A Few Notes On The Dirac Delta Function And The Laplace Transform

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1 Introduction

These notes began life as some thoughts on the Dirac Delta Function and evolved into notes on several related topics including Laplace Transforms. The Dirac Delta function has all kinds of crazy and interesting properties. More TBD.

2 The Dirac Delta Function

The Dirac Delta Function is defined as shown in Figure 1. In the limit ($\epsilon \to 0$) the Dirac Delta function is written $\delta_a(t)$ or sometimes $\delta(t-a)$. As we will see in a moment, the $\delta_{a,\epsilon}(t)$ form of the delta function is useful when we want to use the Mean Value Theorem for Integrals [2] to evaluate integrals involving the delta function.

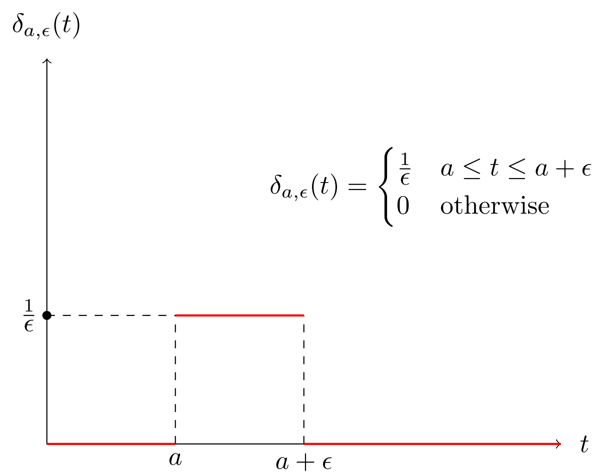

Figure 1: The Dirac Delta Function $\delta_{a,\epsilon}(t)$

So $\delta_{a,\epsilon}(t)$ is defined to be

$$\delta_{a,\epsilon}(t) = \begin{cases} \frac{1}{\epsilon} & a \leq t \leq a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$

and has the constraint that

$$\int_0^\infty \delta_{a,\epsilon}(t) = 1$$

That is, $\delta_{a,\epsilon}(t)$ is in some sense a probability density.

In the limit the Dirac Delta Function looks like

$$\lim_{\epsilon \to 0} \delta_{a,\epsilon}(t) = \delta_a(t) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$$

or sometimes

$$\delta(t-a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$$

$\delta_a(t)$ also has the constraint that

$$\int_0^\infty \delta_a(t) = 1$$

and so is also a probability density. $\delta_a(t)$ is shown in Figure 2.

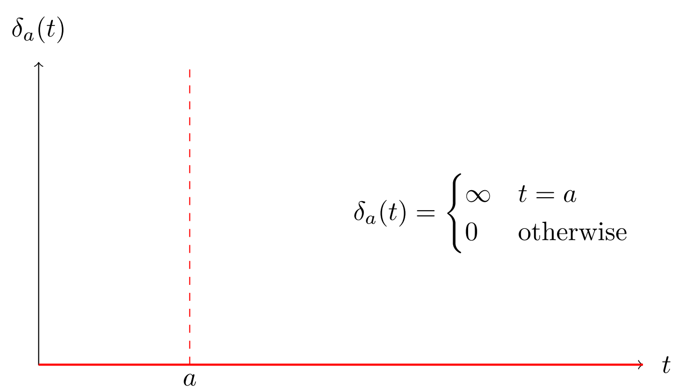

Figure 2: The Dirac Delta Function $\delta_a(t)$

2.1 Integrals involving $\delta_{a,\epsilon}(t)$

$\delta_{a,\epsilon}(t)$ has all kinds of interesting properties. One of them involves the integral of the product $\delta_{a,\epsilon}(t)$ with some function $g(t)$. Here we would like to evaluate integrals of the form

$$\int_0^\infty \delta_{a,\epsilon}(t)g(t)dt \tag{1}$$

where $g(t)$ is continuous on the interval $[a, a+\epsilon]$.

The Mean Value Theorem for Integrals [2] tells us that

$$\int_a^b g(t)dt = (b-a)g(c) \tag{2}$$

where the point c lies in the interval $[a, a+\epsilon]$. Now, since we know that $\delta_{a,\epsilon}(t)$ is zero everywhere except on the interval $[a, a+\epsilon]$ we can rewrite the improper integral in Equation 1 as the proper integral

$$\int_a^{a+\epsilon} \delta_{a,\epsilon}(t)g(t)dt$$

Here we can notice that $\delta_{a,\epsilon}(t) = \frac{1}{\epsilon}$ on the interval $[a, a+\epsilon]$ so we can rewrite our integral as

$$\int_a^{a+\epsilon} \frac{1}{\epsilon}g(t)dt = \frac{1}{\epsilon}\int_a^{a+\epsilon} g(t)dt$$

Now we can use Equation 2, the Mean Value Theorem for Integrals[1], by setting $b = a+\epsilon$ and $a = a$ so that $b - a = \epsilon$. Then

$$\frac{1}{\epsilon}\int_a^{a+\epsilon} g(t)dt = \frac{1}{\epsilon}\underbrace{\Big[(a+\epsilon)-a\Big]}_{b-a} g(c) = \frac{1}{\epsilon}\epsilon g(c) = g(c)$$

where $c \in [a, a+\epsilon]$.

