A Few Notes on the Brachistochrone Problem

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1 Introduction

Johann Bernoulli posed the "problem of the brachistochrone" to the readers of *Acta Eruditorum* in June, 1696 [3], which asks the following question:

> Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time?

The scenario Bernoulli envisioned is depicted in Figure 1.

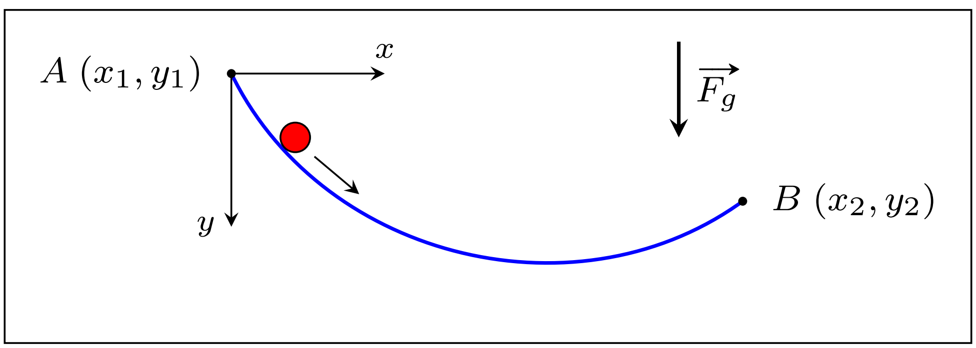

Figure 1: The Brachistochrone Problem Setup

2 A Bit of History

The problem posed by Johann Bernoulli was the called the Brachistochrone[1] problem, or the path of fastest descent. Bernoulli issued his challenge in *Acta Eruditorum* [3], writing

> I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.

[1] The word brachistochrone comes from the Ancient Greek: *brachis* (shortest) + *chronos* (time) [15].

Bernoulli wrote the problem statement as:

> Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.

Figure 1 attempts to capture Bernoulli's problem statement.

Galileo had attempted to solve this problem in his *Two New Sciences* [14] and had concluded, based on geometric arguments, that the solution was a circular path. Galileo's approach is shown in Figure 2. Apparently Galileo hedged a bit on his solution. In fact, it is said that he had reservations about his conclusion and suggested that a "higher mathematics" would possibly find a better solution [4].

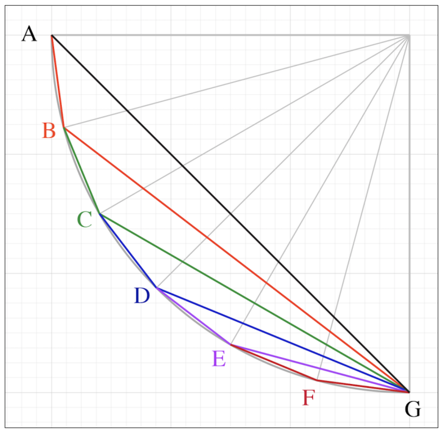

Figure 2: Galileo considered a mass falling along different chords of a circle starting at A. He proved that the path along ABG was quicker than along AG, and ABCG was quicker than ABG, and ABCDG was quicker than ABCG, etc. In this way he showed that the path along the circular arc was quicker than any set of chords. From this he (incorrectly) inferred that the circle was the path of quickest descent. Galileo held out reservations about his solution, and rightfully so.

Then in 1659, while studying the physics of pendulums and time keeping, Christiaan Huygens [10] recognized that a perfect harmonic oscillator, one whose restoring force was linear in the displacement of the oscillator, would produce the perfect time piece. Unfortunately, while the pendulum (as proposed by Galileo) was the simplest oscillator to construct, Huygens knew that it was not a perfect harmonic oscillator. In particular, he knew that the period of oscillation became smaller when the amplitude of the oscillation became larger. In order to "fix" the pendulum, he searched for a curve of equal time, called the *tautochrone* [13], that would allow all amplitudes of the pendulum to have the same period. He found the solution and recognized it to be a cycloid [11]. The cycloid is shown in Figure 3.

