A Few Notes On The Dirac Delta Function

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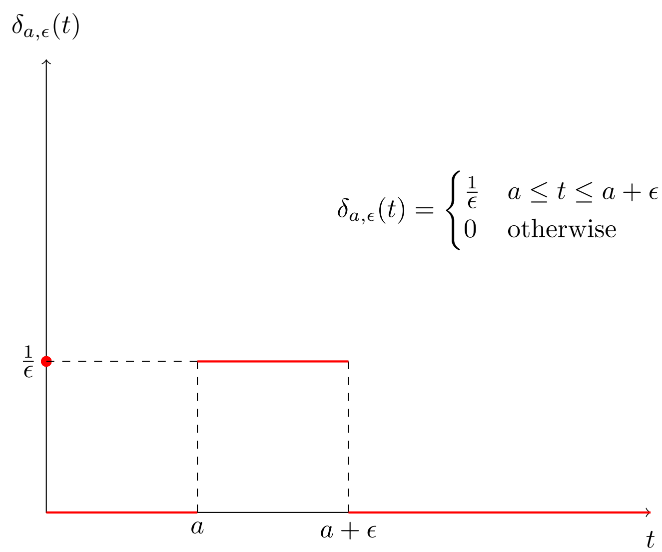
1 Introduction

These notes began life as some thoughts on the Dirac Delta Function and evolved into notes on several related topics including Laplace Transforms. The Dirac Delta function has all kinds of crazy and interesting properties. More TBD.

2 The Dirac Delta Function

The Dirac Delta Function is defined as shown in Figure 1. In the limit ($\epsilon \to 0$) the Dirac Delta function is written $\delta_a(t)$ or sometimes $\delta(t-a)$. As we will see in a moment, the $\delta_{a,\epsilon}(t)$ form of the delta function is useful when we want to use the Mean Value Theorem for Integrals [1] to evaluate integrals involving the delta function.



$$\delta_{a,\epsilon}(t) = \begin{cases} \frac{1}{\epsilon} & a \le t \le a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$

Figure 1: The $\delta_{a,\epsilon}(t)$ function

So $\delta_{a,\epsilon}(t)$ is defined to be

$$\delta_{a,\epsilon}(t) = \begin{cases} \frac{1}{\epsilon} & a \leq t \leq a + \epsilon \\ 0 & \text{otherwise} \end{cases}$$

and has the constraint that

$$\int_0^\infty \delta_{a,\epsilon}(t) = 1$$

That is, $\delta_{a,\epsilon}(t)$ is in some sense a probability density.

In the limit the Dirac Delta Function looks like

$$\lim_{\epsilon \to 0} \delta_{a,\epsilon}(t) = \delta_a(t) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$$

or sometimes

$$\delta(t-a) = \begin{cases} 0 & t \neq a \\ \infty & t = a \end{cases}$$

$\delta_a(t)$ also has the constraint that

$$\int_0^\infty \delta_a(t) = 1$$

and so is also a probability density. $\delta_a(t)$ is shown in Figure 2.

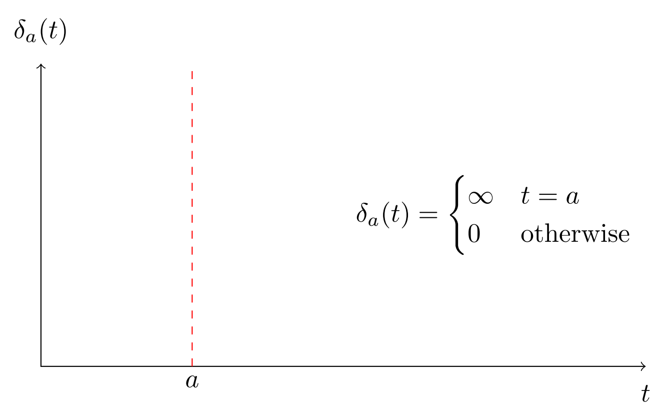


Figure 2: The $\delta_a(t)$ function

2.1 Integrals Involving $\delta_{a,\epsilon}(t)$

$\delta_{a,\epsilon}(t)$ has all kinds of interesting properties. One of them involves the integral of the product $\delta_{a,\epsilon}(t)$ with some function $g(t)$. Here we would like to evaluate integrals of the form

$$\int_0^\infty \delta_{a,\epsilon}(t)g(t)dt \tag{1}$$

where $g(t)$ is continuous on the interval $[a, a+\epsilon]$.

The Mean Value Theorem for Integrals [1] tells us that

$$\int_a^b g(t)dt = (b-a)g(c) \tag{2}$$

where the point c lies in the interval $[a, a+\epsilon]$. Now, since we know that $\delta_{a,\epsilon}(t)$ is zero everywhere except on the interval $[a, a+\epsilon]$ we can rewrite the improper integral in Equation 1 as the proper integral

$$\int_a^{a+\epsilon} \delta_{a,\epsilon}(t)g(t)dt$$

Here we can notice that $\delta_{a,\epsilon}(t) = \frac{1}{\epsilon}$ on the interval $[a, a+\epsilon]$ so we can rewrite our integral as

$$\int_a^{a+\epsilon} \frac{1}{\epsilon}g(t)dt = \frac{1}{\epsilon}\int_a^{a+\epsilon} g(t)dt$$

Now we can use Equation 2, the Mean Value Theorem for Integrals[1], by setting $b = a+\epsilon$ and $a = a$ so that $b - a = \epsilon$. Then by the Mean Value Theorem for Integrals

$$\frac{1}{\epsilon}\int_a^{a+\epsilon} g(t)dt = \frac{1}{\epsilon}\underbrace{\Big[(a+\epsilon)-a\Big]}_{b-a}g(c) = \frac{1}{\epsilon}\epsilon g(c) = g(c)$$

where $c \in [a, a+\epsilon]$. Finally, if we look at the limit as $\epsilon \to 0$ we see that $\lim_{\epsilon\to 0} c = a$ so that

$$\int_0^\infty \delta_a(t)g(t)dt = g(a) \tag{3}$$

Essentially $\delta_a(t)$ pulls out the value of g at a, that is, $g(a)$.

[1] This is where the $\delta_{a,\epsilon}(t)$ form of the delta function comes in handy.

Another way to get this result is to notice that the integrand of

$$\int_0^\infty \delta(t-a)g(t)dt$$

is zero everywhere except where $t = a$, so we can rewrite our integral as $\int_0^\infty \delta(t-a)g(a)dt = g(a)\int_0^\infty \delta(t-a)dt$ (since $g(a)$ doesn't depend on t). Then since by definition $\int_0^\infty \delta(t-a)dt = 1$ we get

$$\int_0^\infty \delta(t-a)g(t)dt = g(a)$$

Acknowledgements

References

[1] Proof Wiki Contributors. Mean Value Theorem For Integrals. `https://proofwiki.org/wiki/Mean_Value_Theorem_for_Integrals`, 2020. [Online; accessed 11-May-2021].