11/20/2019 ①

Statistically stationary processes in the **strict sense**

→ N-fold pdf is <u>independent</u> of the <u>time origi</u>

↳ depends on time differences.

→ <u>mean</u> and <u>variance</u> : independent of time

→ <u>covariance</u> : depends on time difference

Statistically stationary processes in the **wide-sense**

→ N-fold pdf <u>depends</u> on the time origin

→ <u>mean</u> and <u>variance</u> : independent of time.

→ <u>covariance</u> : depends on time difference

Random variable $\xrightarrow{\text{output}}$ number

Random process → function of time
(Stochastic)

Partial Description of Random PRocesses

Mean: $m_X(t) = E(X(t)) = \overline{X(t)}$

Variance: $\sigma_X^2 = E\{[X(t) - \overline{X(t)}]^2\}$

$$= \overline{X^2(t)} - \overline{X(t)}^2$$

Covariance: $\mu_X(t, t+\tau) = E\{[X(t) - \overline{X(t)}] \cdot [X(t+\tau) - \overline{X(t+\tau)}]\} =$

$$= \underbrace{E[X(t) \cdot X(t+\tau)]}_{\text{autocorrelation function}} - \overline{X(t)} \cdot \overline{X(t+\tau)}$$

$$R_X(t_1, t_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1 x_2 \, f_{X_1 X_2}(x_1, t_1 ; x_2, t_2)\, dx_1\, dx_2$$

Ergodic Processes

time and ensemble averages are interchangeable

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cdots \qquad \int_{-\infty}^{+\infty}\cdots$$

$$m_x = E[X(t)] = \langle X(t)\rangle$$

$$\sigma_x^2 = E\left\{\left[X(t)-\overline{X(t)}\right]^2\right\} = \left\langle\left[X(t)-\overline{X(t)}\right]^2\right\rangle$$

$$R_X(\tau) = E\left[X(t)\cdot X(t+\tau)\right] = \langle X(t)\cdot X(t+\tau)\rangle$$

Example

Random Process: $n(t) = A\cdot\cos(2\pi \underbrace{f_0}_{\text{constant}} t + \underbrace{\theta}_{\text{random variable}})$

pdf: $$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & |\theta| \le \pi \\ 0, & \text{otherwise} \end{cases}$$

Statistical averages

(ensemble)

First moment:

$$\overline{n(t)} = \int_{-\infty}^{+\infty} A\cos(2\pi f_0 t + \theta)\, f_\theta(\theta)\, d\theta =$$

$$= \int_{-\pi}^{\pi} \frac{A\cos(2\pi f_0 t + \theta)}{2\pi}\, d\theta = 0$$

Second moment:

$$\overline{n^2(t)} = \int_{-\infty}^{+\infty} A^2 \cos^2(2\pi f_0 t + \theta)\, f_\theta(\theta)\, d\theta$$

$$= \int_{-\pi}^{\pi} \frac{A^2\cos^2(2\pi f_0 t + \theta)}{2\pi}\, d\theta =$$

$$= \frac{A^2}{4\pi} \int_{-\pi}^{\pi} \left[1 + \cos(4\pi f_0 t + 2\theta)\right] d\theta$$

$$= \frac{A^2}{2}$$

$$\overline{\sigma_x^2(t)} = \overline{x^2} - \overline{x}^2 = \frac{A^2}{2}$$

statistically stationary

Time Averages:

(5)

$$\langle n(t)\rangle = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} A\cos(2\pi f_0 t+\theta)\,dt = 0$$

$$\langle n^2(t)\rangle = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} A^2\cos^2(2\pi f_0 t+\theta)\,dt =$$

$$= \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \frac{A^2}{2}\left[1+\cos(4\pi f_0 t+2\theta)\right]dt$$

$$= \frac{A^2}{2}$$

$$\overline{\sigma_x^2(t)} = \overline{X^2} - \overline{X}^2 = \frac{A^2}{2} \longrightarrow \text{ergodic process}$$

Example:

$$\text{pdf} \quad f_\theta(\theta) = \begin{cases} \frac{2}{\pi}, & |\theta| \le \frac{1}{4}\pi \\ 0, & \text{otherwise} \end{cases}$$

$$\overline{n(t)} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} A \cdot \cos(2\pi f_0 t + \theta) \cdot \frac{2}{\pi} \, d\theta$$

$$= \frac{2}{\pi} \Big[A \cdot \sin(2\pi f_0 t + \theta) \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{2\sqrt{2}}{\pi} A \cdot \cos(2\pi f_0 t)$$

