ECE 345 / ME 380: Introduction to Control Systems
Final Exam

Dr. Oishi

Due December 11, 2020, by 11:59am (before noon)

This midterm is open note, open book, and Matlab and electronic resources are allowed. **No communication of any sort regarding the content of the exam is allowed with anyone other than Dr. Oishi.**

Read the questions carefully! For full credit, show all of your work.

Please provide your written response on the exam .pdf if possible, using additional sheets as necessary.

Academic dishonesty is a violation of the UNM Student Code of Conduct. Students suspected of academic dishonesty will be referred for disciplinary action in accordance with University procedures.

By signing below, I affirm that I have completed the midterm independently, under the conditions stated above.

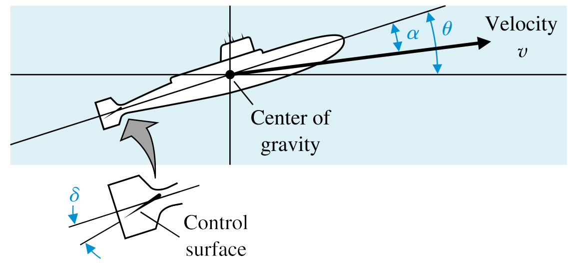
David Kirby — 101652098

Student Name — Student ID #

Problem #	Actual points	Possible points
1	30	30
2	13	20
3	33	45
4	29	35
5	37	40
Total:	142	170

1 Pitch control of a submarine (30 points)

Consider the problem of pitch control of a small submarine. We consider the longitudinal dynamics, in which the pitch of the submarine is indicated by $\theta(t)$, and the angle of attack is $\alpha(t)$. The input $u(t)$ is the deflection of the stern plane, and the output is the pitch angle $\theta(t)$. The constants m, k, b, v are all positive values.



The dynamics of the submarine are described by two coupled differential equations:

$$\ddot{\theta}(t) = -k\theta(t) - b\dot{\theta}(t) - v\alpha(t) + u(t) \quad (1)$$

$$\dot{\alpha}(t) = \frac{2}{v}u(t) \quad (2)$$

1. (+10 points) Find the transfer function with input $U(s)$ and output $\theta(s)$. For full credit, put the transfer function into standard form. (Presume that the initial conditions are all zero.)

$$s^2\Theta(s) = -bs\Theta(s) - k\Theta(s) - vA(s) + U(s)$$

$$sA(s) = \frac{2}{v}U(s) \rightarrow A(s) = \frac{2}{sv}U(s)$$

$$U(s) = (s^2 + bs + k)\Theta(s) + \cancel{v}\left(\frac{2}{s\cancel{v}}\right)U(s)$$

$$U(s) - \frac{2}{s}U(s) = (s^2 + bs + k)\Theta(s)$$

$$U(s)\left[1 - \frac{2}{s}\right] = (s^2 + bs + k)\Theta(s)$$

$$\frac{\Theta(s)}{U(s)} = \frac{1 - \frac{2}{s}}{s^2 + sb + k}$$

$$= \frac{s-2}{s(s^2 + bs + k)}$$

Recall that $\ddot{\theta}(t) = -k\theta(t) - b\dot{\theta}(t) - v\alpha(t) + u(t)$ and $\dot{\alpha}(t) = \frac{2}{v}u(t)$

2. (+10 points) Find a state-space representation, either by defining the state to be $x = \begin{bmatrix} \theta \\ \dot{\theta} \\ \alpha \end{bmatrix}$, or by using phase variable form with $\dfrac{\theta(s)}{U(s)}$. That is, what are the matrices A, B, C, D?

3. (+10 points) Which of the following describes the relationship between the transfer function $\dfrac{\theta(s)}{U(s)}$ in Question 1.1, and the state-space dscription (A, B, C, D) in Question 1.2? *Note: Additional calculations are not needed to answer this.*

   (a) The zeros of the transfer function are the roots of the characteristic equation of A.

   (b) The poles of the transfer function are the same as the eigenvalues of the A matrix.

   (c) The number of zeros is the same as the size of either of the number of rows or columns of the square matrix A.

