Problem 4

Starter Code from Kernel_starter.ipynb

In [ ]:
```
import pickle
import os
from sklearn.model_selection import train_test_split
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.optimize import approx_fprime
from numpy.linalg import norm
```

In [ ]:
```
def check_gradient(model, X, y, dimensionality, verbose=True):
    # This checks that the gradient implementation is correct
    w = np.random.rand(dimensionality)
    f, g = model.funObj(w, X, y)

    # Check the gradient
    estimated_gradient = approx_fprime(w,
                                       lambda w: model.funObj(w,X,y)[0],
                                       epsilon=1e-6)

    implemented_gradient = model.funObj(w, X, y)[1]

    if np.max(np.abs(estimated_gradient - implemented_gradient) > 1e-3):
        raise Exception('User and numerical derivatives differ:\n%s\n%s' %
             (estimated_gradient[:5], implemented_gradient[:5]))
    else:
        if verbose:
            print('User and numerical derivatives agree.')
```

In [ ]:
```
def plotClassifier(model, X, y):
    """plots the decision boundary of the model and the scatterpoints
       of the target values 'y'.

    Assumptions
    -----------
    y : it should contain two classes: '1' and '2'

    Parameters
    ----------
    model : the trained model which has the predict function

    X : the N by D feature array

    y : the N element vector corresponding to the target values

    """
    x1 = X[:, 0]
    x2 = X[:, 1]

    x1_min, x1_max = int(x1.min()) - 1, int(x1.max()) + 1
    x2_min, x2_max = int(x2.min()) - 1, int(x2.max()) + 1
```

```
    x1_line =  np.linspace(x1_min, x1_max, 200)
    x2_line =  np.linspace(x2_min, x2_max, 200)

    x1_mesh, x2_mesh = np.meshgrid(x1_line, x2_line)

    mesh_data = np.c_[x1_mesh.ravel(), x2_mesh.ravel()]

    y_pred = model.predict(mesh_data)
    y_pred = np.reshape(y_pred, x1_mesh.shape)

    plt.figure()
    plt.xlim([x1_mesh.min(), x1_mesh.max()])
    plt.ylim([x2_mesh.min(), x2_mesh.max()])

    plt.contourf(x1_mesh, x2_mesh, -y_pred.astype(int), # unsigned int causes
                cmap=plt.cm.RdBu, alpha=0.6)


    y_vals = np.unique(y)
    plt.scatter(x1[y==y_vals[0]], x2[y==y_vals[0]], color="b", label="class %+
    plt.scatter(x1[y==y_vals[1]], x2[y==y_vals[1]], color="r", label="class %+
    plt.legend()
```

In [ ]:

```
def findMin(funObj, w, maxEvals, *args, verbose=0):
    """
    Uses gradient descent to optimize the objective function

    This uses quadratic interpolation in its line search to
    determine the step size alpha
    """
    # Parameters of the Optimization
    optTol = 1e-2
    gamma = 1e-4

    # Evaluate the initial function value and gradient
    f, g = funObj(w,*args)
    funEvals = 1

    alpha = 1.
    while True:
        # Line-search using quadratic interpolation to
        # find an acceptable value of alpha
        gg = g.T.dot(g)

        while True:
            w_new = w - alpha * g
            f_new, g_new = funObj(w_new, *args)

            funEvals += 1
            if f_new <= f - gamma * alpha*gg:
                break

            if verbose > 1:
                print("f_new: %.3f - f: %.3f - Backtracking..." % (f_new, f))

            # Update step size alpha
            alpha = (alpha**2) * gg/(2.*(f_new - f + alpha*gg))
```

```
        # Print progress
        if verbose > 0:
            print("%d - loss: %.3f" % (funEvals, f_new))

        # Update step-size for next iteration
        y = g_new - g
        alpha = -alpha*np.dot(y.T,g) / np.dot(y.T,y)

        # Safety guards
        if np.isnan(alpha) or alpha < 1e-10 or alpha > 1e10:
            alpha = 1.

        if verbose > 1:
            print("alpha: %.3f" % (alpha))

        # Update parameters/function/gradient
        w = w_new
        f = f_new
        g = g_new

        # Test termination conditions
        optCond = norm(g, float('inf'))

        if optCond < optTol:
            if verbose:
                print("Problem solved up to optimality tolerance %.3f" % optTo
            break

        if funEvals >= maxEvals:
            if verbose:
                print("Reached maximum number of function evaluations %d" % ma
            break

    return w, f
```

In [ ]:
```
def log_1_plus_exp_safe(x):
    out = np.log(1+np.exp(x))
    out[x > 100] = x[x>100]
    out[x < -100] = np.exp(x[x < -100])
    return out
```

In [ ]:
```
def kernel_linear(X1, X2):
    return X1@X2.T
```

In [ ]:
```
class kernelLogRegL2():
    def __init__(self, lammy=1.0, verbose=0, maxEvals=100, kernel_fun=kernel_l
        self.verbose = verbose
        self.lammy = lammy
        self.maxEvals = maxEvals
        self.kernel_fun = kernel_fun
        self.kernel_args = kernel_args

    def funObj(self, u, K, y):
        yKu = y * (K@u)

        # Calculate the function value
        # f = np.sum(np.log(1. + np.exp(-yKu)))
        f = np.sum(log_1_plus_exp_safe(-yKu))
```

```
        # Add L2 regularization
        f += 0.5 * self.lammy * u.T@K@u

        # Calculate the gradient value
        res = - y / (1. + np.exp(yKu))
        g = (K.T@res) + self.lammy * K@u

        return f, g


    def fit(self, X, y):
        n, d = X.shape
        self.X = X

        K = self.kernel_fun(X,X, **self.kernel_args)

        check_gradient(self, K, y, n, verbose=self.verbose)
        self.u, f = findMin(self.funObj, np.zeros(n), self.maxEvals, K, y, ver

    def predict(self, Xtest):
        Ktest = self.kernel_fun(Xtest, self.X, **self.kernel_args)
        return np.sign(Ktest@self.u)
```

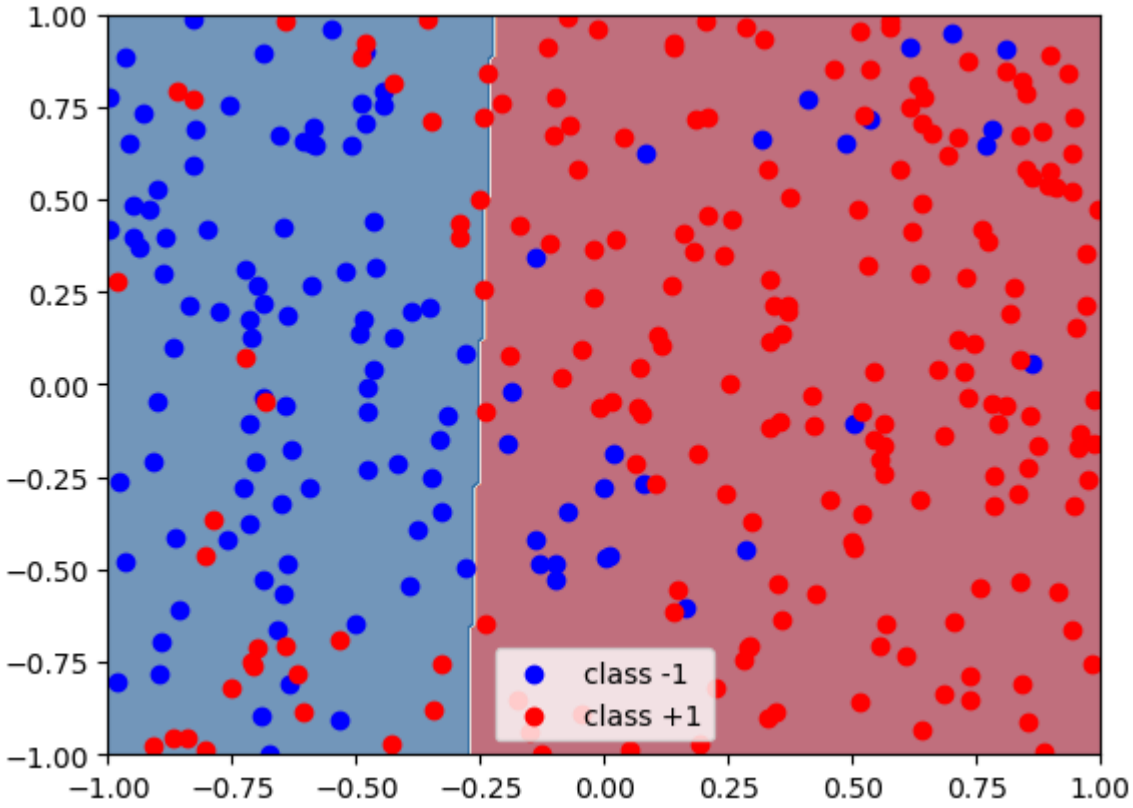
Start of my own code

In [ ]:
```
import pandas as pd
import numpy as np
from sklearn.model_selection import train_test_split
from sklearn.svm import SVC
import matplotlib.pyplot as plt

#import the nonlineardata.csv file
df = pd.read_csv("nonlineardata.csv",header=None)

#access the x and y data sets
data = df.to_numpy()
X = data[:,0:2]
Y = data[:,2]

#split the data into 80% training and 20% testing with random state to 2022
X_train, X_test, Y_train, Y_test = train_test_split(X,Y, test_size=0.2,random_
```

4.1 SVM with linear Kernel, C = 100

In [ ]:
```
clf = SVC(C=100,kernel="linear")
clf.fit(X_train,Y_train)

train_accuracy = clf.score(X_train,Y_train)
test_accuracy = clf.score(X_test,Y_test)

out_str = "Test Accuracy: {}\nTrain Accuracy: {}".format(test_accuracy,train_a
print(out_str)

plotClassifier(clf,X_train,Y_train)
```

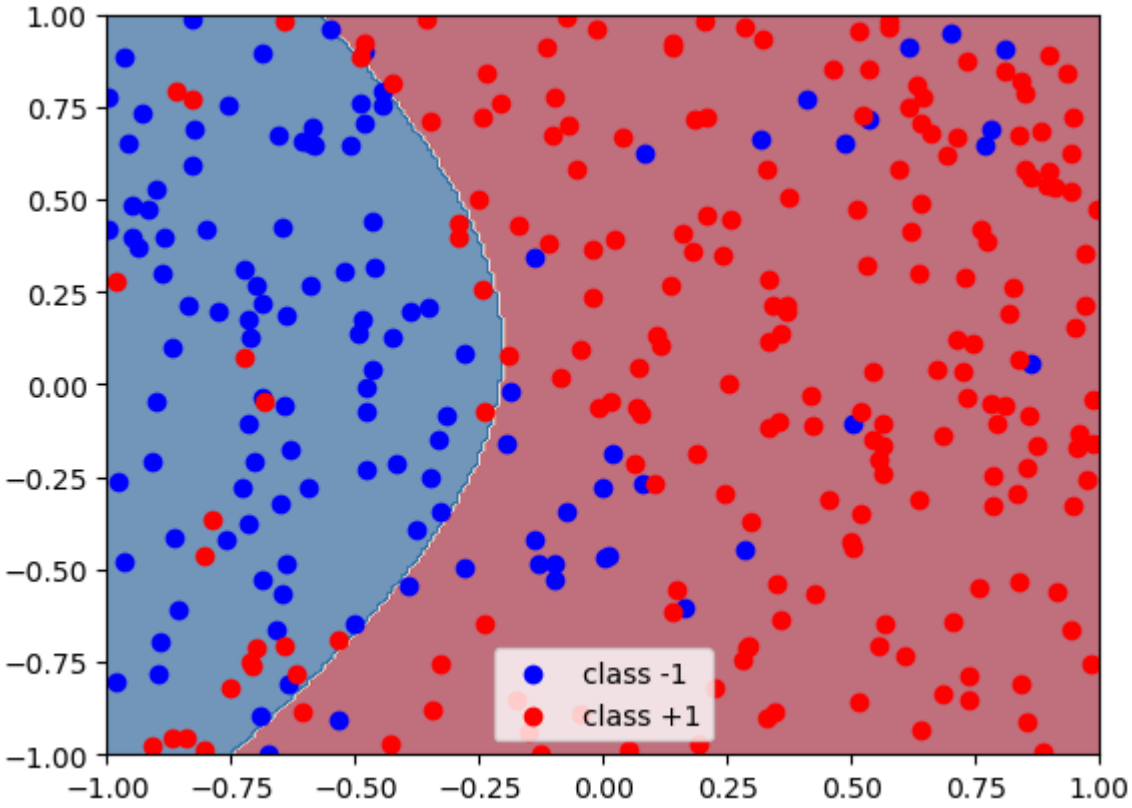
```
Test Accuracy: 0.7625
Train Accuracy: 0.8125
```



