A Few Notes on the Bloch Sphere

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1 Introduction

The fundamental building block of classical computational devices is the two-state system. We typically think of these systems as being built from bits which can take values in $\{0, 1\}$. Quantum mechanics, on the other hand, tells us that any such system can exist in a superposition of states. A *qubit* is a quantum system in which the Boolean states 0 and 1 are represented[1] by a prescribed pair of normalized and mutually orthogonal quantum states labeled as $\{|0\rangle, |1\rangle\}$.

More specifically, a qubit is defined as a 2-dimensional Hilbert space H_2 and we label an orthonormal basis of H_2 by $\{|0\rangle, |1\rangle\}$. The state of the qubit is an associated unit length vector in H_2. If a state is equal to a basis vector then we say it is a *pure* state. If a state is any other linear combination of the basis vectors we say it is a *mixed* state, or that the state is a superposition of $|0\rangle$ and $|1\rangle$.

The state of a qubit bit, $|\psi\rangle$, is described by

$$|\psi\rangle = \alpha\,|0\rangle + \beta|\,|1\rangle$$

where $\alpha, \beta \in \mathbb{C}$, $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$, $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$ and $|\alpha|^2 + |\beta|^2 = 1$.

Here $|x|^2 = xx^*$, where x^* is the complex conjugate of x [8]. The complex plane is shown in Figure 1. Note: the Wikipedia uses "bar" ($\bar{x}$) to denote the complex conjugate as opposed to "star" (x^*).

[1] I use Dirac or braket notion [6] in this document.

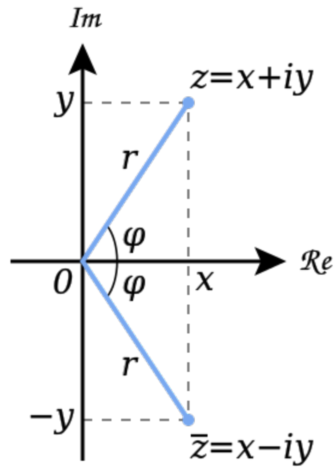

Figure 1: The Complex Plane (Image courtesy Wikipedia [9])

More generally, a *quantum system* is an n-dimensional Hilbert Space H_n. We label an orthonormal basis on H_n by $\{|x_1\rangle, |x_2\rangle, \ldots, |x_n\rangle\}$ such that $x_i \in X$ for some finite X. The associated state of the system is a unit length vector in H_n. If the state S is $|\phi\rangle$ at some moment in time, then we say that S has state $|\phi\rangle$ or S is in state $|\phi\rangle$.

We can also see that a general form for the state of a quantum system is

$$\sum_{i=1}^{n} \alpha_i |x_i\rangle \text{ where } \alpha_i \in \mathbb{C} \text{ and } \sum_{i=1}^{n} |\alpha_i|^2 = 1$$

Remark: We can already see an important technical problem that must be solved if quantum computers are ever to be practical. A superposition of states $|0\rangle$ and $|1\rangle$ is known as a *coherent* state. A coherent state is extremely "unstable"; it inevitably interacts with its environment and collapses into a *pure* state (see discussion below). This process is known as *decoherence*. Algorithms that exploit quantum effects such as superposition are known as quantum algorithms. To apply these quantum algorithms in the real world, the *decoherence time* must be longer than the time to run the algorithm. As a result methods for increasing the decoherence time are of great interest (see, for example [4]).

2 The Bloch Sphere

One question that we might ask, given the complex plane representation of the state of a qubit, is "how does the real, 3-dimensional space in which we live correspond to the 2-

dimensional representation of a qubit's state (as represented in the complex plane)"? The *Bloch Sphere* gives us an elegant way to think about this question.