   (d) Every pole satisfies the equation $|sI - A| = 0$.

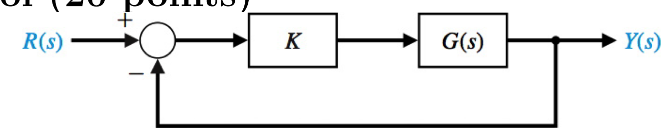
2.

$$\ddot{\theta} = -k\theta - b\dot{\theta} - v\alpha + u, \quad \dot{\alpha} = \frac{2}{v}u, \quad x = \begin{bmatrix} \theta \\ \dot{\theta} \\ \alpha \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -k & -b & -v \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \frac{2}{v} \end{bmatrix} u(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -k & -b & -v \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ \frac{2}{v} \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad D = 0$$

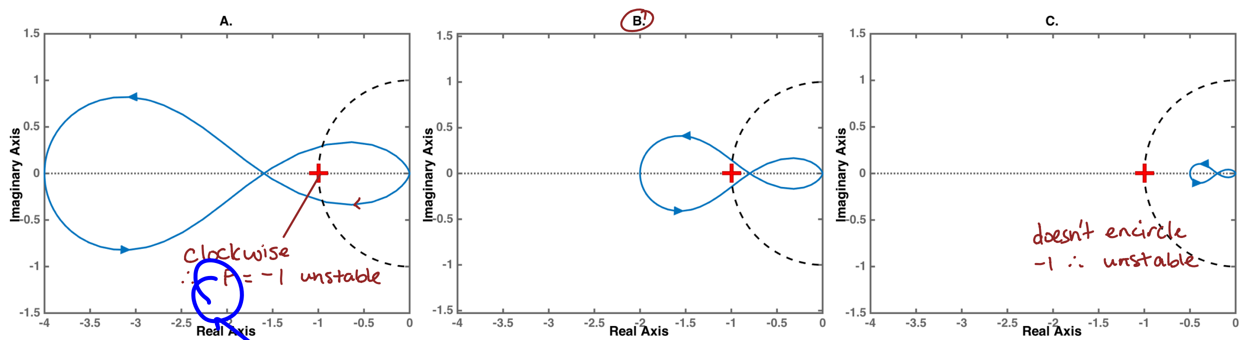
2 Space shuttle attitude control (20 points)



The attitude dynamics of a space shuttle, described by the transfer function

$$G(s) = \frac{2}{(s - \frac{1}{2})(s^2 + 2s + 2)}, \tag{3}$$

are unstable. A control system is designed to stabilize the system using negative unity feedback. Consider the Nyquist diagram of $KG(s)$ for three values of K, as shown below.



$K = a,\ a > 1$ $\qquad$ $K = 1$ $\qquad$ $K = c,\ c < 1$

1. (+10 points) Using the Nyquist criterion, determine which figure(s), if any, depict an asymptotically stable closed-loop system.

   (a) A.

   (b) B.

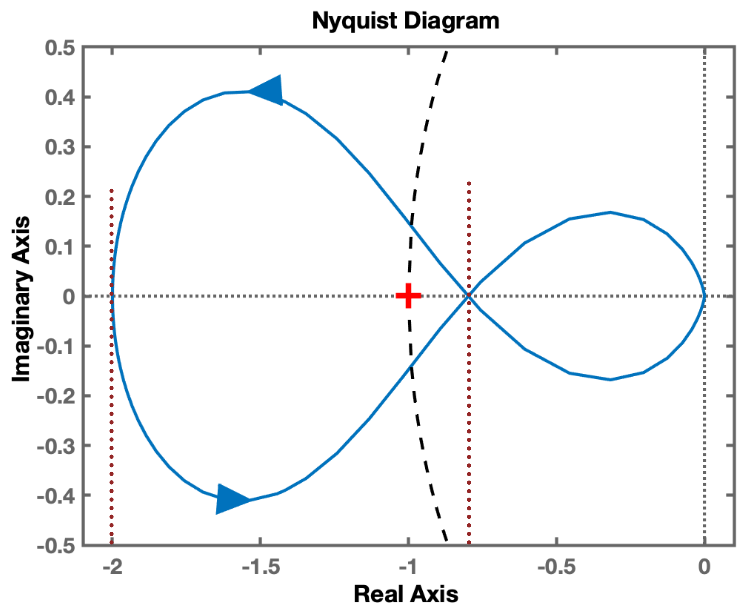
   (c) C.