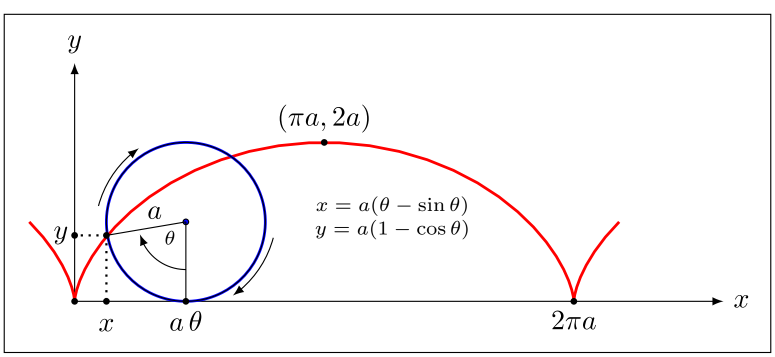

Figure 3: The Cycloid Curve

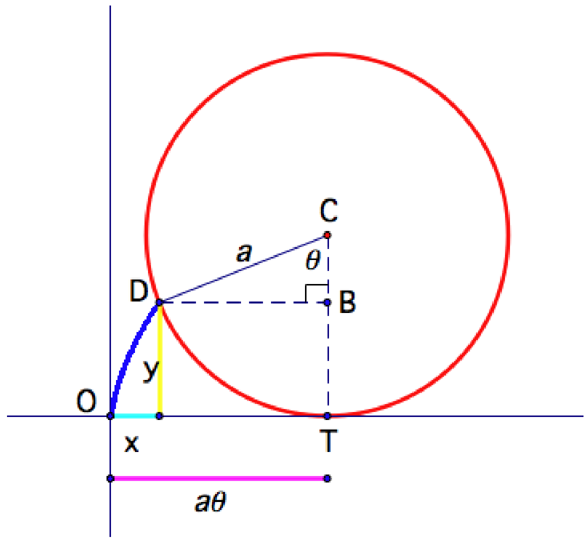

Figure 4: Finding the Coordinates of the Point D on a Cycloid

BTW, how are the cycloid's (x, y) coordinates computed? Consider the scenario depicted in Figure 4 [5]. To find the (x, y) coordinates of the point D we look at the distances involved. We can see from the figure that

- The x-coordinate is the length of OT minus the length of DB
- The y-coordinate is the length of CT minus the length of CB (note that CT $= a$ is the radius of C)
- The length of an arc that is subtended by a central angle is equal to the radius times the central angle

Now, since the distance that circle C has rolled must be equal to the length of arc DT and the arc DT is equal to $a\theta$ we know that $a\theta =$ arc DT $=$ OT. We also know from trigonometry that DB $= a \sin\theta$ and CB $= a\cos\theta$.

If we work out these lengths we see that

$$\begin{aligned} x &= \text{OT - DB} && \text{\# See Figure 4} \\ &= a\theta - a\sin\theta && \text{\# OT = radius times the central angle} = a\theta \text{ and DB} = a\sin\theta \\ &= a(\theta - \sin\theta) && \text{\#}\ x = a(\theta - \sin\theta) \end{aligned}$$

and

$$\begin{aligned} y &= \text{CT - CB} && \text{\# Figure 4} \\ &= a - a\cos\theta && \text{\# CT} = a \text{ and CB} = a\cos\theta \\ &= a(1 - \cos\theta) && \text{\#}\ y = a(1 - \cos\theta) \end{aligned}$$

So as we saw in Figure 3, $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$.

Ok, back to the history. So almost thirty years later, during the infancy of the infinitesimal calculus, the Tautochrone problem was held up as a master example of an "optimal" solution whose derivation should yield to the much more powerful and elegant methods of the calculus. Jakob Bernoulli, Johann's brother, succeeded in deriving the Tautochrone problem in 1690 using the calculus, using the term "integral" for the first time in print, but it was not at first clear what other problems could yield in a similar way.

Then, in 1696, Johann Bernoulli posed the Brachistrochrone problem in the pages of *Acta Eruditorum*.

