Proof that the square root of 2 is irrational

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1 Introduction

Here we prove by contradiction that $\sqrt{2} \notin \mathbb{Q}$. This approach to proving the irrationality of $\sqrt{2}$ is sometimes called "Proof by infinite descent, not involving factoring" [1].

Theorem 1.1. $\sqrt{2} \notin \mathbb{Q}$

Proof: As mentioned above, in this proof we assume that, for contradiction, $\sqrt{2} \in \mathbb{Q}$. Then $\exists a, b \in \mathbb{Z}$ such that $\frac{a}{b} = \sqrt{2}$ and we assume that $\frac{a}{b}$ is simplified to *lowest terms*. The contradiction we will find is that $\frac{a}{b}$ cannot be in lowest term and in fact, our assumptions imply that 2 divides a (written $2|a$) *and* 2 divides b. Ok, but why?

Well, consider our assumption $\sqrt{2} \in \mathbb{Q}$, which implies that for $a, b \in \mathbb{Z}, b \neq 0$

$$\frac{a}{b} = \sqrt{2}$$

Then

$$a = \sqrt{2}b$$

and therefore

$$a^2 = 2b^2 \tag{1}$$

So $2|a^2$. So a^2 is even which implies that a is even[1]. Now, since a is even we can write

[1]Note that if a is not even, then a^2 is not even.

$$a = 2n \tag{2}$$

for some $n \in \mathbb{Z}$. If we plug this into Equation 1 we get

$$4n^2 = 2b^2$$

and therefore $2n^2 = b^2$. So b^2 and hence b is even. So we can write

$$b = 2m \tag{3}$$

for some $m \in \mathbb{Z}$. So for some $a, b \in \mathbb{Z}$ and $b \neq 0$ our assumption implies

$$\begin{array}{lcll}
\sqrt{2} \in \mathbb{Q} & \Rightarrow & \frac{a}{b} = \sqrt{2} & \text{\# assume } \frac{a}{b} \text{ is in lowest terms} \\
 & \Rightarrow & a = 2n & \text{\# Equation 2} \\
 & \Rightarrow & b = 2m & \text{\# Equation 3} \\
 & \Rightarrow & \frac{a}{b} = \frac{2n}{2m} & \text{\# substitute for } a \text{ and } b \\
 & \Rightarrow & 2|a \ \mathit{and} \ 2|b & \text{\# the contradiction } (\frac{a}{b} \text{ not in lowest terms})
\end{array}$$

So we assumed that $\frac{a}{b}$ was in lowest terms and we showed that this leads to a contradiction, namely that 2 divides both a and b and thus $\frac{a}{b}$ is not in lowest terms (the contradiction). Hence our original assumption that $\sqrt{2} \in \mathbb{Q}$ is false and $\sqrt{2} \notin \mathbb{Q}$. $\square$

2 Acknowledgements

References

[1] Wikipedia contributors. Square root of 2 — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Square_root_of_2&oldid=897400199, 2019. [Online; accessed 30-May-2019].