A Few Notes On The Dirac Delta Function

David Meyer

dmm@{1-4-5.net,uoregon.edu,...}

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1 Introduction

These notes began life as some thoughts on the Dirac Delta Function and evolved into notes on several related topics including Laplace Transforms. The Dirac Delta function has all kinds of crazy and interesting properties. More TBD.

2 The Dirac Delta Function

The Dirac Delta Function is defined as shown in Figure 1. In the limit ($\epsilon \to 0$) the Dirac Delta function is written $\delta_a(t)$ or sometimes $\delta(t-a)$. As we will see in a moment, the $\delta_{a,\epsilon}(t)$ form of the delta function is useful when we want to use the Mean Value Theorem for Integrals [2] to evaluate integrals involving the delta function.

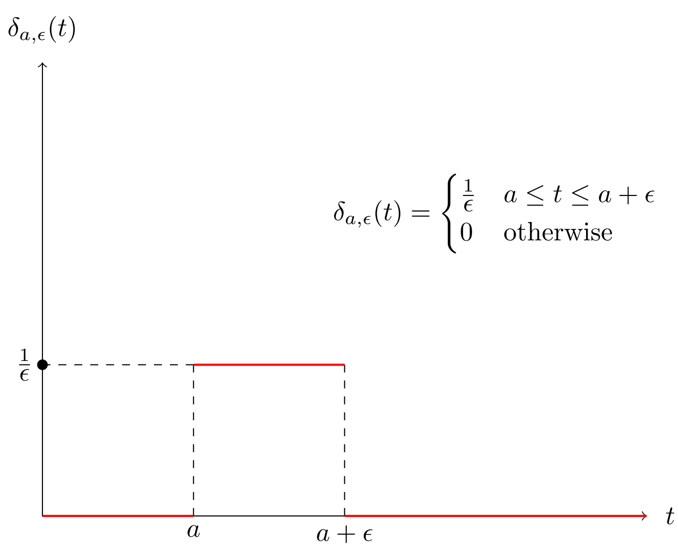

Figure 1: The Dirac Delta Function $\delta_{a,\epsilon}(t)$

So $\delta_{a,\epsilon}(t)$ is defined to be

$$\delta_{a,\epsilon}(t) = \begin{cases} \frac{1}{\epsilon} & a \leq t \leq a + \epsilon \\ 0 & \text{otherwise} \end{cases}$$

and has the constraint that

$$\int_0^\infty \delta_{a,\epsilon}(t) = 1$$

That is, $\delta_{a,\epsilon}(t)$ is in some sense a probability density.

In the limit the Dirac Delta Function looks like

$$\lim_{\epsilon \to 0} \delta_{a,\epsilon}(t) = \delta_a(t) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$$

or sometimes

$$\delta(t - a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$$

$\delta_a(t)$ also has the constraint that

$$\int_0^\infty \delta_a(t) = 1$$

and so is also a probability density. $\delta_a(t)$ is shown in Figure 2.

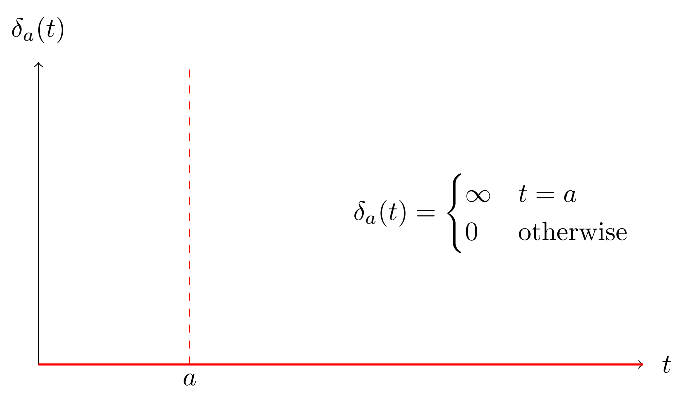

Figure 2: The Dirac Delta Function $\delta_a(t)$

2.1 Integrals Involving $\delta_{a,\epsilon}(t)$

$\delta_{a,\epsilon}(t)$ has all kinds of interesting properties. One of them involves the integral of the product $\delta_{a,\epsilon}(t)$ with some function $g(t)$. Here we would like to evaluate integrals of the form

$$\int_0^\infty \delta_{a,\epsilon}(t)g(t)dt \tag{1}$$

where $g(t)$ is continuous on the interval $[a, a+\epsilon]$.

