A Few Notes on Bell States, Superdense Coding, and Quantum Teleportation

David Meyer

dmm613@gmail.com

Last Update: October 7, 2024

Initial Version: October 8, 2020

1 Introduction

The Bell Circuit, shown in Figure 1, is comprised of two gates, H and CNOT, which are defined as follows:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and results in two maximally entangled qubits[1]. How does this work?

First, recall that

$$H\,|0\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle + |1\rangle\,\big) \text{ and } H\,|1\rangle = \frac{1}{\sqrt{2}}\big(\,|0\rangle - |1\rangle\,\big)$$

The Bell Circuit, shown in Figure 1, applies H to $|b_0\rangle$ and then applies the CNOT gate to $H\,|b_0\rangle$ (control qubit) and $|b_1\rangle$ (target qubit). The inputs and evolution of the Bell Circuit are shown are Table 1.

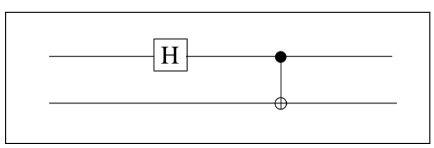

Figure 1: Bell Circuit

$b_0 b_1$	$H\,\vert b_0\rangle$	$\vert b_1\rangle$	Bell Circuit evolution with inputs $\vert b_0\rangle$ and $\vert b_1\rangle$	Bell State
00	$H\,\vert 0\rangle$	$\vert 0\rangle$	$\vert 0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(\vert 0\rangle + \vert 1\rangle) \xrightarrow{\otimes\vert 0\rangle} \frac{1}{\sqrt{2}}(\vert 0\rangle + \vert 1\rangle)\,\vert 0\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(\vert 00\rangle + \vert 11\rangle)$	$\vert \phi^+\rangle$
01	$H\,\vert 0\rangle$	$\vert 1\rangle$	$\vert 0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(\vert 0\rangle + \vert 1\rangle) \xrightarrow{\otimes\vert 1\rangle} \frac{1}{\sqrt{2}}(\vert 0\rangle + \vert 1\rangle)\,\vert 1\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(\vert 01\rangle + \vert 10\rangle)$	$\vert \psi^+\rangle$
10	$H\,\vert 1\rangle$	$\vert 0\rangle$	$\vert 1\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(\vert 0\rangle - \vert 1\rangle) \xrightarrow{\otimes\vert 0\rangle} \frac{1}{\sqrt{2}}(\vert 0\rangle - \vert 1\rangle)\,\vert 0\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(\vert 00\rangle - \vert 11\rangle)$	$\vert \phi^-\rangle$
11	$H\,\vert 1\rangle$	$\vert 1\rangle$	$\vert 1\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(\vert 0\rangle - \vert 1\rangle) \xrightarrow{\otimes\vert 1\rangle} \frac{1}{\sqrt{2}}(\vert 0\rangle - \vert 1\rangle)\,\vert 1\rangle \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(\vert 01\rangle - \vert 10\rangle)$	$\vert \psi^-\rangle$

Table 1: Bell States

[1]This state is sometimes called an *EPR* state.

What we can see from Table 1 is that b_0 selects the "bit" ($|\phi\rangle$ or $|\psi\rangle$), and b_1 selects the "sign" ($|+\rangle$ or $|-\rangle$). Since there are four orthonormal states, the *Bell basis*, we can encode two bits (b_0 and b_1) in the four Bell States.

Now, if Alice wants to send two classical bits to Bob using one qubit (superdense coding), she need only transform her qubit[2] into the Bell State corresponding to the two bits she wants to send, then send her half to Bob (this requires a *quantum* channel). Bob can then recover Alice's two bit message.

But how can Bob recover Alice's message? Recall that unitary quantum operations are reversible. So Bob can use the Reverse Bell Circuit shown in Figure 2 to recover Alice's 2 bit message.

