ECE 345 / ME 380
Introduction to Control Systems
Lecture Notes 9

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November 17, 2020

Learning Objectives

- Define and plot the frequency response of a system
- Use the Nyquist criterion to determine the stability of a system
- Find stability and gain and phase margins via
  - Bode plots
  - Nyquist diagrams
- Find the closed-loop time response parameters of peak time, settling time, and percent overshoot given the open-loop frequency response

References:

- Nise, Chapter 10.1–10.3, 10.5–10.8, 10.11. Recommended: 10.12–10.13.

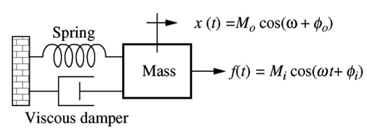
Outline

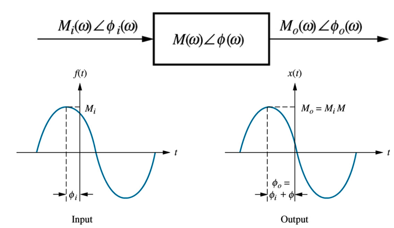
- What is the frequency response?
- Review: Log magnitude
- Sketching the Bode plot
- The Nyquist criterion
- Gain and Phase Margin
- Transient characteristics in frequency response
- Steady-state characteristics in frequency response

What is the frequency response?

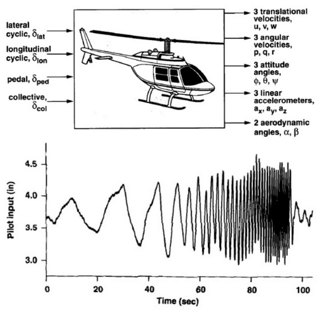
For a stable system,

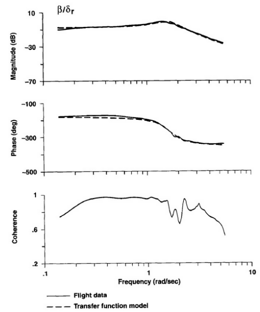
- A sinusoidal input generates a sinusoidal output
- Magnitude and phase depend on the frequency of the input signal
- A Bode plot is the frequency response described in magnitude and phase





What is the frequency response?





Images from "Aircraft and Rotorcraft System Identification", M. Tischler and R. Remple, AIAA Education Series, 2006.

Review: Magnitude and phase

Rewrite the transfer function as

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(z+z_m)}{s^k(s+p_1)(s+p_2)\cdots(s+p_{n-k})}$$

Then take the magnitude of $G(s)$ and convert it into dB

$$\begin{aligned} 20\log|G(j\omega)| &= 20\log K + 20\log|(s+z_1)| + \cdots 20\log|(s+z_m)| \\ &\quad -20\log|s^k| - 20\log|(s+p_1)| - \cdots \\ &\quad -20\log|(s+p_{n-k})| \end{aligned}\Bigg|_{s=j\omega}$$

Note that phase is similarly additive

$$\begin{aligned} \angle G(j\omega) &= \angle K + \angle(s+z_1) + \cdots + \angle(s+z_m) \\ &\quad -\angle s^k - \angle(s+p_1) - \cdots - \angle(s+p_{n-k}) \end{aligned}\Bigg|_{s=j\omega}$$

Sketching the Bode plot

What this additivity means: A Bode plot can be sketched by adding individual Bode plots of each of its elements.

For example:

- The Bode plot of $G(s) = \frac{1}{(s+1)(s+20)}$ looks like the sum of the Bode plots of $G_1(s) = \frac{1}{s+1}$ and the Bode plot of $G_2(s) = \frac{1}{s+20}$.
- This is equivalent to

$$\begin{aligned} 20\log|G(s)| &= 20\log|1| - 20\log|s+1| - 20\log|s+20| \\ \angle G(s) &= \angle(1) - \angle(s+1) - \angle(s+20) \end{aligned}$$

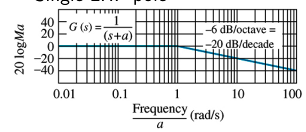
- Nise Example 10.2.

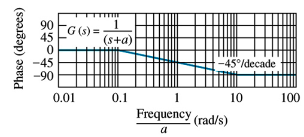
Straight-line approximations of these elements are often very good for back-of-the-envelope sketches.

