ECE 595
Network Economics
WNP - Chapter 6
Game Theory

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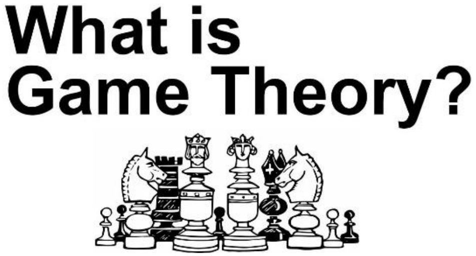
Oligopoly Pricing & Game Theory

- Many self-interested individuals (including firms and consumers) make *interdependent* interactions
  - the payoff of each individual depends not only on his own choices, but also on the choices of other individuals
  - such an interaction can be analyzed by *game theory*
- Basic concepts of game theory
- Multiple classical market competition models
  - the Cournot competition based on output quantities
  - the Bertrand competition based on pricing
  - the Hotelling model that captures the location information in the competition

What is a game?



- A game is a formal representation of a situation in which a number of individuals interact with *strategic interdependence*
- Each individual's welfare depends not only on his own choices but also on the choices of other individuals
- A game consists of
  - **Players**: Who are involved in the game?
  - **Rules**: What actions can players choose? How and when do they make decisions? What information do players know about each other when making decisions?
  - **Outcomes**:What is the outcome of the game for each possible action combinations chosen by players?
  - **Payoffs**:What are the players' preferences (i.e., utilities) over the possible outcomes?
- Each player is *rational* (*self-interested*), whose goal is to choose the actions that produce his most preferred outcome
- When facing potential uncertainty over multiple outcomes, a rational player chooses actions that maximize his expected utility
- Identify the stable outcome(s) of the game: *equilibrium(s)*.

Strategic Form Game $\langle \mathcal{I}, (\mathcal{S}_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$

- All players make decisions simultaneously without knowing each other's choices
- $\mathcal{I} = \{1, 2, \ldots, I\}$: a ***finite set of players***
- S_i a ***set of available actions*** (pure strategies) for player $i \in I$
  - $s_i \in S_i$: an action for player I
  - $\mathbf{s}_{-i} = (s_j, \forall j \neq i)$ a vector of actions for all players except i
  - $\mathbb{S} \triangleq \Pi_i \mathcal{S}_i$ the set of all action profiles
- $u_i : \mathbb{S} \rightarrow \mathbb{R}$: the ***payoff (utility) function*** of player i, which maps every possible action profile in $\mathbb{S}$ to a real number, the utility
- *Strictly dominated strategy*: a strategy that is always worse than another strategy of the same player regardless of the strategy choices of other players

$$u_i(s_i, \boldsymbol{s}_{-i}) < u_i(s_i', \boldsymbol{s}_{-i}), \quad \forall \boldsymbol{s}_{-i} \in \mathbb{S}_{-i}$$

- When a strategy is strictly dominated, it can be safely removed from player i's strategy set without changing the game outcome, as a rational player will never choose a strictly dominated strategy to maximize his payoff

Prisoner’s Dilemma

		PLAYER 2	
		SILENT	CONFESS
PLAYER 1	SILENT	$(-2,-2)$	$(-5,-1)$
	CONFESS	$(-1,-5)$	$(-4,-4)$

- Two players are arrested for a crime and placed in separate rooms. The authorities try to extract a confession from them.
- If they both remain silent, then the authorities will not be able to press serious charges against them and they will both serve a short prison term, say two years ($u_i = -2$ for both players i = 1, 2), for some minor offenses.
- If only one of them (say, player 1) confesses, his term will be reduced to one year ($u_1 = -1$ for player 1), as a reward for him to serve as a witness against the other person, who will get a sentence of five years ($u_2 = -5$ for player 2).
- If they both confess, then both of them get a smaller sentence of four years ($u_i = -4$ for both players i = 1, 2) comparing with the worst case of five years.
- “SILENT” is a strictly dominated strategy for both players
  - When player 2 chooses “SILENT” (the first column), then player 1 obtains a worse payoff of -2 when he chooses “SILENT,” comparing with the payoff of -1 if he chooses “CONFESS”
  - When player 2 chooses “CONFESS” (the second column), then player 1 obtains a worse payoff of -5 when he chooses “SILENT,” comparing with the payoff of -4 if he chooses “CONFESS”
  - For player 1, the strategy “SILENT” is always worse than “CONFESS,” and can be eliminated from his strategy set
- This game is symmetric: the same conclusion is true for player 2
- The unique game result is (CONFESS, CONFESS), and the payoffs of both players are $(-4,-4)$
- Most of the time we cannot predict a game’s outcome by eliminating strictly dominated strategies
- The more general method of predicting the game outcome by looking at the *Best Response Correspondence*

$$B_i(s_{-i}) = \{s_i \in \mathcal{S}_i \mid u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in \mathcal{S}_i\}$$

