

What Does The Series $\sum_{n=1}^{\infty} (\frac{a}{b})^n$ Converge To?

Well, $\sum_{n=1}^{\infty} (\frac{a}{b})^n = \frac{a}{b-a}$ where $a, b \in \mathbb{N}$, $b \neq 0$ and $a < b$ (so $b - a \neq 0$). OK, but why?

Here's one way to think about it:

$$\begin{aligned}
 S &= \sum_{n=1}^{\infty} (\frac{a}{b})^n && \# \text{ define } S \\
 &= (\frac{a}{b})^1 + (\frac{a}{b})^2 + (\frac{a}{b})^3 + \dots && \# \text{ expand series} \\
 \Rightarrow (\frac{b}{a}) \cdot S &= (\frac{b}{a}) \cdot \left[(\frac{a}{b})^1 + (\frac{a}{b})^2 + (\frac{a}{b})^3 + \dots \right] && \# \text{ multiply both sides by } \frac{b}{a} \\
 \Rightarrow (\frac{b}{a}) \cdot S &= 1 + (\frac{a}{b})^1 + (\frac{a}{b})^2 + (\frac{a}{b})^3 + \dots && \# \text{ multiply through on right side} \\
 \Rightarrow (\frac{b}{a}) \cdot S &= 1 + S && \# \text{ definition of } S \\
 \Rightarrow (\frac{b}{a}) \cdot S - S &= 1 && \# \text{ subtract } S \text{ from both sides} \\
 \Rightarrow (\frac{b}{a}) \cdot S - (\frac{a}{a}) \cdot S &= 1 && \# \text{ multiply } S \text{ by } 1 = \frac{a}{a} \\
 \Rightarrow S \cdot \left[\frac{b}{a} - \frac{a}{a} \right] &= 1 && \# \text{ factor out } S \\
 \Rightarrow S \cdot \left[\frac{b-a}{a} \right] &= 1 && \# \text{ simplify} \\
 \Rightarrow S = \frac{a}{b-a} &&& \# \text{ multiply both sides by } \frac{a}{b-a}
 \end{aligned}$$

So $S = \sum_{n=1}^{\infty} (\frac{a}{b})^n = \frac{a}{b-a}$ where $a, b \in \mathbb{N}$, $b \neq 0$ and $a < b$.

For example, if we let $a = 1$ and $b = 2$ then $\sum_{n=1}^{\infty} (\frac{1}{2})^n = \frac{1}{2-1} = 1$. Similarly, if $a = 1$ and $b = 3$ then $\sum_{n=1}^{\infty} (\frac{1}{3})^n = \frac{1}{3-1} = \frac{1}{2}$.

Acknowledgements

Thanks to Dave Neary who pointed out that this series diverges if $a \geq b$.