ECE 345 / ME 380
Introduction to Control Systems
Lecture Notes 6

Dr. Oishi
oishi@unm.edu

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Learning Objectives

- Characterize the stability of a system
  - Asymptotic stability
  - BIBO stability
- Relate asymptotic stability and BIBO stability
- Use Routh-Hurwitz criterion or Routh table to determine existence of RHP poles

References:

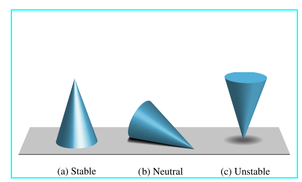
- Nise Chapter 6

Outline

- Introduction
- Asymptotic stability
- BIBO stability
- Routh-Hurwitz Criterion
- Routh tables

Stability

Two types of stability

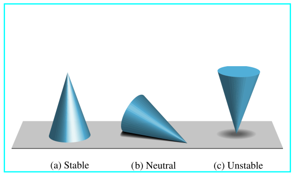


- What does "stability" mean for the cone on a flat surface?
- What does "stability" mean for the car in a slalom course?

Can define stability of a system in response to *an initial condition* or in response to *an input signal*.

Stability

Two types of stability

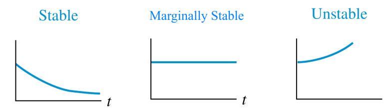


- Asymptotic stability (in response to a non-zero initial condition)
- Bounded Input-Bounded Output (BIBO) stability (in response to input signals)

Addresses different problems, but these types of stability are closely related!

Asymptotic Stability

- A system is *asymptotically stable* if the natural response converges to zero as time approaches infinity, $x(t) \to 0$ as $t \to \infty$
- A system is *unstable* if the natural response grows without bound as time approaches infinity, $|x(t)| \to \infty$ as $t \to \infty$
- A system is *marginally stable* if the natural response neither decays nor grows but remains constant or oscillates as time approaches infinity, $|x(t)| < c$ as $t \to \infty$, for some $c > 0$



Asymptotic Stability

Stability depends on location of the poles in the complex plane.

Stability	Pole location
Asymptotic Stability	All poles in the open LHP
Marginal Stability	At least one non-repeated pole on the imaginary axis, and remaining poles in the open LHP
Unstable	At least one pole in the open RHP, or repeated poles on the imaginary axis

Examples

- $G(s) = \frac{s-2}{(s+1)(s+3)}$
- $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$

Asymptotic Stability

Clicker question

Consider the effect of parameter K on stability of the transfer function

$$G(s) = \frac{1}{s^2 + Ks + 4}$$

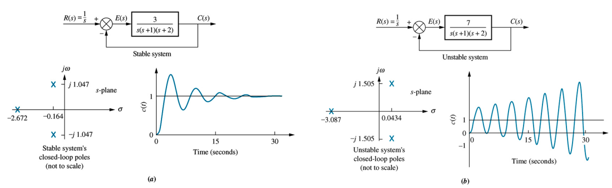
Which of the following statements is most correct?

A. The system is asymptotically stable for $K > 2$, marginally stable for $K = 2$, and unstable for $K < 2$.
B. The system is asymptotically stable for $K > 0$, and unstable for $K \leq 0$.
C. The system is asymptotically stable for $K > 0$, marginally stable for $K = 0$, and unstable for $K < 0$.
D. The system is asymptotically stable for $K < 0$, marginally stable for $K = 0$, and unstable for $K > 0$.

Asymptotic Stability

Role of feedback on stability

- Feedback is often used to stabilize a system
- In some cases, a sufficiently high gain can actually destabilize a system



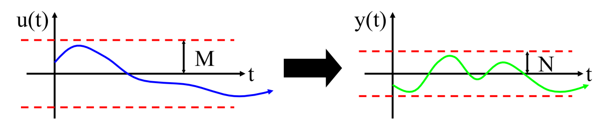
- However, there are many systems for which increasing the gain will *not* destabilize the system

BIBO Stability

- A system is *BIBO stable* if **every** bounded input yields a bounded output
- A system is *unstable* otherwise, e.g., if there exists a bounded input that yields an unbounded output

Bounded signals

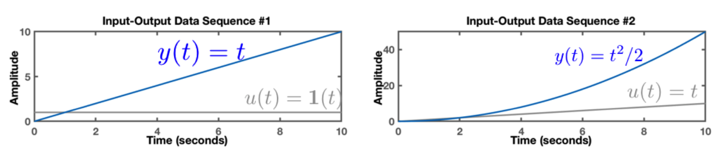
- A bounded signal $u(t)$ is one for which $|u(t)| < M$ (with M some finite, positive number)
- E.g., $\mathbf{1}(t)$; $e^{j\omega t}$; $e^{-at}\mathbf{1}(t), a > 0$.



BIBO Stability

Clicker question

For a system described by the transfer function $G(s)$, we obtain the following input-output data:



Which of the following statements is most correct?

A. The system is BIBO stable because a *bounded* input leads to a *bounded* output.
B. The system is BIBO unstable because a *bounded* input leads to an *unbounded* output.
C. The system is BIBO unstable because an *unbounded* input leads to an *unbounded* output.
D. The system may be BIBO stable, but not enough information is provided to be sure.

BIBO Stability

Assuming no pole-zero cancellation,

- If a system is asymptotically stable, it is BIBO stable
- The converse is NOT true!

Disproving BIBO stability can be easier than proving BIBO stability.

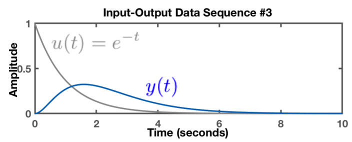
- Consider $G(s) = \frac{1}{s^2+\omega^2}$. What output results when the input is $u(t) = \sin \omega t$?