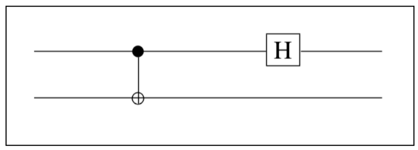

Figure 2: Reverse Bell Circuit

2 Superdense Coding

Suppose Alice wants to send Bob the message 00. Alice can perform one or more unitary operations on her qubit (her half of the entangled pair) that will allow Bob, when presented with Alice's qubit, to reconstruct Alice's message b_0b_1. If we run $|\phi^+\rangle$ through the circuit in Figure 2, that is, $|\phi^+\rangle \xrightarrow{\text{CNOT}} \xrightarrow{\text{H}} |b_0b_1\rangle$, Bob will recover Alice's message ($b_0b_1 = 00$). Why is this?

$$
\begin{aligned}
|\phi^+\rangle = &\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longrightarrow \\
&\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \longrightarrow \\
&\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{\text{H}} \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|0\rangle\Big) \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\Big((|0\rangle + |1\rangle)|0\rangle + (|0\rangle - |1\rangle)|0\rangle\Big) \\
&= \frac{1}{2}(\,|00\rangle + |10\rangle + |00\rangle - |10\rangle\,) \\
&= \frac{1}{2}(2\,|00\rangle + (|10\rangle - |10\rangle)) \\
&= \frac{1}{2}\cdot 2\,|00\rangle \\
&= |00\rangle
\end{aligned}
$$

Bob can now measure both qubits and recover Alice's message ($b_0b_1 = 00$).

In general, Alice notices that

- To send **00**, apply the Identity matrix $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to her half of the EPR pair
- To send **01**, apply the matrix $\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to her half of the EPR pair

[2]Her half of the EPR pair, the two entangled qubits.

- To send **10**, apply the matrix $\mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to her half of the EPR pair
- To send **11**, apply $i\mathbf{Y} = i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, i.e. both **X** and **Z**, to her half of the EPR pair

where **I**, **X**, **Y** and **Z** are the *Pauli* matrices [6].

This transforms the EPR pair $|\phi^+\rangle$ into the four Bell States $|\phi^+\rangle$, $|\psi^+\rangle$, $|\phi^-\rangle$ and $|\psi^-\rangle$ respectively:

- **00**: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle \longrightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |\phi^+\rangle$
- **01:** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle \longrightarrow \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = |\psi^+\rangle$
- **10:** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle \longrightarrow \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = |\phi^-\rangle$
- **11:** $i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle \longrightarrow \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = |\psi^-\rangle$

The four Bell states $|\phi^+\rangle$, $|\psi^+\rangle$, $|\phi^-\rangle$ and $|\psi^-\rangle$ are orthonormal and are hence distinguishable by quantum measurement. Thus after receiving Alice's transformed qubit (her half of the EPR pair), Bob can measure both qubits and recover $b_0 b_1$. Hence one qubit carries two classical bits of information; this is superdense coding. We saw an example of this above in which Bob recovered $|00\rangle$ from $|\phi^+\rangle$ using the Reverse Bell Circuit depicted in Figure 2.

2.1 Aside: Spectral Decomposition of Pauli Matrices

So far we've interpreted the Pauli matrices as a quantum gates. But note that a gate such **Z** is a Hermitian operator and as a result can be interpreted as an observable. Somewhat surprisingly (notice the symmetry), the spectral decomposition [4] of **Z** is

$$\mathbf{Z} = |0\rangle \langle 0| - |1\rangle \langle 1|$$

where $|u\rangle \langle v|$ is Dirac notation [3] for the outer product $\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^{\mathrm{T}}$ of $m \times 1$ vector $\mathbf{u}$ and $n \times 1$ vector $\mathbf{v}$, which yields a $m \times n$ matrix[3].

We can see that the eigenvalues of **Z** are 1 and -1, corresponding to eigenvectors $|0\rangle$ and $|1\rangle$ respectively. So the measurement operators are the projectors $|0\rangle \langle 0|$ and $|1\rangle \langle 1|$, This means that a measurement of the Pauli observable **Z** is a measurement in the computational basis that has eigenvalue +1 corresponding to $|0\rangle$ and eigenvalue -1 corresponding to $|1\rangle$.

[3] The outer product is of vectors **u** and **v** is a special case of the tensor product $\mathbf{u} \otimes \mathbf{v}$. More generally, the outer product is an instance of a Kronecker product [5].

So ok, but why does $\mathbf{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$? Well, we know that the outer product $\mathbf{u} \otimes \mathbf{v}$ of a $m \times 1$ vector $\mathbf{u}$ and a $n \times 1$ vector $\mathbf{v}$ is defined to be the $m \times n$ matrix[4] $\mathbf{u}\mathbf{v}^{\mathrm{T}}$.