The Mean Value Theorem for Integrals [2] tells us that

$$\int_a^b g(t)dt = (b-a)g(c) \tag{2}$$

where the point c lies in the interval $[a, a+\epsilon]$. Now, since we know that $\delta_{a,\epsilon}(t)$ is zero everywhere except on the interval $[a, a+\epsilon]$ we can rewrite the improper integral in Equation 1 as the proper integral

$$\int_a^{a+\epsilon} \delta_{a,\epsilon}(t)g(t)dt$$

Here we can notice that $\delta_{a,\epsilon}(t) = \frac{1}{\epsilon}$ on the interval $[a, a+\epsilon]$ so we can rewrite our integral as

$$\int_a^{a+\epsilon} \frac{1}{\epsilon} g(t)dt = \frac{1}{\epsilon}\int_a^{a+\epsilon} g(t)dt$$

Now we can use Equation 2, the Mean Value Theorem for Integrals[1], by setting $b = a+\epsilon$ and $a = a$ so that $b - a = \epsilon$. Then

$$\frac{1}{\epsilon}\int_a^{a+\epsilon} g(t)dt = \frac{1}{\epsilon}\underbrace{\Big[(a+\epsilon)-a\Big]}_{b-a} g(c) = \frac{1}{\epsilon}\epsilon g(c) = g(c)$$

where $c \in [a, a+\epsilon]$.

Finally, if we look at what happens to c as $\epsilon \to 0$ we see that $\lim\limits_{\epsilon\to 0} c = a$ (sorry about the notation abuse) so that

$$\int_0^\infty \delta_a(t)g(t)dt = g(a) \tag{3}$$

[1] This is where the $\delta_{a,\epsilon}(t)$ form of the delta function comes in handy.

Essentially $\delta_a(t)$ pulls out the value of g at a, that is, $g(a)$.

Another way to get this result [1] is to notice that the integrand of

$$\int_0^\infty \delta(t-a)g(t)dt$$

is zero everywhere except where $t = a$, so we can rewrite our integral as $\int_0^\infty \delta(t-a)g(a)dt = g(a)\int_0^\infty \delta(t-a)dt$ (since $g(a)$ doesn't depend on t). Then since by definition $\int_0^\infty \delta(t-a)dt = 1$ we get

$$\int_0^\infty \delta(t-a)g(t)dt = g(a)$$

3 The Laplace Transform

We start by defining the integral transform of some function $f(t)$.

Definition 3.1. Integral Transform

If a function $f(t)$ is defined on $[0,\infty)$ then we can define an integral transform to be the improper integral

$$F(s) = \int_0^\infty K(s,t)f(t)dt$$

If the improper integral converges then we say that $F(s)$ is the *integral transform* of $f(t)$. The function $K(s,t)$ is called the *kernel* of the transform. When $K(s,t) = e^{-st}$ the transform is called the **Laplace Transform**.