Sketching the Bode plot

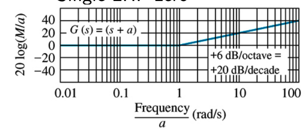
Common elements

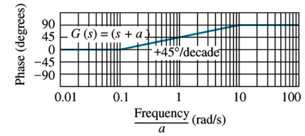
- $G(s) = \frac{1}{s+a}, a > 0$
  Single LHP pole





$G(s) = s+a, a > 0$
Single LHP zero

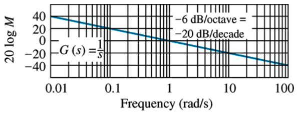


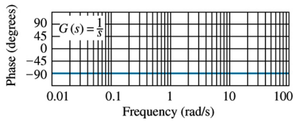


Sketching the Bode plot

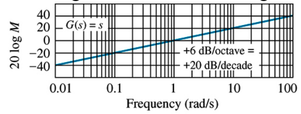
Common elements

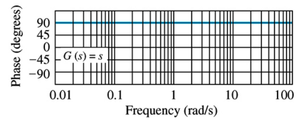
- $G(s) = \frac{1}{s}$
  Single LHP pole at the origin





$G(s) = s$
Single LHP zero at the origin



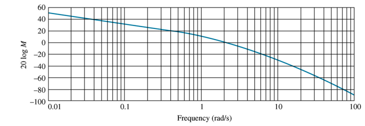


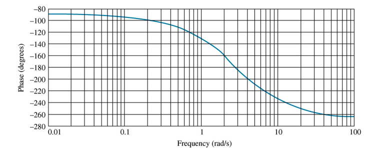
Sketching the Bode plot

Clicker question

Consider a negative unity feedback system with *open-loop* transfer function $G(s)$, shown in the Bode plot to the right.

What is the **type number** of the *closed-loop system* $Y(s)/R(s)$?





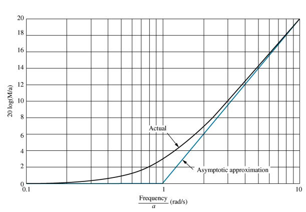
A. 0.
B. 1.
C. 2.
D. Type number cannot be deduced from the plot.

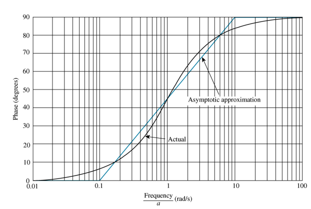
Exercise: What feature of this diagram indicates the magnitude of steady-state error?

Sketching the Bode plot

How reasonable are straight line approximations?

- $G(s) = s + a, a > 0$ (Single LHP zero)

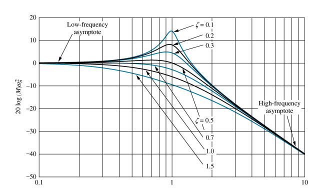


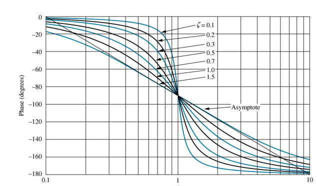


Sketching the Bode plot

Common elements

- $G(s) = \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$, (Complex conjugate LHP pole pair)





- Peak magnitude $M_p = \frac{1}{2\sqrt{1-\zeta^2}}$ occurs at frequency $\omega_p = \omega_n\sqrt{1-2\zeta^2}$

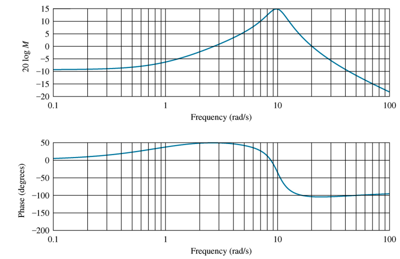
Sketching the Bode plot

1. Straight-line approximations
   - Drop-off in slope of -20dB/dec for each pole
   - Increase in slope of +20dB/dec for each zero
   - Flat slope for frequencies *lower* than the pole / zero
     when pole / zero is not located at the origin
   - 3dB rule
2. Complex conjugate pair of poles
   - Break-point occurs near ω_n
   - Steepness of phase change increases as ζ decreases
   - Peak of magnitude increases as ζ decreases

Sketching the Bode plot

Clicker question

Consider the Bode plot of a system $G(s)$. What is the *order* of $G(s)$, that is, what is the difference between the number of poles and the number of zeros $(n-m)$?



A. $G(s)$ is of order 1.
B. $G(s)$ is of order 2.
C. $G(s)$ is of order 3.
D. The number of poles can be deduced, but not the number of zeros.
E. No information about the number of poles or zeros of can be deduced.

Exercise: What features of this diagram indicate the existence of a zero?