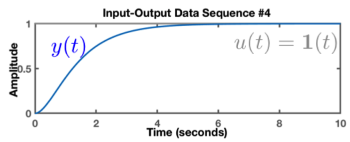
(Only one counterexample is needed to disprove BIBO stability.)

BIBO Stability

What constitutes 'proof'?

For $G(s) = \dfrac{2}{s^2+2s+3}$, why aren't either of the following two input-output trajectories adequate proof of BIBO stability?





**Use asymptotic stability to *prove* BIBO stability, and counterexamples to *disprove* BIBO stability.

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BIBO Stability

Clicker question

Which of the following is *not* BIBO stable?

A. $G(s) = \frac{1}{s^2+2s+1}$
B. $G(s) = \frac{s+1}{(s+2)(s+1)}$
C. $G(s) = \frac{s-1}{s^2+2s+1}$
D. $G(s) = \frac{1}{s^2+2s}$
E. Both C and D

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BIBO Stability

Clicker question

Which of the following is most consistent with a system that is BIBO stable?

A. A sinusoidal input generates a sinusoidal output.
B. A step response generates an output for which $y(t) \to \infty$ as $t \to \infty$
C. An impulse input generates an output $|y(t)| \leq 1$ for all $t \geq 0$.
D. In response to a non-zero initial state, all elements of the state satisfy $|x_i(t)| \leq 1$, $i \in \{1, \cdots, n\}$
E. Both A and D are both consistent with BIBO stability.

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Routh-Hurwitz Criterion

Tool to determine system stability, by checking for RHP poles

1. Routh arrays
   - Uses coefficients of characteristic equation $\Delta(s)$ to determine *how many* roots of the polynomial lie in the open RHP.
   - Does not indicate *where* poles are located.
   - Analytic solution indicates how certain coefficients affect stability

Question: The Matlab command `roots` finds numerical values for poles, so why use more complicated Routh arrays?

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Routh-Hurwitz Criterion

Main idea: Number of roots of $\Delta(s)$ in the RHP is equal to the number of sign changes in the first column of the Routh table (=Routh array).

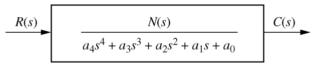


TABLE 6.2 Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Routh-Hurwitz Criterion

What happens when there is a zero in the first column?

Example:

$$\frac{Y(s)}{R(s)} = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

TABLE 6.4 Completed Routh table for Example 6.2

				$\epsilon = +$	$\epsilon = -$
s^5	1	3	5	$+$	$+$
s^4	2	6	3	$+$	$+$
s^3	$\cancel{0}\ \epsilon$	$\frac{7}{2}$	0	$+$	$-$
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0	$-$	$+$
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0	$+$	$+$
s^0	3	0	0	$+$	$+$

Routh-Hurwitz Criterion

What happens when there is a row of zeros?

Example:

$$\frac{Y(s)}{R(s)} = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

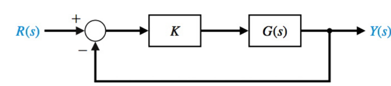
TABLE 6.7 Routh table for Example 6.4

s^5	1	6	8
s^4	$\cancel{7}\ 1$	$\cancel{42}\ 6$	$\cancel{56}\ 8$
s^3	$\cancel{0}\ \cancel{4}\ 1$	$\cancel{0}\ \cancel{12}\ 3$	$\cancel{0}\ \cancel{0}\ 0$
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

Routh-Hurwitz Criterion

Clicker question

Consider the open-loop system with $G(s) = \frac{1}{(s-1)(s+2)(s+3)}$ under negative unity feedback as shown below, with positive gain K.



Which of the following most accurately describes the role of K on the stability of the closed-loop system $\frac{Y(s)}{R(s)}$?

A. The closed-loop system is asymptotically stable for for $10 > K > 6$.
B. The closed-loop system has *one* pole in the RHP for $K > 10$.
C. The closed-loop system has *one* pole in the RHP for $K < 6$.
D. Both A and B are correct.
E. Both A and C are correct.

Routh-Hurwitz Criterion

2. Hurwitz conditions
   - When criteria are satisfied, all roots of $\Delta(s)$ lie in open LHP
   - When criteria fail, does not indicate *how many* poles lie in RHP
   - Analytic solution indicates how certain coefficients affect stability

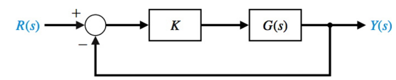
System	$\Delta(s) =$	Conditions for stability
First-order	$s + a_0$	$a_0 > 0$
Second-order	$s^2 + a_1 s + a_0$	
Third-order	$s^3 + a_2 s^2 + a_1 s + a_0$	

Exercise: Set up Routh arrays for 1st, 2nd, and 3rd order systems. Derive the above equations when $a_n = 1$ and all coefficients are positive.

Routh-Hurwitz Criterion

Clicker question

Consider the open-loop system with $G(s) = \frac{1}{s(s+1)(s+2)}$ under negative unity feedback as shown below, with positive gain K.



Which of the following most accurately describes the role of K on the stability of the closed-loop system $\frac{Y(s)}{R(s)}$?

A. The closed-loop system is unstable for for $K > 6$.
B. The closed-loop system is BIBO stable for $K < 6$.
C. The open-loop system $KG(s)$ is marginally stable for all K.
D. Both A and C are correct.
E. Both A, B, and C are correct.

Summary

- Asymptotic stability: *all* poles in the open LHP
- BIBO stability: *all* bounded inputs generate bounded outputs
- Asymptotic stability $\longrightarrow$ BIBO stability
- Use Routh-Hurwitz criterion or Routh table to determine existence of RHP poles