2.1 A (Very) Brief Timeline of Brachistrochrone Related Events

- 1638 – Galileo proposes arc of a circle as the least time solution (Figure 2)
- 1658 – Pascal puts forth a challenge involving finding the area under a segment of a cycloid
- 1659 – Huygens shows experimentally that the cycloid is the solution to the Tautochrone problem
- 1662 – Fermat's proposes the Principle of Least Time [12]
- 1696 – Johann Bernoulli poses the Brachistrochrone problem in *Acta Eruditorum* (edited by Leibniz)
- 1697 – Solutions provided by the Bernoulli brothers, L'Hôpital, Leibniz, and Newton (anonymously)
- 1707 – Euler is born
- 1726 – Euler completes his Ph.D under Johann Bernoulli
- 1727 – Newton dies
- 1755 – Lagrange, at age 19, finds an analytic solution to the Tautochrone problem
- 1755 – Euler drops his own approach in favor of Lagrange's purely analytic approach
- 1756 – Euler renames the subject the "Calculus of Variations" in *Elementa Calculi Variationum* [6]

3 The Calculus of Variations and the Brachistrochrone Problem

Consider a definite integral that depends on an unknown function $y(x)$ and its derivative $y'(x) = \frac{dy}{dx}$

$$I[y(x)] = \int_a^b F(y(x), y'(x))\, dx \tag{1}$$

A typical problem in the Calculus of Variations involves finding a particular function $y(x)$ that maximizes or minimizes the integral $I[y(x)]$ (Equation 1), subject to the boundary conditions $y(a) = A$ and $y(b) = B$. The integral $I[y(x)]$ is an example of a *functional*, which (more generally) is a mapping from a set of allowable functions to the reals. We say that $I[y(x)]$ has an *extremum* when $I[y(x)]$ takes its maximum or minimum value. Note that we use square brackets to indicate that I is a functional (rather than a function) and we frequently see $y(t)$ abbreviated as y and $I[y(x)]$ as $I[y]$ (or just I).

3.1 The Euler-Lagrange Equation

Let $C^k[a, b]$ denote the set of continuous functions defined on the interval $a \leq x \leq b$ whose first k-derivatives are also continuous on $a \leq x \leq b$. Now, suppose that $I[Y(x)]$ is an extremum of the functional

$$I[y(x)] = \int_a^b L(y(x), y'(x))\, dx$$

that is defined defined on all functions $y(x) \in C^2[a, b]$ such that $y(a) = A$ and $y(b) = B$. Then its solution $Y(x)$ satisfies the second order ordinary differential equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0 \tag{2}$$

Equation 2 is known as the Euler-Lagrange equation (see [1] for a proof of the Equation 2). It is the constraint imposed by the Euler-Lagrange equation that allows us to find the extremum of the functional.

Recall the Brachistochrone problem setup shown in Figure 1. Here we are trying to find the path that takes the *shortest time* from point A to B. In the language of the calculus of variations [8] the Brachistochrone problem is to minimize the functional

$$I[y] = \int_0^t dt$$

Consider a curve over in which we call an differential segment of the path ds. Then we know that the velocity of the ball (or bead, in Bernoulli's original problem statement) is

$$V = \frac{ds}{dt} \tag{3}$$

Now, treating $\frac{ds}{dt}$ as a fraction[2] we can see from Equation 3 that

$$dt = \frac{ds}{V}$$

and so

$$I[y] = \int_0^t dt = \int_A^B \frac{ds}{V} \tag{4}$$

Can we say more about ds? Well, if some function $f(x)$ is differentiable in the neighborhood of ds and if ds is of differential length ($ds \to 0$) then we can build a right triangle on $f(x)$ with ds as the hypotenuse. Then from Pythagoras we know that $ds^2 = dx^2 + dy^2$ and so $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$. This scenario is depicted in Figure 5.

