$\sqrt{?} = \sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\cdots}}}}$

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Consider:

$$
\begin{array}{lll}
\sqrt{y} := & \sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\cdots}}}} & \text{\# define } y \\
\Rightarrow & y^{\frac{1}{2}} = x^{\frac{1}{3}}\, x^{\frac{1}{9}}\, x^{\frac{1}{27}} \cdots & \text{\# rewrite radicals as fractions} \\
\Rightarrow & y^{\frac{1}{2}} = x^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots} & \text{\# rewrite product as a sum } (x^a x^b = x^{a+b}) \\
\Rightarrow & y^{\frac{1}{2}} = x^{\sum\limits_{k=1}^{\infty}(\frac{1}{3})^k} & \text{\# } \sum\limits_{k=1}^{\infty}\left(\frac{1}{3}\right)^k = \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots \\
\Rightarrow & y^{\frac{1}{2}} = x^{\sum\limits_{k=0}^{\infty}(\frac{1}{3})^k - 1} & \text{\# } \sum\limits_{k=1}^{\infty}\left(\frac{1}{3}\right)^k = \sum\limits_{k=0}^{\infty}\left(\frac{1}{3}\right)^k - \left(\frac{1}{3}\right)^0 = \sum\limits_{k=0}^{\infty}\left(\frac{1}{3}\right)^k - 1 \\
\Rightarrow & y^{\frac{1}{2}} = x^{\frac{1}{1-\frac{1}{3}} - 1} & \text{\# geometric series: } \sum\limits_{k=0}^{\infty} ar^k = \dfrac{a}{1-r} \text{ for } |r| < 1\text{; here } a = 1 \text{ and } r = \frac{1}{3} \\
\Rightarrow & y^{\frac{1}{2}} = x^{\frac{3}{2}-1} & \text{\# } \dfrac{1}{1-\frac{1}{3}} = \frac{3}{2} \\
\Rightarrow & y^{\frac{1}{2}} = x^{\frac{1}{2}} & \text{\# } \frac{3}{2} - 1 = \frac{1}{2} \\
\Rightarrow & y = x & \text{\# square both sides}
\end{array}
$$

and so apparently $\sqrt{x} = \sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\cdots}}}}$