Convolution :

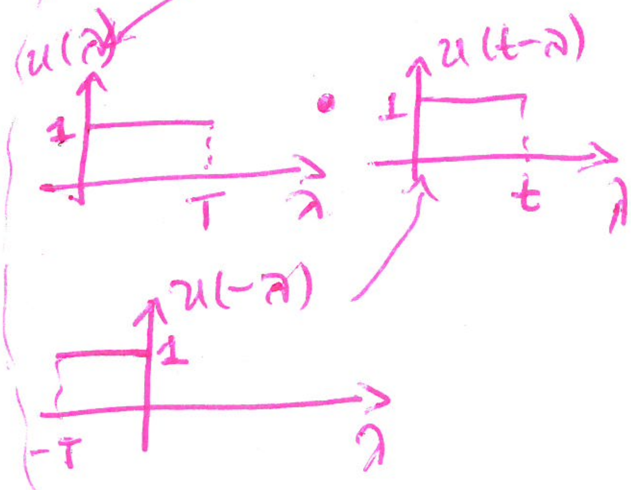
$$x(t) = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda) \cdot x_2(t-\lambda)\, d\lambda$$

Example :

$$x_1(t) = e^{-\alpha t} \cdot u(t) \qquad , \qquad \alpha, \beta > 0$$
$$x_2(t) = e^{-\beta t} \cdot u(t)$$

$$x(t) = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda) \cdot x_2(t-\lambda)\, d\lambda$$

$$= \int_{-\infty}^{+\infty} e^{-\alpha\lambda} \cdot u(\lambda) \cdot e^{-\beta(t-\lambda)} \boxed{u(t-\lambda)}\, d\lambda$$



$$= \begin{cases} 0 & , \lambda < 0 \\ 1 & , 0 < \lambda < t \\ 0 & , \lambda > t \end{cases}$$

$$= \begin{cases} 0 & , t < 0 \\ \int_{-\infty}^{+\infty} e^{-\beta t} \cdot e^{-(\alpha-\beta)\lambda}\, d\lambda & , t \geq 0 \end{cases}$$

$$= e^{-\beta t} \cdot \left[-\frac{1}{a-\beta}\right] \cdot \left[e^{-(a-\beta)\lambda}\right]_0^t$$

$$= -\frac{1}{a-\beta} e^{-\beta t} \left[e^{-(a-\beta)t} - 1\right]$$

$$= -\frac{1}{a-\beta} \left[e^{-at \cancel{+\beta t - \beta t}} - e^{-\beta t}\right]$$

$$= -\frac{1}{a-\beta} \left[e^{-at} - e^{-\beta t}\right]$$

Properties & Transform Theorems

* Superposition Theorem

$$a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 X_1(f) + a_2 X_2(f)$$

* Time - Delay Theorem **

$$x(t - t_0) \longleftrightarrow X(f) \cdot e^{-j2\pi f t_0}$$

* Scale - Change Theorem

$$x(a \cdot t) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

* Duality Theorem **

$$X(t) \longleftrightarrow x(-f)$$

* Frequency-Translation Theorem

$$x(t)\cdot e^{-j2\pi f_0 t} \longleftrightarrow X(f-f_0)$$

* Modulation Theorem ***

$$x(t)\cdot \cos(2\pi f_0 t) \longleftrightarrow \frac{1}{2}X(f-f_0)+\frac{1}{2}X(f+f_0)$$

* Differentiation Theorem

$$\frac{d^n x(t)}{dt^n} \longleftrightarrow (j2\pi f)^n X(f)$$

* Integration Theorem

$$\int_{-\infty}^{t} x(\lambda)\,d\lambda \longleftrightarrow (j2\pi f)^{-1} X(f) + \frac{1}{2}X(0)\,\delta(f)$$

* Convolution Theorem

$$\int_{-\infty}^{+\infty} x_1(\lambda)\,x_2(t-\lambda)\,d\lambda \triangleq \int_{-\infty}^{+\infty} x_1(t-\lambda)\,x_2(\lambda)\,d\lambda \longleftrightarrow X_1(f)\cdot X_2(f)$$

* Multiplication Theorem

$$x_1(t) \cdot x_2(t) \longleftrightarrow X_1(f) * X_2(f) =$$

$$\int_{-\infty}^{+\infty} X_1(\lambda) \cdot X_2(f-\lambda)\, d\lambda$$

Example 1:

$$2AW \cdot \mathrm{sinc}(2Wt) \longleftrightarrow A\Pi\left(\frac{f}{2W}\right)$$

from Tables.