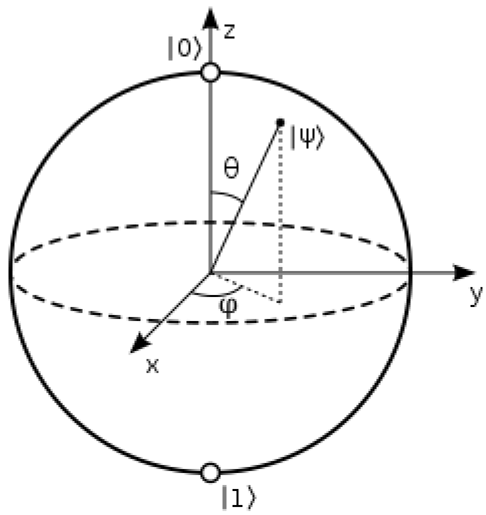

Figure 2: The Bloch Sphere (Image courtesy Wikipedia [7])

The Bloch Sphere, shown in Figure 2, is named for the physicist Felix Bloch [1] and gives us a beautiful way to think about and visualize the state of a qubit in 3-dimensional space.

In the Bloch Sphere the *pure* state of a qubit $|\psi\rangle$ is represented as a point on the surface of the sphere. *Mixed* states are represented as points in the interior of the sphere. Somewhat surprisingly, specification of a point on the Bloch Sphere requires only two parameters, θ and ϕ. This itself is remarkable; you might imagine that describing a vector in 3-dimensional space might take three or more parameters.

The representation of a classical bits on the Bloch Sphere is given by the poles of the sphere. The representation of the probabilistic classical bit, that is, a bit that is 0 with probability p and 1 with probability $1 - p$, is given by the point in z-axis with coordinate $2p - 1$. The interior of the Bloch Sphere is used to describe the states of a qubit in the presence of decoherence.

The standard way to write the state of a qubit using the Bloch Sphere is

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle \tag{1}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. See Section 2.3 for a derivation of this equation.

2.1 Bloch Sphere Coordinate System

Figure 2 shows the coordinate system of the Bloch Sphere. Consider the x axis. Here we can see that

$$
\begin{aligned}
& x_+ : \theta = \frac{\pi}{2}, \phi = 0 \longrightarrow \\
& \cos\frac{\pi}{4}\left|0\right\rangle + e^{i\cdot 0}\sin\frac{\pi}{4}\left|1\right\rangle && \# \; e^{i\cdot 0} = e^0 = 1 \\
& = \frac{1}{\sqrt{2}}\left|0\right\rangle + (1)\frac{1}{\sqrt{2}}\left|1\right\rangle \\
& = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle && \# \; H\left|0\right\rangle = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle \\
& \equiv \left|+\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
& x_- : \theta = \frac{\pi}{2}, \phi = \pi \longrightarrow \\
& \cos\frac{\pi}{4}\left|0\right\rangle + e^{i\pi}\sin\frac{\pi}{4}\left|1\right\rangle && \# \text{ Euler's Identity [11]: } e^{i\pi} + 1 = 0, \; e^{i\pi} = -1 \\
& = \frac{1}{\sqrt{2}}\left|0\right\rangle + (-1)\frac{1}{\sqrt{2}}\left|1\right\rangle \\
& = \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle && \# \; H\left|1\right\rangle = \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle \\
& \equiv \left|-\right\rangle
\end{aligned}
$$

On the y axis of the Bloch Sphere we can see that

$$
\begin{aligned}
& y_+ : \theta = \frac{\pi}{2}, \phi = \frac{\pi}{2} \longrightarrow \\
& \cos\frac{\pi}{4}\left|0\right\rangle + e^{i\frac{\pi}{2}}\sin\frac{\pi}{4}\left|1\right\rangle && \# \text{ Euler's Formula [10]: } e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i\cdot 1 = i \\
& = \frac{1}{\sqrt{2}}\left|0\right\rangle + i\frac{1}{\sqrt{2}}\left|1\right\rangle \\
& = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{i}{\sqrt{2}}\left|1\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&y_{-} : \theta = \frac{\pi}{2}, \phi = \frac{3\pi}{2} \longrightarrow \\
&\cos\frac{\pi}{4}\left|0\right\rangle + e^{i\frac{3\pi}{2}}\sin\frac{\pi}{4}\left|1\right\rangle && \# \; e^{i\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 - i\cdot 1 = -i \\
&= \frac{1}{\sqrt{2}}\left|0\right\rangle - i\frac{1}{\sqrt{2}}\left|1\right\rangle \\
&= \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{i}{\sqrt{2}}\left|1\right\rangle
\end{aligned}
$$