   (d) A. and C.

   (e) None of the figures show an asymptotically stable closed-loop system.

Recall that $G(s) = \frac{2}{(s-\frac{1}{2})(s^2+2s+2)}$.

2. (+10 points) Using Nyquist diagram B. (enlarged below to aid in your analysis), determine the approximate value of K that would result in a closed-loop system $\frac{Y(s)}{R(s)}$ that is marginally stable, with exactly *one* pole on the imaginary axis.



$Z = P - N$, $P = 1$ $\therefore$ Need $N = 1$ for stability

1 counterclockwise encirclement of -1.

$$\Delta(s) = D(s) + KN(s)$$
$$= (s-\tfrac{1}{2})(s^2+2s+2) + 2K$$
$$= 2s^3 + 3s^2 + 2s + (4K-2)$$

$$\begin{array}{c|cc} s^3 & 2 \ {}_{a_3} & 2 \ {}_{a_1} \\ s^2 & 3 \ {}_{a_2} & 4K-2 \ {}_{a_0} \\ s^1 & 10-8K \ {}_{b_1} & 0 \\ s^0 & 4K-2 \ {}_{c_1} & 0 \end{array}$$

$$\frac{(3)(2) \quad -(2(4K-2))}{3}$$
$$=$$
$$= \frac{10-8K}{3}$$

$10 - 8K > 0 \longrightarrow K < \frac{5}{4}$

$4K - 2 > 0 \longrightarrow K > \frac{1}{2}$

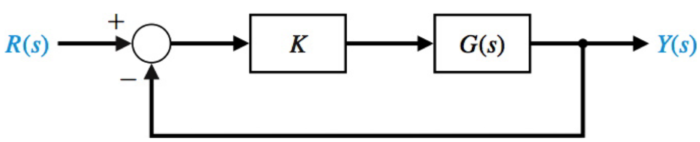
$\times$

-6

$\therefore$ $K = \frac{5}{4}$ will lead to marginal stability with one pole on the axis.

3 Aircraft Carrier Landing (45 points)

A simple model for lateral aircraft dynamics during landing on an aircraft carrier is described by the negative unity feedback system with input that is the center line of the carrier deck, and output that is the lateral position of the aircraft. The open-loop transfer function is $G(s) = \dfrac{s+4}{(s-1)(s+1)}$.



The characteristic equation of the closed-loop system is $\Delta(s) = s^2 + Ks + (4K - 1)$. Note that the Hurwitz criteria requires $K > 1/4$ for asymptotic stability, and that an underdamped response occurs for $8 - 2\sqrt{15} < K < 8 + 2\sqrt{15}$, or approximately $0.25 \lesssim K \lesssim 15.75$. *You do not need to recompute these values on your own.*

1. (+10 points) What values of K that are associated with an underdamped response (i.e., $8 - 2\sqrt{15} < K < 8 + 2\sqrt{15}$) will assure that the closed-loop system $\frac{Y(s)}{R(s)}$ has settling time of *at least* 1 second, and natural frequency of *at least* 4 rad/s?

$$T_s = \frac{4}{\zeta \omega_n} \geq 1 \qquad \omega_n \geq 4 \ \frac{\text{rad}}{s}$$

$$\Delta(s) = s^2 + Ks + (4K-1) \rightarrow s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$2\zeta\omega_n = K \qquad \omega_n^2 = 4K - 1$$

