Narrowband Angle Modulation

$$x_c(t) = A_c \cos(2\pi f_c t + \phi(t))$$

$$x_c(t) = Re[\underbrace{A_c \cdot e^{j\phi(t)}}_{\text{power series}} \cdot e^{j2\pi f_c t}]$$

$$x_c(t) = Re[A_c[\underbrace{1 + j\phi(t)} - \frac{\phi^2(t)}{2!} \ldots\ldots] e^{j2\pi f_c t}]$$

Approximation of the angle-modulated carrier

$$x_c(t) \cong Re[A_c e^{j2\pi f_c t} + A_c \phi(t) j e^{j2\pi f_c t}]$$

$$x_c(t) = A_c \cos(2\pi f_c t) - A_c \phi(t) \cdot \sin(2\pi f_c t)$$

Example: $m(t) = A \cdot \cos(2\pi f_m t)$

$\phi_0 = 0$ FM modulation

$$\phi(t) = k_f \int_{t_0}^{t} m(a)\,da = k_f \int_0^t A \cos(2\pi f_m a)\,da$$

$$= \frac{k_f A}{2\pi f_m} \sin(2\pi f_m t) \xrightarrow{k_f = 2\pi f_d} \frac{A f_d}{f_m} \sin(2\pi f_m t)$$

$\beta = \frac{A \cdot f_d}{f_m}$ Index of the FM modulation

FM modulated signal

$$x_c(t) = A_c \cos\Big[2\pi f_c t + \underbrace{\beta \cdot \sin(2\pi f_m t)}_{\phi(t)}\Big]$$

Spectrum of an Angle-Modulated Signal

$$\phi(t) = \beta \sin(2\pi f_m t)$$

$$x_c(t) = Re\left[A_c\, e^{j2\pi f_c t} \cdot e^{j\beta \sin(2\pi f_m t)}\right]$$

$\tilde{x}_c(t)$

$\downarrow$ periodic signal with frequency f_m

$$x_c(t) = Re\left[\tilde{x}_c(t) \cdot e^{j2\pi f_c t}\right]$$

Fourier coefficients

$$f_m \int_{-1/2f_m}^{1/2f_m} e^{j\beta \sin(2\pi f_m t)} \cdot e^{-j2\pi n \cdot f_m t}\, dt$$

$x = 2\pi f_m t$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-[jnx - \beta \cdot \sin x]} \, dx$$

Bessel function

$J_n(\beta)$

$$x_c(t) = \mathrm{Re}\Big[\underbrace{\Big(A_c \sum_{n=-\infty}^{+\infty} J_n(\beta) \cdot e^{j 2\pi n f_m t} \Big)}_{\tilde{x}_c(t)} \cdot e^{j 2\pi f_c t} \Big]$$

$$x_c(t) = A_c \sum_{n=-\infty}^{+\infty} J_n(\beta) \cdot \cos\big(2\pi \underbrace{(n f_m + f_c)}_{\text{spectrum}} t\big)$$

Properties of Bessel Function

$n \gg$

$$\boxed{J_n(\beta) = \frac{\beta^n}{2^n \cdot n!}}$$

$J_{-n}(\beta) = J_n(\beta)$, n: even

$J_{-n}(\beta) = -J_n(\beta)$, n: odd

$$\boxed{J_{n+1}(\beta) = \frac{2n}{\beta} J_n(\beta) - J_{n-1}(\beta)}$$

Power in Angle-Modulated Signals

$$\langle x_c^2(t) \rangle = \langle A_c^2 \cos^2(2\pi f_c t + \phi(t)) \rangle$$

$$= \left\langle \frac{A_c^2}{2} + \frac{A_c^2}{2}\cos(2\pi f_c t + \phi(t)) \right\rangle$$

$f_c \gg$ $$\langle x_c^2(t) \rangle = \frac{A_c^2}{2}$$

Bandwidth in Angle-Modulation

Power ratio

$$P_r = \frac{\cancel{\frac{1}{2}A_c^2} \sum_{n=-\infty}^{+\infty} J_n^2(\beta)}{\cancel{\frac{1}{2}A_c^2}}$$

$$= \sum_{n=-\infty}^{+\infty} J_n^2(\beta)$$

$$= J_0^2(\beta) + \sum_{n=1}^{+\infty} 2 J_n^2(\beta)$$

Special Case:

$P_r \geq 0.98$ || $$\boxed{B \simeq 2(\beta + 1) f_m}$$

$\beta = \frac{A \cdot f_d}{f_m}$

Bandwidth.

Deviation Ratio || $$D = \frac{\text{peak frequency deviation}}{\text{bandwidth of } m(t)}$$

General case of the bandwidth in angle-modulation signals.

$$\boxed{B = 2(D+1)\cdot W}$$

W: bandwidth of the message signal.

$D \ll 1$ $\quad \boxed{B = 2W}$ narrowband angle modulation

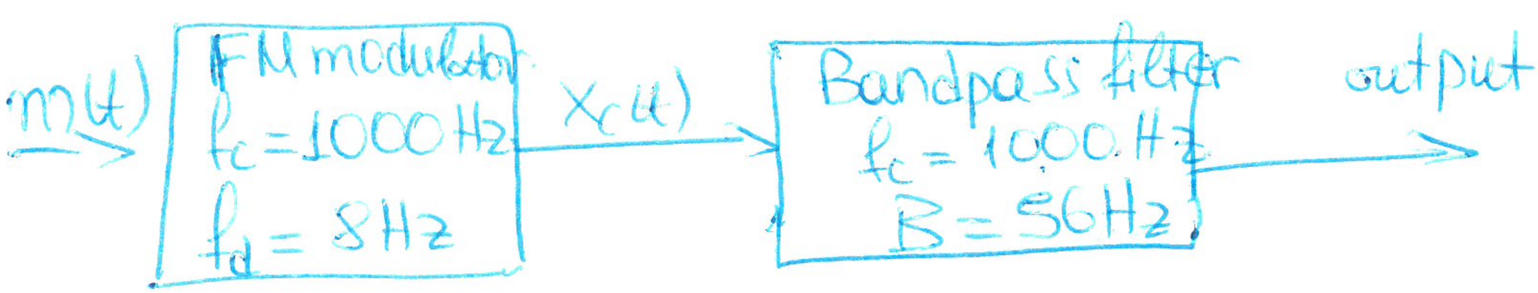
$D \gg 1$ $\quad \boxed{B = 2DW}$ wideband angle modulation