Finally, on the z axis of the Bloch Sphere we can see that

$$
\begin{aligned}
&z_{+} : \theta = 0, \phi = 0 \longrightarrow \\
&\cos\frac{0}{2}\left|0\right\rangle + e^{i0}\sin\frac{0}{2}\left|1\right\rangle \\
&= \cos 0\left|0\right\rangle + (1)\sin 0\left|1\right\rangle \\
&= 1\left|0\right\rangle + 0\left|1\right\rangle \\
&= \left|0\right\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
&z_{-} : \theta = \pi, \phi = 0 \longrightarrow \\
&\cos\frac{\pi}{2}\left|0\right\rangle + e^{i0}\sin\frac{\pi}{2}\left|1\right\rangle \\
&= \cos 0\left|0\right\rangle + (1)\sin\frac{\pi}{2}\left|1\right\rangle \\
&= 0\left|0\right\rangle + 1\left|1\right\rangle \\
&= \left|1\right\rangle
\end{aligned}
$$

So we can write the state of a qubit in many different ways. Perhaps the most generic way to write $|\psi\rangle$ is

$$|\psi\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$$

The maximally superimposed state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle \tag{2}$$

Here the probability of measuring $|0\rangle$ is $|a|^2 = aa^* = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$. Similarly, and the probability of measuring $|1\rangle$ is $\frac{1}{2}$.

Now, note that

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle \tag{3}$$

Why does the second equality in Equation 3 hold? In short its because $|\frac{e^{i\theta}}{\sqrt{2}}|^2 = |\frac{1}{\sqrt{2}}|^2$. But still why? Consider $|e^{i\theta}|^2$. By Euler's Formula[2]

$$\begin{aligned}
|e^{i\theta}|^2 &= |(\cos\theta + i\sin\theta)|^2 \\
&= (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) \\
&= \cos^2\theta - \cos\theta\, i\sin\theta + i\sin\theta\cos\theta - i^2\sin^2\theta \\
&= \cos^2\theta - i^2\sin^2\theta \\
&= \cos^2\theta + \sin^2\theta \\
&= 1
\end{aligned}$$

So $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle$. This implies that the statistics of any measurements we could perform on the state $e^{i\theta}|\psi\rangle$ would be exactly the same as they would be for the state $|\psi\rangle$. This explains the claim in Section 2.3.1 that global phases have no physical significance.

We have seen that measuring $|\psi\rangle$ in the standard basis ($\{|0\rangle, |1\rangle\}$) yields a probability of $\frac{1}{2}$ for measuring either $|0\rangle$ or $|1\rangle$. Is there any measurement that yields information about the phase θ?

Consider a measurement in a different basis, the $\{|+\rangle, |-\rangle\}$ basis. As we saw above, $|+\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. What does $|\psi\rangle$ look like in this new basis? The basic approach is to first represent $|0\rangle$ and $|1\rangle$ in the new basis. Here $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$. Then

[2] $e^{i\theta} = \cos\theta + i\sin\theta$

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle \\
&= \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}\big(|+\rangle + |-\rangle\big) + \frac{e^{i\theta}}{\sqrt{2}}\big(|+\rangle - |-\rangle\big)\Big) \\
&= \frac{1}{2}\big(|+\rangle + |-\rangle\big) + \frac{e^{i\theta}}{2}\big(|+\rangle - |-\rangle\big) \\
&= \frac{1+e^{i\theta}}{2}|+\rangle + \frac{1-e^{i\theta}}{2}|-\rangle
\end{aligned}
$$

Since $e^{i\theta} = \cos\theta + i\sin\theta$ we can write

$$
= \frac{\big(1+\cos\theta + i\sin\theta\big)}{2}|+\rangle + \frac{\big(1-(\cos\theta + i\sin\theta)\big)}{2}|-\rangle
$$

