What Does The Series $\sum_{n=1}^{\infty}(\frac{a}{b})^n$ Converge To?

Well, $\sum_{n=1}^{\infty}(\frac{a}{b})^n = \frac{a}{b-a}$ where $a, b \in \mathbb{N}$, $b \neq 0$ and $a < b$ (so $b - a \neq 0$). OK, but why?

Here's one way to think about it:

$$
\begin{aligned}
S \;=\; & \sum_{n=1}^{\infty}(\tfrac{a}{b})^n = (\tfrac{a}{b})^1 + (\tfrac{a}{b})^2 + (\tfrac{a}{b})^3 + \cdots && \text{\# define } S \\
\Rightarrow\; & (\tfrac{b}{a}) \cdot S = (\tfrac{b}{a}) \cdot \left[(\tfrac{a}{b})^1 + (\tfrac{a}{b})^2 + (\tfrac{a}{b})^3 + \cdots\right] && \text{\# multiply both sides by } \tfrac{b}{a} \\
\Rightarrow\; & (\tfrac{b}{a}) \cdot S = 1 + \left[(\tfrac{a}{b})^1 + (\tfrac{a}{b})^2 + (\tfrac{a}{b})^3 + \cdots\right] && \text{\# multiply through on right side} \\
\Rightarrow\; & (\tfrac{b}{a}) \cdot S = 1 + S && \text{\# definition of } S \\
\Rightarrow\; & (\tfrac{b}{a}) \cdot S - S = 1 && \text{\# subtract } S \text{ from both sides} \\
\Rightarrow\; & (\tfrac{b}{a}) \cdot S - (\tfrac{a}{a}) \cdot S = 1 && \text{\# multiply } S \text{ by } 1 = \tfrac{a}{a} \\
\Rightarrow\; & S \cdot \left[\frac{b}{a} - \frac{a}{a}\right] = 1 && \text{\# factor out } S \\
\Rightarrow\; & S \cdot \left[\frac{b-a}{a}\right] = 1 && \text{\# simplify} \\
\Rightarrow\; & S = \tfrac{a}{b-a} && \text{\# multiply both sides by } \tfrac{a}{b-a}
\end{aligned}
$$

So $S = \sum_{n=1}^{\infty}(\frac{a}{b})^n = \frac{a}{b-a}$ where $a, b \in \mathbb{N}$, $b \neq 0$ and $a < b$.

For example, if we let $a = 1$ and $b = 2$ then $\sum_{n=1}^{\infty}(\frac{1}{2})^n = \frac{1}{2-1} = 1$. Similarly, if $a = 1$ and $b = 3$ then $\sum_{n=1}^{\infty}(\frac{1}{3})^n = \frac{1}{3-1} = \frac{1}{2}$.

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