

Definition 0.1. Let I be a subset of a ring R . Then an additive subgroup of R having the property that

$$ra \in I \text{ for } a \in I, r \in R$$

is called a left ideal of R . Similarly

$$ar \in I \text{ for } a \in I, r \in R$$

is called a right ideal of R . If an ideal is both a right and a left ideal then we call it a two-sided ideal of R , or simply an ideal of R .

We say that an ideal I of R is proper if $I \neq R$. We say that it is non-trivial if $I \neq R$ and $I \neq 0$.

I've seen several different notations for ideals including, among others: $rI = \{ri \mid r \in R, i \in I\}$ and $rI \subset I \forall r \in R$.

So in words: An *ideal* I is a subset of a ring R such that

- I is a subgroup of R under addition (so $0 \in I$ and so $I \neq \emptyset$)
- I is not only closed under multiplication but also satisfies the *stronger* property that it "absorbs" all of the elements of R under multiplication: $\forall r \in R \ rI \subset I$

Groups

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| • $\phi : G \xrightarrow{hom} H$ | # ϕ is a group homomorphism: $G \simeq H$ |
| • $\ker \phi = N$ where $N \triangleleft G$ | # N is a normal subgroup of G |
| • if ϕ is onto then $H \simeq G/N$ | # First Isomorphism Theorem for groups |

Rings

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| • $\phi : R \xrightarrow{hom} S$ | # ϕ is a ring homomorphism: $R \simeq S$ |
| • $\ker \phi = I$ | # I is a two-sided ideal in R with $1 \notin I$ |
| • if ϕ is onto then $S \simeq R/I$ | # First Isomorphism Theorem for rings |

Notes

- An ideal is the same thing as the kernel of a ring homomorphism ϕ
- Ideals are to rings as normal subgroups are to groups