VARIANCE OF A RANDOM VARIABLE.

- Definition: $\sigma_x^2 \triangleq E\{[X - E[X]]^2\} = E\{[X - \bar{X}]^2\}$.

- σ_x: is called the standard deviation of X and is a measure of the concentration of the pdf of X about the mean.

  ↳ It is a measure of the dispersion of a set of values.

For example: a low σ_x means that the values of X are close to the mean

- $\sigma_x^2 \equiv var\{X\}$

In order to compute the variance:

$$\sigma_x^2 = E[X^2] - E^2[X].$$

(second moment).

⇒.

PROOF

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx$$

Let $E[X] = m_X$.

Then from $\sigma_X^2 \triangleq E\{[X - E[X]]^2\}$ we have that:

$$\sigma_X^2 = \int_{-\infty}^{+\infty} (x - m_X)^2 f_X(x)\,dx = \int_{-\infty}^{+\infty} (x^2 - 2x m_X + m_X^2) f_X(x)\,dx$$

$$= \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx - 2\int_{-\infty}^{+\infty} m_X x f_X(x)\,dx + \int_{-\infty}^{+\infty} m_X^2 f_X(x)\,dx$$

$$= E[X^2] - 2 \cdot m_X \cdot m_X + m_X^2$$

$$= E[X^2] - 2 \cdot m_X^2 + m_X^2$$

$$= E[X^2] - E^2[X]. \;//$$

Average of Linear Combination of N Random Variables

- The expected value (average) of an arbitrary linear combination of random variables is the same as the linear combination of their respective means.

$$E\left[\sum_{i=1}^{N} a_i X_i\right] = \sum_{i=1}^{N} a_i E[X_i].$$

$X_1, \dots, X_i, \dots, X_N \rightarrow$ Random Variables

$a_1, \dots, a_i, \dots, a_N \rightarrow$ Arbitrary Constants.

PROOF $(N=2)$

Let $f_{X_1X_2}(x_1,x_2)$ be the joint pdf of X_1 and X_2 random variables.

$$\boxed{E[X]=\int_{-\infty}^{\infty} x f_X(x)\,dx}$$

So:

$$E[a_1X_1+a_2X_2] \triangleq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(a_1x_1+a_2x_2)f_{X_1X_2}(x_1,x_2)\,dx_1dx_2.$$

$$= a_1\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 f_{X_1X_2}(x_1,x_2)\,dx_1x_2 + a_2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_2 f_{X_1X_2}(x_1,x_2)\,dx_1x_2.$$

$$= a_1\int_{-\infty}^{+\infty} x_1\left\{\int_{-\infty}^{+\infty} f_{X_1X_2}(x_1,x_2)\,dx_2\right\}dx_1. +$$

$$+ a_2\int_{-\infty}^{+\infty} x_2\left\{\int_{-\infty}^{+\infty} f_{X_1X_2}(x_1,x_2)\,dx_1\right\}dx_2.$$

$$= a_1\underbrace{\int_{-\infty}^{+\infty} x_1 f_{X_1}(x_1)\,dx_1} + a_2\underbrace{\int_{-\infty}^{+\infty} x_2 f_{X_2}(x_2)\,dx_2}.$$

$$= a_1E[X_1] + a_2E[X_2].$$

↳ This proof can be generalized for $N \geq 2$.

Variance of a Linear Combination of Independent Random Variables

If $X_1, \dots, X_N$ are statistically independent variables:

$$var\left[\sum_{i=1}^{N} a_i X_i\right] = \sum_{i=1}^{N} a_i^2 \cdot var\{X_i\}$$

$a_1, \dots, a_i, \dots, a_N$: arbitrary constants.

$var\{X_i\} \triangleq E[(X_i - \bar{X}_i)^2]$.

PROOF $(N=2)$

Let $Z = a_1 X_1 + a_2 X_2$ and let $f_{X_i}(x_i)$ be the pdf of X_i. However we know that our random variables X_1 and X_2 are statistically independent $\Rightarrow$ This means that the joint pdf of X_1 and X_2 is: $f_{X_1}(x_1) f_{X_2}(x_2)$.