Finally, if we look at what happens to c as $\epsilon \to 0$ we see that $\lim\limits_{\epsilon\to 0} c = a$ (sorry about the notation abuse) so that

$$\int_0^\infty \delta_a(t)g(t)dt = g(a) \tag{3}$$

[1] This is where the $\delta_{a,\epsilon}(t)$ form of the delta function comes in handy.

Essentially $\delta_a(t)$ pulls out the value of g at a, that is, $g(a)$.

Another way to look at this [1] is to notice that the integrand of

$$\int_0^\infty \delta(t-a)g(t)dt$$

is zero everywhere except where $t = a$, so we can rewrite our integral as $\int_0^\infty \delta(t-a)g(a)dt = g(a)\int_0^\infty \delta(t-a)dt$ (since $g(a)$ doesn't depend on t). Then since by definition $\int_0^\infty \delta(t-a)dt = 1$ we get

$$\int_0^\infty \delta(t-a)g(t)dt = g(a)$$

3 The Laplace Transform

We start by defining the integral transform of some function $f(t)$.

Definition 3.1. Integral Transform: If a function $f(t)$ is defined on $[0,\infty)$ then we can define an integral transform to be the improper integral

$$F(s) = \int_0^\infty K(s,t)f(t)dt$$

If the improper integral converges then we say that $F(s)$ is the *integral transform* of $f(t)$. The function $K(s,t)$ is called the *kernel* of the transform. When $K(s,t) = e^{-st}$ the transform is called the **Laplace Transform**.

Definition 3.2. Laplace Transform: The Laplace Transform of a function $f(t)$ is defined to be

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)dt \tag{4}$$

The Laplace Transform will turn out to be useful when solving ordinary differential equations (ODEs). Interestingly, the Laplace Transform of the Dirac Delta Function turns out to be

$$\begin{aligned}
\mathcal{L}\{\delta_a(t)\} &= \textstyle\int_0^\infty e^{-st}\delta_a(t)dt && \text{\# Equation 4 with } f(t) = \delta_a(t) \\
&= \textstyle\int_0^\infty g(t)\delta_a(t)dt && \text{\# set } g(t) = e^{-st} \\
&= g(a) && \text{\# by Equation 3} \\
&= e^{-sa} && \text{\# } g(a) = e^{-sa}
\end{aligned}$$

3.1 The Linearity Property of the Laplace Transform

Definition 3.3. Linearity Property: $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$

Interestingly, $\mathcal{L}$ is what is called a linear operator in vector space parlance [5].

All good, but why does $\mathcal{L}$ have this property? Here's one way to think about it:

$$\begin{aligned}
\mathcal{L}\{af(t) + bg(t)\} &= \int_0^\infty e^{-st}\big[af(t) + bg(t)\big]dt && \text{\# definition of the Laplace Transform (Definition 3.2)} \\
&= \int_0^\infty e^{-st}af(t)dt + \int_0^\infty e^{-st}bg(t)dt && \text{\# by the linearity of improper integrals [3]} \\
&= a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt && \text{\# neither } a \text{ nor } b \text{ depends on } t \\
&= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} && \text{\# definition of the Laplace Transform (Definition 3.2)}
\end{aligned}$$

3.2 So does every function have a Laplace Transform?

The answer is no (consider a function like $f(t) = t^{-1}$). Ok, then what are the properties that $f(t)$ must have in order to have a Laplace Transform? First, f must be of "exponential order".

Definition 3.4. Exponential Order: A function f is said to be of exponential order c if there exist constants $c, M,$ and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$.

Said another way, in order for $f(t)$ to be of exponential order c we require that

$$\lim_{t\to\infty} \frac{f(t)}{e^{ct}} = 0$$

Basically this is saying that in order for $f(t)$ to have a Laplace Transform then in a race between $|f(t)|$ and e^{ct} as $t \to \infty$ e^{ct} must approach its limit first. This situation is depicted in Figure 3.