$$\omega_n = \sqrt{4K-1}$$

$$2\zeta\sqrt{4K-1} = K$$

$$\zeta = \frac{K}{2\sqrt{4K-1}}$$

$$T_s = \frac{4}{\frac{K}{2}} \geq 1 \qquad \omega_n = \sqrt{4K-1} \geq 4$$

$$\frac{8}{K} \geq 1 \qquad 4K - 1 \geq 16$$

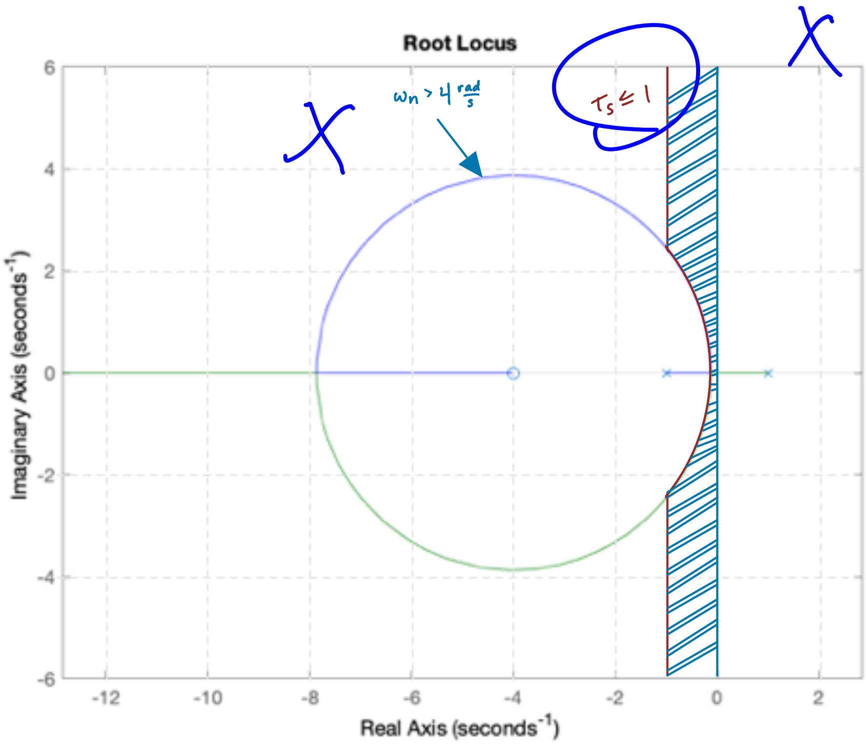
$$K \geq \frac{17}{4}$$

$$K \leq 8 \qquad K \geq 4.25$$

$$4.25 \leq K \leq 8$$

Recall that the desired transient response specifications are: Settling time of at least 1 second, natural frequency of at least 4 rad/s, and that the system be underdamped.

2. (+10 points) On the root locus plot of $G(s)$ below, accurately shade the region of the complex plane which meets the settling time and natural frequency specifications.



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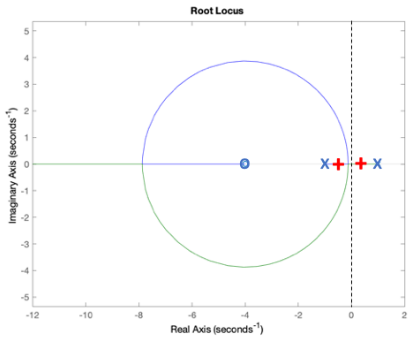
3. (+10 points) Imagine that you are tasked with designing the landing control system to assure stability and the desired transient response specifications. Of the four possible pairs of pole locations shown on the root locus plots on *the next page*, which one(s) would you choose for the landing system? *Describe your reasoning in a single sentence.*

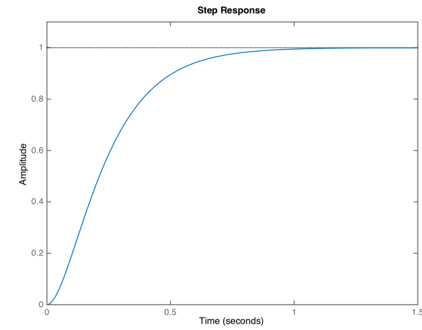
Plot A has a pole in the R.H.P. and is therefore unstable; Plot B has two poles close to the imaginary axis and therefore oscillates, no fun for the pilot; Plot D is overdamped; therefore, I would choose Plot C.

