2.2.2.4 *Maximization of inner product over norm balls.* Given a nonzero vector $y \in \mathbb{R}^n$, consider the problem of finding some vector $x \in \mathcal{B}_p$ (the unit ball in ℓ_p norm) that maximizes the inner product $x^\top y$: that is, solve

$$\max_{\|x\|_p \leq 1} x^\top y.$$

For $p = 2$ the solution is readily obtained from Eq. (2.3): x should be aligned (parallel) to y, so as to form a zero angle with it, and have the largest possible norm, that is, a norm equal to one. Therefore the unique solution is

$$x_2^* = \frac{y}{\|y\|_2},$$

hence $\max_{\|x\|_2 \leq 1} x^\top y = \|y\|_2$.

Consider next the case with $p = \infty$: since $x^\top y = \sum_{i=1}^n x_i y_i$, where each element x_i is such that $|x_i| \leq 1$, then the maximum in the sum is achieved by setting $x_i = \text{sgn}(y_i)$,[7] so that $x_i y_i = |y_i|$. Hence,

$$x_\infty^* = \text{sgn}(y),$$

and $\max_{\|x\|_\infty \leq 1} x^\top y = \sum_{i=1}^n |y_i| = \|y\|_1$. The optimal solution may not be unique, since corresponding to any $y_i = 0$ any value $x_i \in [-1, 1]$ could be selected without modifying the optimal objective.

Finally, we consider the case with $p = 1$: the inner product $x^\top y = \sum_{i=1}^n x_i y_i$ can now be interpreted as a weighted average of the y_is, where the x_is are the weights, whose absolute values must sum up to one. The maximum of the weighted average is achieved by first finding the y_i having the largest absolute value, that is by finding one index m such that $|y_i| \leq |y_m|$ for all $i = 1, \ldots, n$, and then setting

$$[x_1^*]_i = \begin{cases} \text{sgn}(y_i) & \text{if } i = m \\ 0 & \text{otherwise} \end{cases}, \quad i = 1, \ldots, n.$$

We thus have $\max_{\|x\|_1 \leq 1} x^\top y = \max_i |y_i| = \|y\|_\infty$. Again, the optimal solution may not be unique since in the case when vector y has several entries with identical maximum absolute value then m can be chosen to be any of the indices corresponding to these entries.

[7] sgn denotes the sign function, which, by definition, takes values $\text{sgn}(x) = 1$, if $x > 0$, $\text{sgn}(x) = -1$, if $x < 0$, and $\text{sgn}(x) = 0$, if $x = 0$.

Example 2.9 (*Production margin*) Consider a production process involving two raw materials r_1, r_2 and one finished product. The unit cost for the raw materials is subject to variability, and it is given by

$$c_i = \bar{c}_i + \alpha_i x_i, \quad i = 1, 2,$$

where, for $i = 1, 2$, $\bar{c}_i$ is the nominal unit cost of material r_i, $\alpha_i \geq 0$ is the cost spread, and $|x_i| \leq 1$ is an unknown term accounting for cost uncertainty. Production of one unit of the finished product requires a fixed