ECE 345 / ME 380
Introduction to Control Systems
Lecture Notes 2

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Learning Objectives

- Find the Laplace transform and inverse Laplace transform
- Find the transfer function of a system from a differential equation
- Solve a differential equation by using its transfer function
- Find the transfer function for
  - LTI electrical networks
  - LTI mechanical systems (translational and mechanical)
  - LTI electromechanical systems
- Linearize a nonlinear system to find its transfer function

References:

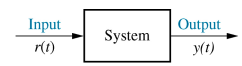
- Nise Chapter 2: 2.1–2.5, 2.9–2.11 (2.6–2.8 recommended but optional)

Outline

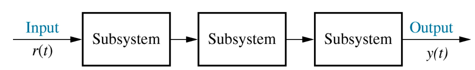
- The Laplace transform
- Transfer functions
  - Electrical systems
  - Mechanical systems
  - Geared systems
- Mechanical-electrical system analogs
- Linearization

The Laplace Transform

Modeling



- Three elements: input, output, and the system (process)
- Differential equations often model systems of interest, but can be cumbersome mathematically
- Laplace transforms facilitate block-diagram modeling of systems and subsystems



The Laplace Transform

- The method of Laplace transforms converts a calculus problem (the linear differential equation) into an algebra problem.
- Pierre-Simon Laplace (1749-1827)
- Integral transformation (similar to the Fourier transform)

$$\mathcal{L}\left[f(t)\right] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

- Multiplication by the Laplace variable s corresponds to differentiation in the time domain
- Inverse Laplace transformation

$$\mathcal{L}^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t)$$

The Laplace Transform

Properties of the Laplace transform

- Linearity
- Differentiation
- *Final value theorem*
- Initial value theorem
- Region of convergence

Important Laplace transform pairs

- Impulse function
- Step function
- Exponential decay
- Sine and cosine
- Damped oscillations

The Laplace Transform

Clicker question

What is the inverse Laplace transform of $F(s) = \dfrac{3s+2}{s^2+1^2}$?

A. $f(t) = (3\cos(t) + 2\sin(t)) \cdot \mathbf{u}(t)$
B. $f(t) = (2\cos(t) + 3\sin(t)) \cdot \mathbf{u}(t)$
C. $f(t) = (3\cos(t) + \mathbf{u}(t)) \cdot \mathbf{u}(t)$
D. $f(t) = (\cos(3t) + \sin(2t)) \cdot \mathbf{u}(t)$

The Laplace Transform

Clicker question

Use the method of Laplace transforms to solve the differential equation

$$\frac{dy}{dt} = -y + 2e^{-3t}, \quad y(0) = 1$$

with non-zero initial condition. What is the solution $y(t)$ for $t \geq 0$?

A. $y(t) = -3e^{-t} + e^{-2t}$
B. $y(t) = 2e^{-t} - e^{-3t}$
C. $y(t) = -e^{-t} + e^{-3t}$
D. $y(t) = e^{-t} - 3e^{-2t}$

Hint: Start by taking the Laplace transform of both sides of the equation.

Transfer functions

Goal: Algebraically relate input and output

- Consider an n^{th} order differential equation with initial conditions equal to zero, that relates the input $u(t)$ and output $y(t)$

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_0 u,$$

- Take the Laplace transform of both sides

$$s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_0 Y(s) = b_m s^m U(s) + b_{m-1}s^{m-1}U(s) + \cdots + b_0 U(s)$$

- And rearrange to obtain

$$\frac{Y(s)}{U(s)} = \frac{\left(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0\right)}{\left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right)}$$

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Transfer functions

Goal: Algebraically relate input and output

- The *transfer function* is the ratio $G(s) = \dfrac{Y(s)}{U(s)}$.

$$U(s) \longrightarrow \boxed{\frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}} \longrightarrow Y(s)$$

- Note that this means that $Y(s) = G(s)U(s)$, so the output signal can be calculated in a straightforward manner (multiplication!).

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Transfer functions

Clicker question

Consider the system represented by

$$\frac{dy}{dt} + 2y(t) = 4u(t), \quad y(0) = 0$$

What is the system response $y(t)$ to a step input $u(t) = \mathbf{u}(t)$?

A. $y(t) = 2 + 2e^{-4t}$ for $t \geq 0$, $y(t) = 0$ for $t < 0$
B. $y(t) = 2 - 2e^{-2t}$ for $t \geq 0$, $y(t) = 0$ for $t < 0$
C. $y(t) = \left(2 + 2e^{-2t}\right)\mathbf{u}(t)$
D. $y(t) = 2e^{-t}$ for $t \geq 0$, $y(t) = 0$ for $t < 0$

Hint: First find the transfer function.

