

Notes on policy gradients and the log derivative trick for reinforcement learning

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Last update: November 10, 2020

1 Introduction

The *log derivative trick*¹ is a widely used identity that allows us to find various gradients required for policy learning. For policy-based reinforcement learning, we directly parameterize the policy. In value-based learning, we imagine we have value function approximator (either state-value or action-value) parameterized by θ :

$$V_\theta(s) \approx V^\pi(s) \quad \# \text{ state-value approximation} \quad (1)$$

$$Q_\theta(s, a) \approx Q^\pi(s, a) \quad \# \text{ action-value approximation} \quad (2)$$

Here our goal is to directly parameterize the policy (i.e., *model-free reinforcement learning*):

$$\pi_\theta(s, a) = \mathbb{P}[a|s, \theta] \quad \# \text{ parameterized policy} \quad (3)$$

2 Policy Objective Functions

There are three basic policy objective functions, each of which has the goal of given a policy $\pi_\theta(s, a)$ with parameters θ , find the best θ .

- In episodic environments we can use the *start value*: $J_1(\theta) = V^{\pi_\theta}(s_1) = \mathbb{E}_{\pi_\theta}[v_1]$
- In continuing environments we can use the *average value*: $J_{av}(\theta) = \sum_s d^{\pi_\theta}(s) V^{\pi_\theta}(s)$

¹<http://blog.shakirm.com/2015/11/machine-learning-trick-of-the-day-5-log-derivative-trick>

- Or we can use the *average reward per time step*: $J_{avR}(\theta) = \sum_s d^{\pi_\theta}(s) \sum_a \pi_\theta(s, a) \mathcal{R}_s^a$

where $d^{\pi_\theta}(s)$ is the *stationary distribution* of a Markov chain for π_θ .

So now we're casting policy based reinforcement learning as an optimization problem (e.g., there is a neural network that we want to learn the policy, e.g., via gradient ascent). Now, let $J(\theta)$ be a *policy objective function*. A policy gradient algorithm searches for a local maximum² in $J(\theta)$ by ascending the gradient of the policy with respect to the parameters θ . The update rule for θ is

$$\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta) \quad (4)$$

where $\nabla_\theta J(\theta)$ is the *policy gradient* and α is the *learning rate* (sometimes step-size parameter). In particular

$$\nabla_\theta J(\theta) = \begin{bmatrix} \frac{\partial J(\theta)}{\partial \theta_1} \\ \frac{\partial J(\theta)}{\partial \theta_2} \\ \vdots \\ \frac{\partial J(\theta)}{\partial \theta_n} \end{bmatrix} \quad (5)$$

3 Computing the gradient analytically

First, we assume that the policy π_θ is differentiable wherever it is non-zero (this is a softer requirement than requiring π_θ be differentiable *everywhere*). In addition, we know the gradient: $\nabla_\theta J(\theta)$. In this case, let $p(\mathbf{x}; \theta)$ be the likelihood parametrized by θ and let $\log p(\mathbf{x}; \theta)$ be the *log likelihood*. Define

$$\begin{array}{ll} y &= p(\mathbf{x}; \theta) & \# \text{ define } y \text{ to be the likelihood parametrized by } \theta \\ z &= \log y = \log p(\mathbf{x}; \theta) & \# z \text{ is the log likelihood} \end{array}$$

Then

²in the case that $J(\theta)$ is *non-convex*.

$$\begin{aligned}
\frac{dz}{d\theta} &= \frac{dz}{dy} \cdot \frac{dy}{d\theta} && \# \text{ chain rule} \\
\frac{dz}{dy} &= \frac{1}{p(\mathbf{x}; \theta)} && \# \frac{d}{dx} \log x \approx \frac{1}{x} \\
\frac{dy}{d\theta} &= \frac{d}{d\theta} p(\mathbf{x}; \theta) = \nabla_{\theta} p(\mathbf{x}; \theta) && \# \text{ definition (chain rule, again)} \\
\frac{dz}{d\theta} &= \frac{dz}{dy} \cdot \frac{dy}{d\theta} = \frac{\nabla_{\theta} p(\mathbf{x}; \theta)}{p(\mathbf{x}; \theta)} && \# \text{ apply chain rule} \\
\frac{dz}{d\theta} &= \nabla_{\theta} \log p(\mathbf{x}; \theta) && \# \text{ set } w = p(\mathbf{x}; \theta) \text{ and note that } \frac{1}{w} \nabla_{\theta} w = \nabla_{\theta} \log w
\end{aligned}$$

Here $\nabla_{\theta} \log p(\mathbf{x}; \theta)$ is known as the score or sometimes the *Fischer* information. So the *log derivative trick* (sometimes *likelihood ratio*) is

$$\nabla_{\theta} \log p(\mathbf{x}; \theta) = \frac{\nabla_{\theta} p(\mathbf{x}; \theta)}{p(\mathbf{x}; \theta)}$$

