# A Few Notes On The Riemann Zeta Function

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Last update: June 25, 2021

## 1 Introduction

TBD

## 2 Euler's Product Formula

Recall that the Riemann zeta function is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

In 1737 Leonhard Euler [2] discovered the beautiful connection between the zeta function and the prime numbers and proved this identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \tag{1}$$

The left side of Equation 1 is by definition $\zeta(s)$. The infinite product on the right side of Equation 1 extends over all prime numbers $p$ and is called an Euler Product [1], which is the expansion of a Dirichlet series into an infinite product indexed by prime numbers:

$$\prod_{p \text{ prime}} \frac{1}{1-p^{-s}} = \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \frac{1}{1-7^{-s}} \cdot \frac{1}{1-11^{-s}} \cdots \frac{1}{1-p^{-s}} \cdots$$

Both sides of the Euler product formula converge for $\mathrm{Re}(s) > 1$.

# References

[1] Noam D. Elkies. Math 259: Introduction to Analytic Number Theory. `http://abel.math.harvard.edu/~elkies/M259.06/euler.pdf`, 1998. [Online; accessed 25-June-2021].

[2] Timothy Murphy. Euler's Product Formula. `https://www.maths.tcd.ie/pub/Maths/Courseware/428/Primes-II.pdf`, 2006. [Online; accessed 25-June-2021].