CS 355 Homework #2:
More With Points, Vectors, and Lines

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1. A line passes through the points $\mathbf{p}_1 = (10, 20)$ and $\mathbf{p}_2 = (30, 40)$.

   (a) Express this line in parametric form as described in Section 9.2.1 of your book.
   ANSWER:
   The vector between the points is $\mathbf{p}_2 - \mathbf{p}_1 = [20, 20]$*, so one way to write it is as*

   $$\mathbf{p}(t) = \mathbf{p}_1 + t(\mathbf{p}_2 - \mathbf{p}_1) = (10, 20) + t[20, 20]$$

   Using a unit vector instead, the line direction is $\frac{(1,1)}{\sqrt{2}} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.
   With this, another way to write the line is

   $$\mathbf{p}(t) = \mathbf{p}_1 + t\frac{\mathbf{p}_2 - \mathbf{p}_1}{\|\mathbf{p}_2 - \mathbf{p}_1\|} = (10, 20) + t\left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$$

   (b) Express the line in implicit form as described in Section 9.2.2 of your book.
   ANSWER:
   The line normal is either $\hat{\mathbf{n}} = \frac{[-1,1]}{\sqrt{2}}$ *or* $\frac{[1,-1]}{\sqrt{2}}$.
   Taking the dot product of this with $\mathbf{p}_1$ *gives* $d = ((-1)(10) + (1)(20))/\sqrt{2}) = 5\sqrt{2}$. *(Or similarly* $-5\sqrt{2}$*, depending on which normal you choose.)*
   The equation is thus $\mathbf{p} \cdot \hat{\mathbf{n}} = d$*, which is either*

   $$\mathbf{p} \cdot \frac{[-1, 1]}{\sqrt{2}} = 5\sqrt{2}$$

   or

   $$\mathbf{p} \cdot \frac{[1, -1]}{\sqrt{2}} = -5\sqrt{2}$$

   Alternative, you could write this as

   $$\frac{-\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y - 5\sqrt{2} = 0$$

   or

   $$\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y + 5\sqrt{2} = 0$$

Any scalar multiple of this also works. If you do this without using a unit-length normal, you might get this:

$$-x + y - 10 = 0$$

or this

$$x - y + 10 = 0$$

(c) How close is the point $\mathbf{q} = (22, 29)$ from the line? (You may use either the implicit or parametric form of the line to do this.)

ANSWER:

Using the implicit form with a unit-length normal, the distance is

$$\begin{aligned}
|\mathbf{q} \cdot \hat{\mathbf{n}} - d| &= \left|(22, 29) \cdot \frac{[-1, 1]}{\sqrt{2}} - 5\sqrt{2}\right| \\
&= \frac{3}{\sqrt{2}} \\
&= \frac{3\sqrt{2}}{2}
\end{aligned}$$

(d) Does the closest point on the line to the point $\mathbf{q} = (22, 29)$ fall within the endpoints of this line segment?

ANSWER: *The value of t at which the line is closest to $\mathbf{q}$ can be found using*

$$\begin{aligned}
t &= (\mathbf{q} - \mathbf{p}_1) \cdot \frac{[1, 1]}{\sqrt{2}} \\
&= (12, 9) \cdot \frac{(1, 1)}{\sqrt{2}} \\
&= \frac{21\sqrt{2}}{2} \\
&\approx 14.85
\end{aligned}$$

Since this is less than the separation between the two endpoints ($\|p_2 - \mathbf{p}_1\| = 20\sqrt{2} \approx 28.28$), the closet point does lie within the endpoints.

2. A circle has center at $\mathbf{c} = (10, 12)$ and radius $r = 3$.

(a) Show mathematically whether the point $\mathbf{q} = (12, 13)$ is within the circle.

ANSWER:

Yes,

$$\|\mathbf{p} - \mathbf{c}\| = \sqrt{5} < 3$$

or, if you want to do it faster:

$$\|\mathbf{p} - \mathbf{c}\|^2 = 5 < 3^2$$

(b) What point on the circle is closest to the point $\mathbf{q} = (20, 15)$?
ANSWER:

$$\begin{aligned}\mathbf{c} + r\,\frac{\mathbf{p}-\mathbf{c}}{\|\mathbf{p}-\mathbf{c}\|} &= (10,12) + 3\left[\frac{10}{\sqrt{109}}, \frac{3}{\sqrt{109}}\right] \\ &= (10 + \frac{30}{\sqrt{109}}, 12 + \frac{9}{\sqrt{109}})\end{aligned}$$

3. An ellipse has center at $\mathbf{c} = (10, 12)$ with width 20 and height 10.

(a) Show mathematically whether the point $\mathbf{q} = (19, 13)$ is within the ellipse.
ANSWER:
The major and minor axes are 10 and 5 respectively, so

$$\left(\frac{19-10}{10}\right)^2 + \left(\frac{13-12}{5}\right)^2 = \frac{17}{20} < 1$$

(b) What are the corners of the bounding box for this shape?
ANSWER:

$$\mathbf{c} + (\pm a, \pm b)$$

The corners are at $(20, 17)$*,* $(0, 17)$*,* $(20, 7)$*,* $(0, 7)$*.*

4. A square with length 10 on each side is centered at position $\mathbf{c} = (60, 80)$. Show mathematically the steps you would do to determine whether the point $\mathbf{q} = (64, 74)$ is within the square.

ANSWER:

The point p *is 4 off of center in the* x *axis and* 6 *off of center in the* y *axis. Because* $6 > 10/2$*, the point is outside the square.*

5. A triangle has corners at $p_1 = (10, 20)$, $p_2 = (30, 40)$, $p_3 = (20, 50)$. Show mathematically whether the point $q = (20, 40)$ is within the triangle.

ANSWER:

There are three tests:

$$\begin{aligned}(\mathbf{q}-\mathbf{p}_1)\cdot(\mathbf{p}_2-\mathbf{p}_1)_\perp &= (10,20)\cdot(-20,20) &= 200 \\ (\mathbf{q}-\mathbf{p}_2)\cdot(\mathbf{p}_3-\mathbf{p}_2)_\perp &= (-10,0)\cdot(-10,-10) &= 100 \\ (\mathbf{q}-\mathbf{p}_3)\cdot(\mathbf{p}_1-\mathbf{p}_3)_\perp &= (0,-10)\cdot(30,-10) &= 100\end{aligned}$$

Since all of these are positive, the point is inside the triangle.