$$\sqrt{?} = \sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\cdots}}}}$$

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Consider:

$$\sqrt{y} = \sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\cdots}}}} \qquad \text{\# define } y$$

$$\Rightarrow \quad y^{\frac{1}{2}} = x^{\frac{1}{3}}\, x^{\frac{1}{9}}\, x^{\frac{1}{27}} \cdots \qquad \text{\# rewrite radicals as fractions}$$

$$\Rightarrow \quad y^{\frac{1}{2}} = x^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots} \qquad \text{\# rewrite product as a sum } (x^a x^b = x^{a+b})$$

$$\Rightarrow \quad y^{\frac{1}{2}} = x^{\sum\limits_{k=1}^{\infty}\left(\frac{1}{3}\right)^k} \qquad \text{\# } \sum\limits_{k=1}^{\infty}\left(\frac{1}{3}\right)^k = \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$$

$$\Rightarrow \quad y^{\frac{1}{2}} = x^{\sum\limits_{k=0}^{\infty}\left(\frac{1}{3}\right)^k - 1} \qquad \text{\# } \sum\limits_{k=1}^{\infty}\left(\frac{1}{3}\right)^k = \sum\limits_{k=0}^{\infty}\left(\frac{1}{3}\right)^k - \left(\frac{1}{3}\right)^0 = \sum\limits_{k=0}^{\infty}\left(\frac{1}{3}\right)^k - 1$$

$$\Rightarrow \quad y^{\frac{1}{2}} = x^{\frac{1}{1-\frac{1}{3}} - 1} \qquad \text{\# geometric series: } \sum\limits_{k=0}^{\infty} ar^k = \frac{a}{1-r} \text{ for } |r| < 1; \text{ here } a = 1 \text{ and } r = \frac{1}{3}$$

$$\Rightarrow \quad y^{\frac{1}{2}} = x^{\frac{3}{2} - 1} \qquad \text{\# } \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

$$\Rightarrow \quad y^{\frac{1}{2}} = x^{\frac{1}{2}} \qquad \text{\# } \frac{3}{2} - 1 = \frac{1}{2}$$

$$\Rightarrow \quad y = x \qquad \text{\# square both sides}$$

and so apparently $\sqrt{x} = \sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\cdots}}}}$