$$x(t) = A\Pi\left(\frac{t}{\tau}\right) \longleftrightarrow A\tau \cdot \mathrm{sinc}(f\tau) = X(f)$$

$$X(t) = A\tau \cdot \mathrm{sinc}(t\tau) \longleftrightarrow \underbrace{A\Pi\left(-\frac{f}{\tau}\right)}_{x(-f)} =$$

Even function
$\Pi(-x) = \Pi(x)$

$$= A\Pi\left(\frac{f}{\tau}\right) \underline{\underline{\tau = 2W}}$$

$$= A\Pi\left(\frac{f}{2W}\right)$$

Example 2:

$$A\delta(t) \longleftrightarrow A$$

$$\mathcal{F}[A\delta(t)] = A\underbrace{\int_{-\infty}^{+\infty}\delta(t)\cdot e^{-j2\pi ft}dt}$$

$$= A \cdot 1 = A$$

Example 3:

$$A\delta(t-t_0) \longleftrightarrow A e^{-j2\pi f t_0}$$

Time-delay theorem to $A\delta(t) \longleftrightarrow A$.

Example 4:

$$A \longleftrightarrow A\delta(f) \qquad X(f)$$

Duality Theorem

$$x(t) = A\delta(t) \longleftrightarrow A$$

Example 5:

$$\left(f_s = \frac{1}{T_s}\right)$$

Fourier Series

$$y_s = \sum_{m=-\infty}^{+\infty}\delta(t-mT_s) = \sum_{m=-\infty}^{+\infty} Y_m e^{-j2\pi m f_s t}$$

Fourier Coefficient

$$Y_m = \frac{1}{T_s}\underbrace{\int_0^{T_s}\delta(t)\cdot e^{-j2\pi m f_s t}dt}_{1} = \frac{1}{T_s} = f_s$$

$$y_s = f_s \sum_{m=-\infty}^{+\infty} e^{-j2\pi m f_s t}$$ (time domain) ⑥

$$Y_s(f) = f_s \sum_{m=-\infty}^{+\infty} F\left[1 \cdot e^{-j2\pi m f_s t}\right]$$

$$= f_s \cdot \sum_{m=-\infty}^{+\infty} \delta(f - m f_s)$$

$$\boxed{\sum_{m=-\infty}^{+\infty} \delta(t - mT_s) \Longleftrightarrow f_s \sum_{m=-\infty}^{+\infty} \delta(f - m f_s)}$$

Fourier Transforms for Periodic Signals

$$x(t) = \left[\sum_{m=-\infty}^{+\infty} \delta(t - mT_s)\right] * p(t) = \sum_{m=-\infty}^{+\infty} \left[\delta(t - mT_s) * p(t)\right]$$

$$= \sum_{m=-\infty}^{+\infty} \left[p(t - mT_s)\right]$$

$$X(f) = F\left[\sum_{m=-\infty}^{+\infty} \delta(t - mT_s)\right] \cdot P(f)$$

$$= \left[f_s \cdot \sum_{m=-\infty}^{+\infty} \delta(f - m f_s)\right] \cdot P(f)$$

$$= f_s \sum_{m=-\infty}^{+\infty} \delta(f - m f_s) \cdot P(f)$$

$$= f_s \cdot \sum_{m=-\infty}^{+\infty} P(m f_s) \cdot \delta(f - m f_s)$$

$$P(f) \cdot \delta(f - n f_s) = P(n f_s) \cdot \delta(f - n f_s)$$

__Example:__

$$y_s(t) = \left[\sum_{n=-\infty}^{+\infty} \delta(t - m T_s) \right] * \overbrace{\Pi\left(\frac{t}{\tau}\right) \cos(2\pi f_0 t)}^{P(t)}$$

$$= \sum_{n=-\infty}^{+\infty} \Pi\left(\frac{t - m T_s}{\tau}\right) \cos\left(2\pi f_0 (t - n T_s)\right)$$

$$Y(f) = \sum_{n=-\infty}^{+\infty} \frac{A f_s \tau}{2} \Big[\mathrm{sinc}(n f_s - f_0)\tau +$$

$$\mathrm{sinc}(n f_s + f_0)\tau \Big] \cdot \delta(f - n f_s)$$

Poisson Sum Formula

$$\sum_{m=-\infty}^{+\infty} p(t - mT_s) = f_s \sum_{m=-\infty}^{+\infty} P(mf_s)\, e^{-j2\pi m f_s t}$$

p ↑ pulse-type signal.

Power Spectral Density $S(f)$

Total Power $\quad P = \int_{-\infty}^{+\infty} S(f)\, df = \langle x^2(t) \rangle$

$$= \lim_{T\to\infty} \int_{-T}^{T} \frac{1}{2T} x^2(t)\, dt.$$