Stag Hunt

		HUNTER 2	
		STAG	HARE
HUNTER 1	STAG	(10, 10)	(0, 2)
	HARE	(2, 0)	(2, 2)

- Two hunters decide to hunt together in a forest, and each of them chooses one animal to hunt: stag or hare. Hunting a stag is challenging and requires cooperation between the hunters to succeed. Hunting a hare is easy and can be done by a single hunter.
- When both hunters choose to hunt a stag, each of them will get a payoff of 10 (pounds of stag meat)
- When hunter 1 hunts a stag but hunter 2 hunts a hare, then hunter 1 gets nothing due to the lack of cooperation, and hunter 2 gets a payoff of 2 (pounds of hare meat). The situation is similar when hunter 1 hunts a hare and hunter 2 hunts a stag
- When both hunters hunt hares, then each of them will get a payoff of 2 as there are enough hares around in the forest
- There is no strictly dominated strategy in this game, as choosing “HARE” is worst than “STAG” for hunter 1 (row player) when hunter 2 chooses “STAG” (the first column), but choosing “STAG” is worst than “HARE” for hunter 1 when hunter 2 chooses “HARE” (the second column)
- Hunter 1’s best response functions are $B_1(STAG) = STAG$ and $B_1(HARE) = HARE$
- The game is symmetric: $B_2(STAG) = STAG$ and $B_2(HARE) = HARE$
- Two strategy profiles (STAG, STAG) and (HARE,HARE) that they are mutual best responses of both players

Pure Strategy Nash Equilibrium

- A pure strategy Nash Equilibrium of a strategic form game $\langle \mathcal{I}, (\mathcal{S}_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ is a strategy profile $s^* \in \mathbb{S}$ such that for each player $i \in I$ the following condition holds

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*), \quad \forall s_i' \in \mathcal{S}_i$$

- or equivalently $s_i^* \in B_i(s_{-i}^*), \quad \forall i \in \mathcal{I}$
- The *Stag Hunt* game has two pure strategy Nash equilibria: (STAG,STAG) and (HARE, HARE)
- The *Prisoner's Dilemma* has a unique pure strategy Nash equilibrium: (CONFESS, CONFESS)
- A pure strategy Nash Equilibrium may not be a Pareto optimal solution
  - For example, in the *Prisoner's Dilemma*, choosing (SILENT, SILENT) will lead to payoffs of $(-2,-2)$, which are better than the payoffs of $(-4,-4)$ under the Nash equilibrium (CONFESS, CONFESS)
  - Such a loss is due to the selfish nature of the players!!!
- ***NOT*** every game possesses a pure strategy Nash Equilibrium

Matching Pennies

		PLAYER 2	
		HEADS	TAILS
PLAYER 1	HEADS	$(1, -1)$	$(-1, 1)$
	TAILS	$(-1, 1)$	$(1, -1)$

- The game is played between two players. Each player has a penny and turns his penny to “HEADS” or “TAILS” secretly and simultaneously with the other player.
- If the pennies match (both heads or both tails), player 1 keeps both pennies, so wins one from player 2 ($u_1 = 1$ for player 1, $u_2 = -1$ for player 2)
- If the pennies do not match (one heads and one tails), player 2 keeps both pennies, so receives one from player 1 ($u_1 = -1$ for player 1, $u_2 = 1$ for player 2)
- *Zero-sum game*, where one player’s gain is exactly the other player’s loss
- It is easy to verify that player 1’s best response is “HEADS” if player 2 selects “HEADS” and “TAILS” if player 2 selects “TAILS”
- Player 2’s best response functions are exactly the opposite
- There is no strategy profile that corresponds mutual best responses of both players: NO pure strategy Nash Equilibrium
- When player 1 chooses “HEADS” player 2 will choose “TAILS” as his best response. In response to this, player 1 will choose “TAILS” which makes player 2 choose “HEADS.” Because of this, player 1 will switch to “HEADS,” hence enters a loop.
- When a game does not have a pure strategy Nash Equilibrium, what kind of outcome will emerge as an “equilibrium”?
- *Mixed strategy Nash Equilibrium*