What then is the probability of measuring, say, $|+\rangle$, in the $\{|+\rangle, |-\rangle\}$ basis? Well, we know that the probability is the amplitude squared, so,

$$
\begin{aligned}
P(|+\rangle) &= \left|\frac{\big(1+\cos\theta + i\sin\theta\big)}{2}\right|^2 \\
&= \frac{1}{4}\big((1+\cos\theta + i\sin\theta)(1+\cos\theta - i\sin\theta)\big) \\
&= \frac{1}{4}\big(1+\cos\theta - i\sin\theta + \cos\theta + \cos^2\theta - \cos\theta i\sin\theta + i\sin\theta + \cos\theta i\sin\theta - i^2\sin^2\theta\big) \\
&= \frac{1}{4}\big(1+2\cos\theta + \cos^2\theta + \sin^2\theta\big) && \#\ \cos^2\theta + \sin^2\theta = 1 \\
&= \frac{1}{4}\big(1+2\cos\theta + 1\big) \\
&= \frac{1}{4}\big(2+2\cos\theta\big) \\
&= \frac{1}{2}\big(1+\cos\theta\big) && \#\ \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \implies 1+\cos\theta = 2\cdot\cos^2\frac{\theta}{2} \\
&= \frac{1}{2}\Big(2\cdot\cos^2\frac{\theta}{2}\Big) \\
&= \cos^2\frac{\theta}{2} && \#\ P(|+\rangle) = \cos^2\frac{\theta}{2}
\end{aligned}
$$

Similarly, we know that the probability of measuring $|-\rangle$, $P(|-\rangle) = \sin^2 \frac{\theta}{2}$. So measuring the qubit in the $\{|+\rangle, |-\rangle\}$ basis does reveal some information about the phase θ. More about this in Section 2.3.

2.2 Why does θ vary between 0 and π in Equation 1?

In the complex plane representation of $|\psi\rangle$, the basis vectors $|0\rangle$ and $|1\rangle$ are orthogonal, that is, there is a $\frac{\pi}{2}$ angle between them and hence their inner product $\langle 0|1\rangle = 0$. In the Bloch Sphere the angle θ between $|0\rangle$ and $|1\rangle$ is π. That is, in the Bloch Sphere $|0\rangle$ and $|1\rangle$ are *antipodal* rather than orthogonal.

Further, in the general form of $|\psi\rangle$ (Equation 4), α and β are known as *probability amplitudes* [3] and thus $0 \leq |\alpha| \leq 1$ (likewise for β). However, in the Bloch Sphere we have $0 \leq \theta \leq \pi$. So to keep the values of the probability amplitudes in the right range ($0 \leq |\alpha| \leq 1$) we need to divide θ by 2 (I'll just note here that there are many other ways to think about this). This is one reason that we see the angle $\frac{\theta}{2}$ in Equation 1. Having $|0\rangle$ and $|1\rangle$ represented on the same axis (z) also saves a dimension in the Bloch Sphere representation. We will see further implications of $|0\rangle$ and $|1\rangle$ being antipodal rather than orthogonal in Section 3.

2.3 Deriving Equation 1

Let's take a look at why Equation 1 looks the way it does. First, remember that the general state of a qubit is

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle \tag{4}$$

where $\alpha, \beta \in \mathbb{C}$ and $\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1$. $|\alpha|^2 + |\beta|^2 = 1$ is known as a *normalization constraint*. Here $\langle\psi|\psi\rangle$ is the inner product of ψ with itself, namely $\begin{bmatrix}\alpha & \beta\end{bmatrix} \begin{bmatrix}\alpha \\ \beta\end{bmatrix} = |\alpha|^2 + |\beta|^2$. In general for two $n \times 1$ vectors $\mathbf{u}$ and $\mathbf{v}$, the inner product $\langle \mathbf{u}|\mathbf{v}\rangle$ is defined as

$$\langle \mathbf{u}|\mathbf{v}\rangle = \mathbf{u}^{\mathrm{T}}\mathbf{v}$$

2.3.1 What can we say about α and β?

Since α and β are complex numbers, we can write α (or β, with different a and b) as $\alpha = a + ib$. This is depicted in Figure 3. There are a few things we can notice about α. First, $a = r\cos\phi$; similarly, $b = r\sin\phi$. So we can write

