Definition 0.1. Let I be a subset of a ring R. Then an additive subgroup of R having the property that

$$ra \in I \text{ for } a \in I,\ r \in R$$

is called a left ideal of R. Similarly

$$ar \in I \text{ for } a \in I,\ r \in R$$

is called a right ideal of R. If an ideal is both a right and a left ideal then we call it a two-sided ideal of R, or simply an ideal of R.

We say that an ideal I of R is proper if $I \neq R$. We say that is it non-trivial if $I \neq R$ and $I \neq 0$.

I've seen several different notations for ideals including, among others: $rI = \{ri \mid r \in R, i \in I\}$ and $rI \subset I$ for $\forall r \in R$.

So in words: An *ideal* I is a subset of a ring R such that

- I is a subgroup of R under addition (so $0 \in I$ and so $I \neq \emptyset$)
- I is not only closed under multiplication but also satisfies the *stronger* property that it "absorbs" all of the elements of R under multiplication: $\forall r \in R \Rightarrow rI \subset I$

Groups

- $\phi : G \xrightarrow{hom} H$ — # ϕ is a group homomorphism: $G \simeq H$
- $\ker \phi = N$ — # N is a **normal subgroup** of G ($N \lhd G$)
- if ϕ is onto then $H \simeq G/N$ — # First Isomorphism Theorem for groups

Rings

- $\phi : R \xrightarrow{hom} S$ — # ϕ is a ring homomorphism: $R \simeq S$
- $\ker \phi = I$ — # I is a two-sided **ideal** in R with $1 \notin I$
- if ϕ is onto then $S \simeq R/I$ — # First Isomorphism Theorem for rings

Notes

- An ideal is the same thing as the kernel of a ring homomorphism ϕ
- Ideals are to rings as normal subgroups are to groups