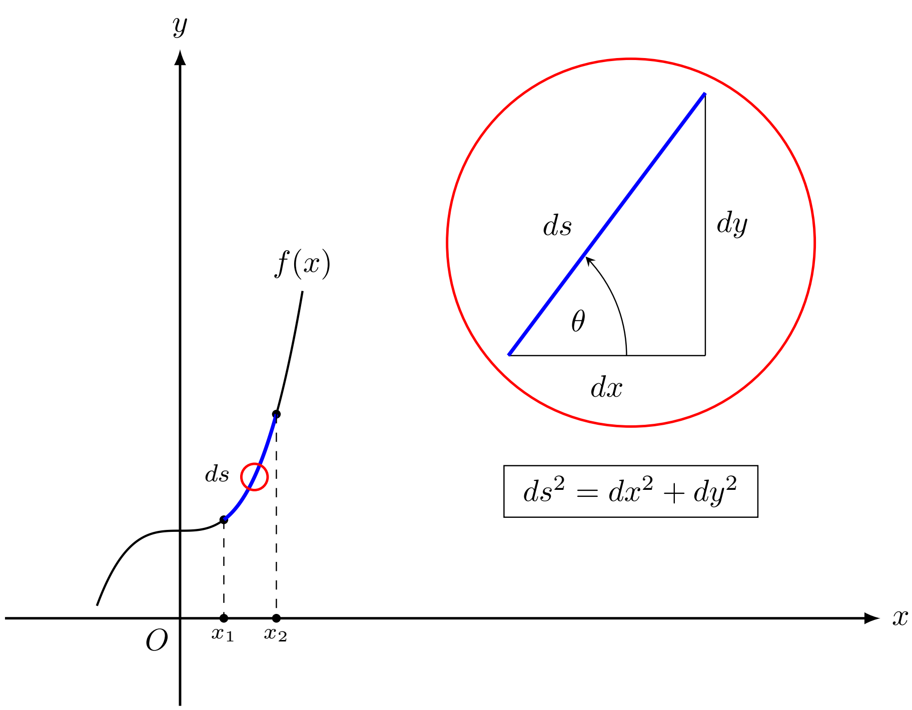

Figure 5: $f(x)$, ds and the Pythagorean Theorem

[2]Care needs to be taken when treating a derivative as a fraction.

Armed with this information we can solve for ds in terms of y':

$$
\begin{aligned}
ds^2 &= dx^2 + dy^2 && \text{\# Pythagorean Theorem (Figure 5)} \\
&= \left(1 + \frac{dy^2}{dx^2}\right) dx^2 && \text{\# factor out } dx^2 \\
&= \left(1 + \left(\frac{dy}{dx}\right)^2\right) dx^2 && \text{\# group } \frac{dy}{dx} \\
&= \left(1 + (y')^2\right) dx^2 && \text{\# } y' = \frac{dy}{dx}\text{; note that here we're treating } \frac{dy}{dx} \text{ as a fraction} \\
&\Rightarrow \; ds = \sqrt{1 + (y')^2}\, dx && \text{\# take the square root of both sides}
\end{aligned}
$$

So now we know that

$$ds = \sqrt{1 + (y')^2}\, dx \tag{5}$$

Next, we know from the Law of Conservation of Energy [2] that

$$
\begin{aligned}
\frac{1}{2}mV^2 - mgy &= 0 && \text{\# } \frac{1}{2}mV^2 \text{ is the kinetic energy and } mgy \text{ is the potential energy} \\
&\Rightarrow \; \frac{1}{2}mV^2 = mgy && \text{\# add } mgy \text{ to both sides} \\
&\Rightarrow \; \frac{1}{2}V^2 = gy && \text{\# cancel } m \\
&\Rightarrow \; V^2 = 2gy && \text{\# multiply both sides by 2} \\
&\Rightarrow \; V = \sqrt{2gy} && \text{\# take the square root of both sides}
\end{aligned}
$$

So now we know that

$$V = \sqrt{2gy} \tag{6}$$

and we can write $I[y]$ as follows

$$
\begin{aligned}
I[y] &= \int_A^B \frac{ds}{V} && \text{\# definition of } I[y] \text{ (Equation 4)} \\
&= \int_A^B \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}} dx && \text{\# Equations 5 and 6} \\
&= \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{1 + (y')^2}{y}}\, dx && \text{\# pull } \frac{1}{\sqrt{2g}} \text{ out of integral, consolidate square root}
\end{aligned}
$$