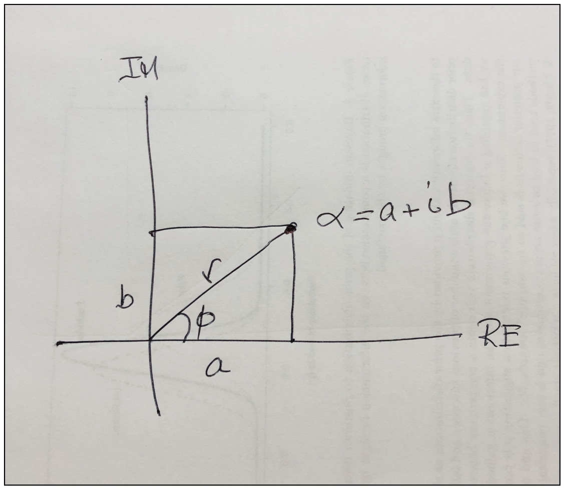

Figure 3: α in the Complex Plane

$$\begin{aligned}\alpha &= a + ib \\ &= r\cos\phi + ir\sin\phi \\ &= r[\cos\phi + i\sin\phi]\end{aligned}$$

Next we can apply Euler's Formula

$$\begin{aligned}\alpha &= a + ib && \text{\# } \alpha \text{ in the complex plane (Figure 3)} \\ &= r[\cos\phi + i\sin\phi] && \text{\# } a = r\cos\phi,\ b = r\sin\phi, \text{ factor out } r \\ &= re^{i\phi} && \text{\# Euler's Formula: } e^{i\phi} = \cos\phi + i\sin\phi\end{aligned}$$

So now let $\alpha = r_0 e^{i\phi_0}$ and $\beta = r_1 e^{i\phi_1}$. Then we have

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = r_0 e^{i\phi_0}|0\rangle + r_1 e^{i\phi_1}|1\rangle$$

If we factor out $e^{i\phi_0}$, we get

$$|\psi\rangle = e^{i\phi_0}[r_0|0\rangle + r_1 e^{i(\phi_1 - \phi_0)}|1\rangle]$$

It turns out that the term $e^{i\phi_0}$, the global phase, has no physical significance [2]; you can rotate the axes any way you like with no effect so long as the x, y and z axes remain perpendicular to one another. So we can drop the $e^{i\phi_0}$ term and write

$$|\psi\rangle = r_0\,|0\rangle + r_1 e^{i(\phi_1-\phi_0)}\,|1\rangle$$

Next let $\phi = \phi_1 - \phi_0$. Then we have

$$|\psi\rangle = r_0\,|0\rangle + r_1 e^{i\phi}\,|1\rangle \tag{5}$$

3 Normalization Constraints

One more observation we can make: Equation 5 tells us that we can write our normalization constraint $|\alpha|^2 + |\beta|^2 = 1$ as $|r_0|^2 + |r_1 e^{i\phi}|^2 = 1$.

Since r_0 has no complex term $|r_0|^2 = r_0^2$ and we can write $r_0^2 + |r_1 e^{i\phi}|^2 = 1$. The term $|r_1 e^{i\phi}|^2$ can be written as $r_1^2|e^{i\phi}|^2$. So now we have $r_0^2 + r_1^2|e^{i\phi}|^2 = 1$.

But we can learn a bit more. Recall that $e^{i\phi} = \cos\phi + i\sin\phi$ (Euler's Formula), so

$$\begin{aligned}
|e^{i\phi}|^2 &= |(\cos\phi + i\sin\phi)|^2 \\
&= (\cos\phi + i\sin\phi)(\cos\phi - i\sin\phi) \\
&= \cos^2\phi - \cos\phi\, i\sin\phi + i\sin\phi\cos\phi - i^2\sin^2\phi \\
&= \cos^2\phi - i^2\sin^2\phi \\
&= \cos^2\phi + \sin^2\phi \\
&= 1
\end{aligned}$$

The result is that the constraint $|\alpha|^2 + |\beta|^2 = 1$ can be written as $r_0^2 + r_1^2 = 1$. If we choose $r_0 = \cos x$ and $r_1 = \sin x$ then we'll have $\cos^2 x + \sin^2 x = 1$ (by the Pythagorean trigonometric identity [14]), which satisfies the normalization constraint.