To see why $\mathbf{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$, first recall that $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$. Then

$$\begin{aligned}
|0\rangle\langle 0| &= \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \text{ and}\\
|1\rangle\langle 1| &= \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \text{ so that}\\
|0\rangle\langle 0| - |1\rangle\langle 1| &= \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \mathbf{Z}
\end{aligned}$$

2.2 Back to Alice wanting to send a message to Bob

Now suppose Alice want's to send Bob the message 01. Alice then applies Pauli matrix X to $|\phi+\rangle$ to get $|\psi^+\rangle$:

$$\mathbf{X}|\phi^+\rangle = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\psi^+\rangle$$

Bob can now recover Alice's message as follows using the Reverse Bell Circuit (Figure 2). That is, Bob can do the the unitary operations $|\psi^+\rangle \xrightarrow{\text{CNOT}}\xrightarrow{\text{H}} |01\rangle$, as follows:

$$\begin{aligned}
|\psi^+\rangle = &\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \longrightarrow\\
&\frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \longrightarrow\\
&\frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \xrightarrow{\text{H}} \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|1\rangle + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|1\rangle\Big)\\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\Big((|0\rangle + |1\rangle)|1\rangle + (|0\rangle - |1\rangle)|1\rangle\Big)\\
&= \frac{1}{2}\big(\,|01\rangle + |11\rangle + |01\rangle - |11\rangle\,\big)\\
&= \frac{1}{2}\big(2|01\rangle + (|11\rangle - |11\rangle)\big)\\
&= \frac{1}{2}\cdot 2|01\rangle\\
&= \frac{2}{2}|01\rangle\\
&= |01\rangle
\end{aligned}$$

Now Bob can measure the two qubits and recover Alice's message ($b_0b_1 = 01$).

Similarly, suppose Alice wants to send the message 10 to Bob. Alice first transforms her qubit as follows

$$\mathbf{Z}|\phi^+\rangle = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\phi^-\rangle$$

[4]Contrast with the scalar inner product $\langle\mathbf{u},\mathbf{v}\rangle = \mathbf{u}^{\mathrm{T}}\mathbf{v}$. Note also that $\langle\mathbf{u},\mathbf{v}\rangle = \mathrm{tr}(\mathbf{u}\otimes\mathbf{v})$, where $\mathrm{tr}(\mathbf{A})$ is the "trace" of matrix $\mathbf{A}$.

Alice now sends her qubit to Bob over a quantum channel. Bob can now recover Alice's message, again using the Reverse Bell Circuit ($|\phi^-\rangle \xrightarrow{\text{CNOT}}\xrightarrow{\text{H}} |10\rangle$). Again, why is this?

$$
\begin{aligned}
|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \longrightarrow \\
&\frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \longrightarrow \\
&\frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \xrightarrow{\text{H}} \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle - \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|0\rangle\Big) \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\Big((|0\rangle + |1\rangle)|0\rangle - (|0\rangle - |1\rangle)|0\rangle\Big) \\
&= \frac{1}{2}\big(\,|00\rangle + |10\rangle - |00\rangle + |10\rangle\,\big) \\
&= \frac{1}{2}\big(2|10\rangle + (|00\rangle - |00\rangle)\big) \\
&= \frac{1}{2}\cdot 2|10\rangle \\
&= |10\rangle
\end{aligned}
$$

Now Bob can measure the two qubits and recover Alice's message ($b_0b_1 = 10$).

Finally, if Alice wants to send 11 to Bob she first transforms her qubit

$$
i\mathbf{Y}|\phi^+\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\psi^-\rangle
$$

Alice now transmits her qubit to Bob and Bob applies the Reverse Bell Circuit to recover Alice's message:

$$
\begin{aligned}
|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \longrightarrow \\
&\frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \longrightarrow \\
&\frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \xrightarrow{\text{H}} \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|1\rangle - \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|1\rangle\Big) \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\Big((|0\rangle + |1\rangle)|1\rangle - (|0\rangle - |1\rangle)|1\rangle\Big) \\
&= \frac{1}{2}\big(\,|01\rangle + |11\rangle - |01\rangle + |11\rangle\,\big) \\
&= \frac{1}{2}\big(2|11\rangle + (|01\rangle - |01\rangle)\big) \\
&= \frac{1}{2}\cdot 2|11\rangle \\
&= \frac{2}{2}|11\rangle \\
&= |11\rangle
\end{aligned}
$$

Now Bob can measure the two qubits and recover Alice's message ($b_0b_1 = 11$).