Setting $\pi_{\theta}(s, a) = p(\mathbf{x}; \theta)$ we see that

$$\begin{aligned}
\nabla_{\theta} \pi_{\theta}(s, a) &= \nabla_{\theta} \pi_{\theta}(s, a) \cdot \left[\frac{\pi_{\theta}(s, a)}{\pi_{\theta}(s, a)} \right] && \# \text{ multiply by } 1 = \frac{\pi_{\theta}(s, a)}{\pi_{\theta}(s, a)} \\
&= \left[\frac{\nabla_{\theta} \pi_{\theta}(s, a)}{\pi_{\theta}(s, a)} \right] \cdot \pi_{\theta}(s, a) && \# \text{ multiplication is associative} \\
&= \nabla_{\theta} \log \pi_{\theta}(s, a) \cdot \pi_{\theta}(s, a) && \# \text{ log derivative trick: } \frac{\nabla_{\theta} \pi_{\theta}(s, a)}{\pi_{\theta}(s, a)} = \nabla_{\theta} \log \pi_{\theta}(s, a)
\end{aligned}$$

and the score function is $\nabla_{\theta} \log \pi_{\theta}(s, a)$.

Another yet similar way to look policy gradients is as follows:³ First, as Andrej points out policy gradients are a special case of a more general *score function gradient estimator*. Here we have an expectation of the form $E_{x \sim p(x|\theta)}[f(x)]$. This is the expectation of a scalar valued function $f(x)$ under some probability distribution $p(x|\theta)$. Here $f(x)$ can be thought of as a *reward function* and $p(x|\theta)$ is the *policy* network.

The problem we want to solve is how we should shift the distribution (though its parameters θ) to increase its score as judged by f . Since the general gradient decent update rule is

$$\theta \leftarrow \theta + \alpha \nabla_{\theta} J(\theta) \tag{6}$$

³h/t Andrej Karpathy <http://karpathy.github.io/2016/05/31/rl/>

our goal is to find $\nabla_{\theta} E_{x \sim p(x|\theta)}[f(x)]$ and update θ in the direction indicated by the gradient. However, what we know is f and p .

$$\begin{aligned}
\nabla_{\theta} E_{x \sim p(x|\theta)}[f(x)] &= \nabla_{\theta} \sum_x p(x) f(x) && \# \text{ defn expectation} \\
&= \sum_x \nabla_{\theta} p(x) f(x) && \# \text{ swap sum and gradient} \\
&= \sum_x p(x) \frac{\nabla_{\theta} p(x)}{p(x)} f(x) && \# \text{ multiply/divide by } p(x) \\
&= \sum_x p(x) \nabla_{\theta} \log p(x) f(x) && \# \frac{1}{z} \nabla_{\theta} z = \nabla_{\theta} \log(z) \\
&= E_{x \sim p(x|\theta)}[f(x) \nabla_{\theta} \log p(x)] && \# \text{ defn expectation again}
\end{aligned}$$

The basic idea here is that we have some distribution $p(x|\theta)$ which we can sample from,⁴ and for each sample we evaluate its score $f(x)$; then the gradient

$$\nabla_{\theta} E_{x \sim p(x|\theta)}[f(x)] = E_{x \sim p(x|\theta)}[f(x) \nabla_{\theta} \log p(x)] \quad (7)$$

is telling us how we should shift the distribution (through its parameters θ) if we wanted its samples to achieve higher scores (as judged by f). The second term $\nabla_{\theta} \log p(x)$ is telling us which direction in parameter space would lead to increase of the probability assigned to a given x . That is, if we were to move θ in the direction $\nabla_{\theta} \log p(x)$ we would see the new probability assigned to some x slightly increase (see Equation 6).

4 The Policy Gradient Theorem

Recall that the the start value policy for episodic environments was

$$J_1(\theta) = V^{\pi_{\theta}}(s_1) = \mathbb{E}_{\pi_{\theta}}[v_1] = \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_{\theta}(s, a) \mathcal{R}_s^a \quad (8)$$

and thus

$$\nabla_{\theta} J(\theta) = \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_{\theta}(s, a) \nabla_{\theta} \log \pi_{\theta}(s, a) \mathcal{R}_s^a \quad (9)$$

$$= \mathbb{E}_{\pi_{\theta}}[\nabla_{\theta} \log \pi_{\theta}(s, a) r] \quad (10)$$

⁴for example, $p(x|\theta)$ could be a Gaussian

A few things to notice:

- The policy gradient theorem generalizes the likelihood ratio
- In the policy gradient theorem we replace r with the long-term value $Q^\pi(s, a)$ (our estimate of r).
- The policy gradient theorem applies to the start-state, average reward and average value objectives

For any differentiable policy $\pi_\theta(s, a)$ and for any policy objective J_1 , J_{avR} or $\frac{1}{1-\gamma}J_{avV}$, the **policy gradient** is:

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta} \left[\nabla_\theta \log \pi_\theta(s, a) Q^{\pi_\theta}(s, a) \right] \quad (11)$$

Now that we have $\nabla_\theta J(\theta)$, we can use this gradient to train a neural network (e.g., the policy networks of AlphaGo).