If we now let $x = \frac{\theta}{2}$ (see Section 2.2), we get Equation 1, namely $|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle$.

4 Qubits, Kronecker/Tensor Products and States

If we consider two binary strings, say 011 and 111, 011 typically represents the (decimal) number 3, while 111 represents the number 7. In general, three physical bits can be prepared in $2^3 = 8$ different configurations that can represent the integers from 0 to 7. However, a register composed of three classical bits can store only one number at a given moment of time. One of the powerful aspects of the quantum world is that a register composed of 3 qubits can represent all 8 configurations at the same time.

As described in Section 1, a qubit is a quantum system in which the Boolean states 0 and 1 are represented by a pair of normalized and mutually orthogonal quantum states labeled as $\{|0\rangle, |1\rangle\}$. These two states form a *computational basis* and any other (pure) state of the qubit can be written as a superposition $\alpha|0\rangle + \beta|1\rangle$ for some α and β such that $|\alpha|^2 + |\beta|^2 = 1$. A collection of n qubits is called a quantum register of size n.

It is typically assumed that information is stored in the registers in binary form. For example, the number 6 is represented by a register in state $|1\rangle \otimes |1\rangle \otimes |0\rangle$. In more compact notation: $|a\rangle$ stands for the tensor product $|a_{n-1}\rangle \otimes |a_{n-2}\rangle \otimes \cdots \otimes |a_1\rangle \otimes |a_0\rangle$, where $a_i \in \{0,1\}$. $|a\rangle$ represents a quantum register prepared with the value $a = 2^0 a_0 + 2^1 a_1 + \cdots + 2^{n-1} a_{n-1}$. There are 2^n states of this kind, representing all binary strings of length n or the numbers from 0 to 2^{n-1} and form a convenient computational basis. Following convention, $a \in \{0,1\}^n$ (a is a binary string of length n) implies that $|a\rangle$ belongs to the computational basis.

4.1 Computational Basis

The computational or standard basis of $\mathbb{C}^n$, denoted by $\{|0\rangle, \cdots, |n-1\rangle\}$, is given by[3]

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots |n-1\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Interestingly, the computational basis forms the set of *one-hot* vectors such that $|i\rangle$ has a 1 in the ith position and 0 everywhere else. The normalized sum of all computational basis vectors defines the vector

$$|D\rangle = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} |i\rangle$$

For a qubit, where $n = 2$, we have

[3]Recall that $\begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}$ is alternate notation for $\begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}$.

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|D\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{1} |i\rangle$$

As mentioned above, multiple qubits states can be prepared in a register. Typically the state of a register is represented by the tensor product [13] of two or more qubit states. Let's take a quick look at how the tensor product works.

4.1.1 Tensor Products

The Kronecker or tensor product, denoted by $\otimes$, is defined as follows: If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is a $p \times q$ matrix, then the tensor product $\mathbf{A} \otimes \mathbf{B}$ is the $mp \times nq$ block matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

For example [13]

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 2 \cdot \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \\ 3 \cdot \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 4 \cdot \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 & 1 \cdot 5 & 2 \cdot 0 & 2 \cdot 5 \\ 1 \cdot 6 & 1 \cdot 7 & 2 \cdot 6 & 2 \cdot 7 \\ 3 \cdot 0 & 3 \cdot 5 & 4 \cdot 0 & 4 \cdot 5 \\ 3 \cdot 6 & 3 \cdot 7 & 4 \cdot 6 & 4 \cdot 7 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{bmatrix}$$

Consider the 3 qubit register $|1\rangle \otimes |0\rangle \otimes |1\rangle = |101\rangle = |5\rangle$. How does the tensor product $|1\rangle \otimes |0\rangle \otimes |1\rangle$ work here?

$$
\begin{aligned}
|1\rangle \otimes |0\rangle \otimes |1\rangle &= |1\rangle \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= |1\rangle \otimes \begin{bmatrix} 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} \\
&= |1\rangle \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \\
&= |101\rangle \\
&= |5\rangle
\end{aligned}
$$

Notice that the vector $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ is of length 2^3 and is the one-hot encoding of $|101\rangle$.