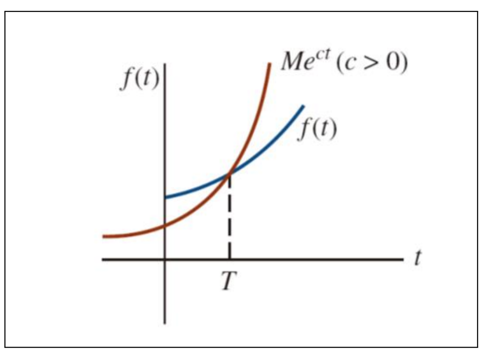

Figure 3: f is of exponential order with constants c, M and T

Now we can answer the question of when f has a Laplace Transfrom:

Theorem 3.1. Existence Theorem for Laplace Transforms: If f is s piecewise continuous on the interval $[0, \infty)$ and is of exponential order c then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

Ok, but why? Consider

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= F(s) && \text{\# definition of the Laplace Transform (Definition 3.2)} \\
&= \int_0^\infty e^{-st} f(t)dt && \text{\# definition of the Laplace Transform (Definition 3.2)} \\
&\leq \int_0^\infty e^{-st} M e^{ct} dt && \text{\# } f \text{ is of exponential order } c \text{ (Definition 3.4)} \\
&= M\int_0^\infty e^{-st} e^{ct} dt && \text{\# } M \text{ doesn't depend on } t \\
&= M\int_0^\infty e^{-st+ct} dt && \text{\# } x^n \cdot x^m = x^{n+m} \\
&= M\int_0^\infty e^{(c-s)t} dt && \text{\# } e^{-st+ct} = e^{ct-st} = e^{(c-s)t} \\
&= M\int_0^\infty e^{u} dt && \text{\# use a } u \text{ substitution with } u = (c-s)t \\
&= M\int_0^\infty e^{u} \tfrac{du}{c-s} && \text{\# } u = (c-s)t \Rightarrow du = (c-s)dt \Rightarrow dt = \frac{du}{c-s} \\
&= \left[\tfrac{M}{c-s}\right]\int_0^\infty e^{u} du && \text{\# neither } c \text{ nor } s \text{ depends on } t \\
&= \left[\tfrac{M}{c-s}\right] e^u \Big|_0^\infty && \text{\# } \textstyle\int_0^\infty e^u du = e^u + C \text{ and the Fundamental Theorem of Calculus} \\
&= \left[\tfrac{M}{c-s}\right] e^{(c-s)t} \Big|_0^\infty && \text{\# } u = (c-s)t \\
&= \lim_{d\to\infty}\left[\tfrac{M}{c-s} e^{(c-s)d}\right] - \tfrac{M}{c-s} e^{(c-s)0} && \text{\# evaluate at the limits} \\
&= \lim_{d\to\infty}\left[\tfrac{M}{c-s} e^{(c-s)d}\right] - \tfrac{M}{c-s} && \text{\# } e^{(c-s)0} = e^0 = 1 \text{ and } \tfrac{M}{c-s}\cdot 1 = \tfrac{M}{c-s} \\
&= 0 - \frac{M}{c-s} && \text{\# } s > c \Rightarrow c - s < 0 \Rightarrow \lim_{d\to\infty}\left[\tfrac{M}{c-s} e^{(c-s)d}\right] = 0 \\
&= \frac{M}{s-c} && \text{\# simplify}
\end{aligned}$$

So if $s = c$ then $\frac{M}{s-c}$ is not defined and $\mathcal{L}\{f(t)\}$ does not exist. Similarly, if $s < c$ then $\lim_{d\to\infty}\left[\frac{M}{c-s}e^{(c-s)d}\right]$ does not converge and $\mathcal{L}\{f(t)\}$ does not exist. All of this implies that functions that do not satisfy the conditions of the Existence Theorem do not have Laplace Transforms.

3.3 Inverse Laplace Transform

Definition 3.5. Inverse Laplace Transform: If $F(s)$ represents the Laplace Transform of a function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$ then the Inverse Laplace Transform of $F(s)$ is $f(t)$, i.e. $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Here are a few examples:

Laplace Transform	
$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\frac{dg}{dt}$	$sG(s) - g(0)$

Inverse Laplace Transform	
$F(s)$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$
$\frac{1}{s}$	1
$\frac{n!}{s^{n+1}}$	t^n
$\frac{1}{s-a}$	e^{at}
$\frac{k}{s^2+k^2}$	$\sin(kt)$
$\frac{s}{s^2+k^2}$	$\cos(kt)$
$\frac{k}{s^2-k^2}$	$\sinh(kt)$
$\frac{s}{s^2-k^2}$	$\cosh(kt)$
$sG(s) - g(0)$	$\frac{dg}{dt}$