Mixed Strategy Nash Equilibrium

- σ_i: mixed strategy for player *i* (a probability distribution function over all pure strategies in set S_i)
- Σ_i the set of all mixed strategies of player *i*, i.e., all probability distributions over S_i
- $\sigma = (\sigma_i)_{i \in \mathcal{I}} \in \Sigma$: a mixed strategy profile for all players, where $\Sigma = \Pi_i \Sigma_i$ is the set of all mixed strategy profiles
- $\sigma_{-i} = (\sigma_j, \forall j \neq i)$: a mixed strategy profile for all players except *i*
- $\Sigma_{-i} = \Pi_{j \neq i} \Sigma_j$ the set of mixed strategy profile for all players except *i*.
- Each player *i*'s payoff under a mixed strategy profile $\boldsymbol{\sigma}$ is given by the expected value of pure strategy payoffs under the distribution σ

$$u_i(\sigma) = \sum_{s \in S} \left(\Pi_{j=1}^{I} \sigma_j(s_j) \right) \cdot u_i(s)$$

- $\text{supp}(\sigma_i) \triangleq \{s_i \in \mathcal{S}_i \mid \sigma_i(s_i) > 0\}$: the support of σ_i is the set of pure strategies which are assigned positive probabilities

A mixed strategy profile $\boldsymbol{\sigma}^*$ is a mixed strategy Nash Equilibrium if and only if for every player $i \in I$, the following two conditions hold:
1. Every chosen action is equally good, that is, the expected payoff given σ^*_{-i} of every $s_i \in \text{supp}(\sigma_i)$ is the same.
2. Every non-chosen action is not good enough, that is, the expected payoff given σ^*_{-i} of every $s_i \notin \text{supp}(\sigma_i)$ must be no larger than the expected payoff of $s_i \in \text{supp}(\sigma_i)$

When a strategic form game possesses a (pure or mixed) strategy Nash equilibrium?

- **Nash 1950:** *Any finite strategic game, i.e., a game that has a finite number of players and each player has a finite number of action choices, has at least one mixed strategy Nash Equilibrium*

- **Debreu-Fan-Glicksburg 1952:** The strategic form game $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ has a pure Nash equilibrium, if for each player $i \in I$ the following conditions hold:
    1. S_i is a non-empty, convex, and compact subset of a finite-dimensional Euclidean space
    2. $u_i(\boldsymbol{s})$ is continuous in $\boldsymbol{s}$ and quasi-concave in s_i

Extensive Form Game - Definition

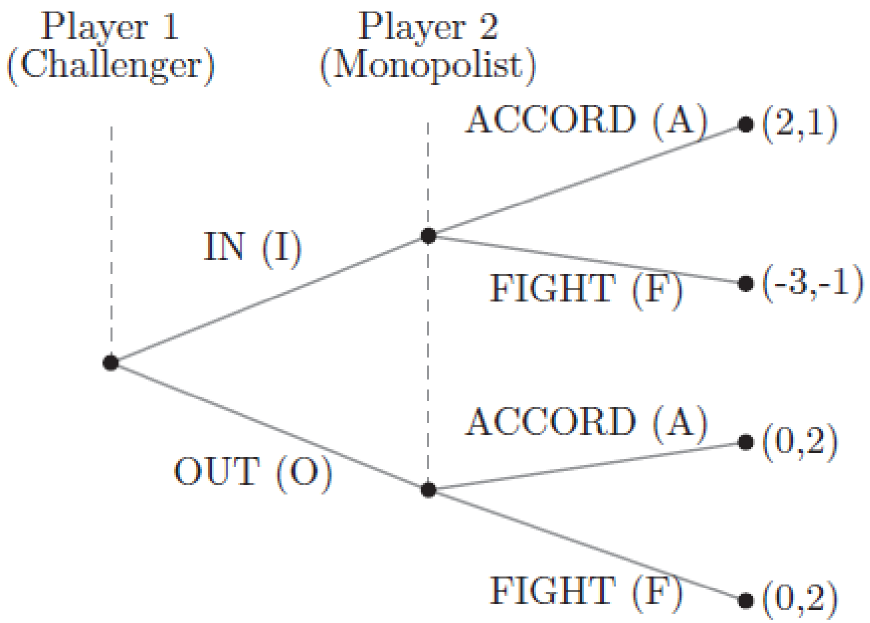
- A set of players $\mathcal{I} = \{1, 2, ..., I\}$
- Histories: A set *H* of sequences which can be finite or infinite, defined by
  - $\mathcal{H}^k = \{h^k\}$: set of all possible histories after stage $k-1$
  - $\bar{\mathcal{H}} = \bigcup_{k=0}^{K+1} \mathcal{H}^k$: set of all possible histories
  - *Market Entry* game: $\mathcal{H}^1 = \{\mathrm{I}, \mathrm{O}\}$ and $\mathcal{H}^2 = \{(\mathrm{I}, \mathrm{A}), (\mathrm{I}, \mathrm{F}), (\mathrm{O}, \mathrm{A}), (\mathrm{O}, \mathrm{F})\}$

