A Few Notes On The Dirac Delta Function

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1 Introduction

These notes began life as some thoughts on the Dirac Delta Function and evolved into notes on several related topics including Laplace Transforms. The Dirac Delta function has all kinds of crazy and interesting properties. More TBD.

2 The Dirac Delta Function

The Dirac Delta Function is defined as shown in Figure 1. In the limit ($\epsilon \to 0$) the Dirac Delta function is written $\delta_a(t)$ or sometimes $\delta(t-a)$. As we will see in a moment, the $\delta_{a,\epsilon}(t)$ form of the delta function is useful when we want to use the Mean Value Theorem for Integrals [2] to evaluate integrals involving the delta function.

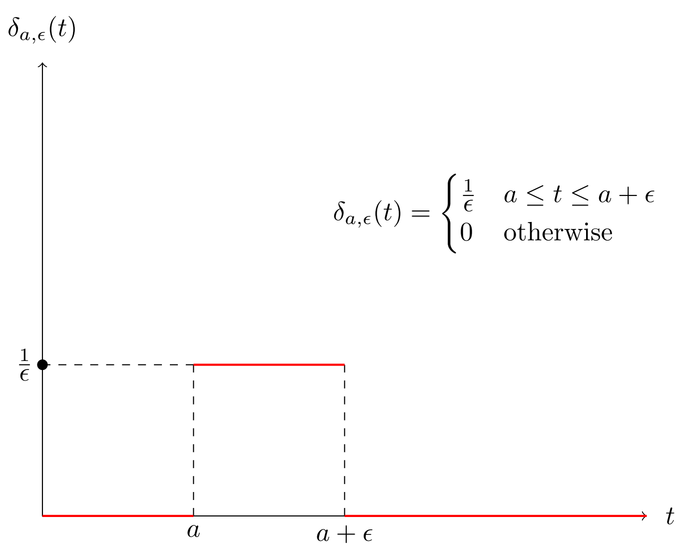

Figure 1: The Dirac Delta Function $\delta_{a,\epsilon}(t)$

So $\delta_{a,\epsilon}(t)$ is defined to be

$$\delta_{a,\epsilon}(t) = \begin{cases} \frac{1}{\epsilon} & a \le t \le a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$

and has the constraint that

$$\int_0^\infty \delta_{a,\epsilon}(t) = 1$$

That is, $\delta_{a,\epsilon}(t)$ is in some sense a probability density.

In the limit the Dirac Delta Function looks like

$$\lim_{\epsilon \to 0} \delta_{a,\epsilon}(t) = \delta_a(t) = \begin{cases} \infty & t = a \\ 0 & t \ne a \end{cases}$$

or sometimes

$$\delta(t-a) = \begin{cases} 0 & t \ne a \\ \infty & t = a \end{cases}$$

$\delta_a(t)$ also has the constraint that

$$\int_0^\infty \delta_a(t) = 1$$

and so is also a probability density. $\delta_a(t)$ is shown in Figure 2.

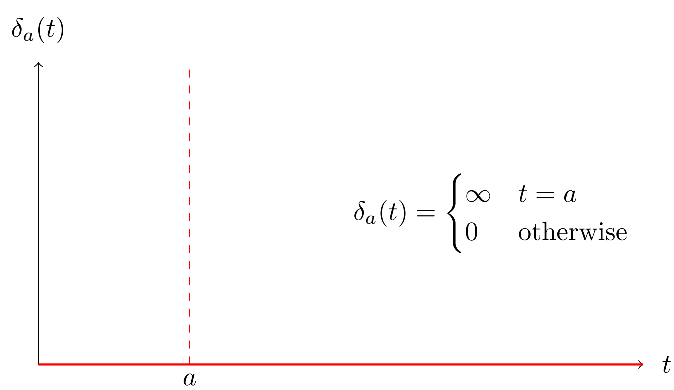

Figure 2: The Dirac Delta Function $\delta_a(t)$

2.1 Integrals Involving $\delta_{a,\epsilon}(t)$

$\delta_{a,\epsilon}(t)$ has all kinds of interesting properties. One of them involves the integral of the product $\delta_{a,\epsilon}(t)$ with some function $g(t)$. Here we would like to evaluate integrals of the form

$$\int_0^\infty \delta_{a,\epsilon}(t)g(t)dt \tag{1}$$

where $g(t)$ is continuous on the interval $[a, a+\epsilon]$.

The Mean Value Theorem for Integrals [2] tells us that

$$\int_a^b g(t)dt = (b-a)g(c) \tag{2}$$

where the point c lies in the interval $[a, a+\epsilon]$. Now, since we know that $\delta_{a,\epsilon}(t)$ is zero everywhere except on the interval $[a, a+\epsilon]$ we can rewrite the improper integral in Equation 1 as the proper integral

$$\int_a^{a+\epsilon} \delta_{a,\epsilon}(t)g(t)dt$$

Here we can notice that $\delta_{a,\epsilon}(t) = \frac{1}{\epsilon}$ on the interval $[a, a+\epsilon]$ so we can rewrite our integral as

$$\int_a^{a+\epsilon} \frac{1}{\epsilon}g(t)dt = \frac{1}{\epsilon}\int_a^{a+\epsilon} g(t)dt$$

Now we can use Equation 2, the Mean Value Theorem for Integrals[1], by setting $b = a+\epsilon$ and $a = a$ so that $b - a = \epsilon$. Then

$$\frac{1}{\epsilon}\int_a^{a+\epsilon} g(t)dt = \frac{1}{\epsilon}\underbrace{\Big[(a+\epsilon)-a\Big]}_{b-a} g(c) = \frac{1}{\epsilon}\epsilon g(c) = g(c)$$

where $c \in [a, a+\epsilon]$.