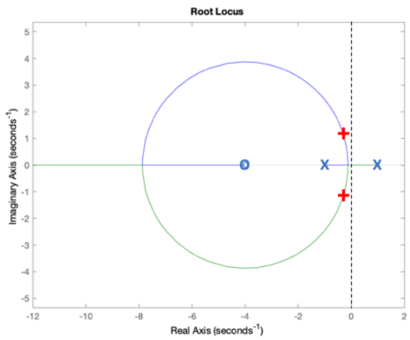
-2.

4. (+15 points) Match the pole locations of the closed-loop system $\frac{Y(s)}{R(s)}$ on the root locus plots of $G(s)$ on the left (indicated by '+') to the appropriate step response of $\frac{Y(s)}{R(s)}$ on the right.

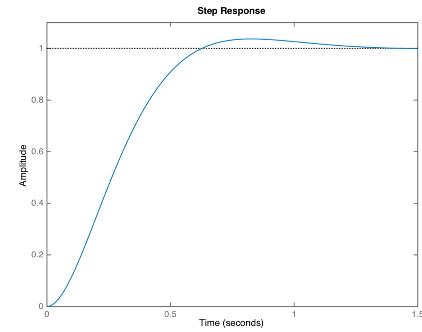
A. 

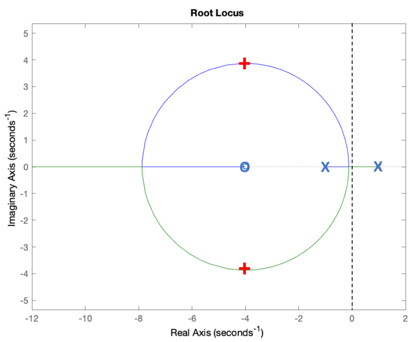
corresponds to 3

1. 

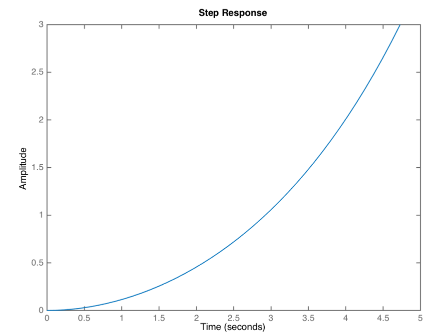
B. 

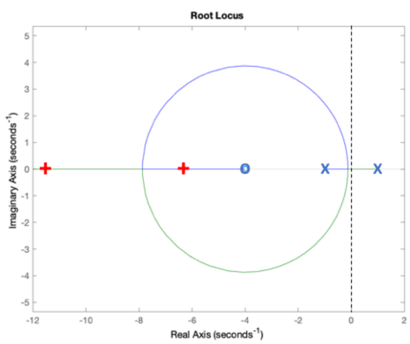
corresponds to 4

2. 

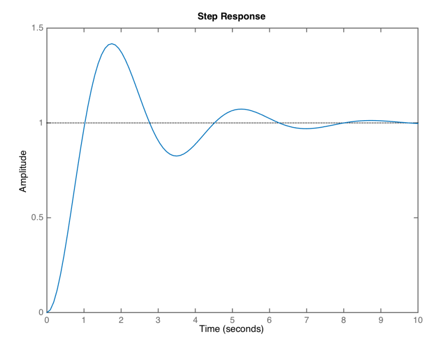
C. 

corresponds to 1

3. 

D. 

corresponds to 2

4. 

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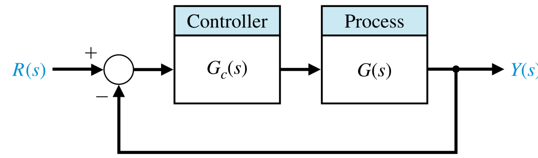
4 Blood Pressure During Anesthesia (35 points)

Blood pressure is thought to be a proxy for depth of anesthesia during surgery. A system has been designed to control mean arterial pressure during anesthesia, with desired blood pressure as input $R(s)$, and actual blood pressure as output $Y(s)$. The plant is described by $G(s) = \frac{1}{(s+2)^2}$. We will explore two controllers: a) $G_c(s) = K$, and b) $G_c(s) = \frac{K}{s}$.



