The Reproducing Property

Manel Martínez-Ramón

ECE, UNM

September 21, 2015

Introduction

- As stated before, a kernel is a function

$$k : X \times X \to \mathbb{R}$$

that can be decomposed as

$$k(\mathbf{x}, \mathbf{z}) = \langle \boldsymbol{\varphi}(\mathbf{x}), \boldsymbol{\varphi}(\mathbf{z}) \rangle$$

where $\boldsymbol{\varphi}(\cdot)$ is a mapping into a Hilbert space.

- This is true if and only if $k(\cdot, \cdot)$ is positive semi-definite.
- Now, we assume that the kernel is positive semi-definite and we derive the properties of mapping $\boldsymbol{\varphi}(\cdot)$ where $\boldsymbol{\varphi}(\cdot)$ is the kernel.

Some abstract notation

- Let the space $\mathcal{F}$ be a Hilbert space endowed with a kernel $k(\mathbf{x}, \mathbf{z}) = \boldsymbol{\varphi}(\mathbf{x})^\top \boldsymbol{\varphi}(\mathbf{z})$
- We define then function $f(\mathbf{x})$ as

$$f(\mathbf{x}) = \sum_{n=1}^{N} \alpha_n k(\mathbf{x}_n, \mathbf{x})$$

- The mapping $\boldsymbol{\varphi}(\mathbf{x})$ has a representation in terms of a coordinate system as

$$\boldsymbol{\varphi}(\mathbf{x}) = \{\varphi_1(\mathbf{x}, \varphi_2(\mathbf{x}), \cdots\}^\top$$

Some abstract notation

- Then, the kernel can be expressed as the sum of products

$$k(\mathbf{x}, \mathbf{z}) = \sum_i \varphi_i(\mathbf{x})\varphi_{\mathbf{i}}(\mathbf{z})$$

- An abstract notation user for vector $\boldsymbol{\varphi}(\mathbf{x})$ in terms of the kernel function is

$$\boldsymbol{\varphi}(\mathbf{x}) = k(\mathbf{x}, \cdot)$$

  that is a vector of the Hilbert space containing all elements $\varphi_i(\mathbf{x})$.
- Then, function $f(\mathbf{x}) = \sum_{n=1}^{N} \alpha_n k(\mathbf{x}_n, \mathbf{x})$ can be considered as the dot product of $\boldsymbol{\varphi}(\mathbf{x})$ with vector $f(\cdot)$

$$f(\cdot) = \sum_{n=1} \alpha_n k(\mathbf{x}_n, \cdot)$$

$\mathcal{F}$ as a function space

- In other words, the elements of the feature space $\mathcal{F}$ are actually functions. Indeed, assume a set of vectors $\mathbf{x}_n$, $1 \leq n \leq N$ that define a subspace in $\mathcal{F}$. Any vector in this subspace has coordinates $\alpha_n k(\mathbf{x}_n, \mathbf{x})$. Then, the feature space can be defined as

$$\mathcal{F} = \left\{ \sum_{n=1}^{N} \alpha_n k(\mathbf{x_i}, \cdot) \right\}$$

where the dot $\cdot$ is used to mark the position of the argument.

$\mathcal{F}$ as a function space

- Now, we define two particular functions into the space

$$f(\mathbf{x}) = \sum_{n=1}^{N} \alpha_n k(\mathbf{x}_n, \mathbf{x})$$

$$g(\mathbf{x}) = \sum_{n=1}^{N} \beta_n k(\mathbf{x}_n, \mathbf{x})$$

- Since $\langle k(\mathbf{x}_n, \cdot), k(\mathbf{x}_m, \cdot) \rangle = k(\mathbf{x}_n, \mathbf{x}_m)$, we can define now the dot product between $f(\cdot)$ and $g(\cdot)$ as

$$\langle f, g \rangle = \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n \beta_n k(\mathbf{x}_n, \mathbf{x}_m) = \sum_{n=1}^{N} \alpha_n g(\mathbf{x}_n) = \sum_{n=1}^{N} \beta_n f(\mathbf{x}_n)$$

The reproducing property

- As a particular case, we can compute $\langle f, f \rangle$ with the result

$$\langle f, f \rangle = \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n k(\mathbf{x}_n, \mathbf{x}_m) \alpha_n = \boldsymbol{\alpha}^\top \mathbf{K} \boldsymbol{\alpha} \geq 0$$

- Also, if we choose $g = k(\mathbf{x}, \cdot)$ we have

$$\langle f, k(\mathbf{x}, .) \rangle = \sum_{n=1}^{N} \alpha_n k(\mathbf{x}, \cdot) = f(\mathbf{x})$$

- Now, define a Cauchy sequence as f_n

  $(f_n(\mathbf{x}) - f_m(\mathbf{x}))^2 = \langle f_n(\mathbf{x}) - f_m(\mathbf{x}), k(\mathbf{x}, \cdot) \rangle \leq \| f_n(\mathbf{x}) - f_m(\mathbf{x}) \|^2 k(\mathbf{x}, \mathbf{x})$

  So the space is complete.