Definition 3.2. Laplace Transform

The Laplace Transform of a function $f(t)$ is defined to be

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)dt \tag{4}$$

and is useful when solving ordinary differential equations (ODEs). Interestingly, the Laplace Transform of the Dirac Delta Function turns out to be

$$\begin{aligned}
\mathcal{L}\{\delta_a(t)\} &= \int_0^\infty e^{-st}\delta_a(t)dt && \text{\# definition (Equation 4 with } f(t) = \delta_a(t)) \\
&= \int_0^\infty g(t)\delta_a(t)dt && \text{\# set } g(t) = e^{-st} \\
&= g(a) && \text{\# by Equation 3} \\
&= e^{-sa} && \text{\# since } g(t) = e^{-st}
\end{aligned}$$

3.1 The Linearity Property Of The Laplace Transform

$\mathcal{L}$ is a linear operator, in other words: $\mathcal{L}\{af(t)+bg(t)\}=a\mathcal{L}\{f(t)\}+b\mathcal{L}\{g(t)\}$. Why? Consider:

$$\begin{aligned}
\mathcal{L}\{af(t)+bg(t)\} &= \int_0^\infty e^{-st}\big[af(t)+bg(t)\big]dt && \text{\# definition of the Laplace Transform (Definition 3.2)}\\
&= \int_0^\infty e^{-st}af(t)dt+\int_0^\infty e^{-st}bg(t)dt && \text{\# by the linearity of improper integrals [3]}\\
&= a\int_0^\infty e^{-st}f(t)dt+b\int_0^\infty e^{-st}g(t)dt && \text{\# constant multiple rule: } \textstyle\int kf(x)dx=k\int f(x)dx\\
&= a\mathcal{L}\{f(t)\}+b\mathcal{L}\{g(t)\} && \text{\# definition of the Laplace Transform (Definition 3.2)}
\end{aligned}$$

One useful rule we can pull from the above is the constant rule for Laplace Transforms:

Definition 3.3. Constant Rule: If a is a constant then $\mathcal{L}\{af(t)\}=a\mathcal{L}\{f(t)\}$.

It is pretty easy to see why the constant rule holds:

$$\begin{aligned}
\mathcal{L}\{af(t)\} &= \int_0^\infty e^{-st}af(t)dt && \text{\# definition of the Laplace Transform (Definition 3.2)}\\
&= a\int_0^\infty e^{-st}f(t)dt && \text{\# } a \text{ doesn't depend on } t\\
&= a\mathcal{L}\{f(t)\} && \text{\# definition of the Laplace Transform (Definition 3.2)}
\end{aligned}$$

3.2 So Does Every Function Have A Laplace Transform?

The answer is no (consider a function like $f(t)=t^{-1}$). OK, then what are the properties that $f(t)$ must have in order to have a Laplace Transform? First, $f(t)$ must be of ”exponential order”.

Definition 3.4. Exponential Order

A function f is said to be of exponential order c if there exist constants $c, M,$ and $T>0$ such that $|f(t)|\le Me^{ct}$ for all $t>T$.

Said another way, in order for $f(t)$ to be of exponential order c we require that

$$\lim_{t\to\infty}\frac{f(t)}{e^{ct}}=0$$

Basically this is saying that in order for $f(t)$ to have a Laplace Transform then in a race between $|f(t)|$ and e^{ct} as $t\to\infty$ e^{ct} must approach its limit first. This situation is depicted in Figure 3.

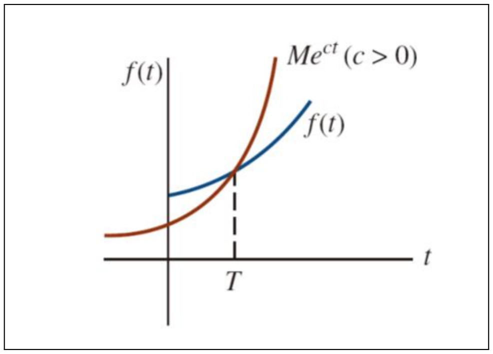

Figure 3: $f(t)$ is of exponential order with constants c, M and T

Next we need the following theorem:

Theorem 3.1. Existence Theorem for Laplace Transforms: If f is s piecewise continuous on the interval $[0, \infty)$ and is of exponential order c then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

