ECE 345 / ME 380
Introduction to Control Systems
Lecture Notes 4

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Learning Objectives

- Use poles and zeros of transfer functions to determine the time response of a control system
- Describe quantitatively the transient response of first-order systems
- Write the general response of a second-order system given the pole location
- Find the damping ratio and natural frequency of a second-order system
- Find the settling time, peak time, percent overshoot, and rise time for an underdamped second-order system
- Approximate higher-order systems and systems with zeros as first- or second-order systems
- Find the time response from the state-space representation

References:

- Nise Chapter 4

Outline

- Transient response
  - Transfer function poles and zeros
  - State-space eigenvalues
- First order systems
- Second order systems
  - Damping ratio and natural frequency
  - Settling time, percent overshoot, peak time, rise time
- Approximations to second-order systems

Transient response

Quantitative analysis of the transfer function

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)} \tag{1}$$

- Basic terminology
  - Zeros: $\{s \mid N(s) = 0\}$
    (Values of s such that the numerator has a value of 0)
  - Poles: $\{s \mid D(s) = 0\}$
    (Values of s such that the denominator has a value of 0)
- The transfer function
  $$G(s) = \frac{s+2}{s+5}$$
  has one zero at $s = -2$ and one pole at $s = -5$.
- Transfer function (1) has m zeros and n poles

Transient response

Standard ways of writing $G(s)$

1. Factored form: $G(s) = \frac{b_m(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$
2. Polynomial form: $G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$

Do not leave in terms of fractions over fractions!

What happens to $G(s)$ at a zero? At a pole?

What if a pole and a zero occur at the same location (e.g., are *collocated*)?

$$G(s) = \frac{s+2}{(s+5)(s+2)}$$

Transient response

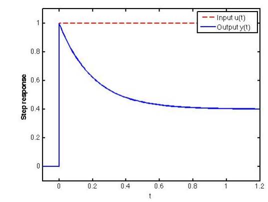
Consider the step response of $G(s) = \frac{s+2}{s+5}$.

- Terms from the input $(1/s)$ and from the plant $(1/(s+5))$ appear in the denominator

$$Y(s) = G(s) \cdot \frac{1}{s} = \frac{s+2}{s(s+5)} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

- The pole at $s = -5$ appears in the exponential decay rate:

$$y(t) = \left(\frac{2}{5} + \frac{3}{5}e^{-5t}\right)\mathbf{u}(t)$$



Transient response

For systems with no pole-zero cancellation,

- **Poles of transfer function = Eigenvalues of state matrix**

With zero initial conditions,

$$G(s) = \frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D, \quad (sI-A)^{-1} = \frac{\mathrm{adj}(A)}{|sI-A|} \tag{2}$$

The *characteristic equation*

$$\Delta(s) = |sI-A| = (s+p_1)(s+p_2)\cdots(s+p_n) = 0 \tag{3}$$

is critical in determining the system response $y(t)$.

Transient response

Clicker question

Which of the following is the characteristic equation of the spring-mass-damper system with output $y(t) = x(t)$ and input $u(t) = f(t)$?

$$m\ddot{x} = -kx - c\dot{x} + f(t) \tag{4}$$

A. $0 = ms^2 + cs + k + F(s)$
B. $F(s) = (ms^2 + cs + k)X(s)$
C. $0 = \frac{1}{ms^2+cs+k}$
D. $0 = ms^2 + cs + k$
E. I have no idea.

Transient response

Relationships between poles, eigenvalues, and the characteristic equation:

1. The poles of the transfer function determine the type of transient response (e.g., exponential growth, exponential decay, sinusoids, ...)
2. These are *the same* as the eigenvalues of the system matrix (for transfer functions without any pole-zero cancellation)
3. The poles of the transfer function are solved though the *characteristic equation*.
4. The order of the characteristic equation indicates the number of solutions (which are the same as the poles of the transfer function)

Recommendations:

- Convince yourself (work through a couple of examples) that the eigenvalues of A are equivalent to the poles of the transfer function.

Transient response

Clicker question

Find the characteristic equation of the system

$$\dot{x} = \begin{bmatrix} 0 & 2 \\ -2 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} x \tag{5}$$

Without any further calculations, which of the following most correctly describes the system response $y(t)$ of the system to an input $u(t) = e^{-2t} \cdot \mathbf{1}(t)$?

