A Few Notes on Wave-Particle Duality (etc)

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1 Introduction

We know from basic electromagnetism [5] that $c = \lambda\nu$, where c is the speed of light, λ is the wavelength, and ν is the frequency. Solving for ν we get

$$\nu = \frac{c}{\lambda} \tag{1}$$

We also know from Planck's work [2] that electromagnetic waves have energy E_{P}, where E_{P} is defined as

$$E_{\text{P}} = h\nu \tag{2}$$

and h is Planck's constant [10]. Substituting Equation (1) into Equation (2) we get

$$E_{\text{P}} = h\frac{c}{\lambda}$$

In addition, we know from Einstein [7] that the energy of a mass m, call it E_{E}, is

$$E_{\text{E}} = mc^2$$

What de Broglie [6] realized was that matter must also behave like a wave[1] [8, 12] and so E_{P} must equal E_{E}. This implies that matter also has a wavelength: $\lambda = \dfrac{h}{p}$, where $p := \|\mathbf{p}\|$ is the Euclidean Norm [1] (magnitude) of the momentum vector $\mathbf{p}$, and $\mathbf{p} = m\mathbf{v}$. λ is called the *de Broglie wavelength*.

Here's one way to think about where λ comes from:

$$\begin{array}{lcll}
h\dfrac{c}{\lambda} & = & mc^2 & \text{\# de Broglie's insight: } E_{\text{P}} = E_{\text{E}} \\
& \Rightarrow & \dfrac{h}{\lambda} = mc & \text{\# cancel } c \\
& \Rightarrow & \dfrac{h}{\lambda} = mv & \text{\# } v := \|\mathbf{v}\| = c \\
& \Rightarrow & \lambda = \dfrac{h}{mv} & \text{\# solve for } \lambda \\
& \Rightarrow & \lambda = \dfrac{h}{p} & \text{\# since } \mathbf{p} = m\mathbf{v} \text{ we have } p := \|\mathbf{p}\| = \|m\mathbf{v}\| = |m|\|\mathbf{v}\| = mv
\end{array}$$

[1] Einstein had already shown that an electromagnetic wave can behave like a particle [9].

1.1 Conservation of Energy in $\mathbb{R}^1$

The first thing to note here is that since we are in one-dimensional space (motion is in $\mathbb{R}^1$), quantities such as displacement, velocity, and acceleration, force, and energy are one-dimensional vectors (aka scalars). So we use, for example, $F = ma$ rather than $\mathbf{F} = m\mathbf{a}$.

Next, we need one definition:

Definition 1.1. Trajectory: A solution x(t) to the equation $F(x(t)) = m\ddot{x}(t)$, Newton's Second Law, is called a trajectory.

Now, consider the case of a general force function $F(x)$. Here we define the kinetic energy of a particle to be $\frac{1}{2}mv^2$. We also define the potential energy of a particle, $V(x)$, to be

$$V(x) = -\int F(x)\,dx \tag{3}$$

so that

$$\frac{d}{dx}V(x) = -F(x) \tag{4}$$

Then the total energy of a particle as a function of displacement and velocity, $E(x, v)$, is defined to be

$$E(x, v) = \frac{1}{2}mv^2 + V(x) \tag{5}$$

One of the main reasons this energy function is important is that it is *conserved*, meaning that its value along any trajectory is constant. Switching notation ($x = x(t)$ and $v = \dot{x}(t)$) and saying this in another way: An energy function is *conserved* if, for each trajectory $x(t)$ conforming to Newton's Second Law, a particle's total energy $E(x(t), \dot{x}(t))$ is independent of t.

Theorem 1.1. Suppose a particle's trajectory conforms to Newton's Second Law in the form $F(x(t)) = m\ddot{x}(t)$ and let V and E be as in Equations (3) and (5). Then the total energy of the particle is conserved.

Proof. One way to prove Theorem 1.1 is to show that the particle's total energy does not change with time, that is, $\frac{d}{dt}E(x(t), \dot{x}(t)) = 0$:

$$
\begin{aligned}
\frac{d}{dt}E(x(t),\dot{x}(t)) &= \frac{d}{dt}\left[\frac{1}{2}m(\dot{x}(t))^2 + V(x(t))\right] && \text{\# definition of } E(x(t),\dot{x}(t)) \text{ (Equation (5))} \\
&= \frac{d}{dt}\left[\frac{1}{2}m(\dot{x}(t))^2\right] + \frac{d}{dt}V(x(t)) && \text{\# derivative is a linear operator [3]} \\
&= m\dot{x}(t)\ddot{x}(t) + \frac{d}{dt}V(x(t)) && \text{\# chain \& power rules [4, 11]: } \frac{d}{dt}\left[\frac{1}{2}m(\dot{x}(t))^2\right] = m\dot{x}(t)\ddot{x}(t) \\
&= m\dot{x}(t)\ddot{x}(t) + \frac{d}{dt}V(u) && \text{\# let } u = x(t)\text{; this implies that } \frac{du}{dt} = \frac{dx}{dt} = \dot{x} \text{ and } \frac{du}{dx} = 1 \\
&= m\dot{x}(t)\ddot{x}(t) + \left[\frac{d}{du}V(u)\right]\dot{x}(t) && \text{\# chain rule: } \frac{dV}{dt} = \frac{dV}{du}\frac{du}{dt} \text{ and } \frac{du}{dt} = \dot{x} \Rightarrow \frac{dV}{dt} = \left[\frac{dV}{du}\right]\dot{x} \\
&= m\dot{x}(t)\ddot{x}(t) + \left[\frac{d}{dx}V(u)\right]\dot{x}(t) && \text{\# chain rule: } \frac{dV}{dx} = \frac{dV}{du}\frac{du}{dx} \text{ and } \frac{du}{dx} = 1 \Rightarrow \frac{dV}{dx} = \frac{dV}{du} \\
&= m\dot{x}(t)\ddot{x}(t) + \left[\frac{d}{dx}V(x(t))\right]\dot{x}(t) && \text{\# since } u = x(t) \\
&= \dot{x}(t)\left[m\ddot{x}(t) + \frac{d}{dx}V(x(t))\right] && \text{\# factor out } \dot{x}(t) \\
&= \dot{x}(t)\Big[m\ddot{x}(t) - F(x(t))\Big] && \text{\# } \frac{d}{dx}V(x(t)) = -F(x(t)) \text{ (Equation (4))} \\
&= 0 && \text{\# Newton's Second Law: } m\ddot{x}(t) - F(x(t)) = 0
\end{aligned}
$$

So we see that the time derivative of the energy along any trajectory is zero, which implies that $E(x(t),\dot{x}(t))$ is independent of t. Energy is sometimes called a *conserved quantity* (or *constant of motion*) because a particle neither gains nor loses energy as it moves according to Newton's Second Law.

2 Conclusions

3 Acknowledgements

LaTeX Source

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