4.2 Polynomial Kernel, r = 1, degree = 2, C = 100

In [ ]:
```
clf = SVC(C=100,kernel="poly",degree = 2, coef0 = 1)
clf.fit(X_train,Y_train)

train_accuracy = clf.score(X_train,Y_train)
test_accuracy = clf.score(X_test,Y_test)

out_str = "Test Accuracy: {}\nTrain Accuracy: {}".format(test_accuracy,train_a
print(out_str)

plotClassifier(clf,X_train,Y_train)
```

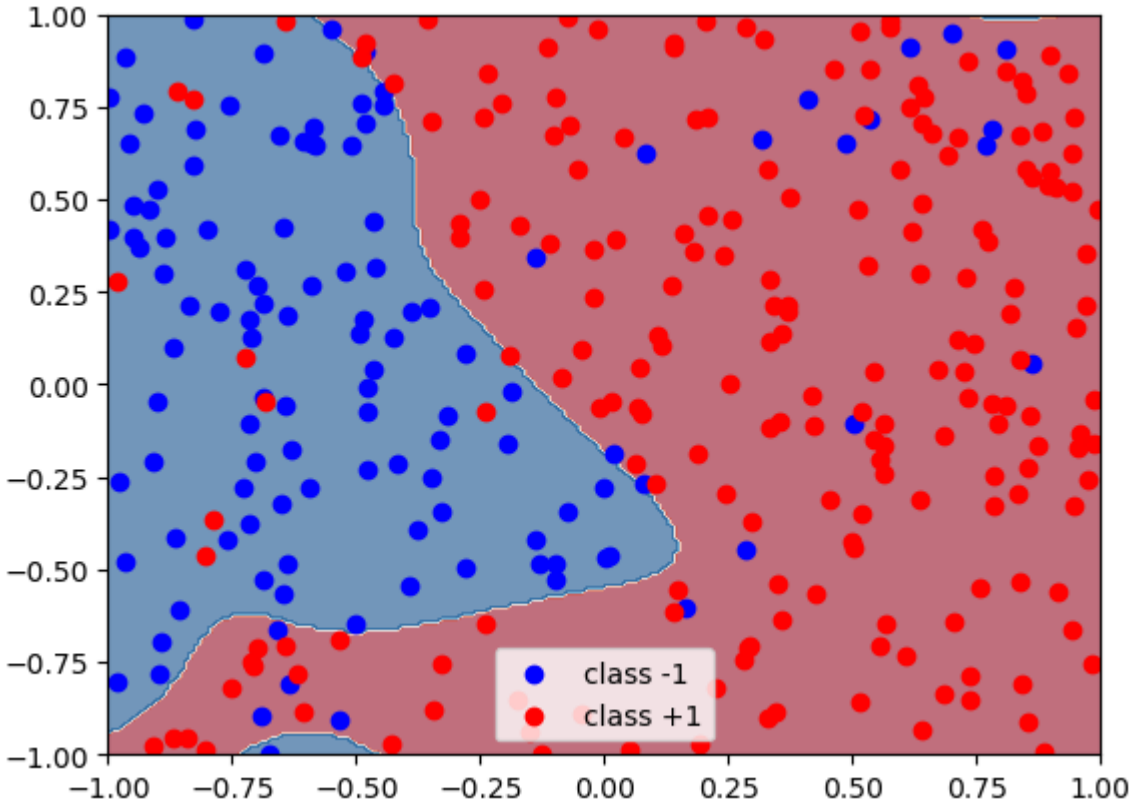
```
Test Accuracy: 0.775
Train Accuracy: 0.828125
```



4.3 RBF Kernel with sigma = 0.5, C = 100

In [ ]:
```
clf = SVC(C=100,kernel="rbf", gamma=1/pow(0.5,2))
clf.fit(X_train,Y_train)

train_accuracy = clf.score(X_train,Y_train)
test_accuracy = clf.score(X_test,Y_test)

out_str = "Test Accuracy: {}\nTrain Accuracy: {}".format(test_accuracy,train_a
print(out_str)

plotClassifier(clf,X_train,Y_train)
```

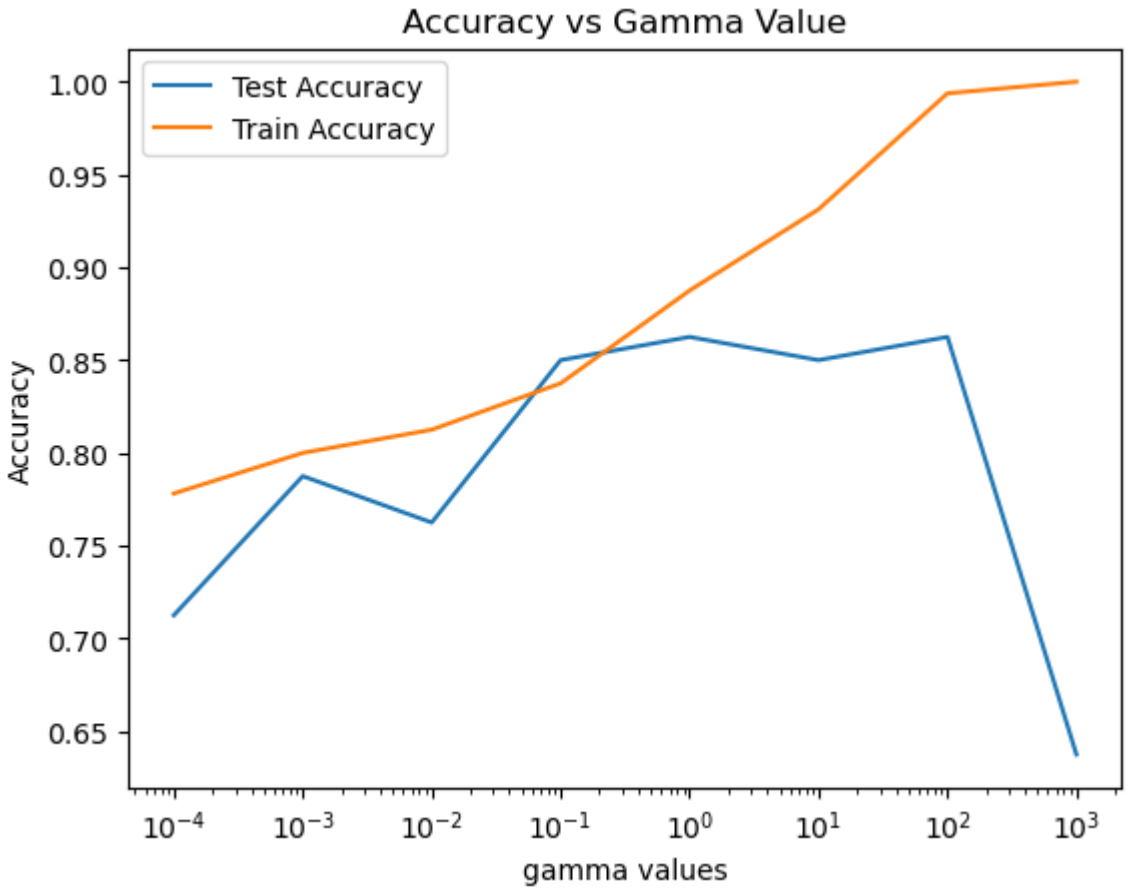
```
Test Accuracy: 0.85
Train Accuracy: 0.9
```



4.4 RBF Kernel Performance over several values of gamma

In [ ]:
```
gamma_vals = np.array([1e-4,1e-3,1e-2,1e-1,1,1e1,1e2,1e3])
test_accuracies = np.zeros(np.shape(gamma_vals))
train_accuracies = np.zeros(np.shape(gamma_vals))

for i in range(np.size(gamma_vals)):
    clf = SVC(C=100,kernel="rbf", gamma=gamma_vals[i])
    clf.fit(X_train,Y_train)

    train_accuracies[i] = clf.score(X_train,Y_train)
    test_accuracies[i] = clf.score(X_test,Y_test)

fig,ax = plt.subplots()
ax.plot(gamma_vals,test_accuracies,label = "Test Accuracy")
ax.plot(gamma_vals,train_accuracies,label = "Train Accuracy")
ax.set_xlabel("gamma values")
ax.set_xscale("log")
ax.set_ylabel("Accuracy")
ax.set_title("Accuracy vs Gamma Value")
ax.legend()
```

Out[ ]:
```
<matplotlib.legend.Legend at 0x7f21aefdbac0>
```



As gamma increases, the training error goes to 100 while the testing error decreases. This indicatest hat the model is overfitting at high values of gamma.

The reason that the model overfits for high values of gamma is due to the fact that the kernel is defined with exp(-gamma ||xi - xl||^(2)). Given this, for large values of gamma, the kernel term basically goes to zero leaving only the regularization term. Given this, the model then tries to exactly fit the training data which results in the model overfitting.

For small values of gamma, there is a high variance. This means that the model is more generalized and not overfit at the expense of not classifying all of the points in the training set perfectly.

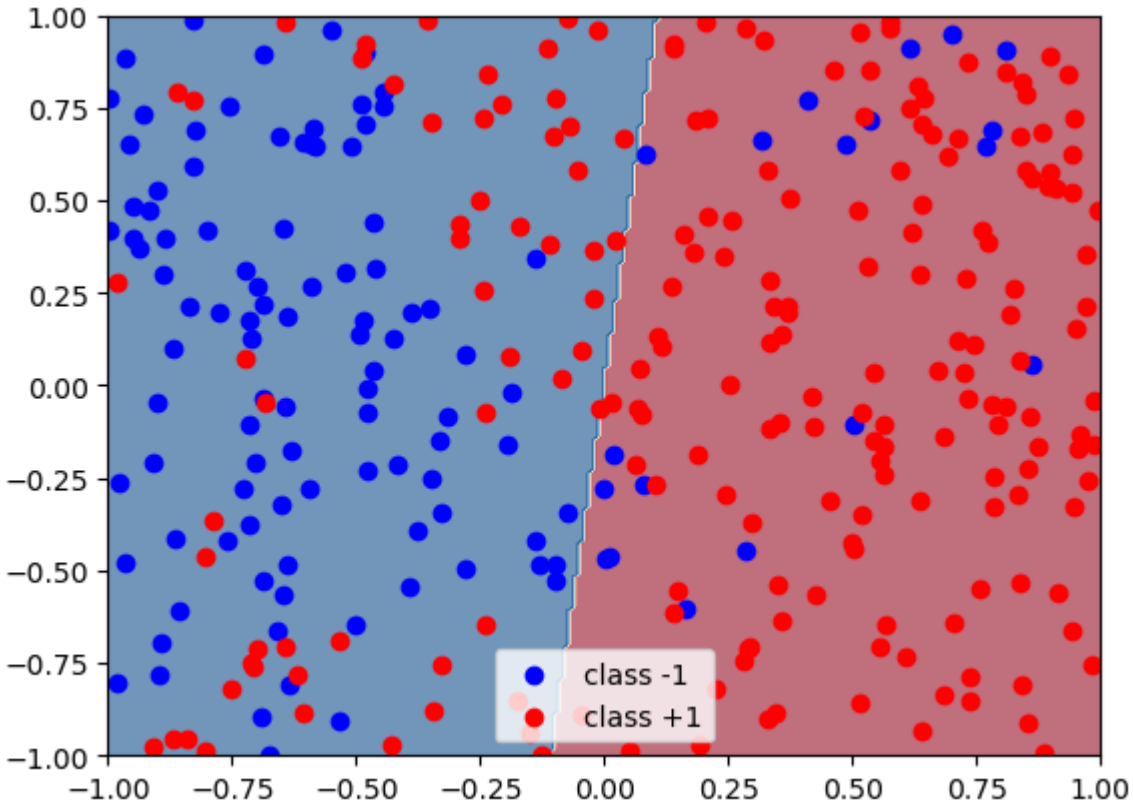
4.5 L2 regularized Logistic Regression, Linear Kernel, lambda = 0.01

In [ ]:
```
clf = kernelLogRegL2(lammy=0.01)
clf.fit(X_train,Y_train)

def get_accuracy(model,X,Y):
    f_x = model.predict(X)
    correct = f_x[f_x == Y]
    return float(np.size(correct))/float(np.size(Y))

train_accuracy = get_accuracy(clf,X_train,Y_train)
test_accuracy = get_accuracy(clf,X_test,Y_test)
```

```
out_str = "Test Accuracy: {}\nTrain Accuracy: {}".format(test_accuracy,train_a
print(out_str)

plotClassifier(clf,X_train,Y_train)
```

```
Test Accuracy: 0.7625
Train Accuracy: 0.7625
```

```
/tmp/ipykernel_11428/470417373.py:2: RuntimeWarning: overflow encountered in e
xp
  out = np.log(1+np.exp(x))
/tmp/ipykernel_11428/2422820963.py:20: RuntimeWarning: overflow encountered in
exp
  res = - y / (1. + np.exp(yKu))
```