So now we have this expression for $I[y]$:

$$I[y] = \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{1+(y')^2}{y}}\, dx$$

The Calculus of Variations is about finding a function (rather than a point) that minimizes or maximizes some other function. The other function, which takes a function as an argument, is called a functional. Here $I[y]$ is the functional and we want to find a function $L(y, y')$ that minimizes $I[y]$. $L(y, y')$ looks like

$$L(y, y') = \sqrt{\frac{1+(y')^2}{y}} \tag{7}$$

The last piece of machinery we need is the Beltrami Identity [7], which is a special case of the Euler–Lagrange equation where $\dfrac{\partial L}{\partial y} = 0$. In this case the Euler-Lagrange equation reduces to

$$L - y' \frac{\partial L}{\partial y'} = C \tag{8}$$

where C is some constant (see [7] for a derivation). Plugging Equation 7 into Equation 8 we get

$$\sqrt{\frac{1+(y')^2}{y}} - y' \frac{\partial}{\partial y'} \left[\sqrt{\frac{1+(y')^2}{y}} \right] = C \tag{9}$$

Taking the partial derivative in Equation 9 we get

$$\sqrt{\frac{1+(y')^2}{y}} - y' \cdot \left[\frac{1}{\sqrt{y}} \cdot \left(\frac{1}{2} \left(1 + (y')^2 \right)^{-\frac{1}{2}} \cdot 2y' \right) \right] = C \quad \text{\# power rule: } \frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx},\ u = 1 + (y')^2,\ n = \frac{1}{2}$$

$$\Rightarrow \sqrt{1+(y')^2} - \frac{(y')^2}{\sqrt{1+(y')^2}} = \sqrt{y} \cdot C \quad \text{\# multiply both sides by } \sqrt{y}\text{, cancel 2 and } \frac{1}{2}$$

$$\Rightarrow 1 + (y')^2 - (y')^2 = \sqrt{y(1+(y')^2)} \cdot C \quad \text{\# multiply both sides by } \sqrt{1+(y')^2}$$

$$\Rightarrow 1 = \sqrt{y(1+(y')^2)} \cdot C \quad \text{\# } (y')^2 - (y')^2 = 0$$

$$\Rightarrow 1 = y(1+(y')^2) \cdot C^2 \quad \text{\# square both sides}$$

$$\Rightarrow \frac{1}{C^2} = y + y(y')^2 \quad \text{\# divide both sides by } C^2\text{, multiply though by } y$$

$$\text{Now, let } \frac{1}{C^2} = C_1 = y + y(y')^2 \quad \text{\# define constant } C_1$$

$$\Rightarrow (y')^2 = \frac{C_1 - y}{y} \quad \text{\# solve for } (y')^2$$

$$\Rightarrow y' = \sqrt{\frac{C_1 - y}{y}} \quad \text{\# take the square root of both sides}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{C_1 - y}{y}} \quad \text{\# } y' = \frac{dy}{dx}$$

So know we know that

$$
\begin{aligned}
\frac{dy}{dx} &= \sqrt{\frac{C_1 - y}{y}} && \text{\# previous derivation} \\
&\Rightarrow\ dx = \sqrt{\frac{y}{C_1 - y}}\, dy && \text{\# solve for } dx \\
&\Rightarrow\ x + C_2 = \int \sqrt{\frac{y}{C_1 - y}}\, dy && \text{\# integrate both sides}
\end{aligned}
$$