A. Exponential growth scaled by e^{+t}, e^{+4t} for $t \geq 0$.
B. Exponential decay scaled by e^{-t}, e^{-4t} for $t \geq 0$.
C. Exponential decay scaled by e^{-t}, e^{-4t}, e^{-2t} for $t \geq 0$.
D. Exponential decay scaled by e^{-t}, e^{-2t}, with oscillations at 2 rad/s, for $t \geq 0$.

Transient response

With non-zero initial conditions, the *output response* is

$$\begin{aligned} Y(s) &= C(sI-A)^{-1}x_0 + \big(C(sI-A)^{-1}B + D\big)\,U(s) \\ y(t) &= Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \end{aligned} \tag{6}$$

In the time-domain, the *state response* is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{7}$$

- The *forced response* is the part of $x(t)$ due to the input $u(t)$
- The *natural response* is the part of $x(t)$ due to initial conditions

Note that $\mathcal{L}\left[e^{At}\right] = (sI-A)^{-1}$, and $\phi(t) \triangleq e^{At}$ is called the *state-transition matrix*. (So is $\Phi(s) \triangleq (sI-A)^{-1} = \mathcal{L}\left[\phi(t)\right]$.)

Transient response

Clicker question

The output response $y(t)$ of the system

$$G(s) = \frac{1}{(s-1)(s+4)} \tag{8}$$

in response to an input $u(t) = e^{-2t} \cdot \mathbf{1}(t)$ is best described by

A. Exponential decay scaled by e^{-t}, e^{+4t} for $t \geq 0$.
B. Exponential growth scaled by e^{+t}, e^{-4t} for $t \geq 0$.
C. Exponential decay scaled by e^{+t}, e^{-4t}, e^{-2t} for $t \geq 0$.
D. Exponential growth scaled by e^{+t}, e^{-4t}, e^{-2t} for $t \geq 0$.
E. Exponential growth scaled by e^{+t}, e^{-2t}, with oscillations at 4 rad/s, for $t \geq 0$.

Transient response

State-transition matrix

- Appears in both the natural response and the forced response
- Contains the eigenvalues of A = poles of the transfer function
- Is defined as the infinite sum:

$$e^{At} \triangleq I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

which is NOT the same as $\begin{bmatrix} e^{a_{11}t} & e^{a_{12}t} & \cdots & e^{a_{1n}t} \\ e^{a_{21}t} & e^{a_{22}t} & \cdots & e^{a_{2n}t} \\ \vdots & \vdots & \ddots & \ddots \\ e^{a_{n1}t} & e^{a_{n2}t} & \cdots & e^{a_{nn}t} \end{bmatrix}$!

- Often easier to compute via inverse Laplace transform $e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$

Transient response

Clicker question

Consider an LTI system whose state transition matrix is

$$\phi(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

Which of the following statements is *incorrect*?

A. The system has a characteristic equation $0 = s^2 + 4s + 3$.

B. The forced response to a step input with $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $X(s) = \begin{bmatrix} \frac{1}{s+3} \\ \frac{1}{s+3} \end{bmatrix}$

C. The poles of the system are in the left half of the complex plane.

D. The natural response to $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is $x(t) = \begin{bmatrix} e^{-t} \\ e^{-3t} \end{bmatrix}$ for $t \geq 0$.

First-order systems

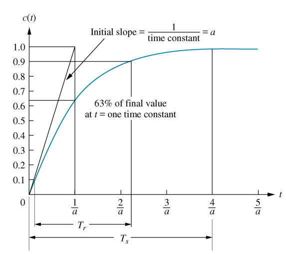
A first-order system

$$G(s) = \frac{a}{s+a}, a \in \mathbb{R}$$

has the step response

$$y(t) = (1 - e^{-at})\mathbf{u}(t)$$

Parameter	Notation	Formula
Rise time	T_r	$2.2/a$
Settling time	T_s	$4/a$



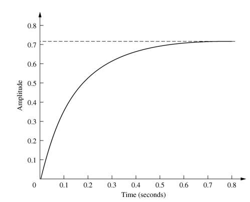
First-order systems

Clicker question

Consider the step response of

$$G(s) = \frac{K}{s+a}$$

with steady-state value 0.72.



Which of the following most correctly describes equations you could use to approximate K and a based solely on the above plot?

A. $K/a = 0.72$, $y(\frac{1}{a}) = 0.63 \cdot 0.72$.

B. $K/a = 0.72$, $y(\frac{2.2}{a}) = 0.9 \cdot 0.72$.

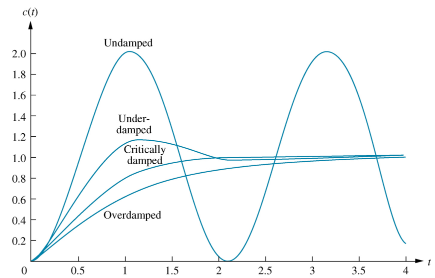
C. $K/a = 0.98 \cdot 0.72$.