The expected value of Z:

$$E[Z] = a_1 E[X_1] + a_2 E[X_2] = a_1 \bar{X}_1 + a_2 \bar{X}_2$$

and the $var\{Z\} = E[(Z - \bar{Z})^2]$.

$$\operatorname{var}[z] = E\left\{\left[(a_1X_1 + a_2X_2) - (a_1\bar{X}_1 + a_2\bar{X}_2)\right]^2\right\}$$

$$= E\left\{\left[a_1(X_1 - \bar{X}_1) + a_2(X_2 - \bar{X}_2)\right]^2\right\}$$

$$= a_1^2\, E\left[(X_1 - \bar{X}_1)^2\right] + 2a_1a_2\, E\left[(X_1 - \bar{X}_1)(X_2 - \bar{X}_2)\right] + a_2^2\, E\left[(X_2 - \bar{X}_2)^2\right]$$

$$= a_1^2 \cdot \operatorname{var}\{X_1\} + 2a_1a_2\, E\left[(X_1 - \bar{X}_1)(X_2 - \bar{X}_2)\right] + a_2^2 \cdot \operatorname{var}\{X_2\}$$

$\cancel{E[(X_1 - X_1)]}$

- $$E\left[(X_1 - \bar{X}_1)(X_2 - \bar{X}_2)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x_1 - \bar{x}_1)(x_2 - \bar{x}_2) f_{X_1}(x_1) f_{X_2}(x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{+\infty} (x_1 - \bar{x}_1) f_{X_1}(x_1)\, dx_1 \int_{-\infty}^{+\infty} (x_2 - \bar{x}_2) f_{X_2}(x_2)\, dx_2$$

$$= \left[\int_{-\infty}^{+\infty} x_1 f_{X_1}(x_1)\, dx_1 - \bar{x}_1 \int_{-\infty}^{+\infty} f_{X_1}(x_1)\, dx_1\right] \int_{-\infty}^{+\infty} (x_2 - \bar{x}_2) f_{X_2}(x_2)\, dx_2$$

$$= (\bar{x}_1 - \bar{x}_1)(\bar{x}_2 - \bar{x}_2)$$

$$= 0.$$

CHARACTERISTIC FUNCTION.

Let $g(X) = e^{juX}$

We obtain the characteristic function of X, or

$M_X(ju)$:

$$M_X(ju) \triangleq E\left[e^{juX}\right] = \int_{-\infty}^{+\infty} e^{juX} f_X(x)\, dx$$

→ So the $M_X(ju)$ would be the Fourier transform of $f_X(x)$ (in the Fourier transform we had minus sign in the exponent instead of a plus sign)!

$\boxed{\text{So, if } j\omega \text{ is replaced by } -ju \text{ in Fourier transform tables, they can be used to obtain characteristic functions from pdfs}}$

⇒ A pdf is obtained from the corresponding characteristic function by the inverse transform relationship:

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{M_X(ju)}\, e^{-jux}\, du.$$

⇒

We can also use the characteristic function to obtain the moments of a random variable

↳ How to find this? ⇒ Differentiation.

- $$\frac{\partial M_x(ju)}{\partial u} = j \underbrace{\int_{-\infty}^{+\infty} x f_x(x) e^{jux} du}_{E[X]}.$$

$$\Rightarrow E[X] = (-j) \left. \frac{\partial M_x(ju)}{\partial u} \right|_{u=0.}$$

- Let's say that I want to find the:

$$\underbrace{E[X^n]}_{} = (-j)^n \left. \frac{\partial^n M_x(ju)}{\partial u} \right|_{u=0.}$$

n^{th} moment of random variable X.

The pdf of the Sum of two Independent Random Variables

- Given two statistically independent variables X and Y.
- Assume that we know pdfs $f_X(x)$ and $f_Y(y)$.
- Goal: Find the pdf of their sum $Z = X+Y$, i.e., the $f_Z(z)$.

$$\boxed{E[X] = \int_{-\infty}^{+\infty} x f_X(x) d}$$

In order to compute this, we will use the characteristic function:

$$M_Z(ju) = E[e^{juZ}] = E[e^{ju(X+Y)}]$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{ju(x+y)} f_X(x) f_Y(y)\, dx dy$$

$$= \underbrace{\int_{-\infty}^{+\infty} f_X(x) e^{jux} dx}\ \underbrace{\int_{-\infty}^{+\infty} f_Y(y) e^{juy} dy}$$

$$= \underbrace{E[e^{juX}]} \cdot \underbrace{E[e^{juY}]}$$

$$= M_X(ju) * M_Y(ju).$$

characteristic function of X

characteristic function of Y.

Important Note

The characteristic function is the Fourier transform of the corresponding pdf and that a product in the frequency domain corresponds to convolution in the time domain:

$$f_Z(z) = f_X(x) * f_Y(y) = \int_{-\infty}^{+\infty} f_X(z-v)\, f_Y(v)\, dv.$$