Now let $y = C_1 \sin^2\theta$ so that $dy = 2\, C_1 \sin\theta\, \cos\theta\, d\theta$. Then

$$
\begin{aligned}
x + C_2 &= \int \sqrt{\frac{C_1 \sin^2\theta}{C_1 - C_1 \sin^2\theta}} \cdot 2\, C_1 \sin\theta\, \cos\theta\, d\theta && \text{\# substitute for } y \text{ and } dy \\
&= \int \sqrt{\frac{C_1 \sin^2\theta}{C_1 (1 - \sin^2\theta)}} \cdot 2\, C_1 \sin\theta\, \cos\theta\, d\theta && \text{\# factor out } C_1 \\
&= \int \sqrt{\frac{C_1 \sin^2\theta}{C_1 \cos^2\theta}} \cdot 2\, C_1 \sin\theta\, \cos\theta\, d\theta && \text{\# } \cos^2\theta = 1 - \sin^2\theta \\
&= \int \frac{\sin\theta}{\cos\theta} \cdot 2\, C_1 \sin\theta\, \cos\theta\, d\theta && \text{\# take square root, cancel } C_1 \\
&= 2\, C_1 \int \sin^2\theta\, d\theta && \text{\# cancel } \cos\theta\text{, factor out } 2\, C_1 \\
&= 2\, C_1 \int \frac{1}{2}(1 - \cos 2\theta)\, d\theta && \text{\# } \cos 2\theta = 1 - 2\sin^2\theta \text{ (double-angle trig identity)} \\
&= C_1 \int (1 - \cos 2\theta)\, d\theta && \text{\# cancel 2 and } \frac{1}{2} \\
&= C_1 \int d\theta - C_1 \int \cos 2\theta\, d\theta && \text{\# integration is a linear operator} \\
&= C_1\theta - C_1 \left(\frac{1}{2} \sin 2\theta\right) && \text{\# integrate} \\
&= C_1\theta - \frac{C_1}{2} \sin 2\theta && \text{\# simplify} \\
&\Rightarrow\ x + C_2 = C_1\theta - \frac{C_1}{2} \sin 2\theta && \text{\# } x = C_1\theta - \frac{C_1}{2} \sin 2\theta - C_2
\end{aligned}
$$

Given these results we can state the solution for the Brachistochrone problem:

$$
x = C_1\theta - \frac{C_1}{2} \sin 2\theta - C_2 \tag{10}
$$

and since $y = C_1 \sin^2\theta$ (definition) and $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ (double-angle trig identity)

$$y = \frac{C_1}{2}\left(1-\cos 2\theta\right) \tag{11}$$

Note that the constants C_1 and C_2 can be found from the boundary conditions.

Finally, recall the cycloid shown in Figure 3 (reproduced here):

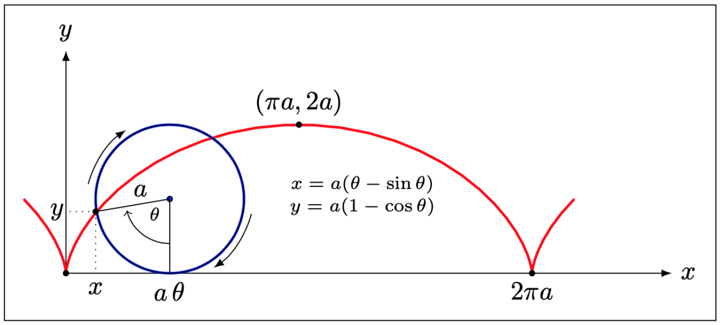

Figure 6: The Cycloid Curve

If we set $C_1 = 2a$, $C_2 = 0$, and $\theta = \frac{\phi}{2}$ in Equations 10 and 11 we see that

$$\begin{aligned}
x &= C_1\theta - \frac{C_1}{2}\sin 2\theta - C_2 && \text{\# Equation 10}\\
&= C_1\theta - \frac{C_1}{2}\sin 2\theta && \text{\# } C_2 = 0\\
&= 2a\theta - \frac{2a}{2}\sin 2\theta && \text{\# } C_1 = 2a\\
&= 2a\frac{\phi}{2} - \frac{2a}{2}\sin\left[2\left(\frac{\phi}{2}\right)\right] && \text{\# } \theta = \frac{\phi}{2}\\
&= a\phi - a\sin\phi && \text{\# cancel } \tfrac{1}{2} \text{ and } 2\\
&= a(\phi - \sin\phi) && \text{\# } x = a(\phi - \sin\phi)
\end{aligned}$$

and

$$\begin{aligned}
y &= \frac{C_1}{2}\left(1-\cos 2\theta\right) && \text{\# Equation 11}\\
&= \frac{2a}{2}\left(1-\cos 2\theta\right) && \text{\# } C_1 = 2a\\
&= \frac{2a}{2}\left(1-\cos\left[2\left(\frac{\phi}{2}\right)\right]\right) && \text{\# } \theta = \frac{\phi}{2}\\
&= a\left(1-\cos\phi\right) && \text{\# } y = a\left(1-\cos\phi\right)
\end{aligned}$$