4.6: L2 Regularized Logistic Regression with polynomial kernel, lambda = 0.01

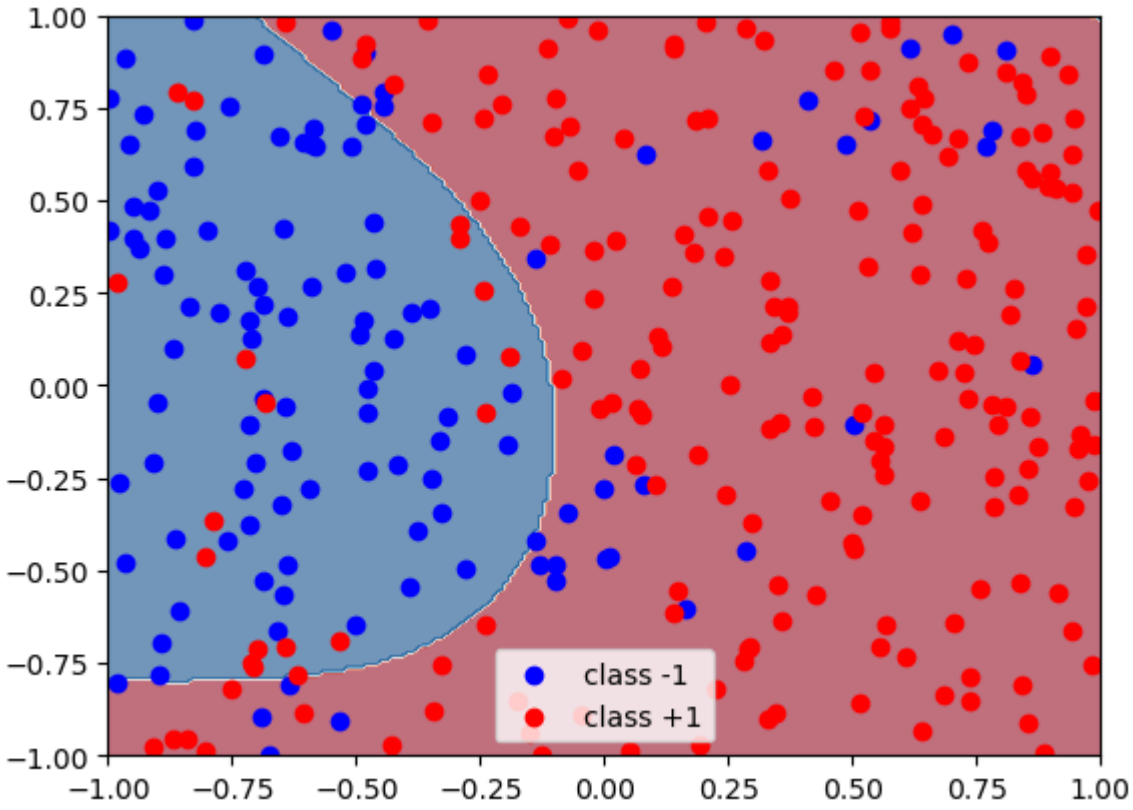
In [ ]:
```
def kernel_poly(X1, X2, p=2):
    return np.power(np.add(1,np.matmul(X1,X2.transpose())),p)
```

In [ ]:
```
clf = kernelLogRegL2(lammy=0.01,kernel_fun=kernel_poly,p=3)
clf.fit(X_train,Y_train)

def get_accuracy(model,X,Y):
    f_x = model.predict(X)
    correct = f_x[f_x == Y]
    return float(np.size(correct))/float(np.size(Y))

train_accuracy = get_accuracy(clf,X_train,Y_train)
test_accuracy = get_accuracy(clf,X_test,Y_test)

out_str = "Test Accuracy: {}\nTrain Accuracy: {}".format(test_accuracy,train_a
```

```
print(out_str)

plotClassifier(clf,X_train,Y_train)
```

```
Test Accuracy: 0.8625
Train Accuracy: 0.834375
```

```
/tmp/ipykernel_11428/470417373.py:2: RuntimeWarning: overflow encountered in e
xp
  out = np.log(1+np.exp(x))
/tmp/ipykernel_11428/2422820963.py:20: RuntimeWarning: overflow encountered in
exp
  res = - y / (1. + np.exp(yKu))
```



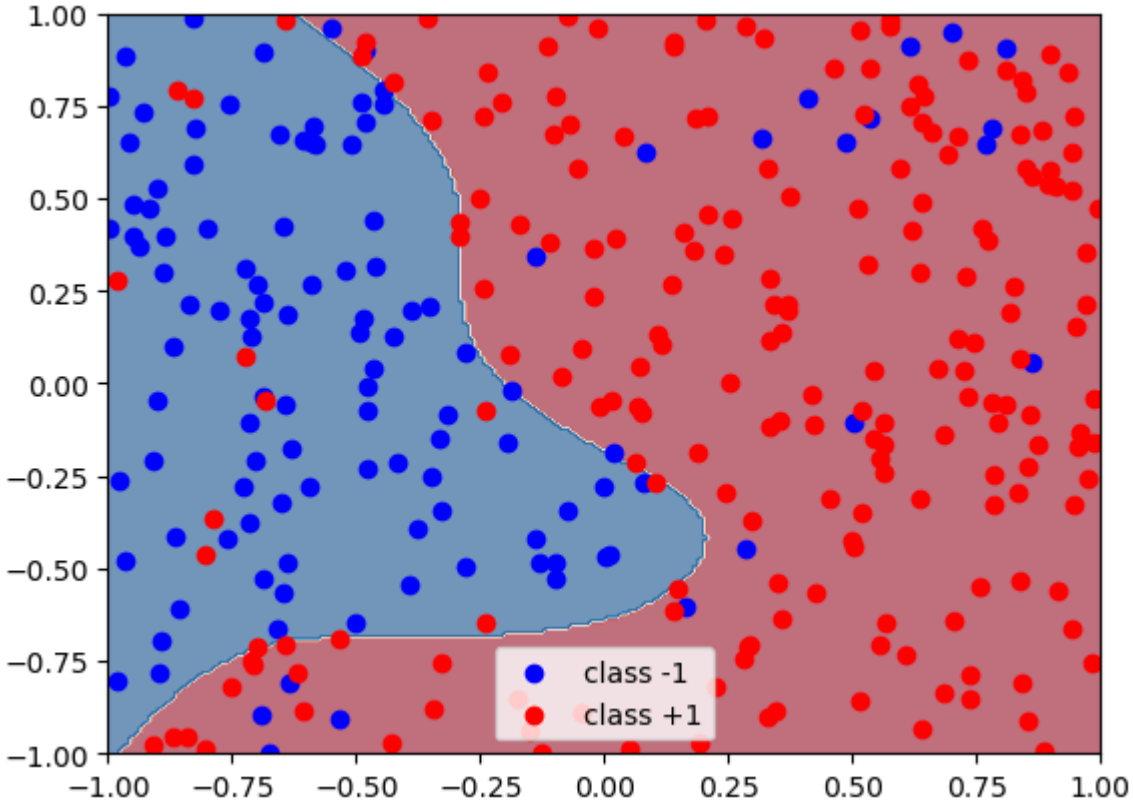
4.7 L2 Regularized Logistic Regression with RBF Kernel, sigma = 0.5, Lambda = 0.01

In [ ]:
```
def kernel_RBF(X1, X2, gamma = 1):
    #Your code here
    n = np.shape(X1)[0]
    p = np.shape(X2)[0]
    K = np.zeros((n,p))

    #while inefficient, use a for loop to compute k
    for r in range(n):
        for c in range(p):
            K[r,c] = np.exp(-1 * gamma * pow(np.linalg.norm(X1[r] - X2[c]),2))

    return K
```

In [ ]:
```
clf = kernelLogRegL2(lammy=0.01,kernel_fun=kernel_RBF,gamma = 1/pow(0.5,2))
clf.fit(X_train,Y_train)
```

```
def get_accuracy(model,X,Y):
    f_x = model.predict(X)
    correct = f_x[f_x == Y]
    return float(np.size(correct))/float(np.size(Y))

train_accuracy = get_accuracy(clf,X_train,Y_train)
test_accuracy = get_accuracy(clf,X_test,Y_test)

out_str = "Test Accuracy: {}\nTrain Accuracy: {}".format(test_accuracy,train_a
print(out_str)

plotClassifier(clf,X_train,Y_train)
```

```
/tmp/ipykernel_11428/470417373.py:2: RuntimeWarning: overflow encountered in e
xp
  out = np.log(1+np.exp(x))
/tmp/ipykernel_11428/2422820963.py:20: RuntimeWarning: overflow encountered in
exp
  res = - y / (1. + np.exp(yKu))
Test Accuracy: 0.85
Train Accuracy: 0.8875
```



4.8 RBF Kernel with various values for gamma

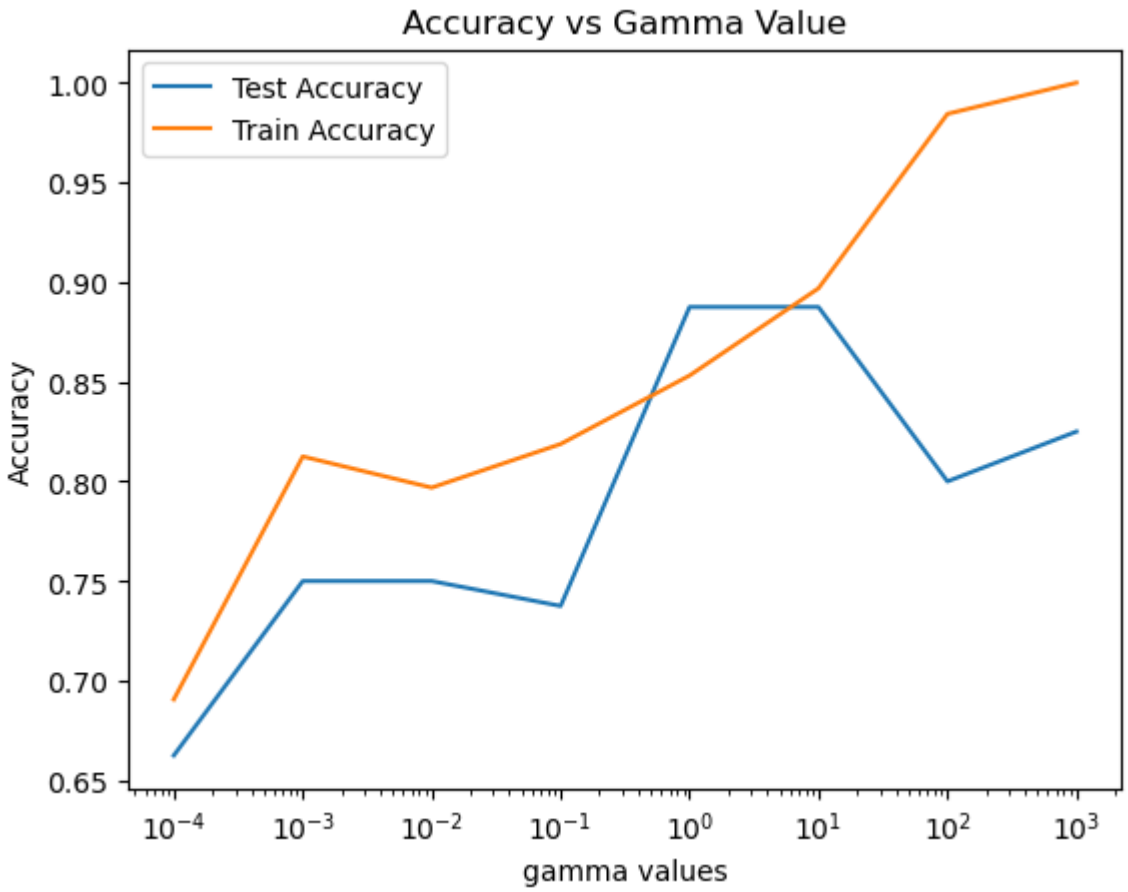
In [ ]:

```
gamma_vals = np.array([1e-4,1e-3,1e-2,1e-1,1,1e1,1e2,1e3])
test_accuracies = np.zeros(np.shape(gamma_vals))
train_accuracies = np.zeros(np.shape(gamma_vals))

for i in range(np.size(gamma_vals)):
    clf = kernelLogRegL2(lammy=0.01,kernel_fun=kernel_RBF,gamma = gamma_vals[i
    clf.fit(X_train,Y_train)

    train_accuracies[i] = get_accuracy(clf,X_train,Y_train)
```

```
    test_accuracies[i] = get_accuracy(clf,X_test,Y_test)

fig,ax = plt.subplots()
ax.plot(gamma_vals,test_accuracies,label = "Test Accuracy")
ax.plot(gamma_vals,train_accuracies,label = "Train Accuracy")
ax.set_xlabel("gamma values")
ax.set_xscale("log")
ax.set_ylabel("Accuracy")
ax.set_title("Accuracy vs Gamma Value")
ax.legend()
```

```
/tmp/ipykernel_11428/470417373.py:2: RuntimeWarning: overflow encountered in e
xp
  out = np.log(1+np.exp(x))
/tmp/ipykernel_11428/2422820963.py:20: RuntimeWarning: overflow encountered in
exp
  res = - y / (1. + np.exp(yKu))
```