$$\begin{cases} h^0 = \emptyset & \text{initial history} \\ h^1 = (s^0) & \text{history after stage } 0 \\ h^2 = (s^0, s^1) & \text{history after stage } 1 \\ \quad \cdots & \quad \cdots \\ h^{k+1} = (s^0, ..., s^k) & \text{history after stage } k \end{cases}$$

- Each pure strategy for player *i* is defined as a contingency plan for every possible history
  - $\mathcal{S}_i(h^k)$: set of actions available to player *i* under history h^k
  - $\mathcal{S}_i(\mathcal{H}^k) = \bigcup_{h^k \in \mathcal{H}^k} \mathcal{S}_i(h^k)$: set of actions available to player *i* under all possible histories at stage *k*
  - $a_i^k : \mathcal{H}^k \to \mathcal{S}_i(\mathcal{H}^k)$: a mapping from $\mathcal{H}^k$ to $\mathcal{S}_i(\mathcal{H}^k)$ such that $a_i^k(h^k) \in \mathcal{S}_i(h^k)$
  - A pure strategy of player *i* is a sequence $s_i = \{a_i^k\}_{k=0}^K$.
- Preferences are defined on the outcome of the game $\mathcal{H}^{K+1}$
  - Utility function $u_i : \mathcal{H}^{K+1} \to \mathbb{R}$

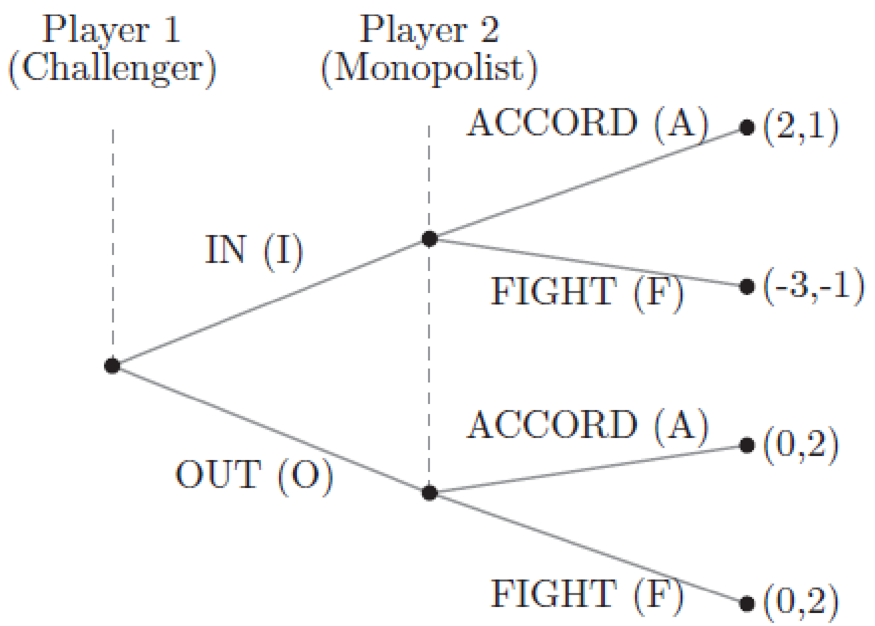
Extensive Form Game – Example (1/2)

- Players engage in sequential decision making - multi-stage games with observed actions where:
  1. All previous actions (history) are observed, i.e., each player is perfectly informed of all previous events
  2. Some players may move simultaneously within the same stage
- Extensive form games can be conveniently represented by tree diagrams
- Example: Market Entry
  - There are two players (firms)
  - Player 1, the challenger, can choose to enter the market (I) or stay out (O)
  - Player 2, the monopolist, after observing the action of the challenger, chooses to accommodate (A) or fight (F)
  - When player 1 chooses "Out," there will be no difference for the player 2 to choose "Fight" or "Accord"
- In an extensive form game, a strategy specifies the action the player chooses for *every* possible history



Extensive Form Game – Example (2/2)