D. $K/a = 0.72$, $y(\frac{4}{a}) = 0.98 \cdot 0.72$.

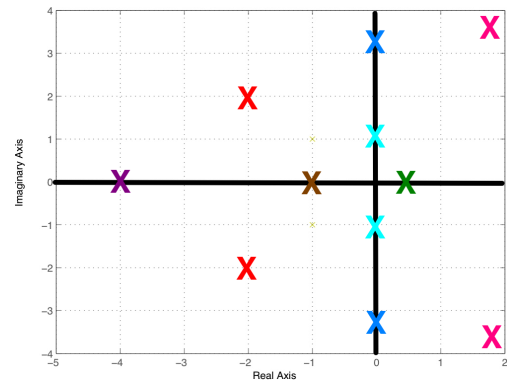
E. Either of A or D

Second-order systems

- Often characterized by step response
- Behaviors largely determined by location of poles
- Very, very widely used (even for higher order systems)



Second-order systems



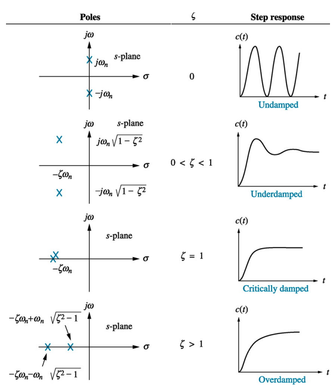
Second-order systems

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Pole locations described by

- Damping ratio ζ
- Natural frequency ω_n

And correspond to differing step responses



Second-order systems

Clicker question

For the transfer function

$$G(s) = \frac{4}{s^2 + 3s + 4} \tag{9}$$

which of the following correctly describes the system's step response characteristics?

A. Overdamped with $\zeta = 2, \omega_n = 1/2$.
B. Underdamped with $\zeta = 1/2, \omega_n = 2$.
C. Critically damped with $\zeta = 1, \omega_n = 2$.
D. Underdamped with $\zeta = 1/4, \omega_n = \sqrt{2}$.
E. Overdamped with $\zeta = 3/4, \omega_n = 2$.

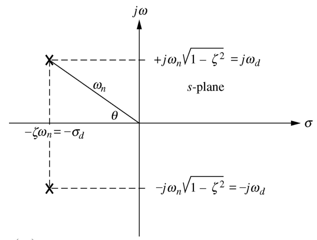
Second-order systems

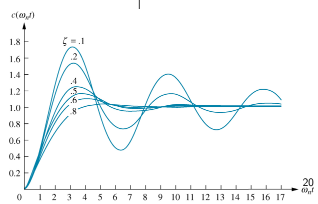
Underdamped systems

- Damping ratio ζ
- Natural frequency ω_n

Geometrical relationships

- $\zeta = \cos\theta$
- Damped frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$
- Decay rate $\zeta\omega_n$





Second-order systems

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{10}$$

Case	Description	ζ	ω_n	Step response for $t \geq 0$
(a)	Undamped	$\zeta = 0$	> 0	$y(t) = 1 - \cos(\omega_n t)$
(b)	Underdamped	$0 < \zeta < 1$	> 0	$y(t) = 1 - e^{-\sigma t}\cos(\omega_d t)$ $-\frac{\sigma}{\omega_n} e^{-\sigma t}\sin(\omega_d t)$
(c)	Critically damped	$\zeta = 1$	> 0	$y(t) = 1 - e^{-\sigma t} - \sigma t e^{-\sigma t}$
(d)	Overdamped	$\zeta > 1$	> 0	$y(t) = 1 - \frac{\sigma_2}{K} e^{-\sigma_1 t} + \frac{\sigma_1}{K} e^{-\sigma_2 t}$

Note that

(b) $\sigma = \zeta\omega_n$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

(c) $\sigma = \omega_n$

(d) $s^2 + 2\zeta\omega_n s + \omega_n^2 = (s+\sigma_1)(s+\sigma_2)$, $K = \sigma_2 - \sigma_1$

Second-order systems

Clicker question

Consider an underdamped spring-mass-damper system. With $m = 1$, $c = 2$, $k = 2$, the poles of

$$G(s) = \frac{1/m}{s^2 + s \cdot c/m + k/m}$$

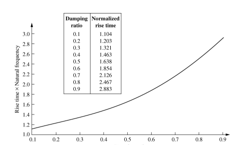
lie at $s = -1 \pm j$. Which one of the following changes could make the system *overdamped*?

