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Course ID: ECE 341 Communication Systems- Fall
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Homework #6

Corresponding to Chapter 6 of Principles of Communications, Rodger E. Zimmer and William H. Tranter, John Wiley, 7th Edition.

1. An honest coin is flipped 10 times. (a) Determine the probability of the occurrence of either 5 or 6 heads. (b) Determine the probability of the first head occurring at toss number 5.

2. Passwords in a computer installation take the form $X_1X_2X_3X_4$, where each character X_i id one of the 26 letters of the alphabet. Determine the maximum possible number of different passwords available for assignment for the following condition: (a) A given letter of the alphabet can be used only once in a password. (b) If selection of letters for a given password is completely random, what is the probability that your competitor could access, on a single try, your computer in part (a)?

3. Assume that 20 honest coins are tossed. By applying the binomial distribution, find the probability that there will be fewer than 3 heads.

4. A digital data transmission system has an error probability of 10^{-5} per digit, find the probability of exactly 1 error in 10^5 digits.

5. If the random variable X is Gaussian, with zero mean and variance σ^2, obtain numerical values for the following probability: $P(|X|>\sigma)$.

6. Two jointly Gaussian zero-mean random variables, X and Y, have respective variances of 3 and 4 and correlation coefficient $\rho_{XY}=-0.4$. A new random variable is defined as $Z=X+2Y$. Write down an expression for the pdf of Z.

7. Two Gaussian random variables, X and Y, are independent. Their respective means are 4 and 2, and their respective variances are 3 and 5. Write down expressions for the marginal pdfs. Write down an expression for their joint pdf. What is the mean of $Z_1=3X+Y$? $Z_2=3X-Y$? What is the variance of $Z_1=3X+Y$? $Z_2=3X-Y$? Write down an expression for the pdf of $Z_1=3X+Y$.

To be delivered at instructor’s office: 27 November 2019

Good Luck!

1. An honest coin is flipped 10 times.

(a) Determine the probability of the occurrence of either 5 or 6 heads.

$n = 10$

$P(H) = P(T) = \frac{1}{2}$

$$P(H=5) = \frac{n!}{p!(1-p)!} = \frac{10!}{5!5!} = 252$$

$$P(H=6) = \frac{n!}{p!(1-p)!} = \frac{10!}{6!4!} = 210$$

$$\frac{252+210}{2^{10}} = \frac{231}{531} \approx \boxed{0.451}$$

(b) Determine the probability of the first head occurring at toss number 5.

$$P(H) = \frac{1}{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4} = \frac{1}{32} = \boxed{0.0313}$$

2. Passwords in a computer installation take the form $X_1X_2X_3X_4$, where each character X_i is one of the 26 letters of the alphabet. Determine the maximum possible number of different passwords available for assignment for the following condition:

(a) A given letter of the alphabet can be used only once in a password.

$$n = (26)(25)(24)(23) = \boxed{358800}$$

(b) If selection of letters for a given password is completely random, what is the probability that your competitor could access, on a single try, your computer in part (a)?

$$P(X) = \frac{1}{358800} \approx \boxed{2.7871 \times 10^{-6}}$$

3. Assume that 20 honest coins are tossed. By applying the binomial distribution, find the probability that there will be fewer than 3 heads.

$$P(X) = \sum_{k=0}^{2} \binom{20}{k}\left(\frac{1}{2}\right)^{10} = \frac{211}{1048576} \approx \boxed{2.0123 \times 10^{-4}}$$

4. A digital data transmission system has an error probability of 10^{-5} per digit. Find the probability of exactly 1 error in 10^5 digits.

$$P_E = 10^{-4} \qquad \bar{k} = (10^{-5})(10^{5}) = 1$$

$$P(K=1) = \sum_{k=1} \frac{(\bar{K})^k}{k!} e^{-\bar{k}} = \frac{1}{e} \approx \boxed{0.3679}$$

5. If the random variable X is Gaussian, with zero mean and variance σ^2, obtain a numerical value for the following probability: $P(|X| > \sigma)$.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{x^2}{2\sigma^2}\right] \rightarrow F_X(x) = \int \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{x^2}{2\sigma^2}\right]$$

$$P(|X| > \sigma) = 1 - P(|X| \le \sigma) = 1 - F_X(\sigma) = 1 - \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{\cancel{\sigma}}{\cancel{\sigma}\sqrt{2}}\right)\right]$$

$$= 1 - \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{1}{\sqrt{2}}\right)\right] = \boxed{0.15866}$$