1. (+10 points) Use *type number* to find the steady-state error of the closed-loop system $\frac{Y(s)}{R(s)}$ in response to a step input $u(t) = \cancel{t} \cdot \mathbf{1}(t)$, under each of the controllers:

   (a) $G_c(s) = K$

   (b) $G_c(s) = \frac{K}{s}$

(a) $\boxed{K} \rightarrow \boxed{G_c(s)\,G(s)} = K \cdot \frac{1}{(s+2)^2} \rightarrow$ Type 0

$$e_{ss} = \frac{1}{1+K_p}$$

$$K_p = \lim_{s\to 0} K\, G_c(s)\, G(s) = \lim_{s\to 0} K \frac{1}{(s+2)^2} = \frac{K}{4}$$

$$e_{ss} = \frac{1}{1+\frac{K}{4}} = \frac{4}{4+K}$$

(b) $\boxed{K} \rightarrow \boxed{G_c(s)\,G(s)} = K \cdot \frac{1}{s(s+2)^2} \rightarrow$ Type 1

$$e_{ss} = \frac{1}{1+K_p}$$

$$K_p = \lim_{s\to 0} K\, G_c(s)\, G(s) = \lim_{s\to 0} K \frac{1}{s(s+2)^2} = \infty$$

−4

$$e_{ss} = 0$$

Recall $G(s) = \frac{1}{(s+2)^2}$, and a) $G_c(s) = K$, or b) $G_c(s) = \frac{K}{s}$.

2. (+10 points) Use either the Hurwitz criterion or the Routh table to determine values of $K > 0$ for which the *closed-loop system* $\frac{Y(s)}{R(s)}$ will be asymptotically stable, for each of the two controllers:

   (a) $G_c(s) = K$

   (b) $G_c(s) = \frac{K}{s}$

(a) $\Delta(s) = D(s) + K N(s)$

$= (s+2)^2 + K$

$= s^2 + 4s + (4+K) \longrightarrow a_0 = 4+K, \quad a_1 = 4, \quad a_2 = 1$

$a_1 a_2 - a_0 > 0$

$4 - (4+K) > 0$

$K > 0$

$\checkmark a_2, \checkmark a_1, a_0 > 0$

$4 + K > 0$

$K > -4$

$\therefore$ closed-loop system will be stable for all values $K > 0$.

(b) $\Delta(s) = D(s) + K N(s)$

$= s(s+2)^2 + K$

$= s^3 + 4s^2 + 4s + K \longrightarrow a_0 = K, \quad a_1 = 4, \quad a_2 = 4, \quad a_3 = 1$

$a_1 a_2 - a_0 > 0$

$16 - K > 0$

$K < 16$

$\checkmark a_2, \checkmark a_1, a_0 > 0$

> 0

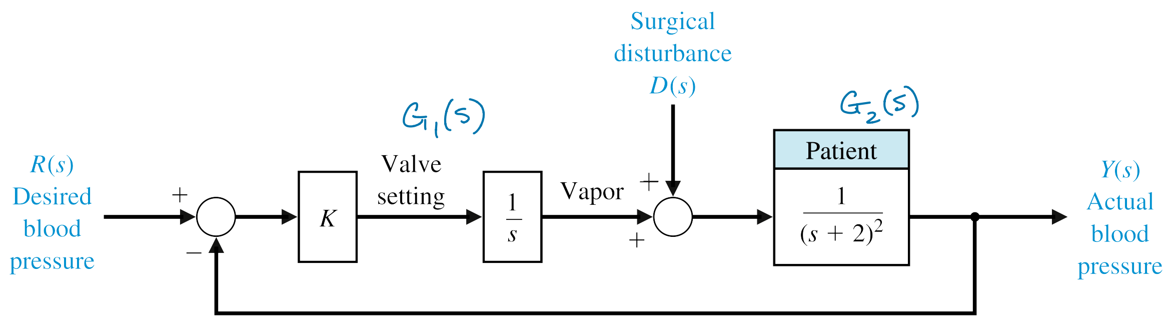
$K > 0$

$\therefore$ closed-loop system will be stable for all values $K < 16$.