Out[ ]:
```
<matplotlib.legend.Legend at 0x7f21ac242e50>
```



Problem 5 Sparse Logistic Regression

5.1 ROC curves for PECTF dataset

In [ ]:
```
import pandas as pd
import numpy as np
from sklearn.model_selection import train_test_split
from sklearn.metrics import roc_curve
from sklearn.metrics import auc
```

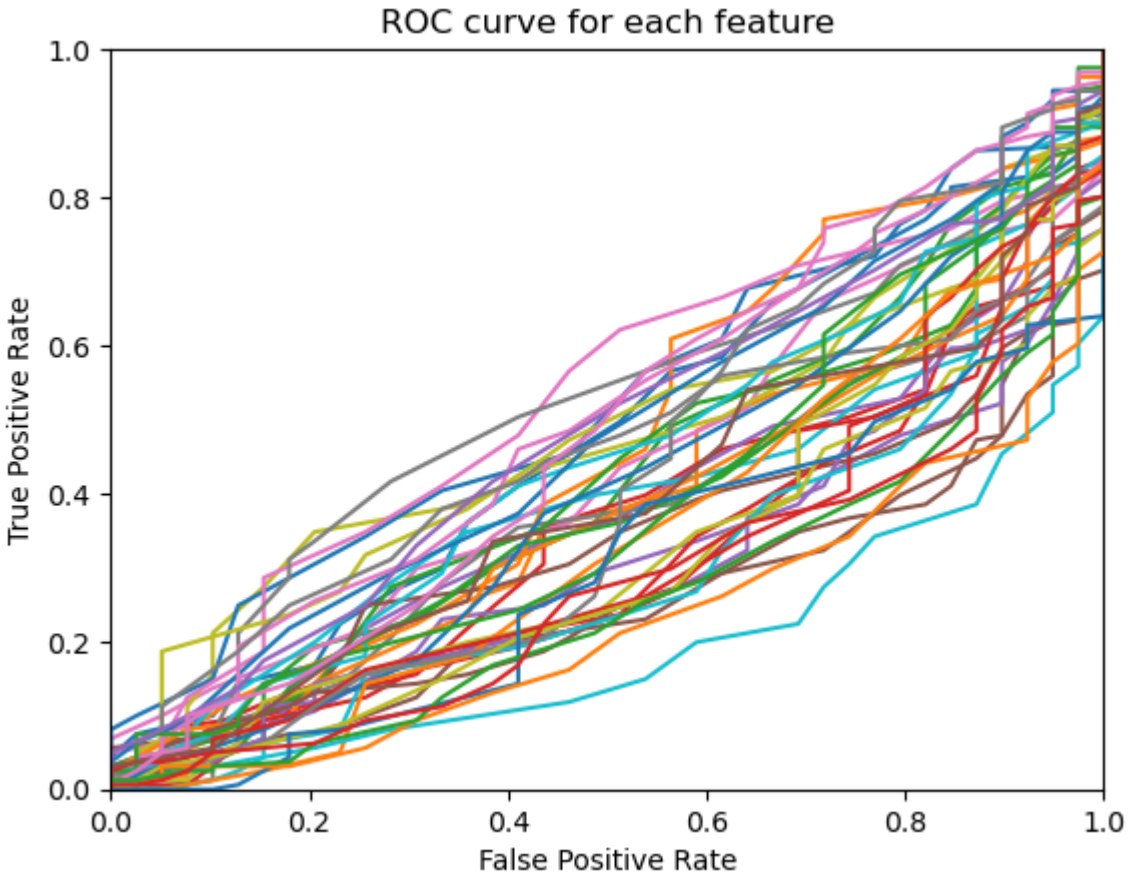
```
import matplotlib.pyplot as plt

#import the nonlineardata.csv file
df = pd.read_csv("spectf.csv",header=None)

#access the x and y data sets
data = df.to_numpy()
X = data[:,1:]
Y = data[:,0]

#split the data into 80% training and 20% testing with random state to 2022
X_train, X_test, Y_train, Y_test = train_test_split(X,Y, test_size=0.25,random
```

In [ ]:

```
num_features = np.shape(X)[1]
feature_AUCs = np.zeros(num_features)

#initialize the plot
fig,ax = plt.subplots()
for i in range(num_features):
    fpr,tpr,thresholds = roc_curve(Y_train,X_train[:,i])
    feature_AUCs[i] = auc(fpr,tpr)
    ax.plot(fpr,tpr)
ax.set_xlabel("False Positive Rate")
ax.set_ylabel("True Positive Rate")
ax.set_xlim(0,1)
ax.set_ylim(0,1)
ax.set_title("ROC curve for each feature")
```

Out[ ]:

```
Text(0.5, 1.0, 'ROC curve for each feature')
```



5.2 Creating 300 different subsets of features

To start, I recognize that the best features will have an AUC furthest from 0.5. For example, on the above plot, there are some features with an AUC well below 0.5, but this just means that the negation of the prediction was right more often than not (which is still useful). Given this, I prioritize the features with AUC furthest from 0.5. To do this, I sorted the list of features with AUC's furthest from 0.5

Next, I decided to create 150 subsets with 2 features and 150 subsets with 3 features.

For the subsets with 2 features, the first feature was randomly selected from the best 25% of features while the second feature was randomly selected from all of the features.

For the subsets with 3 features, the first feature was randomly selected from the best 15%, the second was randomly selected from the best 30%, and the third was randomly selected from all of the features

In [ ]:
```
def get_subset_feature_indicies(num_features,best_AUC_indicies,max_percentile_

    #array to track the feature indicies
    feature_indicies = []

    #variable for the selected index
    selected_index = 0
    for percentile in max_percentile_for_features:
        #for the current percentile, randomly select an index that hasn't yet
        max_idx = np.floor(percentile * num_features)
        idx = np.random.randint(0,max_idx)
        while best_AUC_indicies[idx] in feature_indicies:
            idx = np.random.randint(0,max_idx)

        #once a feature has been found that hasn't been previously selected, a
        feature_indicies.append(best_AUC_indicies[idx])

    return feature_indicies
```

In [ ]:
```
#sort for the features with the AUC furthest away from 0.5
best_AUC_indicies = np.flip(np.argsort(np.absolute(np.subtract(feature_AUCs,0.
print(feature_AUCs[best_AUC_indicies])

feature_indicies = []

#randomly select feature indicies with 2 features
max_percentile_for_features = [0.10,0.25]
for i in range(0,150):
    feature_indicies.append(get_subset_feature_indicies(num_features,best_AUC_

#randomly select feature indicies with 3 features
max_percentile_for_features = [0.05,0.15,0.30]
for i in range(0,150):
    feature_indicies.append(get_subset_feature_indicies(num_features,best_AUC_
```

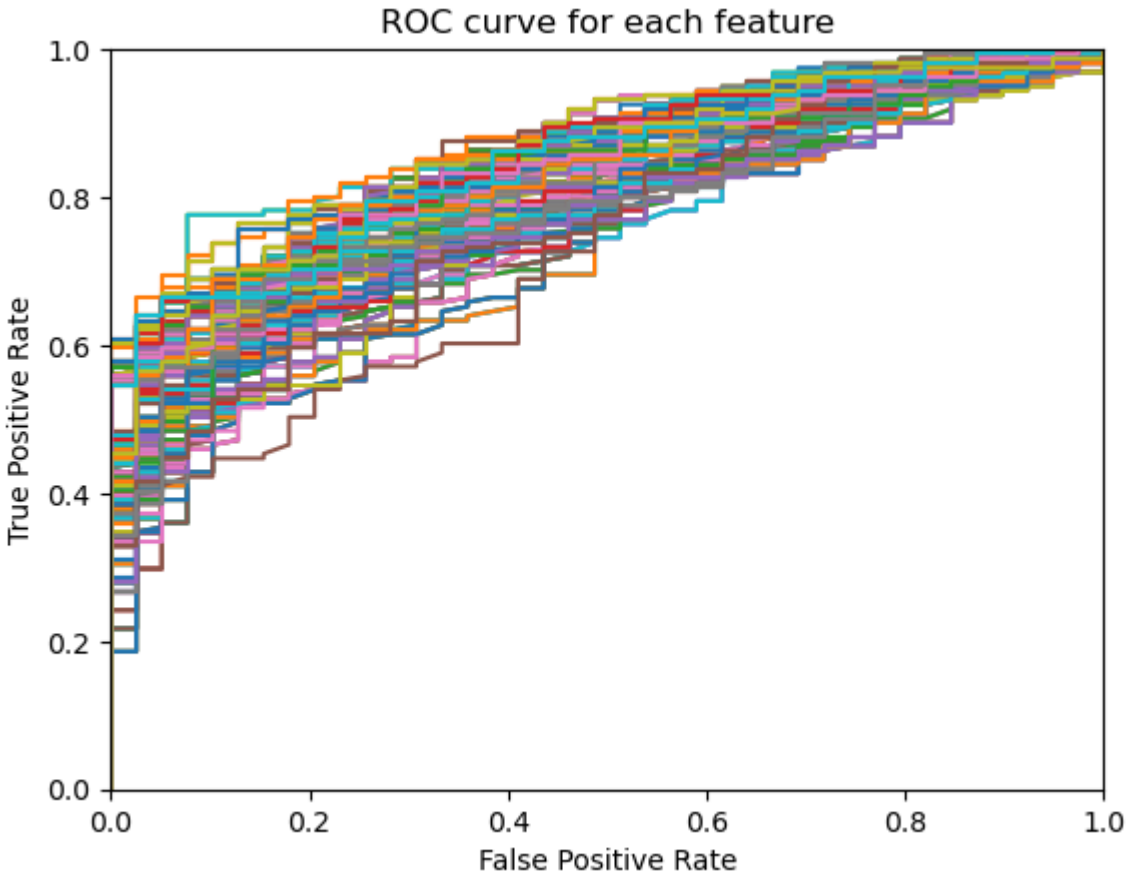
```
[0.19533365 0.22893773 0.25863991 0.25911769 0.26039178 0.27695493
 0.29009396 0.29081064 0.2938366  0.29964963 0.30737379 0.31278866
 0.31692945 0.3212295  0.33548336 0.33580188 0.33882784 0.35570951
 0.36391145 0.37641344 0.37991718 0.38469502 0.38573021 0.39010989
 0.39241918 0.39433031 0.40149705 0.41567129 0.41774168 0.41837872
 0.43629559 0.44051601 0.44792164 0.45859213 0.47396082 0.52500398
 0.47642937 0.5226947  0.47802198 0.48041089 0.48096831 0.48550725
 0.50708712 0.50294633]
```

5.3 Train Logistic regression on all 3 features

In [ ]:

```
from sklearn.linear_model import LogisticRegression

log_regression_models = []
log_regression_AUCs = np.zeros((300,1))

#initialize the plot
fig,ax = plt.subplots()

for i in range(len(feature_indicies)):

    features = feature_indicies[i]
    X_vals = X_train[:,features]
    log_regression_models.append(LogisticRegression(penalty="none"))
    log_regression_models[i].fit(X_vals,Y_train)
    f_x = log_regression_models[i].predict_proba(X_vals)

    #plot the ROC curve and save the AUC value

    fpr,tpr,thresholds = roc_curve(Y_train,f_x[:,1])
    log_regression_AUCs[i] = (auc(fpr,tpr))
    ax.plot(fpr,tpr)

ax.set_xlabel("False Positive Rate")
ax.set_ylabel("True Positive Rate")
ax.set_xlim(0,1)
ax.set_ylim(0,1)
ax.set_title("ROC curve for each feature")

max_auc_idx = np.argmax(log_regression_AUCs)
f = log_regression_models[max_auc_idx]
f_feature_indicies = feature_indicies[max_auc_idx]

out_str = "Best Model: AUC = {}, using feature indicies: {}".format(log_regres
print(out_str)
```

```
Best Model: AUC = [0.87609492], using feature indicies: [39, 25, 33]
```

