique and useful properties: Orthonormality, Completeness, and Projection [3]. We'll briefly look at each of these below[1].

2.1 Orthornomality

We can rewrite Equation 1 as

$$\mathbf{U}^{\dagger}\mathbf{U} = \mathbf{I} \tag{2}$$

where $\mathbf{I}$ is the *identity* matrix. What Equation 2 is really telling us is that the columns of the matrix $\mathbf{U}$ form a set of orthnormal vectors.

[1]I will use the notation $(x_1, \ldots, x_n)^{\mathrm{T}}$ and $[x_1, \ldots, x_n]^{\mathrm{T}}$ interchangably in the following discussion.

Note that we can interpret a matrix as a row vector where the entries are the columns $\mathbf{v}_i$ of $\mathbf{U}$. That is

$$\mathbf{U} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_N \end{bmatrix}$$

Similarly, $\mathbf{U}^{-1}$ can be written as a column vector where the entries are the row vectors $\mathbf{v}_i^\dagger$:

$$\mathbf{U}^{-1} = \mathbf{U}^\dagger = \begin{bmatrix} \mathbf{v}_1^\dagger \\ \mathbf{v}_2^\dagger \\ \vdots \\ \mathbf{v}_N^\dagger \end{bmatrix}$$

Now we can see that

$$\begin{aligned}
\mathbf{U}^\dagger\mathbf{U} &= \begin{bmatrix} \mathbf{v}_1^\dagger \\ \mathbf{v}_2^\dagger \\ \vdots \\ \mathbf{v}_N^\dagger \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_N \end{bmatrix} \\
&= \begin{bmatrix}
\mathbf{v}_1^\dagger \cdot \mathbf{v}_1 & \mathbf{v}_1^\dagger \cdot \mathbf{v}_2 & \mathbf{v}_1^\dagger \cdot \mathbf{v}_3 & \dots & \mathbf{v}_1^\dagger \cdot \mathbf{v}_N \\
\mathbf{v}_2^\dagger \cdot \mathbf{v}_1 & \mathbf{v}_2^\dagger \cdot \mathbf{v}_2 & \mathbf{v}_2^\dagger \cdot \mathbf{v}_3 & \dots & \mathbf{v}_2^\dagger \cdot \mathbf{v}_N \\
\mathbf{v}_3^\dagger \cdot \mathbf{v}_1 & \mathbf{v}_3^\dagger \cdot \mathbf{v}_2 & \mathbf{v}_3^\dagger \cdot \mathbf{v}_3 & \dots & \mathbf{v}_3^\dagger \cdot \mathbf{v}_N \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{v}_N^\dagger \cdot \mathbf{v}_1 & \mathbf{v}_N^\dagger \cdot \mathbf{v}_2 & \mathbf{v}_N^\dagger \cdot \mathbf{v}_3 & \dots & \mathbf{v}_N^\dagger \cdot \mathbf{v}_N
\end{bmatrix} \\
&= \mathbf{I}
\end{aligned}$$

or in Dirac notation [2]

$$
\begin{aligned}
\mathbf{U}^\dagger\mathbf{U} &= \begin{bmatrix} \mathbf{v}_1^\dagger \\ \mathbf{v}_2^\dagger \\ \vdots \\ \mathbf{v}_N^\dagger \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_N \end{bmatrix} \\
&= \begin{bmatrix} \langle v_1| \\ \langle v_2| \\ \vdots \\ \langle v_N| \end{bmatrix} \begin{bmatrix} |v_1\rangle & |v_2\rangle & \dots & |v_N\rangle \end{bmatrix} \\
&= \begin{bmatrix} \langle v_1|v_1\rangle & \langle v_1|v_2\rangle & \langle v_1|v_3\rangle & \dots & \langle v_1|v_N\rangle \\ \langle v_2|v_1\rangle & \langle v_2|v_2\rangle & \langle v_2|v_3\rangle & \dots & \langle v_2|v_N\rangle \\ \langle v_3|v_1\rangle & \langle v_3|v_2\rangle & \langle v_3|v_3\rangle & \dots & \langle v_3|v_N\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle v_N|v_1\rangle & \langle v_N|v_2\rangle & \langle v_N|v_3\rangle & \dots & \langle v_N|v_N\rangle \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \\
&= \mathbf{I}
\end{aligned}
$$

Note here that $\langle v_i|v_i\rangle = 1$ (the v_i are unit vectors) and $\langle v_i|v_j\rangle = 0$ for $i \neq j$ (v_i and v_j are orthogonal). In quantum mechanics two states v_i and v_j are said to be distinguishable or measurable if they are orthogonal, that is, if $\langle v_i|v_j\rangle = 0$.