3.4 Ok, then what is the Laplace Transform of $f'(t)$?

Suppose $f(t)$ is continuous, piecewise smooth and of exponential order, and suppose $f'(t)$ is the derivative of $f(t)$. Then the Laplace Transform of $f'(t)$, $\mathcal{L}\{f'(t)\}$, turns out to be

$$
\begin{aligned}
\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)dt && \text{\# definition Laplace Transform (Definition 3.2)} \\
&= \int_0^\infty \underbrace{e^{-st}}_{v} \underbrace{f'(t)dt}_{du} && \text{\# use integration by parts} \\
&= \underbrace{e^{-st}}_{v} \underbrace{f(t)}_{u} \Big|_0^\infty - \int_0^\infty \underbrace{(-s)e^{-st}}_{dv} \underbrace{f(t)}_{u} dt && \text{\# } \int_0^\infty v\, du = uv - \int_0^\infty u\, dv \\
&= \lim_{d\to\infty} \left[e^{-sd} f(d)\right] - e^{-s\cdot 0} \cdot f(0) - \int_0^\infty (-s)e^{-st} f(t)dt && \text{\# evaluate first term at the limits} \\
&= 0 - e^{-s\cdot 0} \cdot f(0) - \int_0^\infty (-s)e^{-st} f(t)dt && \text{\# } \lim_{d\to\infty} \left[e^{-sd} f(d)\right] = 0 \\
&= 0 - f(0) - \int_0^\infty (-s)e^{-st} f(t)dt && \text{\# } e^{-s\cdot 0} = e^0 = 1 \text{ and } 1 \cdot f(0) = f(0) \\
&= -f(0) + s\int_0^\infty e^{-st} f(t)dt && \text{\# } s \text{ doesn't depend on } t \text{ and simplify} \\
&= -f(0) + s\mathcal{L}\{f(t)\} && \text{\# definition Laplace Transform (Definition 3.2)} \\
&= s\mathcal{L}\{f(t)\} - f(0) && \text{\# rearrange} \\
&= sF(s) - f(0) && \text{\# definition Laplace Transform (Definition 3.2)}
\end{aligned}
$$

One of the interesting things to note here is that by using the Laplace Transform we've taken a statement about the derivative of f, namely $\mathcal{L}\{f'(t)\}$, and converted it into a statement about f itself: $s\mathcal{L}\{f(t)\} - f(0)$. That is, we've converted a differential equation into an algebraic one. This will come in handy later when we want to solve differential equations.

3.5 What about $\mathcal{L}\{f''(t)\}$?

Once we know how to compute $\mathcal{L}\{f'(t)\}$ it is pretty easy to see how to compute $\mathcal{L}\{f''(t)\}$:

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= \mathcal{L}\{g'(t)\} && \text{\# set } g(t) = f'(t) \Rightarrow g'(t) = f''(t) \\
&= s\mathcal{L}\{g(t)\} - g(0) && \text{\# } \mathcal{L}\{g(t)'\} = s\mathcal{L}\{g(t)\} - g(0) \text{ (Section 3.4)} \\
&= s\mathcal{L}\{f'(t)\} - f'(0) && \text{\# } g(t) = f'(t) \\
&= s\big[sF(s) - f(0)\big] - f'(0) && \text{\# } \mathcal{L}\{f(t)'\} = sF(s) - f(0) \text{ (Section 3.4)} \\
&= s^2F(s) - sf(0) - f'(0) && \text{\# simplify}
\end{aligned}$$

3.6 So what is the general form of $\mathcal{L}\{f^{(n)}(t)\}$?

The general form of the Laplace Transform of the n^{th} derivative of some function $f(t)$ is

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0) \tag{5}$$

Here f is assumed to be n-times differentiable and that it's n^{th} derivative, denoted $f^{(n)}$, is piecewise continuous and of exponential order. The result then follows by mathematical induction. Note that $f^{(0)}$, the 0^{th} derivative of f, is just f.

So for example, for the Laplace Transform of the second derivative ($n = 2$) of some function f Equation 5 tells us

$$\begin{aligned}
\mathcal{L}\{f^{(2)}(t)\} &= s^2F(s) - \sum_{k=1}^{2} s^{2-k}f^{(k-1)}(0) && \text{\# Equation 5 with } n = 2 \\
&= s^2F(s) - s^{2-1}f^{(1-1)}(0) - s^{2-2}f^{(2-1)}(0) && \text{\# expand terms} \\
&= s^2F(s) - s^1f^{(0)}(0) - s^0f^{(1)}(0) && \text{\# arithmetic} \\
&= s^2F(s) - sf(0) - f^{(1)}(0) && \text{\# } s^1 = s,\ f^{(0)} = f \text{ and } s^0 = 1 \\
&= s^2F(s) - sf(0) - f'(0) && \text{\# alternate notation (Section 3.5)}
\end{aligned}$$