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Transfer functions

Electrical systems

- Mesh analysis = Kirchhoff's voltage law
- Nodal analysis = Kirchhoff's current law
- Identify integro-differential equations directly, or use Laplace transforms
  - *Impedances* often more convenient for mesh analysis
  - *Admittances* often more convenient for nodal analysis

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

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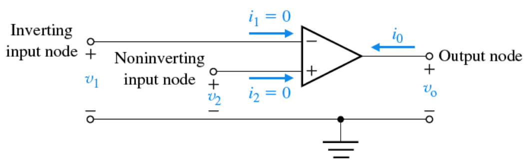
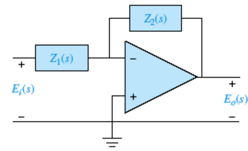
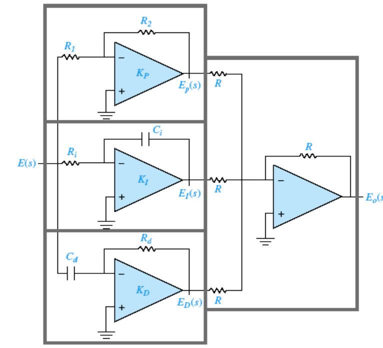
Transfer functions

Electrical systems
- Ideal operational amplifier
  - May be inverting or non-inverting
  - No current at the input terminals that flows into the op-amp (Input impedance is infinite)
  - Voltage at the two input terminals is tied
- Convenient way to build, implement, and realize transfer functions
- Analog controllers



Transfer functions





Transfer functions

Mechanical systems – translational elements
- Spring-mass-damper systems
- Dynamic simulator via Mathworks / Simulink: `https://www.mathworks.com/help/physmod/simscape/ug/creating-and-simulating-a-simple-model.html`
- Choose "positive" direction, then apply Newton's second law.

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring (K)	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper (f_v)	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass (M)	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Transfer functions

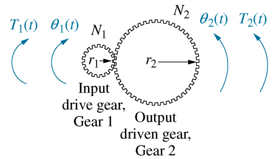
Mechanical systems – rotational elements
- Spring-inertial load-damper systems
- Choose "positive" direction, then apply Newton's second law.

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
Spring (K)	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
Viscous damper (D)	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
Inertia (J)	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Transfer functions

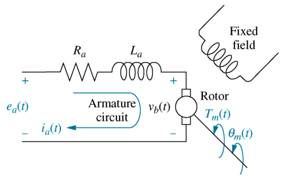
Geared systems

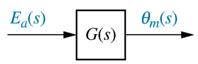
- For lossless gears, $\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$



Electromechanical systems

- DC motor





Mechanical / Electrical System Analogs

Clicker question

Which of the following pairs could be analogs?

A. RC series circuit and a spring-mass system
B. LC series circuit and a spring-mass system
C. RLC parallel circuit and a spring-mass system
D. Not very confident about any of these answers

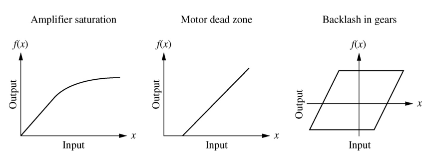
Linearization

Linearity of $f(x)$

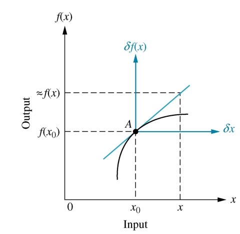
- Scaling: $f(\alpha x) = \alpha f(x)$
- Superposition: $f(x_1 + x_2) = f(x_1) + f(x_2)$

For example: $f(x) = 3x$

Examples of nonlinearities



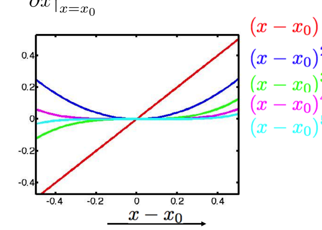
Linearization



Solution:

- Linearize $y = f(x)$ for x near x_0
- Taylor's series approximation
- Works for $(x - x_0)$ "small enough"

$$f(x) - f(x_0) \approx \left.\frac{\partial f}{\partial x}\right|_{x=x_0} \cdot (x - x_0)$$

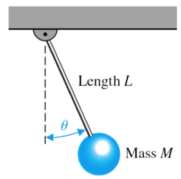


Linearization

Frictionless rigid pendulum

- Linearize the dynamics $\tau(\theta) = MgL \sin\theta$ around $\theta_0 = 0$.

$$\begin{aligned} \tau(\theta) - 0 &\approx MgL \frac{\partial \sin\theta}{\partial\theta}\bigg|_{\theta=0} (\theta - 0) \\ \delta\tau &\approx MgL \cdot \delta\theta \end{aligned}$$



- with $\delta\tau = \tau$, $\delta\theta = \theta$

Linearization

Clicker question

Consider again the pendulum dynamics

$$\tau(\theta) = MgL \sin\theta$$

What is linearization of this system when the pendulum is upright, e.g., around $\theta = \pi$?

A. $\delta\tau \approx -MgL \cdot \delta\theta$, $\delta\tau = \tau$, $\delta\theta = \theta - \pi$
B. $\delta\tau \approx MgL \cdot \delta\theta$, $\delta\tau = \tau - MgL$, $\delta\theta = \theta - \pi$
C. $\delta\tau \approx -MgL \cdot \delta\theta$, $\delta\tau = \tau - \pi$, $\delta\theta = \theta$
D. $\tau \approx -MgL \cdot \delta\theta$, $\delta\theta = \theta - \pi/2$

Key Concepts

1. Laplace transforms
2. Transfer functions
3. Dynamics of spring-mass-damper systems
4. Dynamics of RLC op-amp systems
5. Linearization