Another way to say this to notice[2] that since $(\mathbf{U}^\dagger\mathbf{U})_{ij} = (\mathbf{U}^{-1}\mathbf{U})_{ij} = \delta_{ij}$, the columns of $\mathbf{U}$ can be written as the inner product $\langle v_i|v_j\rangle = \delta_{ij}$. Said another way, the vectors v_i form an orthonormal set. In particular, if $\mathbf{V} = \{v_j\}$ is an orthonormal set, then for $v_i, v_j \in \mathbf{V}$, the inner product $\langle v_i|v_j\rangle = \delta_{ij}$. See Section 4 for a brief discussion on matrix shapes.

2.2 Completeness

From $\mathbf{U}^\dagger\mathbf{U} = \mathbf{I}$ we saw that we could derive orthonormality. But we also expect that $\mathbf{U}\mathbf{U}^\dagger = \mathbf{I}$. It turns out that we can get something interesting by observing this. In particular

[2] δ_{ij} is the Kronecker Delta function [4], $\delta_{ij} = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases}$

$$
\mathbf{U}\mathbf{U}^{\dagger} = \begin{bmatrix} |v_1\rangle & |v_2\rangle & |v_3\rangle & \dots & |v_N\rangle \end{bmatrix} \begin{bmatrix} \langle v_1| \\ \langle v_2| \\ \langle v_3| \\ \vdots \\ \langle v_N| \end{bmatrix}
$$

If we multiply this out we find that

$$
|v_1\rangle \langle v_1| + |v_2\rangle \langle v_2| + \cdots + |v_N\rangle \langle v_N| = \sum_{i=1}^{N} |v_i\rangle \langle v_i| = \mathbf{I} \tag{3}
$$

Equation 3 is known as the *completeness* relation.

Completeness turns out to be useful and is a sort of a "dual" of orthonormality. While orthonormality is kind of an "inner product" ($\mathbf{U}^{\dagger}\mathbf{U}$), completeness is like an outer product in that $\mathbf{U}\mathbf{U}^{\dagger}$ is a sum over i of $|v_i\rangle \langle v_i|$, although the shapes might be seen as reversed (see Section 4 on shapes).

2.3 Projection

To get an idea of what projection is all about, consider the expansion of a vector into components in a basis:

$$
|w\rangle = \sum_{i=1}^{N} w_i |v_i\rangle \tag{4}
$$

Now, if the set of vectors basis vectors $\{v_i\}$ are orthonormal, then we know that

$$
w_i = \langle v_i|w\rangle
$$

and substituting back into Equation 4 we get

$$
|w\rangle = \sum_{i=1}^{N} \langle v_i|w\rangle |v_i\rangle
$$

Interestingly, there is another way to derive this result: use the completeness relation, which is simply a fancy but useful way to write $\mathbf{I}$:

$$|w\rangle = \mathbf{I} \cdot |w\rangle = \left(\sum_{i=1}^{N} |v_i\rangle \langle v_i| \right) |w\rangle = \sum_{i=1}^{N} |v_i\rangle \langle v_i|w\rangle$$

In words, we were able to use the completeness relation to project a vector onto its components in a particular basis.

For example, we know that for vectors $|\alpha\rangle$ and $|\beta\rangle$, we can take the inner product between them by using their components in a basis $\{v_i\}$:

$$\langle \alpha|\beta\rangle = \sum_{i=1}^{N} a_i^* b_i$$

where $a_i = \langle v_i|\alpha\rangle$ and $b_i = \langle v_i|\beta\rangle$. Interestingly, we can again derive this using the completeness relation:

$$\begin{aligned}
\langle \alpha|\beta\rangle &= \langle \alpha| \mathbf{I} |\beta\rangle && \# \ \langle \alpha|\beta\rangle = \langle \alpha| \, |\beta\rangle = \langle \alpha| \mathbf{I} |\beta\rangle \\
&= \langle \alpha| \left(\sum_{i=1}^{N} |v_i\rangle \langle v_i| \right) |\beta\rangle && \# \ \sum_{i=1}^{N} |v_i\rangle \langle v_i| = \mathbf{I} \ \text{ (Equation 3)} \\
&= \sum_{i=1}^{N} \langle \alpha|v_i\rangle \langle v_i|\beta\rangle && \# \ \text{rearrange} \\
&= \sum_{i=1}^{N} \langle v_i|\alpha\rangle^* \langle v_i|\beta\rangle && \# \ \langle a|b\rangle = \langle b|a\rangle^* \text{ so } \langle \alpha|v_i\rangle = \langle v_i|\alpha\rangle^* \\
&= \sum_{i=1}^{N} a_i^* b_i && \# \ a_i^* = \langle v_i|\alpha\rangle^* \ \text{ and } \ b_i = \langle v_i|\beta\rangle
\end{aligned}$$

3 Expectation Values

Consider an observable $\mathbf{A}$ in the pure state $|\psi\rangle$. The expectation value $\langle A\rangle_\psi$ is given by

$$\langle A\rangle_\psi = \langle \psi| A |\psi\rangle \tag{5}$$

where $\dim(\langle \psi|) = 1 \times n$, $\dim(A) = n \times n$, and $\dim(|\psi\rangle) = n \times 1$.

So why is $\langle A\rangle_\psi$ an expectation? Well, first, if A is an observable for a system with a discrete set of values $\{a_1, a_2, \ldots, a_N\}$, then this observable is represented by a Hermitean operator

$\hat{A}$ that has these discrete values as its eigenvalues, and associated eigenstates $\{|a_n\rangle\}$, for $n = 1, 2, 3, \ldots$ satisfying the eigenvalue equation $\hat{A}|a_n\rangle = a_n |a_n\rangle$. I drop the "hat" in most of the below.

First, observe that $\langle a_n| A = a_n \langle a_n|$. Why?

$$
\begin{array}{llll}
A|a_n\rangle & = & a_n|a_n\rangle & \text{\# eigenvalue equation for } A\ (A\mathbf{v} = \lambda\mathbf{v}) \\
& \Longrightarrow & (A|a_n\rangle)^\dagger = (a_n|a_n\rangle)^\dagger & \text{\# conjugate transpose both sides} \\
& \Longrightarrow & |a_n\rangle^\dagger A^\dagger = |a_n\rangle^\dagger a_n^\dagger & \text{\# } (AB)^\dagger = B^\dagger A^\dagger \\
& \Longrightarrow & |a_n\rangle^\dagger A^\dagger = a_n^\dagger |a_n\rangle^\dagger & \text{\# rearrange } (a_n^\dagger \text{ is a scalar}) \\
& \Longrightarrow & |a_n\rangle^\dagger A = a_n^\dagger |a_n\rangle^\dagger & \text{\# } A \text{ is Hermitean so } A = A^\dagger \\
& \Longrightarrow & |a_n\rangle^\dagger A = a_n^* |a_n\rangle^\dagger & \text{\# } a_n^\dagger = a_n^*\ (a_n \text{ is a scalar}) \\
& \Longrightarrow & \langle a_n| A = a_n^* \langle a_n| & \text{\# } |a_n\rangle^\dagger = \langle a_n| \\
& \Longrightarrow & \langle a_n| A = a_n \langle a_n| & \text{\# } a_n* = a_n
\end{array}
\tag{6}
$$

But why does $a_n^* = a_n$ (last line of (6))? Well, consider

$$
\begin{array}{llll}
AX & = & \lambda X & \text{\# eigenvalue equation} \\
& \Longrightarrow & X^\dagger A^\dagger = X^\dagger \lambda^\dagger & \text{\# } (AB)^\dagger = B^\dagger A^\dagger \\
& \Longrightarrow & X^\dagger A^\dagger = \lambda^\dagger X^\dagger & \text{\# rearrange } (\lambda^\dagger \text{ is a scalar}) \\
& \Longrightarrow & X^\dagger A^\dagger = \lambda^* X^\dagger & \text{\# } \lambda^\dagger = \lambda^*\ (\lambda \text{ is a scalar}) \\
& \Longrightarrow & X^\dagger A = \lambda^* X^\dagger & \text{\# } A^\dagger = A \text{ since } A \text{ is Hermitean} \\
& \Longrightarrow & X^\dagger A = X^\dagger \lambda^* & \text{\# rearrange} \\
& \Longrightarrow & X^\dagger A X = X^\dagger \lambda^* X & \text{\# multiply both sides by } X
\end{array}
\tag{7}
$$

