STATISTICAL AVERAGES

In some cases we cannot obtain the pdf ⇒ So for the description of random variables we use statistical averages or mean values.

- **Average of a Discrete Random Variable**

⇒ Random discrete variable X

⇒ Possible values of the X: $x_1, x_2, \ldots, x_m$.

⇒ Respective Probabilities: $P_1, P_2, \ldots, P_m$.

Statistical Average/Expectation

$$\overline{X} = E[X] = \sum_{j=1}^{m} x_j * P_j$$

$X_1 = 100$ $\quad P_1 = 0.8.$

$X_2 = 30.$ $\quad P_2 = 0.2.$

$$\Rightarrow \overline{X} = \sum_{j=1}^{2} X_j * P_j$$

$$= X_1 * P_1 + X_2 * P_2.$$

$$= 100*0.8 + 30*0.2.$$

Proof.

We will use the relative frequency of observations. Let's say that the experiment is repeated a large number of times N, and X_1 is observed n_1 times, X_2 is observed n_2 times etc. In order to get the arethmetical average:

$$\frac{n_1 X_1 + n_2 X_2 + \ldots + n_m X_m}{N} = \sum_{j=1}^{m} X_j \frac{n_j}{N}.$$

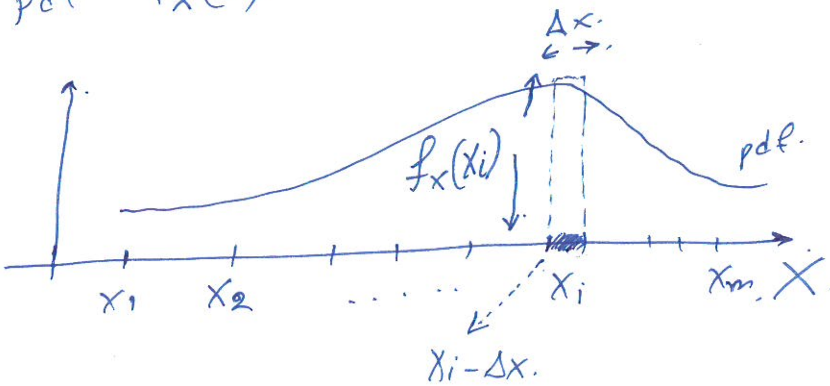
By the relative frequency interpretation when $N \to \infty$ then $\lim\limits_{N\to\infty} \frac{n_j}{N} = P_j.$

- Average of a Coutinous Raudom Variable.

⇒ Raudom Coutinuous Variable X

⇒ pdf $f_X(x)$

⇒



We cousider the range of value that X may take on, say X_0 to X_m, to be broken up into a large uumber of small subintervals of length Δx.

Probability that X lies between $X_i - \Delta X$ and X_i:

$P(X_i - \Delta x \leq X \leq X_i) \cong f_X(X_i) * \Delta X$.

[We have approximated X by a discrete raudom variable that takes on the values $X_0, X_1, \dots, X_m$ with probabilities $f_X(X_0)\Delta X, \dots, f_X(X_m)\Delta$

Expectation

$$E[X] \cong \sum_{i=0}^{M} X_i f_X(X_i) \Delta x.$$

As $\Delta x \to 0$ the $E[X]$ becomes a better approximation

So as $\Delta x \to 0 \Rightarrow \Delta x = dx$

$$\text{So} \quad E[X] = \int_{-\infty}^{+\infty} x f_X(x)\, dx.$$

<u>Note</u>: $E[X]$ → expectation / mean. / first moment of X.

* Average of a Function of a random Variable

⇒ We are interested in statistical averages of functions of X.

⇒ $Y = g(X)$ $[\ Y = X^2 + 1\ ]$.

So: $E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy$

where $\underline{f_Y(y)}$ is the pdf of Y.

⇓

We do not have it!

So:

$$\bar{Y} = E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

EXAMPLE.

Suppose that the random variable Θ has the pdf:

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & |\theta| \leq \pi \\ 0, & \text{otherwise.} \end{cases}$$

(Θ: continuous)

We have to compute:

$$E[\Theta^n] = \int_{-\infty}^{+\infty} \theta^n * f_\Theta(\theta)\, d\theta = \int_{-\pi}^{\pi} \theta^n * \frac{1}{2\pi}\, d\theta$$

$\downarrow$

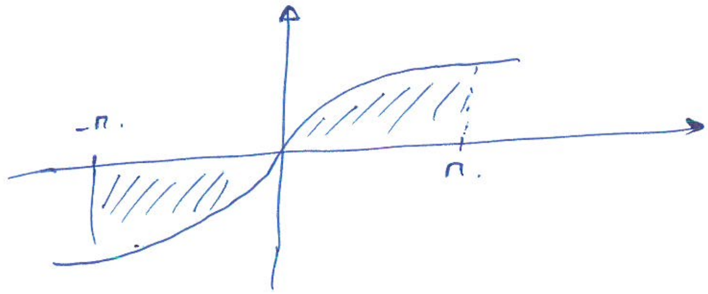
$g(\Theta) = \Theta^n$

$$\Leftrightarrow E[\Theta^n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta^n\, d\theta.$$