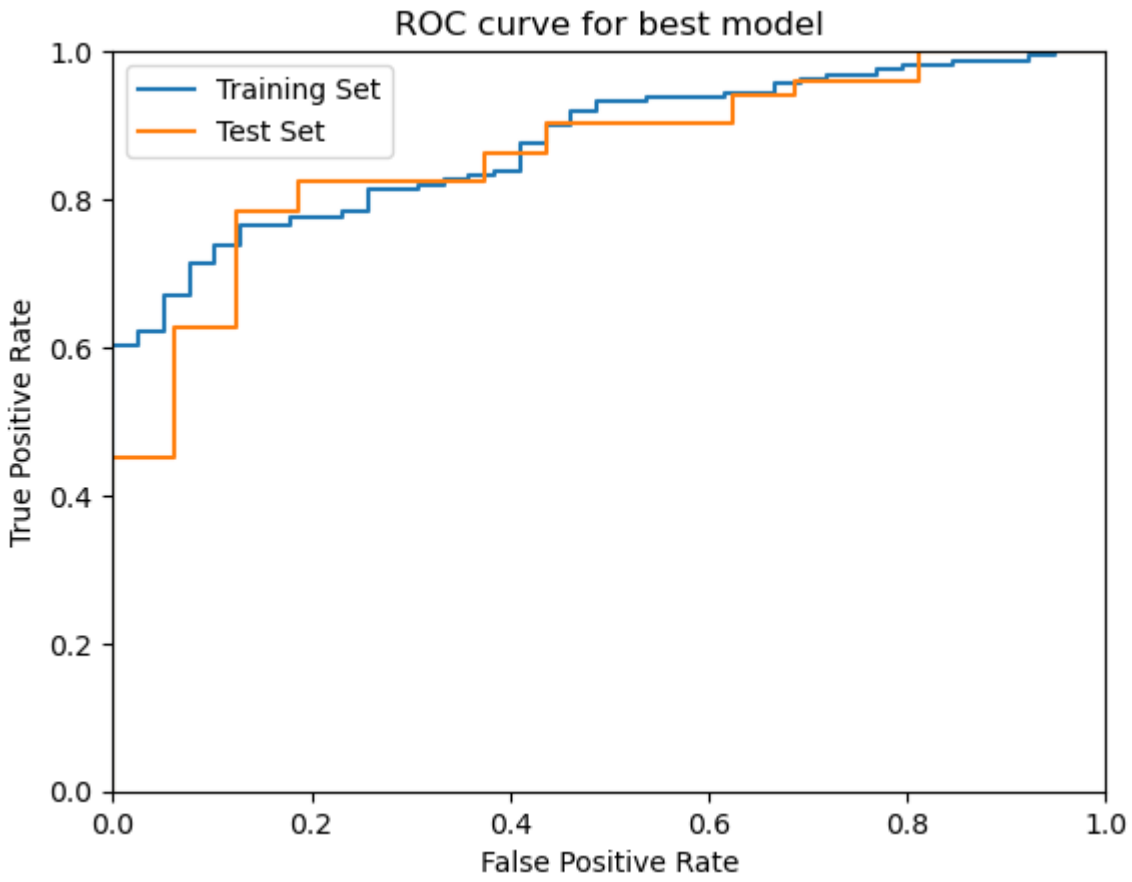


5.4 Analyzing Best Model

In [ ]:
```
# plot the test and train ROC for f on the same figure
X_vals_train = X_train[:,f_feature_indicies]
X_vals_test = X_test[:,f_feature_indicies]

f_x_train = f.predict_proba(X_vals_train)[:,1]
f_x_test = f.predict_proba(X_vals_test)[:,1]

#plot the ROC curve and save the AUC value
fig,ax = plt.subplots()

#plot training ROC
fpr,tpr,thresholds = roc_curve(Y_train,f_x_train)
train_auc = auc(fpr,tpr)
ax.plot(fpr,tpr,label = "Training Set")

fpr,tpr,thresholds = roc_curve(Y_test,f_x_test)
test_auc = auc(fpr,tpr)
ax.plot(fpr,tpr,label = "Test Set")

ax.set_xlabel("False Positive Rate")
ax.set_ylabel("True Positive Rate")
ax.set_xlim(0,1)
ax.set_ylim(0,1)
ax.set_title("ROC curve for best model")
ax.legend()

#report test and train AUC
```

```
out_str = "Test AUC: {} Train AUC: {}".format(test_auc,train_auc)
print(out_str)
```

```
Test AUC: 0.8602941176470589 Train AUC: 0.8760949195731804
```



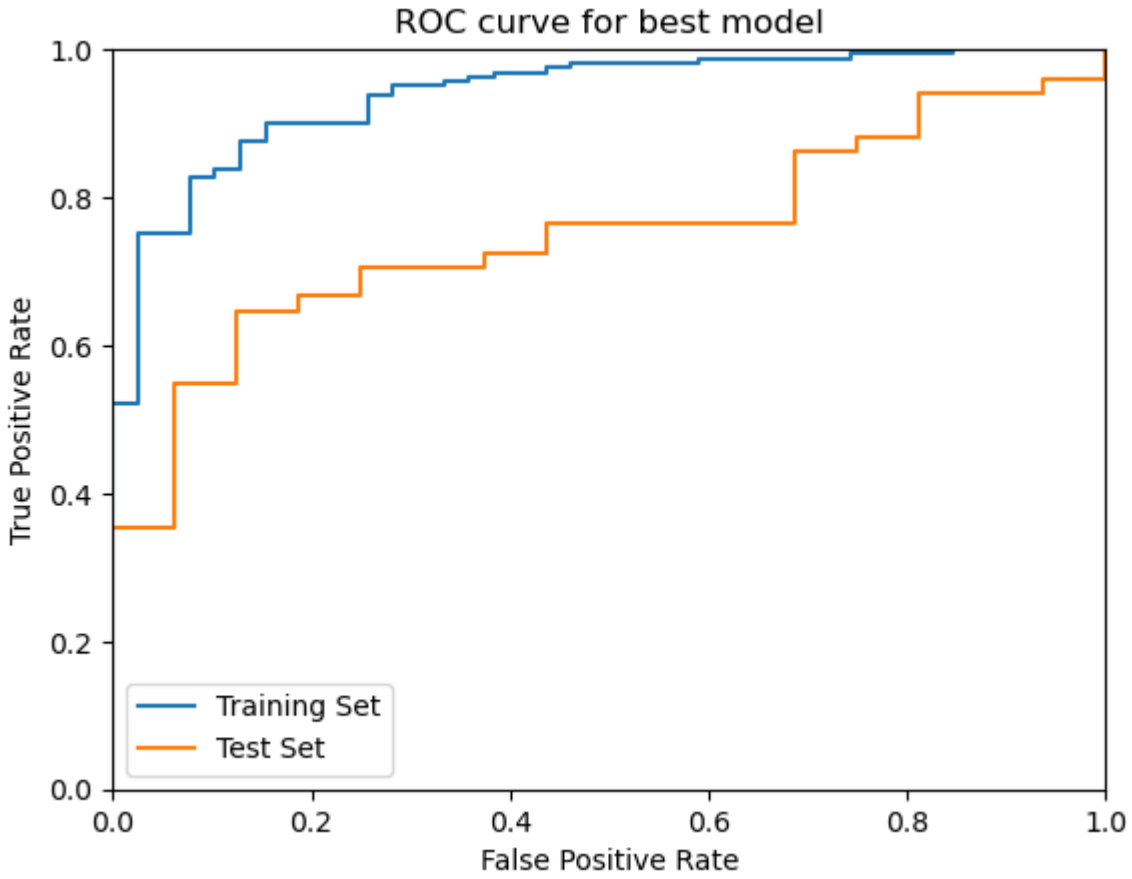
In [ ]:

```
# compare to regularlized logistic regression
clf = LogisticRegression(penalty= "l2",C=0.01)
clf.fit(X_train,Y_train)
f_x_train = clf.predict_proba(X_train)[:,1]
f_x_test = clf.predict_proba(X_test)[:,1]

#plot the ROC curve and save the AUC value
fig,ax = plt.subplots()

#plot training ROC
fpr,tpr,thresholds = roc_curve(Y_train,f_x_train)
train_auc = auc(fpr,tpr)
ax.plot(fpr,tpr,label = "Training Set")

fpr,tpr,thresholds = roc_curve(Y_test,f_x_test)
test_auc = auc(fpr,tpr)
ax.plot(fpr,tpr,label = "Test Set")

ax.set_xlabel("False Positive Rate")
ax.set_ylabel("True Positive Rate")
ax.set_xlim(0,1)
ax.set_ylim(0,1)
ax.set_title("ROC curve for best model")
ax.legend()

#report test and train AUC
```

```
out_str = "Test AUC: {} Train AUC: {}".format(test_auc,train_auc)
print(out_str)
```

```
Test AUC: 0.75 Train AUC: 0.9396400700748527
```

```
/home/david/anaconda3/lib/python3.9/site-packages/sklearn/linear_model/_logist
ic.py:814: ConvergenceWarning: lbfgs failed to converge (status=1):
STOP: TOTAL NO. of ITERATIONS REACHED LIMIT.

Increase the number of iterations (max_iter) or scale the data as shown in:
    https://scikit-learn.org/stable/modules/preprocessing.html
Please also refer to the documentation for alternative solver options:
    https://scikit-learn.org/stable/modules/linear_model.html#logistic-regress
ion
  n_iter_i = _check_optimize_result(
```



My model generalized far better than l2-regularized logistic regression did. Given this, the results show that even with 3 features, sparse-logistic regression will perform just as good (on the test set), if not better, than regularized l2 logistic regression.

6 Ridge Regression and Friends

6.1 Comparing Ridge Regression to Kernel Ridge Regression

In [ ]:
```
#generate a data set with 100 samples with 5e6 features in the distribution
from sklearn.datasets import make_regression
```

```
X,Y = make_regression(
    n_samples = 100,
    n_features= 5000000,
    n_informative= 100,
    noise= 50
)
```

In [ ]:
```
# training with Ridge Regression
from sklearn.linear_model import Ridge

clf = Ridge(alpha = 0.01)
clf.fit(X,Y)
```

Out[ ]:
```
Ridge(alpha=0.01)
```

In [ ]:
```
from sklearn.kernel_ridge import KernelRidge

clf = KernelRidge(kernel="linear",alpha=0.01)
clf.fit(X,Y)
```

Out[ ]:
```
KernelRidge(alpha=0.01)
```

See overleaf part of answer for explanation on times

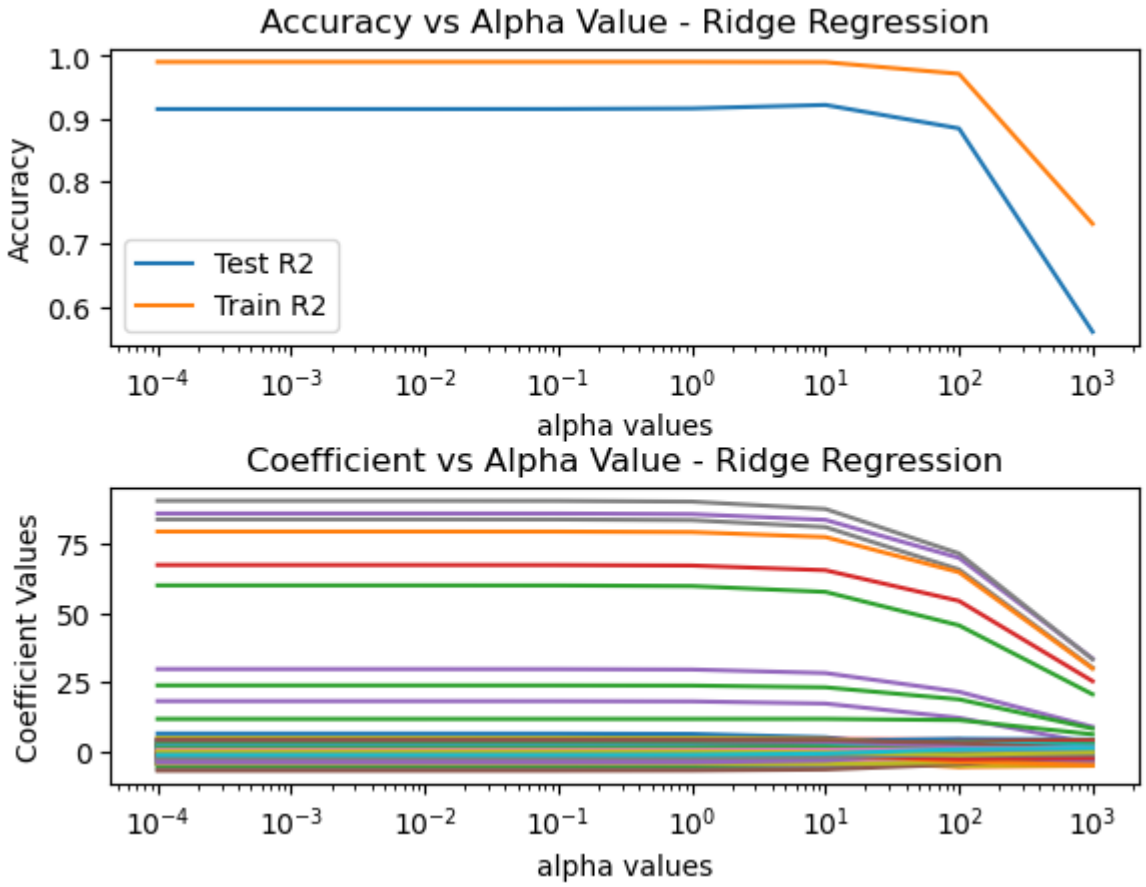
6.2 Ridge Regression and Lasso Regression