3 Quantum Teleportation

Quantum teleportation can be thought of as the dual task to superdense coding. Whereas superdense coding is concerned with conveying classical information via a qubit, quantum teleportation is concerned with conveying quantum information with classical bits [1].

3.1 A high-level view of the quantum teleportation algorithm

1. Alice and Bob share an entangled (EPR) pair $|\phi^+\rangle$
2. Alice chooses a qubit $|\psi\rangle$ as the message she wants to convey to Bob
3. Alice performs operations on $|\psi\rangle$ and $|\phi_A^+\rangle$ (Alice's half of $|\phi^+\rangle$)
4. Alice measures $|\psi\rangle$ and her half of $|\phi_A^+\rangle$, destroying both of her qubits
5. Alice sends the two classical bits that were the results of her measurements to Bob
6. Bob uses the two classical bits to "correct" $|\phi_B^+\rangle$ (his half of $|\phi^+\rangle$) to be $|\psi\rangle$

Alice uses the circuit in Figure 2 to prepare her two qubits (step 3 above). How exactly does this work? First, notice that the input to the Reverse Bell Circuit shown in Figure 2 is $|\psi\rangle \otimes |\phi_A^+\rangle$. To see how this works, first recall that $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Then

$$\begin{aligned}
|\psi\rangle \otimes |\phi_A^+\rangle \quad &= \quad (\alpha|0\rangle + \beta|1\rangle) \otimes \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) && \text{\# defintion of } |\psi\rangle \text{ and } |\phi_A^+\rangle \\
&= \quad \tfrac{1}{\sqrt{2}}\Big(\alpha(|000\rangle + \alpha|011\rangle) + \beta(|100\rangle + |111\rangle)\Big) && \text{\# } |b_0b_1b_2\rangle : b_0 \text{ is the control, } b_1 \text{ is the target} \\
&\xrightarrow{\text{CNOT}} \quad \tfrac{1}{\sqrt{2}}\Big(\alpha|000\rangle + \alpha|011\rangle) + \beta(|110\rangle + |101\rangle)\Big) && \text{\# apply CNOT controlled by } b_0 \text{ (target is } b_1) \\
&\xrightarrow{\text{H}} \quad \tfrac{1}{\sqrt{2}}\Big[\alpha\Big(\tfrac{1}{\sqrt{2}}|0\rangle + |1\rangle\Big)|00\rangle + \alpha\Big(\tfrac{1}{\sqrt{2}}|0\rangle + |1\rangle)\Big)|11\rangle + && \text{\# apply } H = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \\
&\qquad \beta\Big(\tfrac{1}{\sqrt{2}}|0\rangle - |1\rangle\Big)|10\rangle + \beta\Big(\tfrac{1}{\sqrt{2}}|0\rangle - |1\rangle\Big)|01\rangle\Big] \\
&= \quad \tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\Big[\alpha\Big((|0\rangle + |1\rangle)|00\rangle + (|0\rangle + |1\rangle)|11\rangle\Big) + \beta\Big((|0\rangle - |1\rangle)|10\rangle + (|0\rangle - |1\rangle)|01\rangle\Big)\Big] \\
&= \quad \tfrac{1}{2}\Big[\alpha\Big(|000\rangle + |100\rangle + |011\rangle + |111\rangle)\Big) + \beta\Big(|010\rangle - |110\rangle + |001\rangle - |101\rangle\Big)\Big] \\
&= \quad \tfrac{1}{2}\Big[\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\Big]
\end{aligned}$$

Now Alice measures her two qubits ($|\psi\rangle \otimes |\phi_A^+\rangle$) and observes $b_0b_1 \in \{00, 01, 10, 11\}$ with $P(b_0b_1) = \frac{1}{4}$.

Now here's the amazing thing. If Alice observes 00, she communicates this to Bob (over a classical channel). As soon as Bob sees the value 00, he knows that his qubit $|\phi_B^+\rangle = \alpha|0\rangle + \beta|1\rangle$. How does Bob know this?