So we see that Equations 10 and 11 turn out to be the equations for a cycloid (as we saw in Figures 3 and 6) and hence the ”shortest time curve”, that is, the solution to the Brachistochrone problem, is a cycloid.

4 Other Path Minimization Problems

There are many path minimization problems we can solve with the Calculus of Variations. Here are a few examples (WIP).

4.1 Minimize the Distance (Pythagoras)

In Section 3.1 we saw that we can solve for a differential distance ds on some curve using the Pythagorean Theorem; this is shown in Figure 5. Note that this approach requires that we treat $\frac{dy}{dx}$ as a fraction and as mentioned above, care needs to be taken when doing this. That said, as we saw above

$$\begin{aligned}
ds^2 &= dx^2 + dy^2 && \text{\# Pythagorean Theorem (Figure 5)} \\
&= \left(1 + \frac{dy^2}{dx^2}\right) dx^2 && \text{\# factor out } dx^2 \\
&= \left(1 + \left(\frac{dy}{dx}\right)^2\right) dx^2 && \text{\# group } \frac{dy}{dx} \\
&= \left(1 + (y')^2\right) dx^2 && \text{\# } y' = \frac{dy}{dx}\text{; here we're treating } \frac{dy}{dx} \text{ as a fraction} \\
\Rightarrow \quad & ds = \sqrt{1 + (y')^2}\, dx && \text{\# take the square root of both sides}
\end{aligned}$$

So to minimize the distance we want to solve the following integral:

$$I[y] = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$$

4.2 Minimize the Time (Brachistochrone Problem)

We saw above that $ds = \sqrt{1 + (y')^2}\, dx$ (Equation 5) and $V = \sqrt{2gy}$ (Equation 6) so the integral we want to solve to find the curve that minimizes the time taken between two points A and B works out to be

$$I[y] = \int_0^t dt = \int_A^B \frac{ds}{V} = \int_A^B \sqrt{\frac{1 + (y')^2}{2gy}}\, dx = \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{1 + (y')^2}{y}}\, dx$$

4.3 Minimize the Energy (Catenary Problem)

The Catenary Problem [9] (also known as the hanging chain problem) considers the shape of a chain hanging from two stationary points where the only force acting on the chain is gravity. In the problem setup ρ is the density of the chain, g is the acceleration of gravity, and A is the constant cross sectional area of the chain. Note that since the units of ρ are $\frac{\text{mass}}{\text{volume}}$ and $\lim\limits_{ds \to 0} \text{Volume} = \text{Area}$, $\rho A = m$ for a differential piece of the chain of length ds.

In the Catenary Problem the chain is assumed to be stationary so the kinetic energy is zero. So the total energy is the just potential energy mgy. So here $I[y]$ is

$$I[y] = \int_{x_1}^{x_2} mgy\, ds = \int_{x_1}^{x_2} (\rho A) gy\, ds = \rho A g \int_{x_1}^{x_2} y\, ds = \rho A g \int_{x_1}^{x_2} y \sqrt{1 + (y')^2}\, dx$$

4.4 Minimize the Lagrangian (Lagrange's Equations)

This one is discussed in detail in [1] but briefly here $I[y]$ is the integral of the Lagrangian:

$$I[y] = \int_{t_1}^{t_2} L(y(t), y(t)')\ dt$$

Recall that for the case of Lagrangian mechanics the Lagrangian L is defined to be

$$L \equiv T - V$$

where T is an object's kinetic energy and V is its potential energy.

5 Conclusions

Acknowledgements

LaTeX Source

https://www.overleaf.com/read/vpwbqqpnjywj

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