⇒

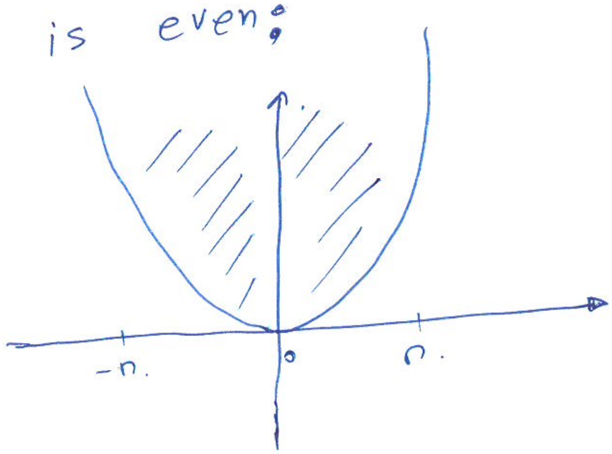
- Let's suppose that $n=3$.

$$\int_{-\pi}^{\pi} \theta^3 d\theta = \frac{\theta^4}{4}\Big|_{-\pi}^{\pi} = \frac{\pi^4}{4} - \frac{(-\pi)^4}{4} = 0.$$



$\Rightarrow$ If n is odd, then $E[\Theta^n] = 0$.

$\Rightarrow$ If n is even:



$$E[\Theta^n] = \cancel{2} * \frac{1}{\cancel{2}\pi} \int_0^{\pi} \theta^n d\theta = \frac{1}{\pi} \cdot \frac{\theta^{n+1}}{n+1}\Big|_0^{\pi} = \frac{\pi^{n+1}}{n+1} - 0 = \frac{\pi^{n+1}}{n+1}.$$

$\Rightarrow$ If $n=1 \Rightarrow E[\Theta] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta \, d\theta = \frac{1}{2\pi} \cdot \frac{\theta^2}{2}\Big|_{-\pi}^{\pi}$

$$= \frac{1}{2\pi} \cdot \left(\frac{\pi^2}{2} - \frac{(-\pi)^2}{2}\right) = 0.$$

Average of a Function of More than One Random Variables

- Function $g(X, Y)$ of two random variables X and Y
- The expectation of g is defined in a manner analogous to the case of a single random variable
- $f_{XY}(x, y)$: Joint pdf of X and Y

→ $$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy.$$

It can be generalized for more than 2 variables.

⇒ The generalization include also the single random variable case.

Suppose that $g(X,Y)$ is replaced by $h(X)$.

So:

$$E[h(X)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x)\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty} h(x) f_X(x)\, dx.$$

where:

$$\int_{-\infty}^{+\infty} f_{XY}(x,y)\, dy = f_X(x)$$

• Average of a Function of More than One Random Variable

⇒ Function $g(X,Y)$ of two random variables X and Y.

⇒ The expectation of g is defined in a manner analogous to the case of a single random variable.

⇒ $f_{XY}(x,y)$: joint pdf of X and Y.

So, for the expectation we have:

$$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

This equation can be generalized for more than 2 random variables.

• The generalization includes also the single random variable case. (Proof).
Suppose that $g(X,Y)$ is replaced by $h(X)$.

So:

$$E[g(X,Y)] = E[h(X)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x) f_{XY}(x,y)\, dx\, dy.$$

$$= \int_{-\infty}^{+\infty} h(x) f_X(x)\, dx$$

where:

$$\int_{-\infty}^{+\infty} f_{XY}(x,y)\, dy = f_X(x).$$

EXAMPLE.

- $g(X,Y) = XY$
- joint pdf : $f_{XY}(x,y) = \begin{cases} 2 * e^{-(2x+y)} & x,y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$

$$E[g(X,Y)] = E[XY] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy f_{XY}(x,y)\, dx\, dy$$

$$= \int_0^{+\infty} \int_0^{+\infty} 2xy e^{-(2x+y)}\, dx\, dy$$

$$= 2 \int_0^{+\infty} x \cdot e^{-2x}\, dx \int_0^{+\infty} y \cdot e^{-y}\, dy.$$

Integration by parts.

$$\int f\, dg = fg - \int g\, df.$$

- $\int_0^{+\infty} x \cdot e^{-2x}\, dx$

let $f = x$ and $dg = \cdot e^{-2x}$

$df = dx$ and $g = -\frac{e^{-2x}}{2}$ $\left[\rightsquigarrow \left(-\frac{e^{-2x}}{2} \right)' = + \frac{2e^{-2x}}{2} = e^{-2x} \right.$

$\Rightarrow$

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$$\int_0^{+\infty} x \cdot e^{-2x}\,dx = -x \cdot \left(\frac{e^{-2x}}{2}\right)\Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-2x}}{2}\,dx$$

$$= \left[-x \cdot \frac{e^{-2x}}{2}\right]_0^{+\infty} + \frac{1}{2}\left[\frac{e^{-2x}}{2}\right]_0^{+\infty}.$$

- $\lim_{x \to \infty}\left[-x \cdot \frac{e^{-2x}}{2}\right] = 0.$
- $\lim_{x \to +\infty}\left[\frac{e^{-2x}}{2}\right]_0^{+\infty} = 0.$
- $\frac{1}{2}\left[\frac{e^{0}}{2}\right] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$

$$\int_0^{+\infty} x \cdot e^{-2x}\,dx = \frac{1}{4}.$$

Similarly, $\int_0^{+\infty} y \cdot e^{-y} = 1.$

$$E\left[g(X,Y)\right] = 2 * \frac{1}{4} * 1 = \frac{1}{2}.$$