Finally, if we look at what happens to c as $\epsilon \to 0$ we see that $\lim\limits_{\epsilon\to 0} c = a$ (sorry about the notation abuse) so that

$$\int_0^\infty \delta_a(t)g(t)dt = g(a) \tag{3}$$

[1]This is where the $\delta_{a,\epsilon}(t)$ form of the delta function comes in handy.

Essentially $\delta_a(t)$ pulls out the value of g at a, that is, $g(a)$.

Another way to get this result [1] is to notice that the integrand of

$$\int_0^\infty \delta(t-a)g(t)dt$$

is zero everywhere except where $t = a$, so we can rewrite our integral as $\int_0^\infty \delta(t-a)g(a)dt = g(a)\int_0^\infty \delta(t-a)dt$ (since $g(a)$ doesn't depend on t). Then since by definition $\int_0^\infty \delta(t-a)dt = 1$ we get

$$\int_0^\infty \delta(t-a)g(t)dt = g(a)$$

3 The Laplace Transform

We start by defining the integral transform of some function $f(t)$.

Definition 3.1. Integral Transform

If a function $f(t)$ is defined on $[0,\infty)$ then we can define an integral transform to be the improper integral

$$F(s) = \int_0^\infty K(s,t)f(t)dt$$

If the improper integral converges then we say that $F(s)$ is the integral transform of $f(t)$. The function $K(s,t)$ is called the *kernel* of the transform. When $K(s,t) = e^{-st}$ the transform is called the **Laplace Transform**.

Definition 3.2. Laplace Transform

The Laplace Transform of a function $f(t)$ is defined to be

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

and is useful when solving ordinary differential equations (ODEs). Interestingly, the Laplace Transform of the Dirac Delta Function turns out to be

$$\begin{array}{rcll}
\mathcal{L}\{\delta_a(t)\} & = & \int_0^\infty \delta_a(t)e^{-st}dt & \text{\# definition of the Laplace Transform} \\
 & = & \int_0^\infty \delta_a(t)g(t)dt & \text{\# set } g(t) = e^{-st} \\
 & = & g(a) & \text{\# by Equation 3} \\
 & = & e^{-sa} & \text{\# since } g(t) = e^{-st}
\end{array}$$

3.1 The Linearity Property Of The Laplace Transform

$\mathcal{L}$ is a linear operator, in other words: $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$. Why? Consider:

$$
\begin{aligned}
\mathcal{L}\{af(t) + bg(t)\} &= \int_0^\infty e^{-st}\big[\mathcal{L}\{af(t) + bg(t)\}\big]dt && \text{\# definition of the Laplace Transform} \\
&= a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt && \text{\# by the linearity of improper integrals [3]} \\
&= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} && \text{\# definition of the Laplace Transform}
\end{aligned}
$$

3.2 So Does Every Function Have A Laplace Transform?

The answer is no (consider a function like $f(t) = t^{-1}$). There are two properties that $f(t)$ must have in order to have a Laplace Transform. First, $f(t)$ must be of "exponential order". So what does that mean?

Definition 3.3. Exponential Order

A function f is said to be of **exponential order** c if there exist constants $c, M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$. Basically this is saying that in order for $f(t)$ to have a Laplace Transform then in a race between $|f(t)|$ and e^{ct} as $t \to \infty$ e^{ct} must approach its limit first. That is, $\lim_{t\to\infty} \frac{f(t)}{e^{ct}} = 0$.

Next, we need the following theorem:

Theorem 3.1. Existence Theorem: If f is s piecewise continuous on $[0, \infty)$ and of exponential order c then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

Ok, but why? Consider

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st}f(t)dt && \text{\# definition of the Laplace Transform} \\
&\leq \int_0^\infty e^{-st}Me^{ct}dt && \text{\# definition of exponential order with constants } M \text{ and } c \\
&= M\int_0^\infty e^{(c-s)t}dt && \text{\# M doesn't depend on } t\text{, consolidate powers of } e \\
&= M\frac{e^{(c-s)t}}{c-s}\Big|_0^\infty && \text{\# } \textstyle\int_0^\infty e^{kx}dx = \frac{1}{k}e^{kx} + C \text{ (ignore } C\text{, use } u = kt \text{ so that } du = kdt...\text{)} \\
&= M\frac{e^{(c-s)\infty}}{c-s} - \frac{Me^{(c-s)0}}{c-s} && \text{\# evaluate at endpoints} \\
&= 0 - \frac{M}{c-s} && \text{\# we assumed that } s > c \text{ so } c - s < 0 \Rightarrow \text{the first term goes to zero} \\
&= \frac{M}{s-c} && \text{\# simplify...}
\end{aligned}
$$

All of this implies that functions $f(t)$ that do not satisfy the Existence Theorem do not have Laplace Transforms since $\mathcal{L}\{f(t)\}$ doesn't converge.

Acknowledgements

References

[1] Leonard Susskind. Lecture 4 — Modern Physics: Quantum Mechanics (Stanford). https://www.youtube.com/watch?v=oWe9brUwOOQ&t=1130s, 2008. [Online; accessed 11-May-2021].

[2] Proof Wiki Contributors. Mean Value Theorem For Integrals. https://proofwiki.org/wiki/Mean_Value_Theorem_for_Integrals, 2020. [Online; accessed 11-May-2021].

[3] Tom Lewis. Improper Integrals. http://math.furman.edu/~tlewis/math450/ash/chap7/sec9.pdf, 2014. [Online; accessed 25-May-2021].