Example

FM output modulator : $x_c(t) = \overset{A_c}{100}\cos[2\pi \overset{f_c}{1000}t + \phi(t)]$

degree of freedom based on Deviation ratio

$m(t) = \underset{A}{5}\cdot\cos(2\pi \underset{f_m}{8}t)$



$x_c(t) \Longleftrightarrow X_c(f)$

$$x_c(t) = 100\cos[2\pi 1000t + \phi(t)]$$

$$k_f \int_{t_0}^{t} m(\alpha)\,d\alpha =$$

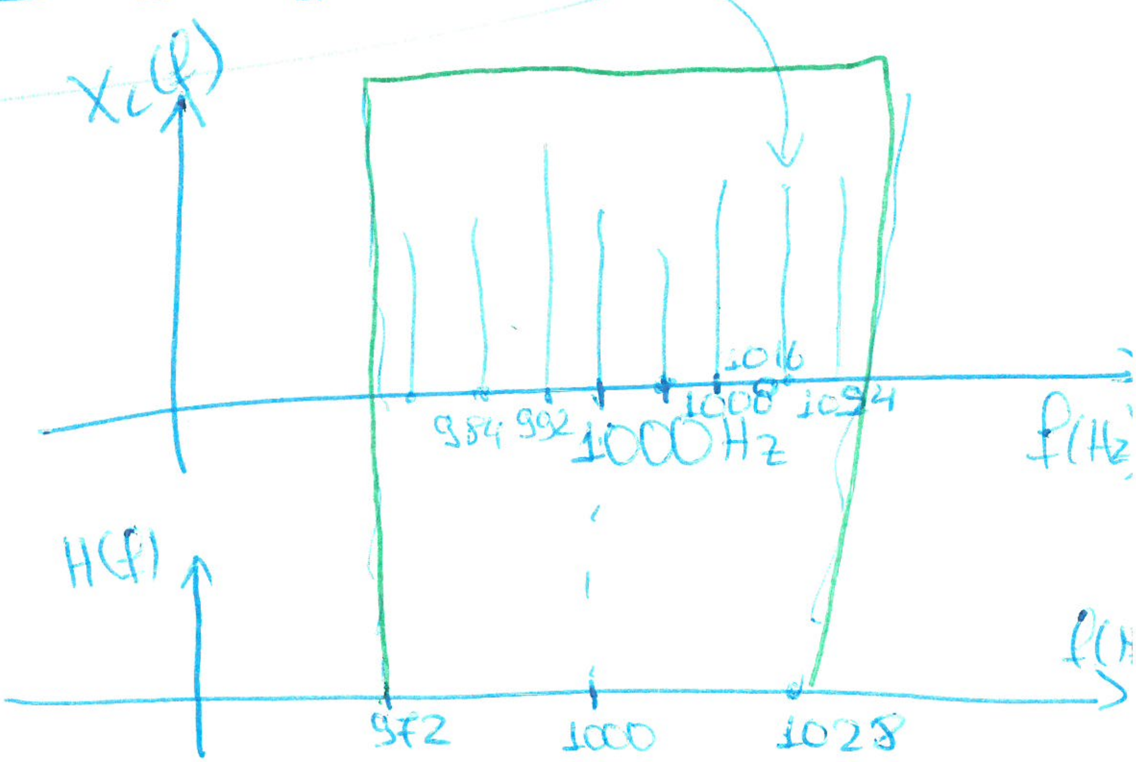
$$5k_f \int_{0}^{t} \cos(2\pi 8\alpha)\,d\alpha =$$

$$= \frac{5k_f}{2\pi 8}\sin(2\pi 8 \cdot t)$$

$$x_c(t) = 100\cos\left[2\pi 1000t + \frac{5k_f}{2\pi 8}\sin(2\pi 8 t)\right]$$

$$\Updownarrow$$

$$X_c(f) = - - - -$$



Example

$$m(t) = A\cos(2\pi f_m(t)) + B\cos(2\pi f_m(t))$$

FM modulation

General

$$\phi(t) = \beta \sin(2\pi f_m t)$$

$$\phi(t) = \beta_1 \sin(2\pi f_1 t) + \beta_2 \sin(2\pi f_2 t)$$

$\beta = \frac{A \cdot f_d}{f_m}$ Index of the FM modulation ②

$$\beta_1 = \frac{A \cdot f_d}{f_1}, \quad \beta_2 = \frac{A \cdot f_d}{f_2}$$

FM modulated signal

$$x_c(t) = A_c \cos(2\pi f_c t + \phi(t))$$

$$x_c(t) = A_c \cos(2\pi f_c t + \beta_1 \sin(2\pi f_c t) + \beta_2 \sin(2\pi f_c t))$$

$$= \mathrm{Re}\left[A_c \cdot e^{j2\pi f_c t} \cdot e^{j\beta_1 \sin(2\pi f_1 t)} \cdot e^{j\beta_2 \sin(2\pi f_2 t)}\right]$$