In [ ]:
```
# create a data set with 1000 samples and 500 features
from sklearn.datasets import make_regression

X,Y = make_regression(
    n_samples = 1000,
    n_features= 500,
    n_informative= 10,
    noise= 30
)

X_train, X_test, Y_train, Y_test = train_test_split(X,Y, test_size=0.2,random_
```

In [ ]:
```
from sklearn.linear_model import Ridge

alpha_vals = np.array([1e-4,1e-3,1e-2,1e-1,1,1e1,1e2,1e3])

# fit ridge regression

#arrays to track accuracies and coefficient vectors
test_accuracies = np.zeros(np.shape(alpha_vals))
train_accuracies = np.zeros(np.shape(alpha_vals))
ridge_coef_vals = np.zeros((np.shape(alpha_vals)[0],np.shape(X_train)[1]))


for i in range(np.size(alpha_vals)):
    clf = Ridge(alpha=alpha_vals[i])
    clf.fit(X_train,Y_train)

    train_accuracies[i] = clf.score(X_train,Y_train)
    test_accuracies[i] = clf.score(X_test,Y_test)
```

```
    ridge_coef_vals[i,:] = clf.coef_

fig,ax = plt.subplots(2)
fig.tight_layout(pad=3.0)

#plot the accuracies
ax[0].plot(alpha_vals,test_accuracies,label = "Test R2")
ax[0].plot(alpha_vals,train_accuracies,label = "Train R2")
ax[0].set_xlabel("alpha values")
ax[0].set_ylabel("Accuracy")
ax[0].set_xscale("log")
ax[0].set_title("Accuracy vs Alpha Value - Ridge Regression")
ax[0].legend()

#plot the coefficients
for i in range(np.shape(X_train)[1]):
    ax[1].plot(alpha_vals,ridge_coef_vals[:,i])

ax[1].set_title("Coefficient vs Alpha Value - Ridge Regression")
ax[1].set_xlabel("alpha values")
ax[1].set_xscale("log")
ax[1].set_ylabel("Coefficient Values")
```

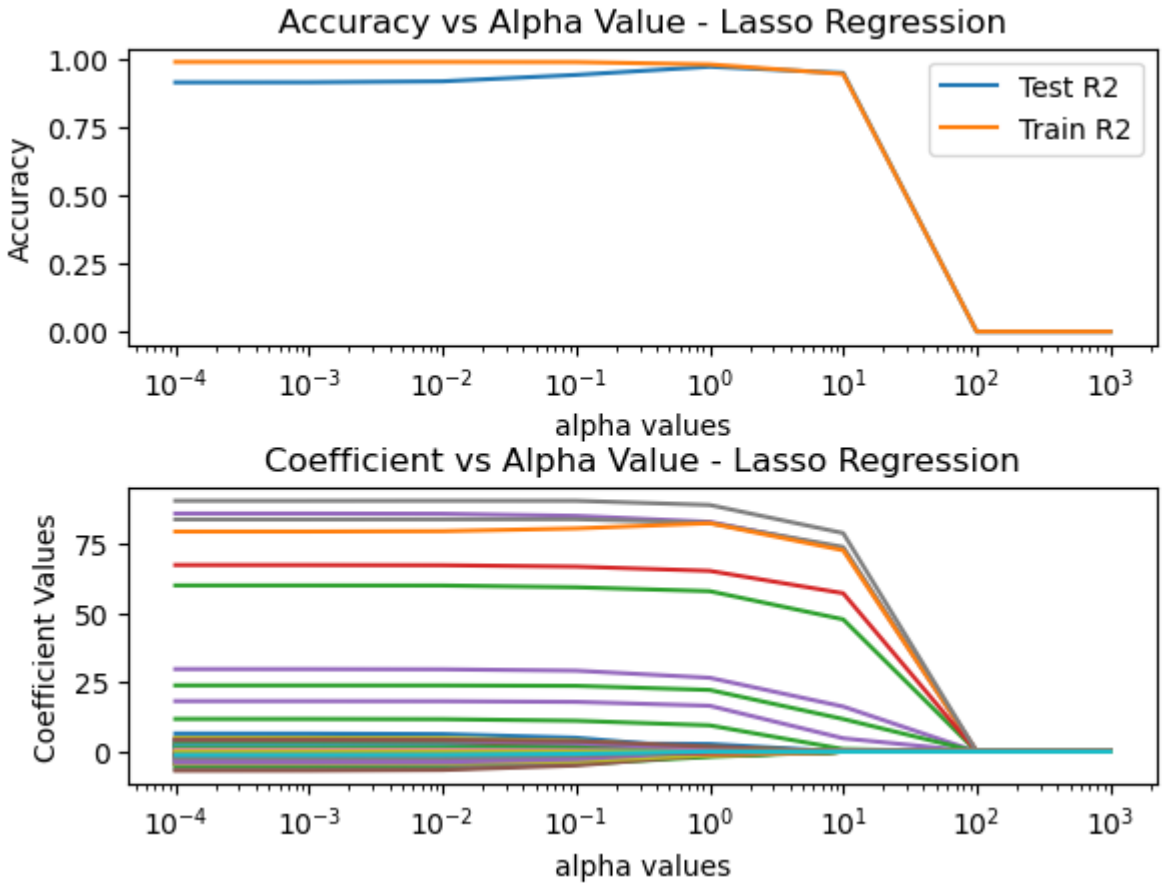
Out[ ]:
```
Text(42.597222222222214, 0.5, 'Coefficient Values')
```



In [ ]:
```
from sklearn.linear_model import Lasso

# fit ridge regression

#arrays to track accuracies and coefficient vectors
test_accuracies = np.zeros(np.shape(alpha_vals))
```

```
train_accuracies = np.zeros(np.shape(alpha_vals))
lasso_coef_vals = np.zeros((np.shape(alpha_vals)[0],np.shape(X_train)[1]))


for i in range(np.size(alpha_vals)):
    clf = Lasso(alpha=alpha_vals[i])
    clf.fit(X_train,Y_train)

    train_accuracies[i] = clf.score(X_train,Y_train)
    test_accuracies[i] = clf.score(X_test,Y_test)
    lasso_coef_vals[i,:] = clf.coef_

fig,ax = plt.subplots(2)
fig.tight_layout(pad=3.0)

#plot the accuracies
ax[0].plot(alpha_vals,test_accuracies,label = "Test R2")
ax[0].plot(alpha_vals,train_accuracies,label = "Train R2")
ax[0].set_xlabel("alpha values")
ax[0].set_ylabel("Accuracy")
ax[0].set_xscale("log")
ax[0].set_title("Accuracy vs Alpha Value - Lasso Regression")
ax[0].legend()

#plot the coefficients
for i in range(np.shape(X_train)[1]):
    ax[1].plot(alpha_vals,lasso_coef_vals[:,i])

ax[1].set_title("Coefficient vs Alpha Value - Lasso Regression")
ax[1].set_xlabel("alpha values")
ax[1].set_xscale("log")
ax[1].set_ylabel("Coefficient Values")
```

Out[ ]:
```
Text(42.597222222222214, 0.5, 'Coefficient Values')
```



I simulated a data set with quite a bit of added noise and only 10 out of the 500 features being informative (done so that I could better see the effect of l1 vs l2 regularization).

As the strength of regularization increased, the coefficients in the lambda decreased.

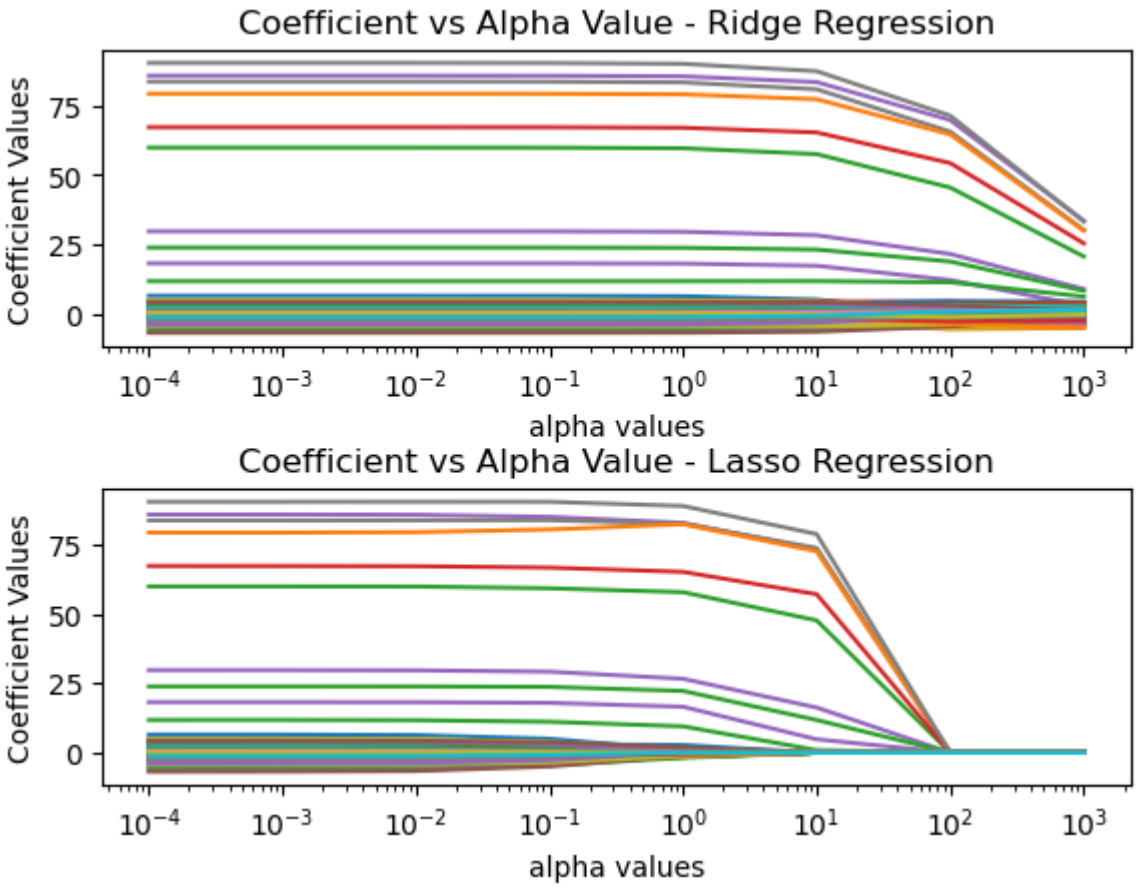
Based on my observations, one possible advantage of Lasso over Ridge Regression is that it can automatically perform feature selection by nature of using the l1 norm instead of the l2 norm. Additionally, even though Lasso uses less features, it still seemed to fit the simulated model quite well.

6.3: Regularization Paths for Lasso and Ridge Regression

In [ ]:
```
fig,ax = plt.subplots(2)
fig.tight_layout(pad=3.0)

#plot the ridge coefficients
for i in range(np.shape(X_train)[1]):
    ax[0].plot(alpha_vals,ridge_coef_vals[:,i])
    ax[1].plot(alpha_vals,lasso_coef_vals[:,i])

ax[0].set_title("Coefficient vs Alpha Value - Ridge Regression")
ax[0].set_xlabel("alpha values")
ax[0].set_xscale("log")
ax[0].set_ylabel("Coefficient Values")
```

```
#plot the Lasso
ax[1].set_title("Coefficient vs Alpha Value - Lasso Regression")
ax[1].set_xlabel("alpha values")
ax[1].set_xscale("log")
ax[1].set_ylabel("Coefficient Values")
```

Out[ ]: Text(42.597222222222214, 0.5, 'Coefficient Values')



COMPSCI 671 **Name:** *David Hunt*
HW 3
15 November, 2022

Agreement 1) This assignment represents my own work. I did not work on this assignment with others. All coding was done by myself.

