Why is i^i a real number?

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Last update: November 20, 2022

1 Introduction

There are a many ways to think about this somewhat perplexing question. But first we look at why a complex number z equals $re^{i\theta}$. Then in Section 3 we use the fact that complex conjugation is an automorphism of $\mathbb{C}$ to show that $i^i \in \mathbb{R}$. In Section 4 we use the power series expansion of i^i to show that $i^i \in \mathbb{R}$, and in Section 5 we rely on Euler's formula to show the same result. In Section 6 we do a bit of arithmetic to show the numerical value of i^i. Finally Section 7 offers a few conclusions.

Before launching into all of this, note that we will make heavy use of the exponential function $\exp(z)$ (and therefore as you might expect, this means heavy use of $\log(z)$. The important point for this discussion is that $\exp(z)$ is injective over $\mathbb{R}$ but not over $\mathbb{C}$. More specifically, for a complex number z the the complex logarithm $\log(z)$ is defined as the inverse function to the exponential function. That is, it satisfies $e^{\log z} \equiv z$. Now since $re^{i\theta} = re^{i(\theta+2k\pi)}$ for all $k \in \mathbb{Z}$ we have that for any choice of k, $\log(re^{i(\theta+2k\pi)})$ is a valid inverse for $re^{i\theta}$. The logarithm is therefore a multivalued function and each value of k defines what is called a different *branch* of the logarithm. The *principal branch* usually refers to the choice $k = 0$ [9]. See Remark 7.1 for a bit more on this point.

One note here: I will use "log" to denote the natural log ($\log_e$) of the principal value of the logarithm of z, noting that some authors "Log" to distinguish the principal value from other logarithms of z [11].

2 First: A Bit of Review

This section reviews the nature of a complex number z, where the length of the line from the origin to the point z, $|z|$, equals r. In particular, why does a complex number $z = re^{i\theta}$? To see why this is the case, first consider the complex plane, shown in Figure 1:

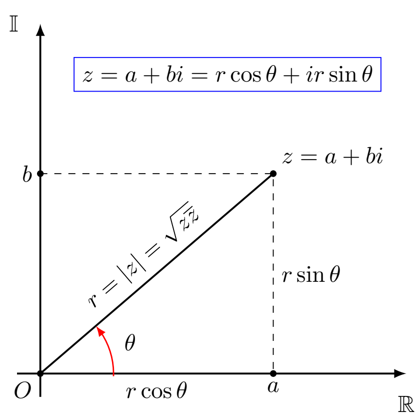


Figure 1: The Complex Plane

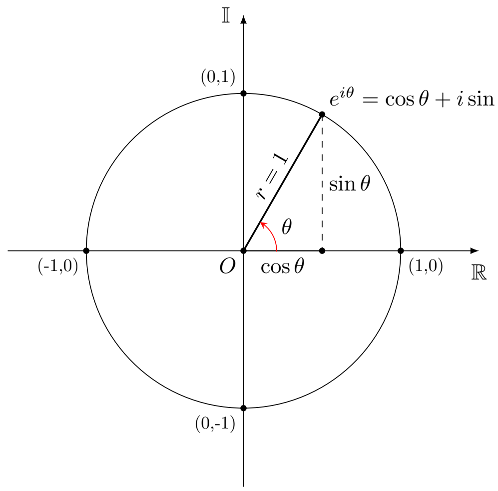


Figure 2: Euler's Formula, the Unit Circle, and the Complex Plane

Figures 1 and 2 give us a pretty way to see why $z = re^{i\theta}$:

$$\begin{aligned} z &= a + ib && \text{\# definition of a point } z \text{ in the complex plane} \\ &= r\cos\theta + ir\sin\theta && \text{\# switch to polar coordinates: } a = r\cos\theta \text{ and } b = r\sin\theta \\ &= r(\cos\theta + i\sin\theta) && \text{\# factor out } r \\ &= re^{i\theta} && \text{\# } e^{i\theta} = \cos\theta + i\sin\theta \text{ (Euler's formula [8])} \end{aligned}$$

So we see that $z = re^{i\theta}$. ■

3 Automorphism of $\mathbb{C}$

In this section we look at solving this puzzle using the fact that complex conjugation is an automorphism of $\mathbb{C}$ [4]. For this approach we need the following facts:

1. If a complex number z equals its complex conjugate $\overline{z}$ then $z \in \mathbb{R}$
2. $-i = i^{-1}$
3. $\overline{z} = re^{-i\theta}$

To see 1., consider the following argument: Let z be a complex number so that $z = a + bi$. Then $\overline{z} = a - bi$. So $z = \overline{z} \Rightarrow a + bi = a - bi$. Subtracting a from both sides of the right hand side of this implication gives us $bi = -bi$ or $2bi = 0$. Since we know that $2 \neq 0$ and $i \neq 0$ it must be the case that $b = 0$. Said another way: $\mathrm{Im}(z) = 0$ and $\mathrm{Re}(z) = z$. That is, $z \in \mathbb{R}$. So if $z = \overline{z}$ we know that $z \in \mathbb{R}$.