Ok, but why? Consider

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= F(s) && \text{\# definition of the Laplace Transform (Definition 3.2)} \\
&= \int_0^\infty e^{-st} f(t)dt && \text{\# definition of the Laplace Transform (Definition 3.2)} \\
&\leq \int_0^\infty e^{-st} M e^{ct} dt && \text{\# } f \text{ is of exponential order } c \text{ (Definition 3.4)} \\
&= M\int_0^\infty e^{-st} e^{ct} dt && \text{\# } M \text{ doesn't depend on } t \\
&= M\int_0^\infty e^{-st+ct} dt && \text{\# } x^n \cdot x^m = x^{n+m} \\
&= M\int_0^\infty e^{(c-s)t} dt && \text{\# } e^{-st+ct} = e^{ct-st} = e^{(c-s)t} \\
&= M\int_0^\infty e^{u} dt && \text{\# use a } u \text{ substitution with } u = (c-s)t \\
&= M\int_0^\infty e^{u} \tfrac{du}{c-s} && \text{\# } u = (c-s)t \Rightarrow du = (c-s)dt \Rightarrow dt = \frac{du}{c-s} \\
&= \left[\tfrac{M}{c-s}\right] \int_0^\infty e^{u} du && \text{\# neither } c \text{ nor } s \text{ depends on } t \\
&= \left[\tfrac{M}{c-s}\right] e^{u}\Big|_0^\infty && \text{\# } \textstyle\int_0^\infty e^u du = e^u + C \text{ and the Fundamental Theorem of Calculus} \\
&= \left[\tfrac{M}{c-s}\right] e^{(c-s)t}\Big|_0^\infty && \text{\# } u = (c-s)t \\
&= \lim_{d\to\infty} \left[\tfrac{M}{c-s} e^{(c-s)d}\right] - \tfrac{M}{c-s} e^{(c-s)0} && \text{\# evaluate at the limits} \\
&= \lim_{d\to\infty} \left[\tfrac{M}{c-s} e^{(c-s)d}\right] - \tfrac{M}{c-s} && \text{\# } e^{(c-s)0} = e^0 = 1 \text{ and } \tfrac{M}{c-s} \cdot 1 = \tfrac{M}{c-s} \\
&= 0 - \tfrac{M}{c-s} && \text{\# } s > c \Rightarrow c - s < 0 \Rightarrow \lim_{d\to\infty} \left[\tfrac{M}{c-s} e^{(c-s)d}\right] = 0 \\
&= \tfrac{M}{s-c} && \text{\# simplify}
\end{aligned}
$$

So if $s = c$ then $\frac{M}{s-c}$ is not defined and $\mathcal{L}\{f(t)\}$ does not exist. Similarly, if $s < c$ then $\lim_{d\to\infty} \left[\frac{M}{c-s} e^{(c-s)d}\right]$ does not converge and $\mathcal{L}\{f(t)\}$ does not exist. All of this implies that functions that do not satisfy the conditions of the Existence Theorem do not have Laplace Transforms.

3.3 Inverse Laplace Transform

Definition 3.5. Inverse Laplace Transform: If $F(s)$ represents the Laplace Transform of a function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$ then the Inverse Laplace Transform of $F(s)$ is $f(t)$, i.e. $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Here are a few examples:

Laplace Transform	
$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\dfrac{dg}{dt}$	$sG(s) - g(0)$

Inverse Laplace Transform	
$F(s)$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$
$\frac{1}{s}$	1
$\frac{n!}{s^{n+1}}$	t^n
$\frac{1}{s-a}$	e^{at}
$\frac{k}{s^2+k^2}$	$\sin(kt)$
$\frac{s}{s^2+k^2}$	$\cos(kt)$
$\frac{k}{s^2-k^2}$	$\sinh(kt)$
$\frac{s}{s^2-k^2}$	$\cosh(kt)$
$sG(s) - g(0)$	$\dfrac{dg}{dt}$