3.7 What is the derivative of $F(s)$?

$$
\begin{aligned}
F'(s) &= \frac{d}{ds}F(s) && \text{\# switch to more a convenient notation} \\
&= \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt && \text{\# definition of the Laplace Transform (Definition 3.2)} \\
&= \int_0^\infty \frac{\partial}{\partial s}e^{-st}f(t)dt && \text{\# swap } \frac{d}{ds} \text{ with } \int \text{ by the Leibniz integral rule} \\
&= \int_0^\infty (-t)e^{-st}f(t)dt && \text{\# } \frac{\partial}{\partial s}e^{-st} = -te^{-st} \text{ by the chain rule} \\
&= \int_0^\infty e^{-st}(-t)f(t)dt && \text{\# rearrange} \\
&= \int_0^\infty e^{-st}g(t)dt && \text{\# set } g(t) = -tf(t) \\
&= \mathcal{L}\{g(t)\} && \text{\# definition of the Laplace Transform (Definition 3.2)} \\
&= \mathcal{L}\{-tf(t)\} && \text{\# } g(t) = -tf(t) \\
&= -\mathcal{L}\{tf(t)\} && \text{\# } -1 \text{ doesn't depend on } t \text{ (see Section 3.1)}
\end{aligned}
$$

So $F'(s) = -\mathcal{L}\{tf(t)\}$. We can also pretty easily see that in general $F^{(n)}(s) = (-1)^n\mathcal{L}\{t^n f(t)\}$.

3.8 Example: Solving Ordinary Differential Equations (ODEs)

Suppose we have the following Initial Value Problem (IVP) [4]:

$$y'' - y' - 2y = 0 \text{ with } y(0) = 1 \text{ and } y'(0) = 0 \tag{6}$$

We can use the Laplace Transform to solve this ODE. The basic idea is to take the Laplace Transform of both sides of Equation 6, set $Y(s) = \mathcal{L}\{y\}$, and then solve for $Y(s)$. Then we can find $y(t)$ by taking the Inverse Laplace Transform of $Y(s)$.

$$
\begin{array}{rcll}
y'' - y' - 2y & = & 0 & \text{\# ODE we want to solve (Equation 6)} \\
& \Rightarrow & \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - \mathcal{L}\{2y\} = \mathcal{L}\{0\} & \text{\# take the LT of both sides} \\
& \Rightarrow & \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0 & \text{\# } \mathcal{L}\{af(t)\} = a\mathcal{L}\{f(t)\} \text{ and } \mathcal{L}\{0\} = 0 \\
& \Rightarrow & \left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0 & \text{\# } \mathcal{L}\{y''\} = \ldots \text{ (Section 3.5)} \\
& \Rightarrow & \left[s^2Y(s) - sy(0) - y'(0)\right] - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0 & \text{\# set } Y(s) = \mathcal{L}\{y\} \\
& \Rightarrow & \left[s^2Y(s) - sy(0) - y'(0)\right] - \left[sY(s) - y(0)\right] - 2\mathcal{L}\{y\} = 0 & \text{\# } \mathcal{L}\{y'\} = sY(s) - y(0) \text{ (Section 3.4)} \\
& \Rightarrow & \left[s^2Y(s) - sy(0) - y'(0)\right] - \left[sY(s) - y(0)\right] - 2Y(s) = 0 & \text{\# } \mathcal{L}\{y\} = Y(s) \\
& \Rightarrow & s^2Y(s) - s\cdot 1 - 0 - sY(s) + 1 - 2Y(s) = 0 & \text{\# IVP: } y(0) = 1 \text{ and } y'(0) = 0 \\
& \Rightarrow & s^2Y(s) - s - sY(s) + 1 - 2Y(s) = 0 & \text{\# simplify} \\
& \Rightarrow & s^2Y(s) - sY(s) - 2Y(s) - (s-1) = 0 & \text{\# collect terms} \\
& \Rightarrow & Y(s)\left[s^2 - s - 2\right] - (s-1) = 0 & \text{\# factor out } Y(s) \\
& \Rightarrow & Y(s)\left[s^2 - s - 2\right] = s - 1 & \text{\# add } s-1 \text{ to both sides} \\
& \Rightarrow & Y(s) = \dfrac{s-1}{s^2 - s - 2} & \text{\# solve for } Y(s)
\end{array}
$$

So $Y(s) = \dfrac{s-1}{s^2-s-2}$. Now we can split $Y(s)$ using partial fractions and solve for $y(t)$ using the Inverse Laplace Transform:

$$
\begin{aligned}
Y(s) &= \frac{s-1}{s^2-s-2} && \text{\# see above}\\
&= \frac{s-1}{(s-2)(s+1)} && \text{\# factor denominator}\\
&\Rightarrow \frac{s-1}{(s-2)(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s+1)} && \text{\# use partial fractions}\\
&\Rightarrow s-1 = A(s+1) + B(s-2) && \text{\# multiply both sides by } (s-2)(s+1)\\
&\Rightarrow -1 = A - 2B && \text{\# coefficients of 1}\\
&\Rightarrow 1 = A + B && \text{\# coefficients of } s\\
&\Rightarrow A = 1 - B && \text{\# solve for } A \text{ in previous equation}\\
&\Rightarrow -1 = (1-B) - 2B && \text{\# plug } A = 1-B \text{ into } -1 = A - 2B\\
&\Rightarrow -2 = -3B && \text{\# simplfy}\\
&\Rightarrow B = \tfrac{2}{3} && \text{\# solve for } B\\
&\Rightarrow A = \tfrac{1}{3} && \text{\# plug } B \text{ into } A = 1 - B\\
&\Rightarrow Y(s) = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}
\end{aligned}
$$