Now notice that if we multiply both sides of the original eigenvalue equation ($AX = \lambda X$) by $X^\dagger$ we get $X^\dagger AX = X^\dagger \lambda X$. We know from (7) that $X^\dagger AX = X^\dagger \lambda^* X$ and therefore that $X^\dagger \lambda^* X = X^\dagger \lambda X$. This implies that $\lambda^* = \lambda$, so $\lambda \in \mathbb{R}$. Similarly $a_n^* = a_n$ so $a_n \in \mathbb{R}$.

Another way to look at this is to assume the computational basis[3] and then

[3] The approach taken in (6) doesn't seem to require this assumption.

$$
\begin{aligned}
\langle a_n| A &= a_n \langle n| A && \# \ \langle a_n| = a_n \begin{bmatrix}0 & \dots & 1 & \dots 0\end{bmatrix} = a_n \langle n| \\
&= a_n \langle n| A^\dagger && \# \ A \text{ is Hermitian so } A = A^\dagger \\
&= a_n \langle n| \begin{bmatrix} \langle a_1| \\ \vdots \\ \langle a_n| \\ \vdots \\ \langle a_N| \end{bmatrix} && \# \quad A^\dagger = \begin{bmatrix} \langle a_1| \\ \vdots \\ \langle a_n| \\ \vdots \\ \langle a_N| \end{bmatrix} \\
&= a_n \begin{bmatrix}0 & \cdots & 1 & \cdots 0\end{bmatrix} \begin{bmatrix} \langle a_1| \\ \vdots \\ \langle a_n| \\ \vdots \\ \langle a_N| \end{bmatrix} && \# \ \langle n| = \begin{bmatrix}0 & \dots & 1 & \dots 0\end{bmatrix} \\
&= a_n \langle a_n| && \# \ \langle n| \text{ selects the } n^{th} \text{ element of } A^\dagger, \langle a_n|
\end{aligned}
$$

In any event, now we have $\langle a_n| A = a_n \langle a_n|$. So we can observe that

$$
\begin{aligned}
\langle A \rangle_\psi &= \langle \psi | A | \psi \rangle && \# \text{ definition of } \langle A \rangle_\psi \text{ for } \textit{pure} \text{ state } |\psi\rangle \\
&= \langle \psi | I A | \psi \rangle && \# \ I \cdot A = A \\
&= \langle \psi | \left(\sum_{n=1}^{N} |a_n\rangle \langle a_n| \right) A | \psi \rangle && \# \ \sum_{n=1}^{N} |a_n\rangle \langle a_n| = \mathbf{I} \ \big(\text{Equation 3}\big) \\
&= \sum_{n=1}^{N} \langle \psi | a_n \rangle \langle a_n | A | \psi \rangle && \# \text{ rearrange} \\
&= \sum_{n=1}^{N} \langle \psi | a_n \rangle a_n \langle a_n | \psi \rangle && \# \ \langle a_n| A = a_n \langle a_n| \ \text{ (see above)} \\
&= \sum_{n=1}^{N} \langle \psi | a_n \rangle \langle a_n | \psi \rangle a_n && \# \text{ rearrange} \\
&= \sum_{n=1}^{N} | \langle \psi | a_n \rangle |^2 a_n && \# \ | \langle \psi | a_n \rangle |^2 = \langle \psi | a_n \rangle \langle \psi | a_n \rangle^* = \langle \psi | a_n \rangle \langle a_n | \psi \rangle \\
&= \sum_{n=1}^{N} p(a_n) a_n && \# \ | \langle \psi | a_n \rangle |^2 = p(a_n), \text{the probability of observing eigenvalue } a_n \\
&= \sum_{n=1}^{N} \tfrac{N_n}{N} a_n && \# \ N_n \text{ is the number of times } a_n \text{ has been measured} \\
&= \mathbb{E}[A] && \# \ \mathbb{E}[X] = \sum_{n=1}^{N} p(X_n) X_n
\end{aligned}
$$

So the expectation value for the result of a measurement represented by a self-adjoint operator A, $\langle A \rangle_\psi$, is the weighted average of all possible outcomes under A, that is, $\mathbb{E}[A]$.