First, as shown above

$$|\psi\rangle \otimes |\phi^+\rangle = \frac{1}{2}\Big[\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\Big] \qquad (1)$$

Alice's measurement of the first two qubits collapses Bob's qubit to the third qubit[5]. The only terms in Equation 1 that are consistent with the first two qubits being $|00\rangle$ (resulting from Alice's measurement) are $\alpha|000\rangle$ and $\beta|001\rangle$. The "collapsed version" is $\alpha|0\rangle$ and $\beta|1\rangle$. Hence Bob knows that his qubit, $|\phi_B^+\rangle$, equals $\alpha|0\rangle + \beta|1\rangle$.

[5] Recall that the original three qubits were $|\psi\rangle \otimes |\phi_{AB}^+\rangle$.

Since Alice sent the two bits she saw to Bob, he knows which operations to perform to transform $|\phi_B^+\rangle \to |\psi\rangle$. In particular, if $b_0 = 1$ Bob should apply Pauli matrix Z to his qubit and I otherwise, and if $b_1 = 1$ he should apply X and I otherwise. This transforms $|\phi_B^+\rangle$, Bob's qubit, into $|\psi\rangle$. This is shown in Table 2.

Amazingly this procedure teleports Alice's qubit $|\psi\rangle$ to Bob using the two classical bits that Alice learned by measuring her two qubits ($|\psi\rangle$ and $|\phi_A^+\rangle$).

b_0b_1	$\vert\phi_B^+\rangle$	Transformation	Computation
00	$\alpha\vert 0\rangle + \beta\vert 1\rangle$	$\mathbf{I}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\alpha\vert 0\rangle+\beta\vert 1\rangle=\vert\psi\rangle$
01	$\beta\vert 0\rangle + \alpha\vert 1\rangle$	$\mathbf{X}\begin{bmatrix}\beta\\ \alpha\end{bmatrix}$	$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\beta\\ \alpha\end{bmatrix}=\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\alpha\vert 0\rangle+\beta\vert 1\rangle=\vert\psi\rangle$
10	$\alpha\vert 0\rangle - \beta\vert 1\rangle$	$\mathbf{Z}\begin{bmatrix}\alpha\\ -\beta\end{bmatrix}$	$\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\alpha\\ -\beta\end{bmatrix}=\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\alpha\vert 0\rangle+\beta\vert 1\rangle=\vert\psi\rangle$
11	$\beta\vert 0\rangle - \alpha\vert 1\rangle$	$\mathbf{XZ}\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}$	$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}=\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\alpha\vert 0\rangle+\beta\vert 1\rangle=\vert\psi\rangle$

Table 2: Bob's transformations on receiving classical bits $\mathbf{b_0b_1}$ from Alice

3.2 Curious Entry for 11 in Table 2?

Note that the row for the result of Alice's measurement **11** in Table 2 is curious. When Bob sees **11** from Alice he knows that his remaining qubit $|\psi_B^+\rangle$, equals $-\beta|0\rangle + \alpha|1\rangle$. Why does the table say $\beta|0\rangle - \alpha|1\rangle$?

Here is one way to look at this: First, recall that when Bob receives classical bits **11** from Alice he knows that his qubit, $|\psi_B^+\rangle$, is

$$|\psi_B^+\rangle = -\beta|0\rangle + \alpha|1\rangle = \begin{bmatrix}-\beta\\ \alpha\end{bmatrix}$$

Now, if Bob now wants to transform $|\psi_B^+\rangle \to |\psi\rangle$, he would apply **ZX** as follows

$$\begin{aligned}\mathbf{ZX}|\psi_B^+\rangle &= \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}-\beta\\ \alpha\end{bmatrix}\\ &= \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\alpha\\ -\beta\end{bmatrix}\\ &= \begin{bmatrix}\alpha\\ \beta\end{bmatrix}\\ &= \alpha|0\rangle + \beta|1\rangle\\ &= |\psi\rangle\end{aligned}$$