4.2 Superpositions

A classical register of size three, like a quantum register of size three, can store one of 2^3 numbers such as 3 or 7, as follows

$$|0\rangle \otimes |1\rangle \otimes |1\rangle \equiv |011\rangle \equiv |3\rangle$$
$$|1\rangle \otimes |1\rangle \otimes |1\rangle \equiv |111\rangle \equiv |7\rangle$$

However, unlike a classical register, a quantum register can be prepared in the superposition of some or all of its qubits. For example, if instead of setting the first qubit to $|0\rangle$ or $|1\rangle$, we prepare it in the superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, then we have

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle \otimes |1\rangle \equiv \frac{1}{\sqrt{2}}(|011\rangle + |111\rangle) \equiv \frac{1}{\sqrt{2}}(|3\rangle + |7\rangle) \tag{6}$$

There is nothing special about Equation 6. We can also prepare a 3 qubit register in a superposition of all eight states by putting each qubit into the superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. This gives us

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

which can also be written as

$$2^{-\frac{3}{2}}\Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle \Big)$$

or in decimal

$$2^{-\frac{3}{2}}\Big(|0\rangle + |1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle + |7\rangle \Big) = 2^{-\frac{3}{2}} \sum_{i=0}^{7} |i\rangle$$

This kind of qubit preparationn and any other manipulation of qubits must be performed by unitary operator (matrix)[4]. A quantum logic gate is a device which performs a fixed unitary operation on selected qubits in a fixed period of time and a quantum network is a device consisting of quantum logic gates whose computational steps are synchronized in time [5].

[4]Recall that a unitary matrix U has the property that $UU^\dagger = U^\dagger U = I$, where $U^\dagger$ is the conjugate transpose, or *adjoint*, of U and I is the identity matrix [15].

This is kind of interesting: The *Hadamard* transform (or gate) [12], H, defined as

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ into $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. That is, $H\left|0\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle)$. This is also called the $\left|+\right\rangle$ state. Similarily, $H\left|1\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle) \equiv \left|-\right\rangle$.

Now, what is the value of $H(H\left|0\right\rangle)$? First, observe that

$$H\left|0\right\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 1 + -1 \cdot 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \# \; H\left|0\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle)$$

Here's an interesting observation. It turns out that the positive and negative amplitudes of $\left|1\right\rangle$ in $H(H\left|0\right\rangle)$ cancel out. This effect is called *interference*, and is analogous to interference patterns between light or sound waves. So why is this?

$$\begin{aligned} H(H\left|0\right\rangle) &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 \\ 1 \cdot 1 + -1 \cdot 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} \cdot 2 \\ \frac{1}{2} \cdot 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \left|0\right\rangle \end{aligned}$$

Now, if we apply the Hadamard gate H to each bit of a n bit register containing all zeros, we get the superposition of all n-bit strings, namely

$$\frac{1}{\sqrt{2^n}} \sum_{j \in \{0,1\}^n} |j\rangle$$

More generally, if we apply $H^{\otimes n}$ to an initial state $|i\rangle$ with $i \in \{0,1\}^n$ we get the n-fold Hadamard transform, denoted $H^{\otimes n}|i\rangle$ and defined as follows

$$H^{\otimes n}|i\rangle = \frac{1}{\sqrt{2^n}} \sum_{j \in \{0,1\}^n} (-1)^{i \cdot j} |j\rangle$$

where $i \cdot j = \sum_{k=1}^{n} i_k j_k$ is the inner product of the n-bit strings $i, j \in \{0,1\}^n$. For example

$$\begin{aligned}
H^{\otimes 2}|01\rangle &= H|0\rangle \otimes H|1\rangle && \# \; H^{\otimes 2}|01\rangle = H|0\rangle \otimes H|1\rangle = |+\rangle \otimes |-\rangle = |\uparrow\rangle \otimes |\downarrow\rangle \\
&= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) && \# \; H = H^{\otimes 1} \\
&= \frac{1}{2}\big(|00\rangle - |01\rangle + |10\rangle - |11\rangle \big) \\
&= \frac{1}{2} \sum_{j \in \{0,1\}^2} (-1)^{01 \cdot j} |j\rangle
\end{aligned}$$

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