To see 2., notice that $-(i \cdot i) = -i^2 = -(-1) = 1$ and $-i \cdot i = 1 \Rightarrow -i = i^{-1}$.

To see 3., consider that for $r, \theta \in \mathbb{R}$ we have

$$\begin{aligned} \overline{z} &= \overline{r \cdot (\cos\theta + i\sin\theta)} && \text{\# since } z = r \cdot (\cos\theta + i\sin\theta) \text{ (Figure 1)} \\ &= \overline{r} \cdot \overline{(\cos\theta + i\sin\theta)} && \text{\# since } \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2} \text{ (product rule for complex conjugation [5])} \\ &= r \cdot \overline{(\cos\theta + i\sin\theta)} && \text{\# since } \overline{r} = \overline{r + 0i} = r - 0i = r \ (\overline{x} = x \text{ for } x \in \mathbb{R}) \\ &= r \cdot (\cos\theta - i\sin\theta) && \text{\# since } \overline{a + bi} = a - bi \\ &= re^{-i\theta} && \text{\# since } e^{-i\theta} = \cos\theta - i\sin\theta \text{ (corollary to Euler's formula [3])} \end{aligned}$$

Now we want to show that $\overline{i^i} = i^i$, which would show that $i^i \in \mathbb{R}$ by the above argument[1]. So consider

$$\begin{aligned}
\overline{i^i} &= \overline{i}^{\overline{i}} && \text{\# see proof below} \\
&= (-i)^{-i} && \text{\# since } i = 0 + 1i \text{ and } \overline{0+1i} = -1i = -i \\
&= (i^{-1})^{-i} && \text{\# since } -i = i^{-1} \text{ (item 2. above)} \\
&= i^{(-1 \cdot -i)} && \text{\# since } (x^m)^n = x^{mn} \\
&= i^i && \text{\# since } -1 \cdot -i = i
\end{aligned}$$

So we see that $\overline{i^i} = i^i$, which implies that $i^i \in \mathbb{R}$. ■

This is all good, but why does $\overline{i^i} = \overline{i}^{\overline{i}}$? To see why first notice that

$$\begin{aligned}
\log(\overline{z}) &= \log(re^{-i\theta}) && \text{\# since } \overline{z} = re^{-i\theta} \text{ (item 3. above)} \\
&= \log\left(\frac{r}{e^{i\theta}}\right) && \text{\# since } a^{-b} = \frac{1}{a^b} \\
&= \log(r) - \log(e^{i\theta}) && \text{\# by the quotient rule for logarithms [7]} \\
&= \log(r) - i\theta && \text{\# since } \log(e^{i\theta}) = i\theta \\
&= \log(r) + \overline{i\theta} && \text{\# since } -i\theta = \overline{0 + i\theta} = \overline{i\theta} \\
&= \log(r) + \overline{\log(e^{i\theta})} && \text{\# since } i\theta = \log(e^{i\theta}) \\
&= \overline{\log(r)} + \overline{\log(e^{i\theta})} && \text{\# since } x = \overline{x} \text{ for } x \in \mathbb{R} \text{ and } \log(r) \in \mathbb{R} \\
&= \overline{\log(r) + \log(e^{i\theta})} && \text{\# by the sum rule for conjugates [6]} \\
&= \overline{\log(re^{i\theta})} && \text{\# by the product rule for logarithms [7]} \\
&= \overline{\log(z)} && \text{\# since } z = re^{i\theta}
\end{aligned}$$

Note that @antoinechambertloir@mathstodon.xyz says that $\log(\overline{z})$ not necessarily equal to $\overline{\log(z)}$ (see Remark 7.1). However, if it were to be correct, then

$$\begin{aligned}
\log(\overline{z^w}) &= \overline{\log(z^w)} && \text{\# this is the result that is in question} \\
&= \overline{w \cdot \log(z)} && \text{\# by the power rule for logarithms} \\
&= \overline{w} \cdot \overline{\log(z)} && \text{\# by the product rule for conjugation} \\
&= \overline{w} \cdot \log(\overline{z}) && \text{\# since } \log(\overline{z}) = \overline{\log(z)} \\
&= \log(\overline{z}^{\overline{w}}) && \text{\# by the power rule for logarithms [7]}
\end{aligned}$$

Since $\log(\overline{z^w}) = \log(\overline{z}^{\overline{w}})$ we know that $e^{\log(\overline{z^w})} = e^{\log(\overline{z}^{\overline{w}})}$ which in turn implies that $\overline{z^w} = \overline{z}^{\overline{w}}$. Setting $z = w = i$ we see that $\overline{i^i} = \overline{i}^{\overline{i}}$. ■

[1]See Remark 7.1 for a bit on what is still open here.

