

What Does The Series $\sum_{n=1}^{\infty} (\frac{a}{b})^n$ Converge To?

Well, $\sum_{n=1}^{\infty} (\frac{a}{b})^n = \frac{a}{b-a}$ where $a, b \in \mathbb{N}$, $b \neq 0$ and $a < b$ (so $b - a \neq 0$). OK, but why?

Here's one way to think about it:

$$\begin{array}{ll}
 S &= \sum_{n=1}^{\infty} (\frac{a}{b})^n && \# \text{ define } S \\
 &= (\frac{a}{b})^1 + (\frac{a}{b})^2 + (\frac{a}{b})^3 + \dots && \# \text{ expand terms} \\
 \Rightarrow &(\frac{b}{a}) \cdot S = (\frac{b}{a}) \cdot [(\frac{a}{b})^1 + (\frac{a}{b})^2 + (\frac{a}{b})^3 + \dots] && \# \text{ multiply both sides by } \frac{b}{a} \\
 \Rightarrow &(\frac{b}{a}) \cdot S = 1 + (\frac{a}{b})^1 + (\frac{a}{b})^2 + (\frac{a}{b})^3 + \dots && \# \text{ multiply through on right side} \\
 \Rightarrow &(\frac{b}{a}) \cdot S = 1 + S && \# \text{ definition of } S \\
 \Rightarrow &(\frac{b}{a}) \cdot S - S = 1 && \# \text{ subtract } S \text{ from both sides} \\
 \Rightarrow &\frac{bS - aS}{a} = 1 && \# \text{ multiply through on left side} \\
 \Rightarrow &S \cdot \frac{(b-a)}{a} = 1 && \# \text{ factor out } S \\
 \Rightarrow &S = \frac{a}{b-a} && \# \text{ solve for } S
 \end{array}$$

So $S = \sum_{n=1}^{\infty} (\frac{a}{b})^n = \frac{a}{b-a}$ where $a, b \in \mathbb{N}$, $b \neq 0$ and $a < b$.

For example, if we let $a = 1$ and $b = 2$ then $\sum_{n=1}^{\infty} (\frac{1}{2})^n = \frac{1}{2-1} = 1$. Similarly, if $a = 1$ and $b = 3$ then $\sum_{n=1}^{\infty} (\frac{1}{3})^n = \frac{1}{3-1} = \frac{1}{2}$.

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