- Player 1 moves in the first stage (i.e., stage 0) and player 2 moves in the second stage (i.e., stage 1)
- The strategy of player 1 is the function $a_1^0 : \mathcal{H}^0 = \emptyset \to \mathcal{S}_1 = \{\mathrm{I}, \mathrm{O}\}$
- The strategy of player 2 is the function $a_2^1 : \mathcal{H}^1 = \{\mathrm{I}, \mathrm{O}\} \to \mathcal{S}_2(\mathcal{H}^1)$
- Four possible strategies for player 2: AA, AF, FA, and FF



player 2 will select "ACCORD" under both histories

player 2 will select "ACCORD" under $h^1 = \{\mathrm{I}\}$ and "FIGHT" $h^1 = \{\mathrm{O}\}$

PLAYER 2

PLAYER 1	AA	AF	FA	FF
I	(2, 1)	(2, 1)	(−3, −1)	(−3, −1)
O	(0, 2)	(0, 2)	(0, 2)	(0, 2)

- Four pure strategy Nash Equilibria in this game: (I, AA), (I, AF), (O, FA), and (O, FF)
- The two equilibria (O, FA) and (O, FF) are problematic

Once player 1 chooses "IN", player 2 will choose "ACCORD", get a payoff of 1 (as "FIGHT" leads to a worse payoff of -1). This will eliminate player 2's two strategies: "FA" and "FF."

The only "reasonable" Nash equilibria will be (I,AA) and (I,AF).

Subgame Perfect Equilibrium

- Requires the strategy of each player to be optimal not only at the start of the game but also after every history
- Game: $G(h^k)$
  - Histories: $h^{K+1} = (h^k, s^k, ..., s^K)$
  - Strategies: $s_{i|h^k}$ is the restriction of s_i to histories in $G(h^k)$
  - Payoffs: $u_i(s_i, s_{-i}|h^k)$ is the payoff of player i after histories in $G(h^k)$
- A strategy profile $\boldsymbol{s}^*$ is a subgame perfect equilibrium for an extensive form game if for every history $\boldsymbol{h}^k$, the restriction $s^*_{i|h^k}$ is an Nash equilibrium of the subgame $G(\boldsymbol{h}^k)$
- The subgame perfect equilibria can be derived using ***backward induction***

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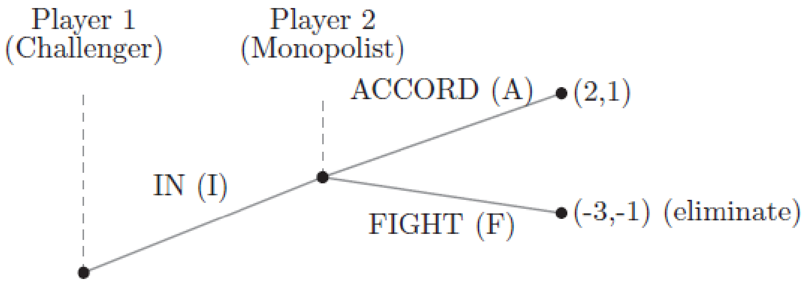
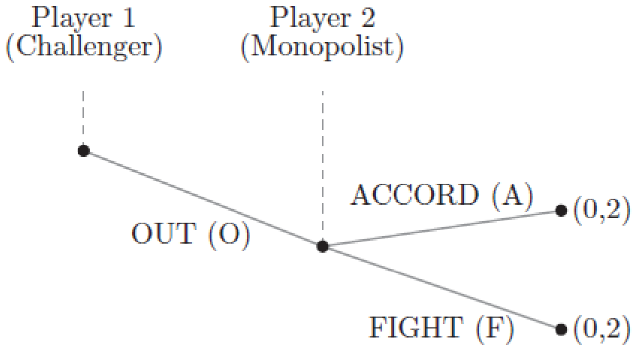
Subgame Perfect Equilibrium - Example

- Two subgames from history $\boldsymbol{h_1}$={I} and $\boldsymbol{h_1}$={O}, which concern the decision of player 2 in the last stage
- First subgame *G*(I): player 2 will choose "ACCORD" to maximize his payoff (as 1 is better than -1), and hence we can eliminate "FIGHT"
- Second subgame *G*(O), player 2 is indifferent from choosing "ACCORD" or "FIGHT" hence we cannot eliminate any action
- The remaining subgame, which concerns player 1's decision in the initial stage
- Player 1 now faces two possible payoffs: he will get 2 if he chooses "IN," and he will get 0 if he chooses "OUT"
- **So clearly he will choose "IN"**
- The SPEs are (I, AA) and (I, AF)
- The final *equilibrium path* would be (player 1 chooses "IN," player 2 chooses "ACCORD"), and the *equilibrium payoffs* are (2, 1)





Backward Induction