So now we know that

$$Y(s) = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}$$

and we also know (Section 3.3) that

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

Putting this all together

$$
\begin{aligned}
Y(s) \quad &= \quad \mathcal{L}\{y(t)\} && \text{\# definition of } Y(s) \text{ (Section 3.8)} \\
&\Rightarrow \quad \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\mathcal{L}\{y(t)\}\} && \text{\# take the Inverse Laplace Transform of both sides} \\
&\Rightarrow \quad \mathcal{L}^{-1}\{Y(s)\} = y(t) && \text{\# } \mathcal{L}^{-1}\{\mathcal{L}\{y(t)\}\} = y(t) \\
&\Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}\right\} = y(t) && \text{\# } Y(s) = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1} \text{ (Section 3.8)} \\
&\Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{2}{3}}{s+1}\right\} = y(t) && \text{\# by the linearity of the Laplace Transform (Section 3.1)} \\
&\Rightarrow \quad \tfrac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \tfrac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = y(t) && \text{\# neither } \tfrac{1}{3} \text{ nor } \tfrac{2}{3} \text{ depends on } t \\
&\Rightarrow \quad \tfrac{1}{3}e^{2t} + \tfrac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = y(t) && \text{\# } \mathcal{L}^{-1}\left\{\tfrac{1}{s-2}\right\} = e^{2t} \text{ (Section 3.3)} \\
&\Rightarrow \quad \tfrac{1}{3}e^{2t} + \tfrac{2}{3}e^{-t} = y(t) && \text{\# } \mathcal{L}^{-1}\left\{\tfrac{1}{s+1}\right\} = e^{-t} \text{ (Section 3.3)} \\
&\Rightarrow \quad y(t) = \tfrac{1}{3}e^{2t} + \tfrac{2}{3}e^{-t} && \text{\# = is symmetric}
\end{aligned}
$$

So we've been able to use the Laplace Transform to solve the ODE in Section 3.8 (Equation 6):

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

4 The Heaviside Function, Convolution, and the Laplace Transform

We saw in Section 3.1 that the Laplace Transform has a linearity property, namely,

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

One natural question is whether there is a "linearity-like" property for multiplication. Unfortunately

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

However, convolution behaves better with respect to the Laplace Transform of a product. Specifically

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} \tag{7}$$

where $*$ is the convolution operator.

To see why Equation 7 holds we need another piece of machinery, the Heaviside function.

4.1 The Heaviside Function

The most basic definition of the Heaviside function is

Definition 4.1. Heaviside Function:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

The Heaviside function is shown in Figure 4.

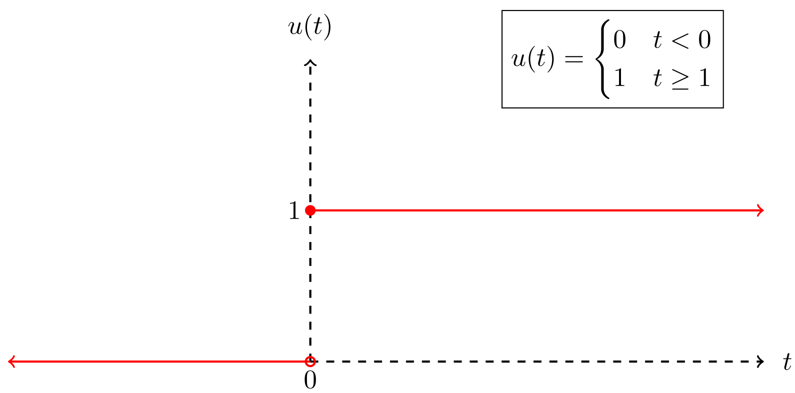

Figure 4: The Heaviside Function $u(t)$

More frequently we are interested in $u(t-a)$, which is sometimes written as $u_a(t)$, is shown in Figure 5.

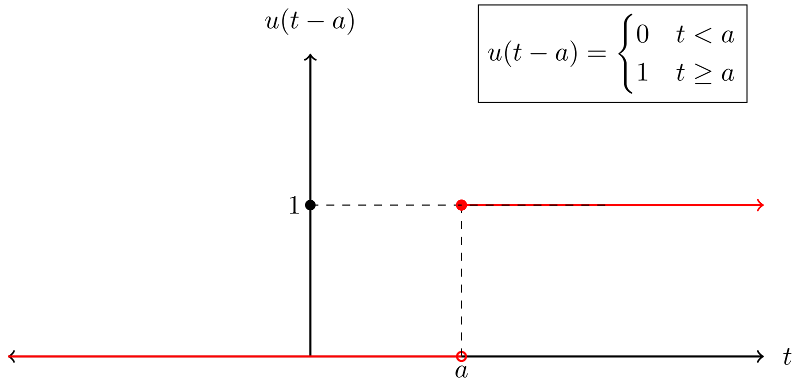

Figure 5: $u(t-a)$

Another useful version of the Heaviside function, $u(a-t)$, is shown in Figure 6.