3. (+5 points) Consider your responses to the two previous questions. Which of the two controllers is 'best'? Why? *Justify your response in a single sentence.*

Controller (b) has 0 steady-state error and would thus be able to track the system better; however, controller (b) is not as robust since it is only stable for gain $K < 16$. Given the choice, I would choose controller (b).

Now consider the effect of an additive disturbance, $d(t)$, which captures the effect of surgical actions on blood pressure, as shown in the figure below. Presume that $G_c(s) = \frac{K}{s}$, as shown below.



4. (+10 points) Calculate the steady-state *response* to a unit *ramp* disturbance input in terms of the gain K. (Presume that the reference input in this case is $r(t) = 0$.) Does increasing K improve steady-state response?

y_{ss} due to $D(s) = \frac{1}{s^2}$, $R(s) = 0$

Let $G_1(s) = \frac{K}{s}$ (our controller) and $G_2(s) = \frac{1}{(s+2)^2}$ (our plant).

$$G_R(s) = \frac{G_1(s)\,G_2(s)}{1 + G_1(s)G_2(s)} = \frac{K}{s^3+4s^2+4s+K}$$

$$G_D(s) = \frac{G_2(s)}{1 + G_1(s)\,G_2(s)} = \frac{s}{s^3+4s^2+4s+K}$$

$$Y(s) = G_D(s) \cdot D(s) = \frac{s}{s^3+4s^2+4s+K} \cdot \frac{1}{s^2} = \frac{1}{s(s^3+4s^2+4s+K)}$$

F.V.T.

$$y_{ss} = \lim_{s \to 0} sY(s) = \lim_{s\to 0} \cancel{s}\left(\frac{1}{\cancel{s}(s^3+4s^2+4s+K)}\right) = \frac{1}{K}$$

∴ increasing K does not improve steady-state reponse, it has an inverse effect.

−2

5 Stability (40 points)

Consider the system in state-space form below.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0 \cdot u \end{aligned} \tag{4}$$

1. (a) (+5 points) Show that the characteristic equation of the system is $\Delta(s) = (s+3)(s+1)$, *without* first converting the system into transfer function form.

   (b) (+5 points) Show that the transfer function for the system is $G(s) = \frac{2}{(s+1)(s+3)}$.

2. (+10 points) Which *one* of the following is most correct?

   (a) The system is BIBO stable because for $u(t) = \mathbf{1}(t)$, $y(t)$ is also bounded.

   (b) The system is BIBO stable because it is asymptotically stable.

   (c) The system is BIBO stable and it is also asymptotically unstable.

   (d) The system is BIBO unstable because it is asymptotically unstable.

   (e) The system is BIBO unstable because for $u(t) = t^2 \cdot \mathbf{1}(t)$, $y(t) \to \infty$ as $t \to \infty$.

1. (a) $A = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$ $D = 0$

$$\Delta(s) = |sI - A| = 0$$

$$= \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} s+3 & 0 \\ 0 & s+1 \end{bmatrix} \right|$$

$$= (s+3)(s+1) - 0$$

$$= (s+3)(s+1)$$

✓

(b) $G(s) = C(sI - A)^{-1} B + D$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+3 & 0 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 0$$

