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Properties of the unit impulse function:

1) $\delta(at) = \frac{1}{|a|}\delta(t)$

2) $\delta(-t) = \delta(t)$

3) $\int_{t_1}^{t_2} x(t)\delta(t-t_0)dt = \begin{cases} x(t_0), & t_1 < t_0 < t_2 \\ 0, & \text{otherwise} \\ \underline{\text{undefined}}, & t_0 = t_1 \\ & t_0 = t_2 \end{cases}$

4) $x(t)\cdot\delta(t-t_0) = x(t_0)\delta(t-t_0)$, $x(t)$ contin. $t=t_0$

5) $\int_{t_1}^{t_2} x(t)\delta^{(n)}(t-t_0)dt = (-1)^n x^{(n)}(t_0)$, $t_1 < t_0 < t_2$

6) If $f(t) = g(t)$

$$\left.\begin{aligned} f(t) &= a_0\delta(t) + a_1\delta^{(1)}(t) + \dots + a_n\delta^{(n)}(t) \\ g(t) &= b_0\delta(t) + b_1\delta^{(1)}(t) + \dots + b_n\delta^{(n)}(t) \end{aligned}\right\} \Rightarrow \begin{aligned} a_0 &= b_0 \\ a_1 &= b_1 \\ &\vdots \\ a_n &= b_n. \end{aligned}$$

Examples:

1) $\int_2^5 \cos(3\pi t)\delta(t-1)dt = 0$

2) $\int_0^5 \cos(3\pi t)\delta(t-1)dt = \cos(3\pi t)\Big|_{t=1} = \cos(3\pi) = -1$

(prop. 3)

3) $\int_0^5 \cos(3\pi t)\, \frac{d\delta(t-1)}{dt}\, dt = (-1)^1 \cdot \frac{d\cos(3\pi t)}{dt}\Big|_{t=1}$

(prop. 5)

$= 3\pi \sin(3\pi) = 0$

4) $\int_{-10}^{10} \cos(3\pi t)\underline{\underline{\delta(2t)}}\, dt = \int_{-20}^{20} \cos(3\pi t)\, \frac{1}{2}\delta(t)\, d$

$\delta(t-t_0)$, $t_0 = 0$

$= \frac{1}{2}\cos(3\pi \cdot 0) = \frac{1}{2}\cos(0) = \frac{1}{2}$ (prop. 1)

5) $2\delta(t) + 3\frac{d\delta(t)}{dt} = a\delta(t) + b\frac{d\delta(t)}{dt} + c\cdot\frac{d^2\delta(t)}{dt^2}$

$a = 2$

$b = 3$ (prop. 6.)

$c = 0$

6) $\frac{d[e^{-4t}u(t)]}{dt} = \frac{de^{-4t}}{dt}\cdot u(t) + e^{-4t}\frac{du(t)}{dt}$

$= -4\cdot e^{-4t}u(t) + e^{-4t}\delta(t)$

Signals

Energy

total energy:

$$E \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

If and only if

$E < \infty$

$P = 0$

Power

Average Power:

$$P \triangleq \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

If and only if:

$0 < P < \infty$

$E = \infty$

Example 1:

$$x_1(t) = A \cdot e^{-at} u(t) \quad , \; a > 0$$

$$E = \int_{-\infty}^{+\infty} |A \cdot e^{-at} u(t)|^2 dt$$

$$= \underbrace{\int_{-\infty}^{0} |(A \cdot e^{-at}) \cdot 0|^2 dt}_{\to 0} + \int_{0}^{+\infty} |A \cdot e^{-at} \cdot 1|^2 dt$$

$$= \int_{0}^{+\infty} A^2 \cdot e^{-2at} dt = \frac{A^2}{2a} < \infty \quad \text{energy signal}$$

$a \to 0$: $\quad x_2 = A \cdot u(t)$ (4)

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |A\,u(t)|^2 dt = \cancel{\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{0} |A\cdot 0|^2 dt} \to 0$$

$$+ \lim_{T\to\infty} \frac{1}{2T}\int_{0}^{T} A^2\, dt = \lim_{T\to\infty} \frac{1}{2\cancel{T}}\cancel{T}\cdot A^2 = \frac{A^2}{2} < \infty.$$

power signal

Example 2: $\quad \tilde{x}(t) = A \cdot e^{j(\omega_0 t + \theta)}$ rotating phasor

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\tilde{x}(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |A\cdot e^{j(\omega_0 t+\theta)}|^2 dt$$

Euler's Theorem

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |A[\cos(\omega_0 t+\theta) + j\sin(\omega_0 t+\theta)]|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2 \left(\sqrt{\underbrace{\cos^2(\omega_0 t+\theta)+\sin^2(\omega_0 t+\theta)}_{1}}\right)^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2\, dt = A^2 < \infty \quad \text{power signal.}$$

Periodic signals $\quad P = \dfrac{1}{T_0}\displaystyle\int_{t_0}^{t_0+T_0} |x_p(t)|^2 dt$

Example:

$$x_p(t) = A \cdot \cos(\omega_0 t + \theta)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |A \cdot \cos(\omega_0 t + \theta)|^2 dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} A^2 \cos^2(\omega_0 t + \theta)\, dt$$

$$= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \frac{A^2}{2} dt + \cancel{\frac{1}{T_0}\int_{t_0}^{t_0+T_0} \frac{A^2}{2}\cos[2(\omega_0 t + \theta)]\, dt} \to 0$$

$$= \frac{A^2}{2}$$

Fourier Series

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j\omega_0 n t}, \quad t_0 \le t < t_0 + T_0.$$

X_n → Fourier Coefficient

$$\boxed{X_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t) \cdot e^{-jn\omega_0 t}\, dt}$$

Example: $x(t) = \cos(\omega_0 t) + \sin^2(2\omega_0 t) =$

$$\sin^2(2\omega_0 t) = \frac{1}{2} - \frac{1}{2}\cos(4\omega_0 t)$$

$$= \cos(\omega_0 t) + \frac{1}{2} - \frac{1}{2}\cos(4\omega_0 t)$$

$$= \frac{1}{2} + \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} - \frac{1}{4}e^{+j4\omega_0 t} - \frac{1}{4}e^{-j4\omega_0 t}$$

$X_0 = \frac{1}{2}$, $X_4 = -\frac{1}{4}$

$X_1 = \frac{1}{2}$, $X_{-4} = -\frac{1}{4}$

$X_{-1} = \frac{1}{2}$

Symmetry Properties

$$X_n^* = X_{-n}$$

$$X_n = |X_n| e^{j\angle X_n}$$

$$|X_n| = |X_{-n}|$$ Amplitude Even Symmetry

$$\angle X_n = -\angle X_{-n}$$ Phase Odd Symmetry.

Even signal $x(t) = x(-t)$

$X_n \rightarrow$ real

Odd signal $x(t) = -x(-t)$

$X_n \rightarrow$ imaginary.

Odd halfwave symmetry :

$$x\left(t \pm \frac{T_0}{2}\right) = -x(t)$$

$$X_n = 0 \,, \quad n = 0, \pm 2, \pm 4, \ldots$$

Read pages 22-27