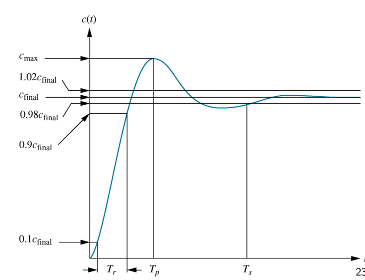
A. Set $c = 0$ and leave k unchanged.
B. Decrease c and leave k unchanged.
C. Increase c and leave k unchanged.
D. Leave c unchanged and increase k.

Second-order systems

Transient response characteristics

- Percent overshoot $M_p = \exp(-\zeta\pi/\sqrt{1-\zeta^2}) \cdot 100$
- Settling time $T_s = 4/(\zeta\omega_n)$
- Rise time $T_r = \left(1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1\right)/\omega_n$
- Peak time $T_p = \pi/\left(\omega_n\sqrt{1-\zeta^2}\right)$



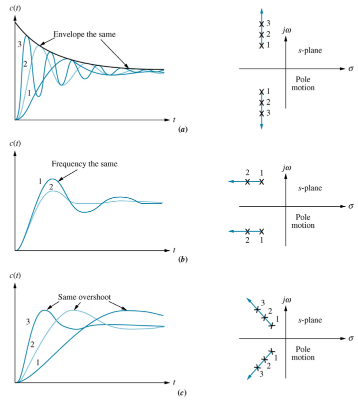


Second-order systems

Transient characteristics are affected by pole location.

Which of these figures shows

- Constant settling time?
- Constant peak time?



Second-order systems

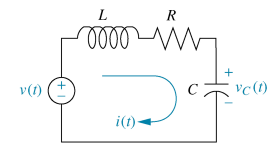
Clicker question

Consider the RLC circuit with input $v(t)$ and output $v_c(t)$, governed by

$$G(s) = \frac{\frac{1}{LC}}{s^2 + s \cdot \frac{R}{L} + \frac{1}{LC}}$$

A fast yet non-reactive *underdamped* response is desired. Which of these parameter combinations will result in a settling time $T_s \leq 2$ seconds, with as little overshoot as possible? *Hint: Log calculations are unnecessary.*

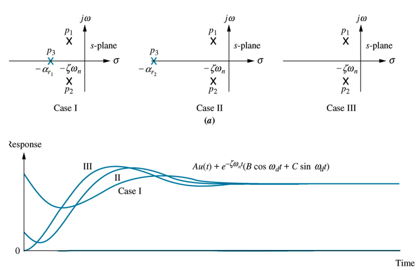
A. $\frac{R}{L} = 2,\ \frac{1}{LC} = 4$
B. $\frac{R}{L} = 4,\ \frac{1}{LC} = 9$
C. $\frac{R}{L} = 4,\ \frac{1}{LC} = 16$
D. $\frac{R}{L} = 6,\ \frac{1}{LC} = 16$



Approximations to second-order systems

Third order systems

- If the pole is sufficiently "far" from the complex conjugate pair, can approximate third order system as a second order system
- Rule of thumb: $\alpha_r > 5 \cdot \zeta\omega_n$ (but more is better)



Approximations to second-order systems

Second order systems with a zero

$$G(s) = \frac{\omega_n^2 (s+a)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{11}$$

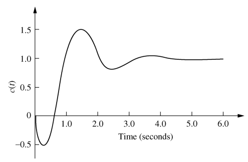
- LHP zero:
$$Y(s)(s+a) = sY(s) + aY(s)$$
  - Derivative of original response (= response of system without the zero)
  - Scaled version of the original response
- Hence usually has *higher overshoot* depending on magnitude of a

Approximations to second-order systems

Second order systems with a zero

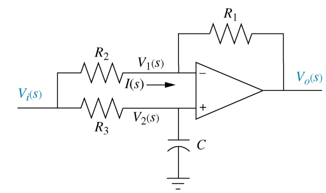
$$G(s) = \frac{\omega_n^2(s+a)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{12}$$

- RHP zero: *Non-minimum phase system*
  - Negative initial derivative outweighs scaled term
- Movement in direction opposite to what would be expected



Approximations to second-order systems

Consider the all-pass filter below (with $R_1 = R_2$).



1. Show that this a non-minimum phase system, by locating its RHP zero.
2. Using Matlab, compute the step response for $R_3C = 1/10$. How is the non-minimum phase nature of the system evident in the transient response?