$$\overline{n^2(t)} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} A^2 \cos^2(2\pi f_0 t + \theta) \frac{2}{\pi} \, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{A^2}{\pi} \left[1 + \cos(4\pi f_0 t + 2\theta) \right] d\theta$$

$$= \frac{A^2}{2} + \frac{A^2}{\pi} \cos(4\pi f_0 t)$$

NOT statistically stationary

$\langle n(t) \rangle = \ldots$

$\langle n^2(t) \rangle = \ldots$

check if the process is not ergodic

Ergodic Processes - Physical Meaning

1) DC Component: $\overline{X(t)} = \langle X(t) \rangle$ mean

2) DC Power: $\overline{X(t)}^2 = \langle X(t) \rangle^2$

3) Total Power: $\overline{X^2(t)} = \langle X^2(t) \rangle$

4) Power in ac component:

$$\sigma_X^2 = \overline{X^2(t)} - \overline{X(t)}^2$$
$$= \langle X^2(t) \rangle - \langle X(t) \rangle^2$$

5) AC + DC Power:

$$\overline{X^2(t)} = \sigma_X^2 + \overline{X(t)}^2 .$$

Power Spectral Density

$$\hookrightarrow S(f) \overset{F}{\longleftrightarrow} R(\tau)$$

↑ autocorrelation function

Average Power: $R_{\underline{X}}(0) = \int_{-\infty}^{+\infty} S(f)\,df$

Random Process: $n(t, \zeta_i)$

$$\downarrow$$

$n_T(t, \zeta_i)$ truncated version

$$n_T(t,\zeta_i) = \begin{cases} n(t,\zeta_i), & |t| \leq \frac{1}{2}T \\ 0, & \text{otherwise} \end{cases}$$

$\downarrow$ fourier

$$N_T(f,\zeta_i) = \int_{-\frac{T}{2}}^{\frac{T}{2}} n(t,\zeta_i)\cdot e^{-j2\pi f t}\,dt$$

Energy Spectral Density: $|N_T(f,\zeta_i)|^2$

Power Spectral Density $[-\frac{T}{2}, \frac{T}{2}]$: $\frac{|N_T(f,\zeta_i)|^2}{T}$

Example :

Random Process $\quad n(t,\theta) = A \cdot \cos(2\pi f_0 t + \theta)$

random variable

$$n_T(t,\theta) = A \cdot \Pi\left(\frac{t}{\tau}\right) \cdot \cos(2\pi f_0 t + \theta)$$

$$\cos(2\pi f_0 t) \Leftrightarrow \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$

$$F\{\cos(2\pi f_0 t + \theta)\} \Leftrightarrow \frac{1}{2}\delta(f-f_0)\cdot e^{j\theta} + \frac{1}{2}\delta(f+f_0)\, e^{-j\theta}$$

$$F\left\{\Pi\left(\frac{t}{\tau}\right)\right\} \Leftrightarrow T \cdot \mathrm{sinc}(Tf)$$

$$N_T(f,\theta) = \frac{1}{2} A \cdot T \cdot \left[e^{j\theta} \cdot \mathrm{sinc}[(f-f_0)T] + e^{-j\theta} \cdot \mathrm{sic}[(f+f_0)\cdot T]\right]$$

Energy spectral Density :

$$|N_T(f,\theta)|^2 = \left(\frac{1}{2}AT\right)^2 \cdot \left\{ \mathrm{sinc}^2[(f-f_0)T] + \right.$$

$$+ e^{j2\theta} \cdot \cancel{\mathrm{sinc}[T(f-f_0)] \cdot \mathrm{sinc}[T(f+f_0)]} + e^{2j\theta} \cdot \cancel{\mathrm{sinc}[T(f-f_0)] \cdot \mathrm{sinc}[T(f+f_0)]} + \mathrm{sinc}^2[T(f+f_0)] \Big\}$$

$$= \left(\frac{1}{2} A T\right)^2 \left[\mathrm{sinc}^2[T(f-f_0)] + \mathrm{sinc}^2[T(f+f_0)] \right]$$

Power Spectral Density:

$$S_n(f) = \lim_{T \to \infty} \frac{|N_T(f,\theta)|^2}{T} =$$

$$\boxed{\lim_{T \to \infty} T\, \mathrm{sinc}^2(Tu) = \delta(u)}$$

$$= \lim_{T \to \infty} \frac{1}{4} A^2 T \cdot \left\{ \mathrm{sinc}^2[T(f-f_0)] + \mathrm{sinc}^2[T(f+f_0)] \right\}$$

$$= \frac{1}{4} A^2 \delta(f-f_0) + \frac{1}{4} A^2 \delta(f+f_0)$$

Average Power: $\int_{-\infty}^{+\infty} S_n(f)\, df = \frac{A^2}{2}$