A Few Notes On The Riemann Zeta Function

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1 Introduction

TBD

2 Euler's Product Formula

Recall that the Riemann zeta function is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

In 1737, Leonhard Euler discovered the beautiful connection between the zeta function and the prime numbers and proved this identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \tag{1}$$

The left side of Equation 1 is by definition $\zeta(s)$. The infinite product on the right side of Equation 1 extends over all prime numbers p and is called an Euler Product [1], which is the expansion of a Dirichlet series into an infinite product indexed by prime numbers:

$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \frac{1}{1 - 2^{-s}} \cdot \frac{1}{1 - 3^{-s}} \cdot \frac{1}{1 - 5^{-s}} \cdot \frac{1}{1 - 7^{-s}} \cdot \frac{1}{1 - 11^{-s}} \cdots \frac{1}{1 - p^{-s}} \cdots$$

Both sides of the Euler product formula converge for $\text{Re}(s) > 1$.

References

[1] Noam D. Elkies. Math 259: Introduction to Analytic Number Theory. `http://abel.math.harvard.edu/~elkies/M259.06/euler.pdf`, 1998. [Online; accessed 25-June-2021].