Agreement 2) I understand that if I struggle with this assignment that I will reevaluate whether this is the correct class for me to take. I understand that the homework only gets harder.

1 Hoeffding and Beyond

1.1 Markov's Inequality

For any non-negative random variable X, we seek to prove the following:
$P(X \geq \epsilon) \leq \frac{E(X)}{\epsilon}$

for any positive real number $\epsilon > 0$

1. Let X be a random variable such that $Pr(X \geq 0) = 1$, and let the probability function of X be $f(x)$.
2. Let $E(X) \triangleq$ mean of X
3. Using the definition of the mean, we can say the following
   $E(X) = \sum_x xf(x) = \sum_{x<t} xf(x) = \sum_{x \geq t} xf(x)$
4. From this, we can say the following:
   $\Rightarrow E(X) \geq \sum_{x \geq t} xf(x) \geq \sum_{x \geq t} tf(x) = tPr(X \geq t)$
5. From this, we can arrive at the final proof statement
   $\Rightarrow Pr(X \geq t) = \frac{E(X)}{t}$

1.2 Chebyschev's Inequality

We seek to prove Chebyschev's inequality which states that for any random variable X with mean μ and variance σ^2, we have:
$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$

1. Let us define a random variable $Y = |X - E(X)|^2$ where $E(x) = \mu \triangleq$ mean of X
2. by definition of variance we can say the following:
   $E(Y) = E(|X - E(X)|^2) = Var(X) = \sigma^2$
3. Applying Markov's inequality to Y, we can say the following
   $P(|X - \mu| \leq \epsilon) = P(|X - \mu|^2 \leq \epsilon^2) = P(Y \leq \epsilon^2) \leq \frac{E(Y)}{\epsilon^2} = \frac{\sigma^2}{\epsilon^2}$

1.3 Polynomial Markov's Inequality

For a random variable X with a mean μ and kth central moment $E[|X-\mu|^k]$, we seek to prove the polynomial version of Markov's inequality which states that:
$P(|X-\mu| \geq \epsilon) \leq \frac{E[|X-\mu|^k]}{\epsilon^k}$

1. Define $Y = |X-\mu|^k$
2. $\Rightarrow E(Y) = E[|X-\mu|^k$
3. From here, we can then apply Markov's Inequality as proved in part 1 to complete the proof:
   $P(|X-\mu| \geq \epsilon) = P(|X-\mu|^k \geq \epsilon^k) = P(Y \geq \epsilon^k) \leq \frac{E(Y)}{\epsilon^k} = \frac{E[|X-\mu|^k}{\epsilon^k}$

1.4 Chernoff Bound

Prove the following:
$P(X-\mu \geq \epsilon) \leq \inf_{\lambda \geq 0} \frac{M_{X-\mu}(\lambda)}{exp(\lambda\epsilon)}$
where μ is the mean of $X, \lambda > 0$ is a positive real number and ϵ is any real number

1. Start by defining a random variable $Y = X - \mu$
2. Next, we can say: $P(X-\mu \geq \epsilon) = P(Y \geq \epsilon) = P(exp(\lambda Y) \geq exp(\lambda\epsilon))$
3. Apply Markov's inequality:
   $P(exp(\lambda Y) \geq exp(\lambda\epsilon)) \leq \frac{E(exp(\lambda Y))}{exp(\lambda\epsilon)} = \frac{M_Y(\lambda)}{exp(\lambda\epsilon)}$
4. Since this holds for all λ it also holds for the infimum:
   $P(exp(\lambda Y) \geq exp(\lambda\epsilon)) \leq \inf_{\lambda \geq 0} \frac{M_Y(\lambda)}{exp(\lambda\epsilon)}$
5. Simplifying and substituting arrives at the final form of the proof
   $P(X-\mu \geq \epsilon) = P(Y \geq \epsilon) = P(exp(\lambda Y) \geq exp(\lambda\epsilon)) \leq \inf_{\lambda \geq 0} \frac{M_Y(\lambda)}{exp(\lambda\epsilon)} = \inf_{\lambda \geq 0} \frac{M_{X-\mu}(\lambda)}{exp(\lambda\epsilon)}$

1.5 Hoeffding's inequality

Fist, Hoeffding's Lema is defined as follows for any random variable X taking values in the interval $[a, b]$:
$E[exp(\lambda(X-\mu))] \leq exp\left(\frac{\lambda^2(b-a)^2}{8}\right)$ Given $X_1..., X_n$ as IID bounded random variables in the range $[a, b]$ with a common mean μ, for any positive integer $n > 0$ and any real number ϵ, prove the following inequality which is a one-sided version of Hoeffding's Inequality:
$P\left(\left(\frac{1}{n}\sum_{i=1}^n X_i\right) - \mu \geq \epsilon\right) \leq exp\left(\frac{-2n\epsilon^2}{(b-a)^2}\right)$

1. Start by multiplying both sides of the probability function by n and simplifying:
   $P\left(\left(\frac{1}{n}\sum_{i=1}^n X_i\right) - \mu \geq \epsilon\right) = P\left(\left(\sum_{i=1}^n X_i\right) - n\mu \geq n\epsilon\right) = P\left(\left(\sum_{i=1}^n X_i - \mu\right) \geq n\epsilon\right)$

2. Apply the Chernoff Bound:
   $P\left(\left(\sum_{i=1}^{n} X_i\right) - n\mu \geq n\epsilon\right) \leq \inf_{\lambda \geq 0} \frac{M_{\left(\sum_{i=1}^{n} X_i\right) - n\mu}(\lambda)}{exp(\lambda n\epsilon)}$

3. Simplify $M_{\left(\sum_{i=1}^{n} X_i\right) - n\mu}(\lambda)$ using Hoeffding's Lema:
   $M_{\left(\sum_{i=1}^{n} X_i\right) - n\mu}(\lambda) = M_{\sum_{i=1}^{n} X_i - \mu}(\lambda) = E[exp(\lambda(\sum_{i=1}^{n} X_i - \mu)] = E[\prod_{i=1}^{n} exp(\lambda(X_i - \mu)]$
   $\leq \prod_{i=1}^{n} exp\left(\frac{\lambda^2(b-a)^2}{8}\right) = exp\left(\frac{\lambda^2 n(b-a)^2}{8}\right)$

4. Subsituting back into the original equation, we get the following:
   $P\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mu \geq \epsilon\right) = P\left(\left(\sum_{i=1}^{n} X_i\right) - n\mu \geq n\epsilon\right) \leq \inf_{\lambda \geq 0} \frac{M_{\left(\sum_{i=1}^{n} X_i\right) - n\mu}(\lambda)}{exp(\lambda n\epsilon)}$
   $\leq \inf_{\lambda \geq 0} \frac{exp\left(\frac{\lambda^2 n(b-a)^2}{8}\right)}{exp(\lambda n\epsilon)} = \inf_{\lambda \geq 0}\left(exp\left(\frac{\lambda^2 n(b-a)^2}{8} - \lambda n\epsilon\right)\right) = \inf_{\lambda \geq 0}\left(exp\left(\frac{\lambda^2 n(b-a)^2 - 8\lambda n\epsilon}{8}\right)\right)$

5. To find the optimatl λ take the derrivative and set it equal to zero
   $\frac{d}{d\lambda}\left(exp\left(\frac{\lambda^2 n(b-a)^2 - 8\lambda n\epsilon}{8}\right)\right) = \frac{2n(b-a)^2\lambda - 8n\epsilon}{8} exp\left(\frac{\lambda^2 n(b-a)^2 - 8\lambda n\epsilon}{8}\right) = 0$
   $\Rightarrow \frac{2n(b-a)^2\lambda}{8} exp\left(\frac{\lambda^2 n(b-a)^2 - 8\lambda n\epsilon}{8}\right) = (n\epsilon) exp\left(\frac{\lambda^2 n(b-a)^2 - 8\lambda n\epsilon}{8}\right)$
   $\Rightarrow \frac{2n(b-a)^2\lambda}{8} = n\epsilon$
   $\Rightarrow \lambda = \frac{8\epsilon}{2(b-a)^2} = \frac{4\epsilon}{(b-a)^2}$

6. Substituting this back into the original equation
   $\inf_{\lambda \geq 0}\left(exp\left(\frac{\lambda^2 n(b-a)^2}{8} - \lambda n\epsilon\right)\right) = exp\left(\frac{(\frac{4\epsilon}{(b-a)^2})^2 n(b-a)^2}{8} - (\frac{4\epsilon}{(b-a)^2})n\epsilon\right)$
   $= exp\left(\frac{2\epsilon^2 n}{(b-a)^2} - \frac{4\epsilon^2 n}{(b-a)^2}\right) = exp\left(\frac{-2n\epsilon^2}{(b-a)^2}\right)$

7. This arrives at the final form of the proof:
   $P\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mu \geq \epsilon\right) \leq exp\left(\frac{-2n\epsilon^2}{(b-a)^2}\right)$

1.6 Applying Concentration inequalities to classifier evaluation

For any $1 > \delta > 0$, and $\epsilon > 0$, use Hoeffding's inequality to determine how many samples n we need to ensure that with probabiliyt at least 1 - δ, $|R_s(g) - R(g)|$ is less than ϵ

1. let $X_i = l_{01}(y, g(X)$ with range $[0, 1]$
2. Define $R(g) := E_D[l_{01}(y, g(x)) = E(X)$
3. Define $R_s(g) := \frac{1}{n}\sum_{i=1}^{n} l_{01}(y_i, g(x_i))$
4. Start by saying: $P(|R_s(g) - R(g)| < \epsilon) = 1 - P(|R_s(g) - R(g)| \geq \epsilon)$
5. Applying Hoeffding's inequality with $a = 0, b = 1$, we can then say:
   $1 - P(|R_s(g) - R(g)| \geq \epsilon) > 1 - 2exp(\frac{-2n\epsilon}{(b-a)^2}) = 1 - 2exp(-2n\epsilon)$

6. define $\delta := 2exp(-2n\epsilon)$
7. Simplifying further:
   $\delta = 2exp(-2n\epsilon^2)$
   $ln(\frac{\delta}{2}) = -2n\epsilon^2$
   $n = \frac{-1}{2\epsilon^2} ln(\frac{\delta}{2})$
   $n = \frac{1}{2\epsilon^2} ln(\frac{2}{\delta})$
8. Thus, in order to be a good approximation we n must meet the following condition
   $n > \frac{1}{2\epsilon^2} ln(\frac{2}{\delta})$

Next, we are asked to do the same thing with Chebychev's inequality

1. for $X = R_s(X)$, with each individual element having a variance of 1. We can use the properties of the variance to say that the variance of $Var(R_S(X)) = \sigma^2/n = \frac{1}{n}$
2. Next, we can say: $P(|R_s(g) - R(g)| < \epsilon) = 1 - P(|R_s(g) - R(g)| \geq \epsilon)$ as done before
3. Next, we can apply Chebyschev's inequality to say the following:
   $1 - P(|R_s(g) - R(g)| \geq \epsilon) > 1 - \frac{1}{n\epsilon^2}$
4. If we let $\delta = \frac{1}{n\epsilon^2}$, we can easily see that $n > \frac{1}{\delta\epsilon^2}$

Why is the sample complexity lower bound/guarantee given by Hoeffding's inequality prefered over the one given by Chebyschev's inequality
It is prefered because it decays faster versus the Chebyschev's Inequality

1.7 Application to Algorithmic Stability

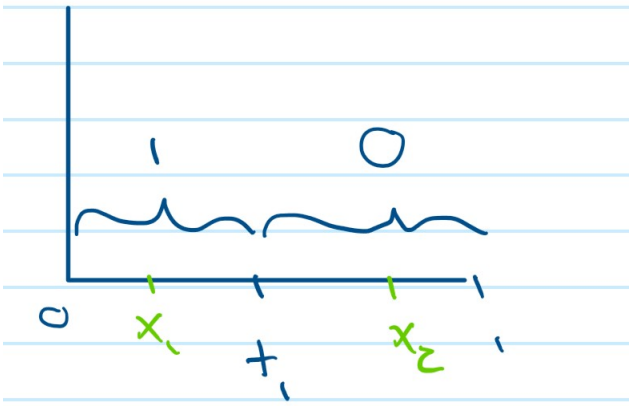
Ran out of time, did not complete

2 Classic Exercises in VC Dimmension

2.1

Find the VC dimension of the following hypothesis class. $\mathcal{F} = \{f : [0,1] \to \{0,1\}, f(x) = \mathbf{1}_{x<t}, t \in [0,1]\}$

The hypothesis class is plotted below

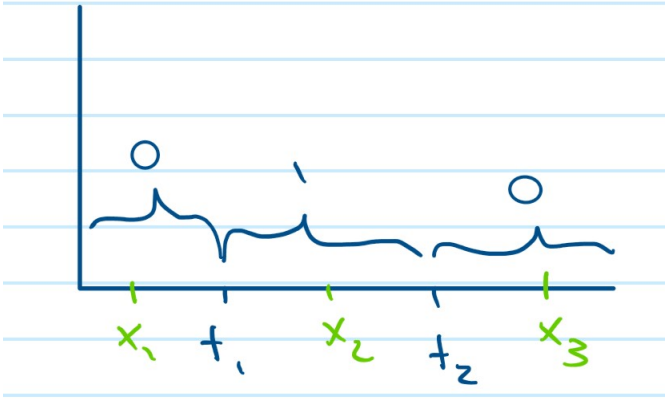


The VC dimension is the size of the largest set that a given classifier can shatter no matter what the labels are. As can be seen above, if $x_1 = 0, x_2 = 1$, the classifier cannot shatter the given set. As a result, the VC dimension is 1.