Sketching the Bode plot

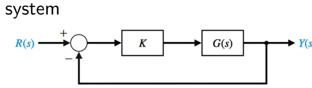
For a given Bode diagram, it is possible to identify certain features of the unknown system $G(s)$:

- Order of $G(s)$ $(= n - m)$
- Type number of the the closed-loop system under negative unity feedback (with open-loop transfer function $G(s)$)
- Steady state error of the closed-loop system under negative unity feedback (with open-loop transfer function $G(s)$)

This can be very useful in designing control laws for systems whose model may be based on experimental data (and not on first principles, e.g., $F = ma$).

Nyquist Criterion

For a negative unity feedback system



The Nyquist criterion *relates the stability of the closed-loop system* to the *open-loop frequency response and open-loop pole location.*

Similar to root locus, in that stability of $\frac{Y(s)}{R(s)}$ can be determined by analyzing $G(s)$ only.

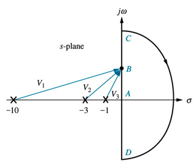
Efficient computational tools in Matlab

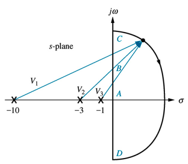
- `bode(sys)`
- `nyquist(sys)`
- `margin(sys)`

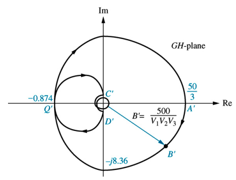
Nyquist Criterion

Main idea:

- Evaluate $G(s)$ for values of s that lie on a specially chosen closed contour that encloses the entire RHP.
  This generates a new closed contour in the complex plane.
- For the new closed contour, count encirclements of the point -1.



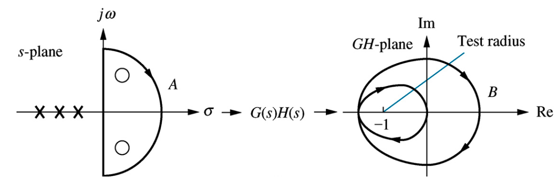




Nyquist Criterion

$$Z = P - N$$

- Z : number of poles of the closed-loop system $Y(s)/R(s)$ **in the RHP**
- N : number of *counter-clockwise* encirclements of -1
- P : number of poles of open-loop system $G(s)$ in the RHP

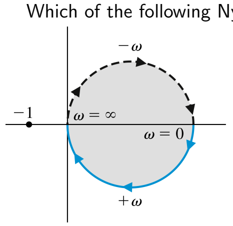


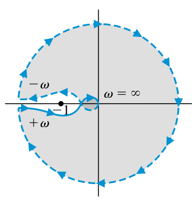
For stable closed-loop systems, we will have $Z = 0$.

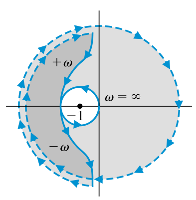
Nyquist Criterion

Clicker question

Which of the following Nyquist diagrams shows 2 clockwise encirclements of -1 (e.g., $N = -2$)?







A. B. C.

D. Both B. and C.

E. None of the above.

Nyquist Criterion

Clicker question

Consider a system whose open-loop transfer function $G(s) = \frac{1}{(s+3)(s+1)^2}$ has the following Nyquist diagram. Is the closed-loop system stable under negative unity feedback with $K = 1$?

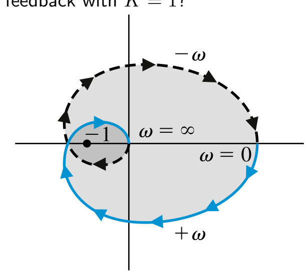
A. Yes, because $Z = P - N = 0 - 0 = 0$.

B. Yes, because $Z = P - N = 2 - 2 = 0$.

C. No, because $Z = P - N = 3 - 2 = 1$.

D. No, because $Z = P - N = 0 - (-2) = +2$.

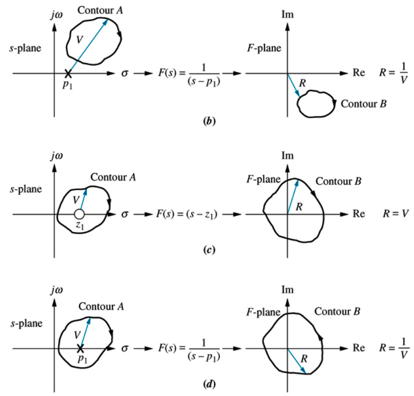
E. Very little confidence in any of the above answers.