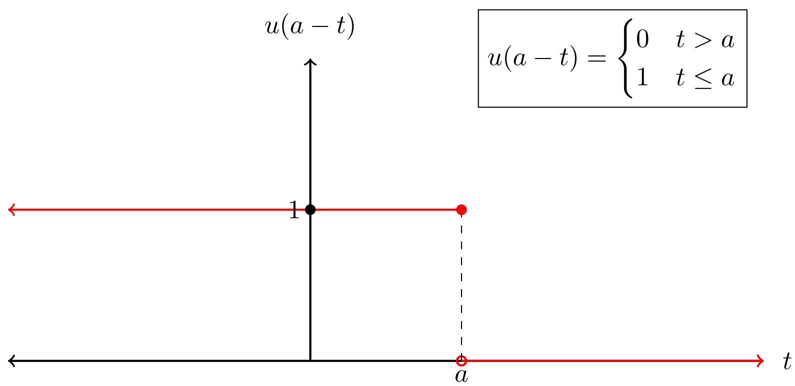

Figure 6: $u(a-t)$

Now the obvious question is how do we compute $\mathcal{L}\{u(t-a)\}$? Recall that by the definition of the Laplace Transform we have

$$F(s) = \int_0^\infty e^{-st} u(t-a) dt$$

The first thing to observe is that $u(t-a)$ is zero for $t < a$ and one otherwise (Figure 5), so we can rewrite our integral with a lower limit of a.

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st}u(t-a)dt && \text{\# definition of } \mathcal{L}\{u(t-a)\} \text{ (Section 3)} \\
&= \int_a^\infty e^{-st}u(t-a)dt && \text{\# we can change the lower integration limit since } u(t-a)=0 \text{ for } t<a \\
&= \int_a^\infty e^{-st}\cdot 1dt && \text{\# } u(t-a)=1 \text{ for } t\geq a \\
&= \int_a^\infty e^{-st}dt && \text{\# simplify} \\
&= \left.\frac{e^{-st}}{-s}\right|_a^\infty && \text{\# } \int e^{cx}dx = \tfrac{e^{cx}}{c}+C \text{ and the Fundamental Theorem of Calculus} \\
&= \lim_{d\to\infty}\frac{e^{-sd}}{-s}-\frac{e^{-sa}}{-s} && \text{\# evaluate at limits} \\
&= 0-\frac{e^{-sa}}{-s} && \text{\# } \lim_{d\to\infty}\frac{e^{-sd}}{-s}=0 \\
&= \frac{e^{-sa}}{s} && \text{\# } \mathcal{L}\{u(t-a)\}=\frac{e^{-sa}}{s}
\end{aligned}
$$

4.2 What about $\mathcal{L}\{u(t-a)f(t-a)\}$?

What can we say about the Laplace Transform of $u(t-a)f(t-a)$, that is, the Heaviside function multiplied by another function? It turns out that

$$e^{-as}F(s) = \mathcal{L}\{u(t-a)f(t-a)\}$$

Ok, but why? First, notice that

$$u(t-a)f(t-a) = \begin{cases} 0 & t<a \\ f(t-a) & t\geq a \end{cases}$$

One way to think about this is that $u(t-a)$ "turns on" $f(t-a)$ at a. So consider

$$
\begin{aligned}
F(s) \quad = \quad & \int_0^\infty e^{-s\tau} f(\tau) d\tau && \text{\# definition of Laplace Transform (Section 3)} \\
\Rightarrow \quad & e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau} f(\tau) d\tau && \text{\# multiply both sides by } e^{-as} \\
\Rightarrow \quad & e^{-as}F(s) = \int_0^\infty e^{-as} e^{-s\tau} f(\tau) d\tau && \text{\# move } e^{-as} \text{ inside integral (} e^{-as} \text{ doesn't depend on } \tau\text{)} \\
\Rightarrow \quad & e^{-as}F(s) = \int_0^\infty e^{-s\tau - sa} f(\tau) d\tau && \text{\# } a^n \cdot a^m = a^{(m+n)} \\
\Rightarrow \quad & e^{-as}F(s) = \int_0^\infty e^{-s(\tau + a)} f(\tau) d\tau && \text{\# factor out } s \\
\Rightarrow \quad & e^{-as}F(s) = \int_a^\infty e^{-st} f(t-a) dt && \text{\# substitution: } t = \tau + a \text{ so } \tau = t - a \text{ and } d\tau = dt \\
\Rightarrow \quad & e^{-as}F(s) = \int_0^\infty e^{-st} u(t-a) f(t-a) dt && \text{\# } u(t-a) = 0 \text{ for } t < a \text{ so adjust lower limit to 0} \\
\Rightarrow \quad & e^{-as}F(s) = \mathcal{L}\{u(t-a)f(t-a)\} && \text{\# definition of Laplace Transform (Section 3)}
\end{aligned}
$$

4.3 Convolution

Next we need the definition of the convolution of two functions f and g.

