Notice that $\sum_{n=1}^{\infty}(\frac{a}{b})^n = \frac{a}{b-a}$ where $a, b \in \mathbb{N}$, $b \neq 0$, $a < b$ and $b - a \neq 0$. OK, but why?

Here's one way to think about it:

$$
\begin{array}{lll}
S & = & \sum_{n=1}^{\infty}(\frac{a}{b})^n & \text{\# define } S \\
 & = & (\frac{a}{b})^1 + (\frac{a}{b})^2 + (\frac{a}{b})^3 + \cdots & \text{\# expand terms} \\
 & \Rightarrow & (\frac{b}{a}) \cdot S = (\frac{b}{a}) \cdot [(\frac{a}{b})^1 + (\frac{a}{b})^2 + (\frac{a}{b})^3 + \cdots] & \text{\# multiply both sides by } \frac{b}{a} \\
 & \Rightarrow & (\frac{b}{a}) \cdot S = 1 + (\frac{a}{b})^1 + (\frac{a}{b})^2 + (\frac{a}{b})^3 + \cdots & \text{\# multiply through on right side} \\
 & \Rightarrow & (\frac{b}{a}) \cdot S = 1 + S & \text{\# definition of } S \\
 & \Rightarrow & (\frac{b}{a}) \cdot S - S = 1 & \text{\# subtract } S \text{ from both sides} \\
 & \Rightarrow & \frac{bS - aS}{a} = 1 & \text{\# multiply through on left side} \\
 & \Rightarrow & S \cdot \frac{(b-a)}{a} = 1 & \text{\# factor out } S \\
 & \Rightarrow & S = \frac{a}{b-a} & \text{\# solve for } S
\end{array}
$$

So $S = \sum_{n=1}^{\infty}(\frac{a}{b})^n = \frac{a}{b-a}$ where $a, b \in \mathbb{N}$, $b \neq 0$, $a < b$ and $b - a \neq 0$.

For example, if we let $a = 1$ and $b = 2$ then $\sum_{n=1}^{\infty}(\frac{1}{2})^n = \frac{1}{2-1} = 1$. Similarly, if $a = 1$ and $b = 3$ then $\sum_{n=1}^{\infty}(\frac{1}{3})^n = \frac{1}{3-1} = \frac{1}{2}$.

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