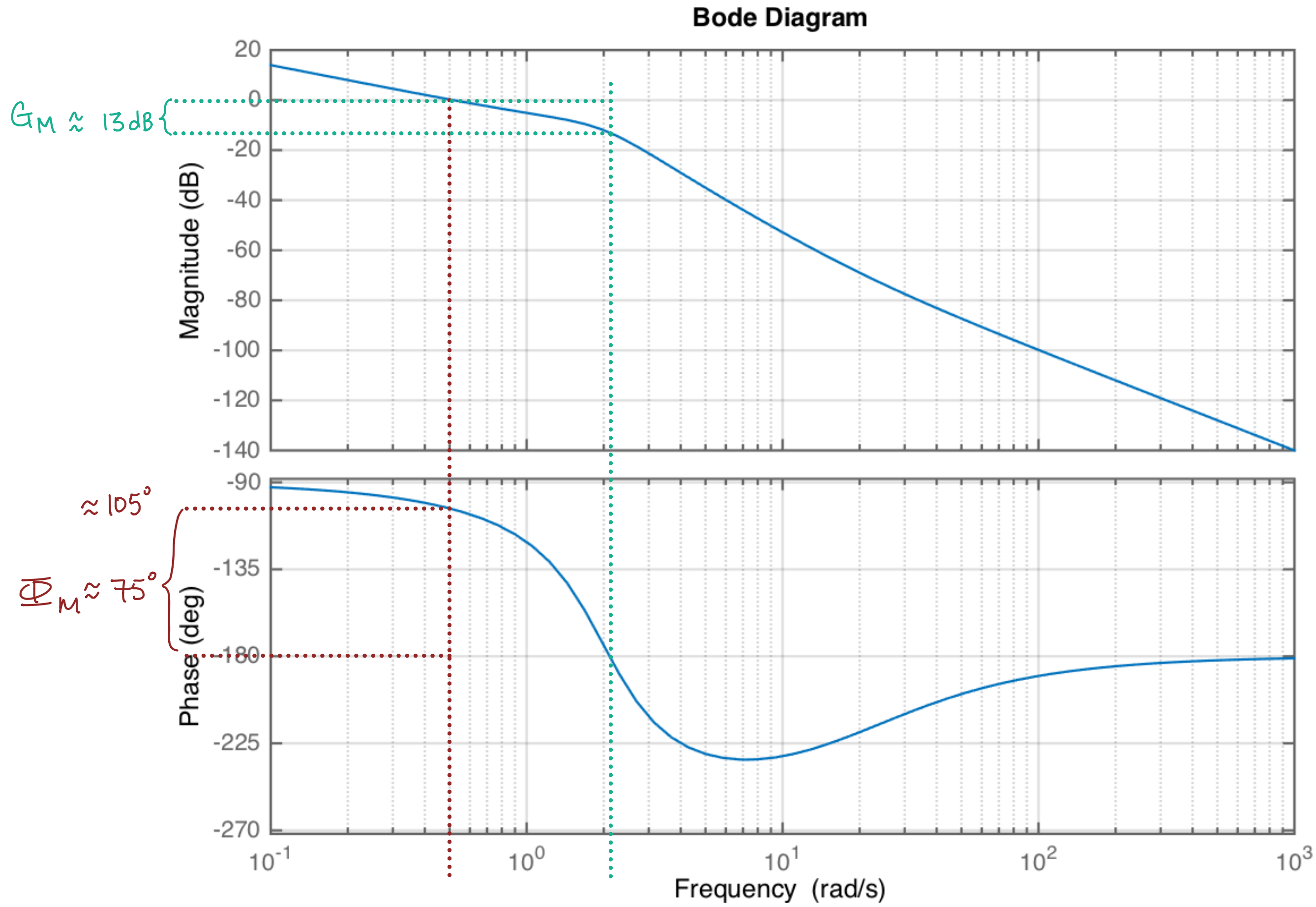
$$= \frac{1}{s+1} \cdot \frac{1}{s+3}$$

?

-3

$$= \frac{2}{(s+1)(s+3)}$$

The following Bode diagram of an open-loop system $G(s)$ is obtained experimentally. For full credit, show your work.



3. (+10 points) Find the phase margin Φ_M, and the gain margin G_M *in dB*. Is the negative unity feedback system stable with $K = 1$? Why or why not?

4. (+10 points) Will the negative unity feedback system be unstable for $K = 10$? Briefly justify your answer in a single sentence.

3.) Please see plot above, $\Phi_M \approx 75°$, $G_M \approx 13dB$.
Yes, the system is stable at $K=1$ because both $G_M > 0dB$ and $\Phi_M > 0°$

4.) Increasing K by 10 shifts the gain margin up 20dB while leaving the phase margin the same, making the $G_M \approx -7dB$ and the system will be unstable.

End of exam.