4 Shapes

One way to visualize $\langle A \rangle_\psi$ is

$$
\begin{aligned}
\langle A \rangle_\psi &\rightarrow \underbrace{\begin{bmatrix} \dots\dots\dots \end{bmatrix}}_{1\times n} \underbrace{\begin{bmatrix} \cdots & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots \end{bmatrix}}_{n\times n} \underbrace{\begin{bmatrix} \cdots \\ \vdots \\ \cdots \end{bmatrix}}_{n\times 1} \\
&\rightarrow c
\end{aligned}
$$

where $c \in \mathbb{C}$.

The *density operator* ρ for pure state $|\psi\rangle$ is given by $\rho = |\psi\rangle \langle\psi|$. The shape of ρ is

$$
\rho \rightarrow \underbrace{\begin{bmatrix} \cdots \\ \vdots \\ \cdots \end{bmatrix}}_{n\times 1} \underbrace{\begin{bmatrix} \dots\dots\dots \end{bmatrix}}_{1\times n} \rightarrow \underbrace{\begin{bmatrix} \cdots & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots \end{bmatrix}}_{n\times n}
$$

The shape of the inner product of two $n \times 1$ column vectors $\langle \mathbf{u}, \mathbf{v} \rangle = \langle u|v \rangle = \mathbf{u}^{\mathrm{T}}\mathbf{v}$ is

$$
\mathbf{u}^{\mathrm{T}}\mathbf{v} \rightarrow \underbrace{\begin{bmatrix} \dots\dots\dots \end{bmatrix}}_{1\times n} \underbrace{\begin{bmatrix} \cdots \\ \vdots \\ \cdots \end{bmatrix}}_{n\times 1} \rightarrow c
$$

where $c \in \mathbb{C}$. The shape of the outer product $\mathbf{u} \otimes \mathbf{v} = |u\rangle \langle v| = \mathbf{u}\mathbf{v}^{\mathrm{T}}$ is

$$
\mathbf{u}\mathbf{v}^{\mathrm{T}} \rightarrow \underbrace{\begin{bmatrix} \cdots \\ \vdots \\ \cdots \end{bmatrix}}_{n\times 1} \underbrace{\begin{bmatrix} \dots\dots\dots \end{bmatrix}}_{1\times n} \rightarrow \underbrace{\begin{bmatrix} \cdots & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots \end{bmatrix}}_{n\times n}
$$

5 The Density ρ and the Trace of an Operator D

So ρ is an $n \times n$ *linear operator* with $\mathrm{Tr}(\rho) = \mathrm{Tr}(|\psi\rangle\langle\psi|) = \langle\psi|\psi\rangle$. In addition, $\mathrm{Tr}(|\psi_i\rangle\langle\psi_i|) = \langle\psi_i|\psi_i\rangle = \delta_{ii} = 1$, and if $\{|\psi_i\rangle\}$ is an orthonormal basis then $\mathrm{Tr}(|\psi_i\rangle\langle\psi_j|) = \langle\psi_i|\psi_j\rangle = \delta_{ij}$.

The density matrix [1] ρ has the following important properties:

Projection:	$\rho^2 = \rho$
Hermiticity:	$\rho^\dagger = \rho$
Normalization:	$\mathrm{Tr}(\rho) = 1$
Positivity:	$\rho \geq 1$

The *trace* of an operator D, $\mathrm{Tr}(D)$, is defined to be $\mathrm{Tr}(D) = \sum_{i=1}^{n} \langle n|D|n\rangle$. Now, suppose $D = |\psi\rangle\langle\phi|$. Then we can see that $\mathrm{Tr}(D) = \mathrm{Tr}(|\psi\rangle\langle\phi|) = \langle\phi|\psi\rangle$ as follows:

$$
\begin{aligned}
\mathrm{Tr}(D) &= \sum_{n=1}^{N} \langle n|D|n\rangle && \#\ \text{definition of } \mathrm{Tr}(D) \\
&= \sum_{n=1}^{N} \langle n|\left(|\psi\rangle\langle\phi|\right)|n\rangle && \#\ D = |\psi\rangle\langle\phi| \\
&= \sum_{n=1}^{N} \langle n|\,|\psi\rangle\langle\phi|\,|n\rangle && \#\ \text{drop parens} \\
&= \sum_{n=1}^{N} \langle n|\psi\rangle\langle\phi|n\rangle && \#\ \langle a|b\rangle = \langle a|\,|b\rangle \\
&= \sum_{n=1}^{N} \langle\phi|n\rangle\langle n|\psi\rangle && \#\ \text{rearrange} \\
&= \langle\phi|\left(\sum_{n=1}^{N} |n\rangle\langle n|\right)|\psi\rangle && \#\ \text{neither } \phi \text{ nor } \psi \text{ depend on } n \\
&= \langle\phi|I|\psi\rangle && \#\ \sum_{n=1}^{N} |n\rangle\langle n| = \mathbf{I}\ \big(\text{Equation 3}\big) \\
&= \langle\phi|\,|\psi\rangle && \#\ \langle\phi|\,I = \langle\phi| \text{ and } I\,|\psi\rangle = |\psi\rangle \\
&= \langle\phi|\psi\rangle && \#\ \langle\phi|\psi\rangle = \langle\phi|\,|\psi\rangle
\end{aligned}
$$

So the trace of the outer product $|\psi\rangle\langle\phi|$, $\mathrm{Tr}(|\psi\rangle\langle\phi|)$, is the inner product $\langle\phi|\psi\rangle$.

A simple theorem relates the expectation value of an observable A in a state represented by a density matrix ρ to the trace of A:

$$\langle A\rangle_\rho = \mathrm{Tr}(\rho A) \tag{8}$$

The proof of Equation 8 is also pretty simple:

$$
\begin{aligned}
\mathrm{Tr}(\rho A) &= \mathrm{Tr}(|\psi\rangle\langle\psi| A) && \#\ \rho \equiv |\psi\rangle\langle\psi| \\
&= \sum_{n=1}^{N} \langle n| |\psi\rangle \langle\psi| A |n\rangle && \#\ \text{definition of } \mathrm{Tr}(\cdot) \\
&= \sum_{n=1}^{N} \langle n|\psi\rangle \langle\psi| A |n\rangle && \#\ \langle n|\psi\rangle = \langle n| |\psi\rangle \\
&= \sum_{n=1}^{N} \langle\psi| A |n\rangle \langle n|\psi\rangle && \#\ \text{rearrange} \\
&= \langle\psi| A \left(\sum_{n=1}^{N} |n\rangle \langle n| \right) |\psi\rangle && \#\ \text{neither } A \text{ nor } \psi \text{ depend on } n \\
&= \langle\psi| A \cdot I |\psi\rangle && \#\ \sum_{n=1}^{N} |n\rangle\langle n| = \mathbf{I}\ \big(\text{Equation 3}\big) \\
&= \langle\psi| A |\psi\rangle && \#\ \mathbf{A} \cdot \mathbf{I} = \mathbf{A} \\
&= \langle A\rangle && \#\ \langle A\rangle_\psi = \langle\psi| A |\psi\rangle\ \big(\text{Equation 5}\big)
\end{aligned}
$$

6 A More General View of the Density Operator

Consider an ensemble of identical quantum systems. The system has probability w_i to be in quantum state $|\psi_i\rangle$. Here $\langle\psi_i|\psi_i\rangle = 1$, but the states $|\psi_i\rangle$ aren't necessarily orthogonal to one another. That means that out of all the examples in the ensemble, a fraction w_i are in state $|\psi_i\rangle$, with $w_i > 0$ and $\sum_i w_i = 1$.

The expectation value for the result of a measurement represented by a self-adjoint operator A is

$$\langle A\rangle_\psi = \sum_i w_i \langle\psi_i|A|\psi_i\rangle \tag{9}$$

We can write the expectation value in a different way using a basis $|K\rangle$ as