2.2

Find the VC dimension of the following hypothesis class. $\mathcal{F} = \{f : [0,1] \to \{0,1\}, f(x) = \mathbf{1}_{t_1<x<t_2}, t_1, t_2 \in [0,1]\}$

The hypothesis class is plotted below:



As can be seen above, if $x_1 = x_3 = 1, x_2 = 0$, the classifier will not be able to shatter the given set. As a result the VC dimension is 2.

2.3

Find the VC dimension of the following hypothesis class $\mathcal{F} = \{\mathbb{R}^d \to \{-1,1\}, f(\underline{x}) = sign(\underline{b}^T\underline{x}), \underline{b} \in \mathbb{R}^d\}$

1. Let $\mathcal{X} = \{\underline{e}_i\}_{i=1}^d$ where $\underline{e}_i$ are $d \times 1$ standard unit vectors.

(a) For a configuration of points as defined by $\mathcal{X}$, we can easily find a $\underline{b}$ that satisfies $f(\underline{x}) = sign(\underline{b}^T \underline{x})$. Let $\underline{b} = \sum_{i=1}^{d} y_i \underline{e}_i$.

(b) $\Rightarrow f(\underline{x}_i) = \sum_{i=1}^{d} y_i \underline{e}_i^T \underline{e}_i = y_i \underline{e}_i^T \underline{e}_i = y_i$

(c) Thus, we've shown that there is a classifier $\underline{b}$ that can shatter the configuration of d points described by $\mathcal{X}$ regardless of what the labels are.

2. However, if we were to add another point, $\underline{x}_{d+1} = \sum_{i=1}^{d} s_i \underline{x}_i$ for $x_i \in \mathcal{X}, s_i \in \mathbb{R}$, that is a linear combination of the elements in $\mathcal{X}$. However, we can show that there does not exist a $\underline{b}$ that can shatter the elements of the set regardless of label.

   (a) If, $\underline{b}$ can shatter the points of this new set, then the following would always hold: $y_{d+1} = f(x_{d+1}) = sign(\sum_{i=1}^{d} s_i) = sign(\underline{b}^T x_{d+1}) = sign(\underline{b}^T \sum_{i=1}^{d} s_i \underline{x}_i)$

   (b) However, this does not always hold. For example, if we say $s_i = \mathbf{1}_{y_i=1}$ and $y_i = -1$.

   (c) In this case $f(x_{d+1}) = sign(\underline{b}^T \sum_{i=1}^{d} s_i \underline{x}_i) > 0$.

   (d) However, since $y_i = -1$ which is a contradiction. Thus, we can say that there isn't a $\underline{b}$ that can shatter this slightly larger set.

3. The VC dimension is d

2.4

What is the VC dimension of the set of all binary decision trees with number of leaves at most l?

A decision tree can perfectly shatter at most l points if each point falls in a different leaf. However, if there are more points than leaves, then it may fail to perfectly classify each point (ex: two points with the same input, but different classifications). As a result, the VC dimension of all binary decision trees with l leaves is l.

What is the VC dimension of the set of all binary decision trees with number of splits at most d?

A decision tree with d splits will have $d + 1$ leaves. From the previous answer, this means that it will have a VC dimmension of $d + 1$.

2.5

Let $\mathcal{F}$ be a finite hypothesis class for binary classification, i.e $|\mathcal{F}| < \infty$. Show that the VC dimension of F is upper bounded by $log_2|\mathcal{F}|$

1. From the notes, the VC dimension of a class $\mathcal{F}$ is the largest n such that $\mathcal{S}_{\mathcal{F}}(n) = 2^n$ where $\mathcal{S}_{\mathcal{F}}(n)$ is the growth function .

2. Thus, for $|\mathcal{F}| < \infty$, the VC dimmension, will at most be the n such that $|\mathcal{F}| = 2^n$.

3. $\Rightarrow n = log_2|\mathcal{F}|$. Thus we have proved the upper bound.

3 Logistic Regression and Kernels

Let $\{(\underline{x}_i, y_i)\}_{i=1}^n$ be a set of training data where $\underline{x}_i \in \mathbb{R}^d$ for all i and $y_i \in \{-1, 1\}$.

Consider the l_2 regularized logistic regression model with parameter $\underline{\theta}$, where we want to find an $f_{\underline{\theta}}$ that minimizes the loss function:

$\sum_{i=1}^n ln(1 + exp(-y_i(f_{\underline{\theta}}(\underline{x}_i)))) + \lambda ||f_{\underline{\theta}}||_{\mathcal{H}}^2$

where $f_{\underline{\theta}}(\underline{x})$ is of the form $\underline{\theta}^T \underline{x}$ and $||f_{\underline{\theta}}||_{\mathcal{H}}^2 = \underline{\theta}^T \underline{\theta}$

3.1

Let $\mathcal{H}$ be the reproducing kernel Hilbert space that corresponds to the above l_2 regularized logistic regression. Using the representer theorem, what is the form of the optimal predictive function?

1. Per the notes on the representer theorem, for a fixed set $\mathcal{X}$, a kernel k and $\mathcal{H}$ as the corresponding RKHS. If $\Omega : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function and any loss function $l : \mathbb{R}^d \to \mathbb{R}$, the solution to the optimation problem:
   $f^* \in argmin_{f \in \mathcal{H}} \sum_{i=1}^n l(f(x_i), y_i) + \Omega(||f||^2)$
   can be expressed in the form $f^* = \sum_{i=1}^n \alpha_i k(x_i, .)$
2. It can easily be seen that the optimization problem described in the problem statement easily aligns with the representer theorem.
3. As such, the form of the optimal predictive function will be $f^* = \sum_{i=1}^n \alpha_i k(x_i, .)$. In this case, based on the definition of $||f_{\underline{\theta}}||_{\mathcal{H}}^2 = \underline{\theta}^T \underline{\theta}$, we can say that the kernel is a linear kernel.

3.2 L2 Regularization

Define the following function g as $g(\zeta) = ln(1 + exp(-\zeta))$.
We are asked to solve the optimization problem specified by:
$\min_{\underline{w}, \underline{\zeta}} \frac{1}{2}||\underline{w}||^2 + C \sum_{i=1}^n g(\zeta_i)$
subject to:
$y_i(\underline{w}^T \underline{x}_i) \geq \zeta_i, \forall i$

1. Start by identifying the $g_i(x)$ constraints for the Lagrangian
   (a) $g_i(x)$ constraints come in the form $g_i(x) \leq 0$
   (b) We can put the given constraints into this form as follows:
   $y_i(\underline{w}^T \underline{x}_i) \geq \zeta_i$
   $0 \geq \zeta_i - y_i(\underline{w}^T \underline{x}_i)$
2. With the $g_i(x)$ constraints in the correct form, we can then form the lagrangian:
   $\mathcal{L}(\underline{w}, \underline{\zeta}, \underline{\alpha}) = \frac{1}{2}||\underline{w}||^2 + C \sum_{i=1}^n g(\zeta_i) + \sum_{i=1}^n \alpha_i [\zeta_i - y_i(\underline{w}^T \underline{x}_i)]$

3. Next we can apply the Lagrangian Stationarity Condition

   (a) With respect to the gradient of $\underline{w}$
   $\nabla_{\underline{w}}\mathcal{L}(\underline{w}, \underline{\zeta}, \underline{\alpha}) = \underline{w} - \sum_{i=1}^{n} \alpha_i y_i \underline{x}_i = 0$
   $\Rightarrow \underline{w} = \sum_{i=1}^{n} \alpha_i y_i \underline{x}_i$

   (b) With respect to the gradient of each ζ_i
   $\nabla_{\zeta_i}\mathcal{L}(\underline{w}, \underline{\zeta}, \underline{\alpha}) = \frac{-C}{1+exp(\zeta_i}$
   $\Rightarrow \alpha_i = \frac{C}{1+exp(\zeta_i)}$
   $\Rightarrow 1 + exp(\zeta_i) = \frac{C}{\alpha_i}$
   $\Rightarrow exp(\zeta_i) = \frac{C}{\alpha_i} - 1 = \frac{C-\alpha_i}{\alpha_i}$
   $\Rightarrow \zeta_i = ln(\frac{c-\alpha_i}{\alpha_i}) = -ln(\frac{\alpha_i}{C-\alpha_i})$
   $\Rightarrow 0 < \alpha_i < C$ where it is important to note that $\alpha_i > 0$

4. Apply the dual feasible condition which says that $\alpha_i \geq 0$. However, its also worth noting that we showed that $\alpha_i > 0$ when applying the Lagrangian Stationarity Condition

5. Apply Complementary Slackness Condition

   (a) $\alpha_i[\zeta_i - y_i(\underline{w}^T \underline{x}_i)] = 0$

   (b) Since, we showed that $\alpha_i > 0$ in the lagrangian stationarity condition, it follows that
   $\Rightarrow [\zeta_i - y_i(\underline{w}^T \underline{x}_i)] = 0$

6. Rewrite the Lagrangian and simplify with the conditions
   $\mathcal{L}(\underline{w}, \underline{\zeta}, \underline{\alpha}) = \frac{1}{2}||\underline{w}||^2 + C\sum_{i=1}^{n} g(\zeta_i) + \sum_{i=1}^{n} \alpha_i[\zeta_i - y_i(\underline{w}^T \underline{x}_i)]$

   (a) Simplify $\frac{1}{2}||\underline{w}||^2$ using the first lagrangian stationarity condition
   $\frac{1}{2}||\underline{w}||^2 = \frac{1}{2}\sum_{j=1}^{n}(w_i)^2 = \frac{1}{2}\sum_{j=1}^{n}(\sum_{i=1}^{n} \alpha_i y_i \underline{x}_{ij})^2 = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{n} \alpha_i\alpha_k y_i y_k \underline{x}_{ij}\underline{x}_{kj}$
   $= \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \alpha_i\alpha_k y_i y_k \underline{x}_i^T \underline{x}_k$

   (b) Simplify $C\sum_{i=1}^{n} g(\zeta_i) = \sum_{i=1}^{n} ln(1+exp(-\zeta_i))$ using the second lagrangian stationary condition (with respect to ζ_i
   $\sum_{i=1}^{n} ln(1 + exp(-\zeta_i)) = \sum_{i=1}^{n} ln(1 + exp(ln(\frac{\alpha_i}{C-\alpha_i})))$
   $\sum_{i=1}^{n} ln(1 + \frac{\alpha_i}{C-\alpha_i})$

   (c) Finally, $\sum_{i=1}^{n} \alpha_i[\zeta_i - y_i(\underline{w}^T \underline{x}_i)]$ is always zero because the complementary slackness showed that $\alpha_i > 0$ which means that $[\zeta_i - y_i(\underline{w}^T \underline{x}_i)] = 0$.

7. Using the final simplifications, we arrive at the final form of the lagrangian for L2 regularization:
   $\mathcal{L}(\underline{\alpha}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \alpha_i\alpha_k y_i y_k \underline{x}_i^T \underline{x}_k + \sum_{i=1}^{n} ln(1 + \frac{\alpha_i}{C-\alpha_i}), 0 < \alpha_i < C$

4 Kernel Power on SVM and Regularized logistic Regression

5 Sparse Logistic Regression

6 Ridge Regression and its friends

6.1 Ridge Regression vs Kernel Ridge Regression

From my experiments, ridge regression took 2.1 seconds while kernel regression only took 0.6 seconds. The reason that ridge regression took so much longer than kernel regression can be seen in the closed form solutions for each solver.

For ridge regression, the closed form solution is $\lambda^* = (X^T X + CI)^{-1} X^T Y$. For an nxd X matrix where n is significantly smaller than d, the big-O complexity is $O(nd^2 + d^3)$

For kernel regression, the closed form solution is $\lambda^* = (XX^T + CI)^{-1} Y$. The big-O complexity is $O(dn^2 + n^3)$. Given that n is significantly larger than d, this helps to explain why kernel regression takes so less time than ridge regression in this example