4 Power Series

We know that the Taylor series [10] for $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. We also know from Euler's formula [1] that $i = e^{i\frac{\pi}{2}}$ (since $e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$). Note that the same result can be obtained with $x \in \left\{\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots\right\}$, so in general $x = \frac{\pi}{2} \cdot (1 + 4n)$ where $n \in \mathbb{N} \cup \{0\}$.

So we see that

$$
\begin{aligned}
i^t &= \left(e^{\frac{i\pi}{2}}\right)^t && \text{\# Euler's formula for } i \text{, raised to the power } t \\
&= e^{\frac{it\pi}{2}} && \text{\# since } (x^m)^n = x^{mn} \\
&= \sum_{k=0}^{\infty} \frac{\left(\frac{it\pi}{2}\right)^k}{k!} && \text{\# since } e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ and } x = \frac{it\pi}{2}
\end{aligned}
$$

If we let $t = i$ we get $i^i = e^{i^2\frac{\pi}{2}} = e^{-\frac{\pi}{2}}$. So $i^i = e^{-\frac{\pi}{2}}$ and since $e^{-\frac{\pi}{2}} \in \mathbb{R}$ we know that $i^i \in \mathbb{R}$. Fortunately this agrees with the result we find in Section 5. ■

5 Euler's Formula

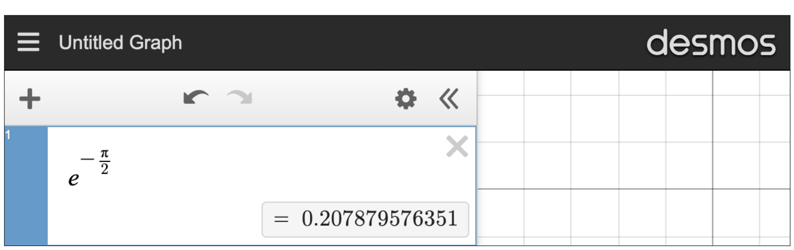
Another approach is to consider Euler's formula evaluated at $x = \frac{\pi}{2}$:

$$
\begin{aligned}
e^{ix} &= \cos x + i\sin x && \text{\# Euler's formula} \\
&\Rightarrow e^{i\frac{\pi}{2}} = \cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2} && \text{\# set } x = \tfrac{\pi}{2} \\
&\Rightarrow e^{i\frac{\pi}{2}} = 0 + i \cdot 1 && \text{\# } \cos\tfrac{\pi}{2} = 0 \text{ and } \sin\tfrac{\pi}{2} = 1 \\
&\Rightarrow e^{i\frac{\pi}{2}} = i && \text{\# simplify} \\
&\Rightarrow \left(e^{i\frac{\pi}{2}}\right)^i = i^i && \text{\# raise both sides to } i \\
&\Rightarrow e^{\frac{i^2\pi}{2}} = i^i && \text{\# } (x^m)^n = x^{mn} \\
&\Rightarrow e^{-\frac{\pi}{2}} = i^i && \text{\# } i^2 = -1 \\
&\Rightarrow e^{-\frac{\pi}{2}} \in \mathbb{R} \Rightarrow i^i \in \mathbb{R} && \text{\# } i^i \text{ is a real number}
\end{aligned}
$$

It is important to notice that this expression is multivalued, and when we evaluate it at $\frac{\pi}{2}$ we are on the principle branch. However, the same result can be obtained with $x \in \left\{\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots\right\}$. In general $x = \frac{\pi}{2} \cdot (1 + 4n)$ for $n \in \mathbb{N} \cup \{0\}$. ■

6 Ok, $i^i \in \mathbb{R}$, but what does it equal

We saw above that $i^i = e^{-\frac{\pi}{2}}$. So $i^i = e^{-\frac{\pi}{2}} \approx 0.20788$.



7 Conclusions

Remark 7.1. Antoine Chambert-Loir (@antoinechambertloir@mathstodon.xyz) tells me that restricting the logarithm to any particular branch causes all algebraic relations break down. The example he gives is that $a^c \times b^c = (ab)^c$ only when the sum of the principal arguments of a and b equals the principal argument of ab [2]. Apparently a similar problem arises for conjugacy when a is strictly negative. I must admit that I don't fully understand this comment.

@antoinechambertloir@mathstodon.xyz also says that "the complex log either is multivalued, or requires the a priori choice of a determination of the argument, so that the formula for $\log(\overline{z})$ is incorrect: with the choice of principal determination, it is the conjugate of $\log(z)$ only when z is not strictly negative. And $\log(-1) = i\pi$ is not its conjugate."

Acknowledgements

Thanks to Dima Pasechnik (@dimpase@mathstodon.xyz), Andrés E. Caicedo (@AndresCaicedo@mathstodon.xyz), Ben Reiniger (@bmreiniger@mathstodon.xyz) and Dave Neary (@dneary@mastodon.ie) for their many helpful comments.

LaTeX Source

https://www.overleaf.com/read/cfjttjzdbfnc

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