Definition 4.2. Convolution:

$$
f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau \tag{8}
$$

where $*$ is the convolution operator. Note that the righthand side of Equation 8 is some function of t (τ is the dummy variable for integration). That is

$$
f(t) * g(t) = \underbrace{\int_0^t f(\tau) g(t-\tau) d\tau}_{\text{some function of } t}
$$

Convolution has quite a few interesting properties including:

- Commutativity: $f * g = g * f$
- Associativity: $f * (g * h) = (f * g) * h$
- Distributivity: $f * (g + h) = f * g + f * h$

It is pretty easy to see why the commutative property holds:

$$\begin{aligned}
f(t) * g(t) &= \int_0^t f(\tau)g(t-\tau)d\tau && \text{\# definition of convolution (Section 4.3)} \\
&= -\int_0^t f(t-u)g(u)du && \text{\# use a u substitution with } u = t-\tau \text{ so } \tau = t-u \text{ and } d\tau = -du \\
&= -\int_t^0 f(t-u)g(u)du && \text{\# lower limit: } u = t-0 = t\text{, upper limit: } u = t-t = 0 \\
&= \int_0^t f(t-u)g(u)du && \text{\# by the Fundamental Theorem of Calculus} \\
&= \int_0^t g(u)f(t-u)du && \text{\# multiplication is commutative} \\
&= g(t) * f(t) && \text{\# definition of convolution (Section 4.3)}
\end{aligned}$$

4.4 Ok, why does $\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ hold?

Now we can use the machinery we've built up to show why Equation 7 holds:

$$
\begin{aligned}
\mathcal{L}\{f(t) * g(t)\} &= \int_0^\infty e^{-st}\big[f(t) * g(t)\big]dt && \text{\# definition of Laplace Transform (Definition 3.2)}\\
&= \int_0^\infty e^{-st}\left[\int_0^t f(t-v)g(v)dv\right]dt && \text{\# definition of convolution (Section 4.3)}\\
&= \int_0^\infty e^{-st}\left[\int_0^\infty f(t-v)g(v)u(t-v)dv\right]dt && \text{\# } u(t-v)=0 \text{ for } v>t \text{ so change upper limit}\\
&= \int_0^\infty \left[\int_0^\infty e^{-st}f(t-v)g(v)u(t-v)dv\right]dt && \text{\# } e^{-st} \text{ doesn't depend on } v\\
&= \int_0^\infty \left[\int_0^\infty e^{-st}f(t-v)g(v)u(t-v)dt\right]dv && \text{\# swap order of integration}\\
&= \int_0^\infty g(v)\left[\int_0^\infty e^{-st}f(t-v)u(t-v)dt\right]dv && \text{\# } g(v) \text{ doesn't depend on } t\\
&= \int_0^\infty g(v)\, e^{-sv}\, F(s)\, dv && \text{\# } \mathcal{L}\{f(t-v)u(t-v)\} = e^{-sv}F(s) \text{ (Section 4.2)}\\
&= F(s)\int_0^\infty g(v)\, e^{-sv}\, dv && \text{\# } F(s) \text{ doesn't depend on } v\\
&= F(s)\int_0^\infty e^{-sv}g(v)dv && \text{\# multiplication is commutative}\\
&= F(s)\, G(s) && \text{\# } G(s) = \int_0^\infty e^{-sv}g(v)dv \text{ (Definition 3.2)}\\
&\Rightarrow \mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} && \text{\# the LT of a convolution is the product of LTs}
\end{aligned}
$$

This result also implies that $\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t)$. It is pretty easy to see why:

$$
\begin{aligned}
\mathcal{L}\{f(t) * g(t)\} &= \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} && \text{\# above result}\\
&\Rightarrow \mathcal{L}^{-1}\{\mathcal{L}\{f(t) * g(t)\}\} = \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}\} && \text{\# take the inverse LT of both sides}\\
&\Rightarrow f(t) * g(t) = \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}\} && \text{\# } \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\} = f(t)\\
&\Rightarrow f(t) * g(t) = \mathcal{L}^{-1}\{F(s)G(s)\} && \text{\# } \mathcal{L}\{f(t)\} = F(s) \text{ (Definition 3.2)}\\
&\Rightarrow \mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) && \text{\# = is symmetric}
\end{aligned}
$$

Acknowledgements

Thanks to Dave Neary for catching a typo: was $-f(0) + \mathcal{L}\{f(t)\}$, should be $-f(0) + s\mathcal{L}\{f(t)\}$.

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