6. Two jointly Gaussian zero-mean random variables, X and Y, have respective variances of 3 and 4 and correlation coefficient $\rho_{XY} = -0.4$. A new random variable is defined as $Z = X + 2Y$. Write down an expression for the pdf of Z.

$$E(Z) = aE(\cancel{X})^{0} + bE(\cancel{Y})^{0} = 0$$

but $Z = X + 2Y$

$\therefore\ a = 1 \quad b = 2$

$$\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_X\sigma_Y\rho_{XY}$$

$$\sigma^2_Z = (3) + (4)(4) + 2(2)(\sqrt{3})(\sqrt{4})(-0.4)$$

$$= 13.45754$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma_Z^2}} \exp\left[\frac{-(z - m_z)^2}{2\sigma_Z^2}\right] = \frac{1}{\sqrt{2\pi\, 13.5}} \exp\left[\frac{-(z-0)^2}{2(13.5)}\right]$$

$$= \boxed{0.10875\, e^{-z^2/26.91487}}$$

7. Two Gaussian random variables, X and Y, are independent. Their respective means are 4 and 2, and their respective variances are 3 and 5.

(a) Write down expressions for their marginal pdfs.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left[\frac{-(x - m_x)^2}{2\sigma_x^2}\right] = \frac{1}{\sqrt{2\pi(3)}} \exp\left[\frac{-(x-4)^2}{2(3)}\right]$$

$$= \boxed{0.2303\, e^{-1/6(x-4)^2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp\left[\frac{-(y - m_y)^2}{2\sigma_y^2}\right] = \frac{1}{\sqrt{2\pi(5)}} \exp\left[\frac{-(y-2)^2}{2(5)}\right]$$

$$= \boxed{0.1784\, e^{-1/10(y-2)^2}}$$

(b) Write down an expression for their joint pdf.

since independent, $f_{XY}(x,y) = f_X(x)\, f_Y(y)$

$$f_{XY}(x,y) = \frac{e^{-1/6(x-4)^2} \times e^{-1/10(y-2)^2}}{\sqrt{6\pi} \times \sqrt{10\pi}}$$

$$= \boxed{\frac{\sqrt{15}\, e^{-1/10 y^2 + \frac{2}{5}y - \frac{2}{5}} \times e^{-1/6 x^2 + \frac{4}{3}x - \frac{8}{3}}}{30\pi}}$$

(c) What is the mean of $Z_1 = 3X + Y$? $Z_2 = 3X - Y$?

$$Z_1 = 3(E(X)) + E(Y) = 3(4) + (2) = \boxed{14}$$

$$Z_2 = 3(E(X)) - E(Y) = 3(4) - (2) = \boxed{10}$$

(d) What is the variance of $Z_1 = 3X + Y$? $Z_2 = 3X - Y$

$$\sigma_x^2 = E(x^2) - E(x)^2 \rightarrow 3 = E(x^2) - (4)^2 \rightarrow E(x^2) = 19$$

$$\sigma_y^2 = E(Y^2) - E(Y)^2 \rightarrow 5 = E(Y^2) - (2)^2 \rightarrow E(Y^2) = 9$$

$$Z_1^2 = (3X + Y)^2 = 9X^2 + 6XY + Y^2$$

$$E(Z_1^2) = 9(E(X^2)) + 6(E(X)E(Y)) + E(Y^2)$$

$$= 9(19) + 6(4)(2) + 9 = 228$$

$$\sigma_{Z_1}^2 = E(Z_1^2) - E(Z_1)^2 = 228 - (14)^2 = \boxed{32}$$

$$Z_2^2 = (3X - Y)^2 \rightarrow 9X^2 - 6XY + Y^2$$

$$E(Z_2^2) = 9(E(X^2)) - 6(E(X)E(Y)) + E(Y^2)$$

$$= 9(19) - 6(4)(2) + 9 = 132$$

$$\sigma_{Z_2}^2 = E(Z_2^2) - E(Z_2)^2 = 132 - (10)^2 = \boxed{32}$$

(e) Write down an expression for the pdf of $Z_1 = 3X + Y$

$$f_{Z_1}(z_1) = \frac{1}{\sqrt{2\pi\sigma_{z_1}^2}} \exp\left[\frac{-(z - m_{z_1})^2}{2\sigma_{z_1}^2}\right] = \frac{1}{\sqrt{2\pi(32)}} \exp\left[\frac{-(z-14)^2}{2(32)}\right]$$

$$= \frac{1}{8\sqrt{\pi}} e^{-1/64 (z-14)^2}$$

$$= \boxed{0.0705\, e^{-1/64 (z-14)^2}}$$