Here's a worked example: Suppose $f(t)$ is continuous, piecewise smooth and of exponential order, and suppose $f'(t)$ is the derivative of $f(t)$. Then the Laplace Transform of $f'(t)$ turns out to be

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)dt && \text{\# Definition 3.2 (Laplace Transform)} \\
&= \int_0^\infty \underbrace{e^{-st}}_{v} \underbrace{f'(t)dt}_{du} && \text{\# use integration by parts} \\
&= \underbrace{e^{-st}}_{v} \underbrace{f(t)}_{u} \Big|_0^\infty - \int_0^\infty \underbrace{(-s)e^{-st}}_{dv} \underbrace{f(t)}_{u} dt && \# \int_0^\infty v\, du = uv - \int_0^\infty u\, dv \\
&= \lim_{d\to\infty} \Big[e^{-sd} f(d)\Big] - e^{-s\cdot 0} \cdot f(0) - \int_0^\infty (-s)e^{-st} f(t)dt && \text{\# evaluate first term at the limits} \\
&= 0 - e^{-s\cdot 0} \cdot f(0) - \int_0^\infty (-s)e^{-st} f(t)dt && \# \lim_{d\to\infty} \Big[e^{-sd} f(d)\Big] = 0 \\
&= 0 - f(0) - \int_0^\infty (-s)e^{-st} f(t)dt && \# e^{-s\cdot 0} = e^0 = 1 \text{ and } 1 \cdot f(0) = f(0) \\
&= -f(0) + s\int_0^\infty e^{-st} f(t)dt && \text{\# simplify, } s \text{ doesn't depend on } t \\
&= -f(0) + sF(s) && \text{\# Definition 3.2 (Laplace Transform)} \\
&= -f(0) + s\mathcal{L}\{f(t)\} && \# \mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f(t)\}
\end{aligned}$$

One of the interesting things to note here is that by using the Laplace Transform we've taken a statement about the derivative of f, namely $\mathcal{L}\{f'(t)\}$, and converted it into a statement about f itself: $-f(0) + s\mathcal{L}\{f(t)\}$. That is, we've converted a differential equation into an algebraic one. This will come in handy later when we want to solve differential equations.

3.4 What is the Derivative of $F(s)$?

$$
\begin{aligned}
F'(s) &= \frac{d}{ds}F(s) && \text{\# switch to more a convenient notation} \\
&= \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt && \text{\# Definition 3.2 (Laplace Transform)} \\
&= \int_0^\infty \frac{\partial}{\partial s}e^{-st}f(t)dt && \text{\# swap } \frac{d}{ds} \text{ with } \int \text{ by the Leibniz integral rule} \\
&= \int_0^\infty (-t)e^{-st}f(t)dt && \text{\# } \frac{\partial}{\partial s}e^{-st} = -te^{-st} \text{ by the chain rule} \\
&= \int_0^\infty e^{-st}(-t)f(t)dt && \text{\# rearrange} \\
&= \int_0^\infty e^{-st}g(t)dt && \text{\# set } g(t) = -tf(t) \\
&= \mathcal{L}\{g(t)\} && \text{\# Definition 3.2 (Laplace Transform)} \\
&= \mathcal{L}\{-tf(t)\} && \text{\# } g(t) = -tf(t) \\
&= -\mathcal{L}\{tf(t)\} && \text{\# by the Constant Rule with } a = -1 \text{ (Definition 3.3)}
\end{aligned}
$$

So $F'(s) = -\mathcal{L}\{tf(t)\}$. We can also pretty easily see that in general $F^{(n)}(s) = (-1)^n\mathcal{L}\{t^n f(t)\}$.

Acknowledgements

Thanks to Dave Neary for catching a typo: was $-f(0) + \mathcal{L}\{f(t)\}$, should be $-f(0) + s\mathcal{L}\{f(t)\}$.

References

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