$$
\begin{array}{rcll}
\langle A\rangle_\psi & = & \sum\limits_i w_i \langle\psi_i|A|\psi_i\rangle & \text{\# defintion of } \langle A\rangle_\psi\text{, Equation 9} \\
& = & \sum\limits_i w_i \langle\psi_i|IAI|\psi_i\rangle & \# \ \mathbf{A} = \mathbf{I}\cdot\mathbf{A}\cdot\mathbf{I} \\
& = & \sum\limits_i w_i \langle\psi_i|\Big(\sum\limits_J |J\rangle\langle J|\Big)|A|\Big(\sum\limits_K |K\rangle\langle K|\Big)|\psi_i\rangle & \# \sum\limits_J |J\rangle\langle J| = \mathbf{I}, \sum\limits_K |K\rangle\langle K| = \mathbf{I} \\
& = & \sum\limits_i w_i \sum\limits_{J,K} \langle\psi_i|J\rangle\langle J|A|K\rangle\langle K|\psi_i\rangle & \text{\# rearrange} \\
& = & \sum\limits_i w_i \sum\limits_{J,K} \langle K|\psi_i\rangle\langle\psi_i|J\rangle\langle J|A|K\rangle & \text{\# rearrange} \\
& = & \sum\limits_{J,K} \sum\limits_i w_i \langle K|\psi_i\rangle\langle\psi_i|J\rangle\langle J|A|K\rangle & \text{\# none of } A,\ J, \text{ or } K \text{ depend on } i \\
& = & \sum\limits_{J,K} \langle K|\Big(\sum\limits_i w_i |\psi_i\rangle\langle\psi_i|\Big)|J\rangle\langle J|A|K\rangle & \text{\# rearrange} \\
& = & \sum\limits_{J,K} \langle K|\rho|J\rangle\langle J|A|K\rangle & \# \ \rho \equiv \sum\limits_i w_i |\psi_i\rangle\langle\psi_i| \\
& = & \sum\limits_K \langle K|\rho IA|K\rangle & \# \sum\limits_J |J\rangle\langle J| = \mathbf{I} \\
& = & \sum\limits_K \langle K|\rho A|K\rangle & \# \ \mathbf{I}\cdot\mathbf{A} = \mathbf{A} \\
& = & \mathrm{Tr}(\rho A) & \# \ \mathrm{Tr}(D) = \sum\limits_n \langle n|D|n\rangle
\end{array}
$$

6.1 Properties of the Density Operator

As mentioned above, there are several important properties of the density operator ρ. The first of these is that $\mathrm{Tr}(\rho) = 1$. This follows from w_i has $w_i > 0$ and $\sum\limits_i w_i = 1$.

Next, ρ is self-adjoint: $\rho^\dagger = \rho$. Because it is self-adjoint, ρ has eigenvectors $|J\rangle$ with eigenvalues λ_J and the eigenvectors form a basis for vector space. Thus ρ has a standard spectral representation

$$\rho = \sum_J \lambda_J |J\rangle\langle J|$$

We can express λ_J as $\lambda_J = \langle J|\rho|J\rangle$. Then

$$
\begin{array}{rcll}
\lambda_J & = & \langle J|\rho|J\rangle & \# \\
& = & \langle J|\Big(\sum\limits_i w_i |\psi_i\rangle\langle\psi_i|\Big)|J\rangle & \# \ \rho = \sum\limits_i w_i |w_i\rangle\langle w_i| \\
& = & \sum\limits_i w_i \langle J|\psi_i\rangle\langle\psi_i|J\rangle & \text{\# rearrange} \\
& = & \sum\limits_i w_i \langle J|\psi_i\rangle\langle J|\psi_i\rangle^* & \# \ \langle J|\psi_i\rangle^* = \langle\psi_i|J\rangle \\
& = & \sum\limits_i w_i |\langle J|\psi_i\rangle|^2 & \# \ \langle J|\psi_i\rangle\langle J|\psi_i\rangle^* = |\langle J|\psi_i\rangle|^2
\end{array}
$$

Since $w_i > 0$ and $| \langle J|\psi_i\rangle |^2 > 0$, each eigenvalue must be non-negative, that is, $\lambda_J \geq 0$. In addition, the trace of ρ is the sum of its eigenvalues, so $\sum_J \lambda_J = 1$. Since each eigenvalue is non-negative, $\lambda_J \leq 1$.

Another way to see why $| \langle a_n|\psi\rangle |^2 = p(a_n)$:

$$\begin{aligned} |\psi\rangle &= I\,|\psi\rangle && \#\ \mathbf{I}\cdot\mathbf{X} = \mathbf{X} \\ &= \sum_n |a_n\rangle \langle a_n|\,|\psi\rangle && \#\ \sum_n |a_n\rangle\langle a_n| = I \\ &= \sum_n |a_n\rangle \langle a_n|\psi\rangle && \#\ \langle a_n|\,|\psi\rangle = \langle a_n|\psi\rangle \end{aligned}$$

So $\langle a_n|\psi\rangle$ is the amplitude of $|a_n\rangle$, making $| \langle a_n|\psi\rangle |^2 = p(a_n)$.

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