But our rule (Table 2) tells Bob to apply **XZ** when he sees **11** from Alice. Why? Notice the following:

$$\begin{aligned}\mathbf{ZX}\begin{bmatrix}x_0\\ x_1\end{bmatrix} &= \mathbf{Z}\begin{bmatrix}x_1\\ x_0\end{bmatrix} = \begin{bmatrix}x_1\\ -x_0\end{bmatrix}\\ \mathbf{XZ}\begin{bmatrix}x_0\\ x_1\end{bmatrix} &= \mathbf{X}\begin{bmatrix}x_0\\ -x_1\end{bmatrix} = \begin{bmatrix}-x_1\\ x_0\end{bmatrix}\end{aligned}$$

which implies that

$$\mathbf{ZX}\begin{bmatrix}x_0\\x_1\end{bmatrix} = -\mathbf{XZ}\begin{bmatrix}x_0\\x_1\end{bmatrix} \tag{2}$$

So now let $x_0 = \beta$ and $x_1 = \alpha$. Then

$$\mathbf{XZ}\Big[\beta\,|0\rangle - \alpha\,|1\rangle\Big] = \mathbf{XZ}\begin{bmatrix}\beta\\-\alpha\end{bmatrix} = \mathbf{X}\begin{bmatrix}\beta\\\alpha\end{bmatrix} = \begin{bmatrix}\alpha\\\beta\end{bmatrix} = \alpha\,|0\rangle + \beta\,|1\rangle = |\psi\rangle$$

and $-\big(\beta\,|0\rangle - \alpha\,|1\rangle\big) = -\beta\,|0\rangle + \alpha\,|1\rangle \longrightarrow$

$$\mathbf{ZX}\Big[-\beta\,|0\rangle + \alpha\,|1\rangle\Big] = \mathbf{ZX}\begin{bmatrix}-\beta\\\alpha\end{bmatrix} = \mathbf{Z}\begin{bmatrix}\alpha\\-\beta\end{bmatrix} = \begin{bmatrix}\alpha\\\beta\end{bmatrix} = \alpha\,|0\rangle + \beta\,|1\rangle = |\psi\rangle$$

The choice of the transformation rules shown in Table 2 and Equation 2 allows us to write $\beta\,|0\rangle - \alpha\,|1\rangle$ rather than $-\beta\,|0\rangle + \alpha\,|1\rangle$.

Why do this? One thing it does is make the symmetry in Table 2 more explicit, but hopefully there is a better reason...

3.2.1 Cloning and/or Faster Than Light Communication?

First, no faster-than-light communication occurs since Bob learns nothing from the changes until Alice actually sends the two classical bits to him (even though Alice operating on $|\phi_A^+\rangle$ instantly affects $|\phi_B^+\rangle$).

The No-Cloning Theorem [2] is not violated since, even though Bob has an exact copy of $|\psi\rangle$, Alice had to destroy her copy (by measuring it).

Finally, an interesting point is that neither Alice or Bob ever "know" what $|\psi\rangle$ is (in terms of its actual amplitudes); all they know is that it was transferred (whatever it was).

4 Bell and CHSH

TBD

Acknowledgements

Thanks to Joel Bion for finding a typo in Section 3.1. Doug Merritt (@dougmerritt@mathstodon.xyz) found many typos and helped clarify many sections of this document.

LaTeX Source

`https://www.overleaf.com/read/hmgrypdcxjhh`

References

[1] Charles H. Bennett, Gilles Brassard, Claude Crepeau, Richard Jozsa, Asher Peres, and William K. Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Phys. Rev. Lett.*, 70:1895–1899, 1993.

[2] S. Epstein. Algorithmic No-Cloning Theorem. *ArXiv e-prints*, August 2018.

[3] F. Gieres. Mathematical surprises and Dirac's formalism in quantum mechanics. *Reports on Progress in Physics*, 63:1893–1931, December 2000.

[4] W.-H. Steeb and Y. Hardy. Spin Operators, Pauli Group, Commutators, Anti-Commutators, Kronecker Product and Applications. *arXiv e-prints*, May 2014.

[5] Wikipedia Contributors. Kronecker product — Wikipedia, the free encyclopedia, 2018. [Online; accessed 15-November-2018].

[6] Wikipedia Contributors. Pauli matrices — Wikipedia, the free encyclopedia. `https://en.wikipedia.org/w/index.php?title=Pauli_matrices&oldid=868880007`, 2018. [Online; accessed 26-November-2018].