Exercise: How does the Nyquist diagram change if $K \neq 1$?

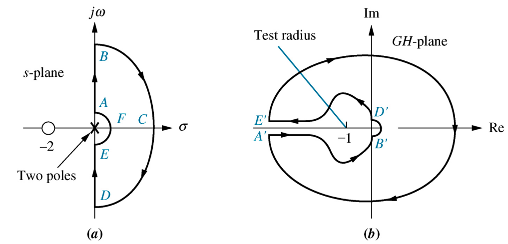
Nyquist Criterion

Where does the Nyquist Criterion come from?



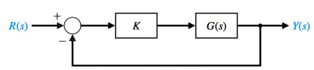
Nyquist Criterion

What if there are poles / zeros on the imaginary axis?



Gain and Phase Margin

Relative measures of stability



- Useful for systems which have branches of the root locus extending into the RHP
- Most often for systems that are stable in open-loop configuration, but can become unstable when gain is increased.

Gain margin: Factor of gain by which you need to would need to increase K in order to destabilize the system

Phase margin: Amount of phase which would needed to be removed from the system in order to destabilize it.

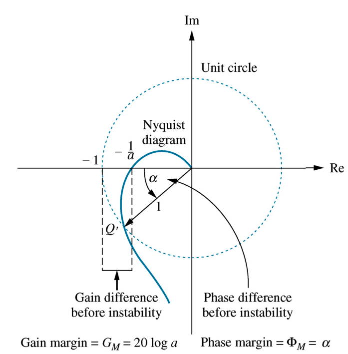
Measures of *how close* the current configuration is to instability.

Gain and Phase Margin

Measured by distance to -1 point in complex plane

Gain margin: Change in open-loop gain, at 180° phase shift, to make the closed-loop system unstable.
(= Reciprocal of real-axis crossing)

Phase margin: Change in open-loop phase shift, required at unity gain, to make the closed-loop system unstable.



Gain and Phase Margin

Clicker question

What is the approximate gain margin and phase margin?

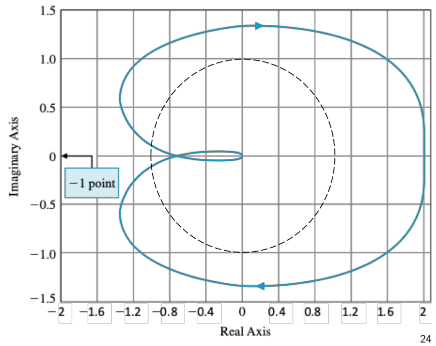
A. $G_M \approx 0.7$, $\Phi_M \approx \tan^{-1}(0.2)$

B. $G_M \approx \frac{1.0}{0.7}$, $\Phi_M \approx \tan^{-1}\left(\frac{0.2}{1.0}\right)$

C. $G_M \approx -\frac{1.0}{0.7}$, $\Phi_M \approx \tan^{-1}\left(-\frac{0.2}{1.0}\right)$

D. $G_M \approx \frac{1.0}{0.7}$, $\Phi_M \approx \tan^{-1}\left(-\frac{0.2}{1.0}\right)$

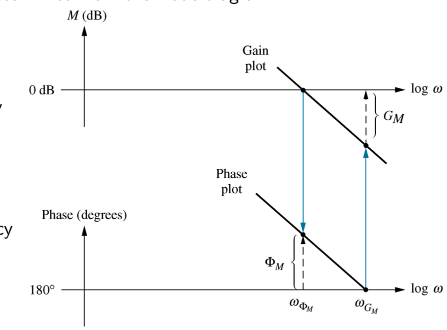
E. $G_M \approx \frac{1.0}{0.3}$, $\Phi_M \approx \tan^{-1}(-0.2)$



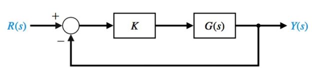
Gain and Phase Margin

Gain margin and phase margin can also be determined from the Bode diagram

- Measure gain margin G_M at the frequency where the phase is $180°$
- Measure phase margin Φ_M at the frequency where the gain is $1 = 0$ dB



Gain and Phase Margin



- Assess *open-loop* transfer function to determine *closed-loop* system stability
- Some systems may have infinite gain margin (e.g., $G(s) = \frac{1}{s+2}$ which *cannot* be destabilized by turning up the gain K)

- ****Both** $G_M > 0$ dB **and** $\Phi_M > 0°$ are required for the closed-loop system to be stable
- The system is unstable if *either* gain margin or phase margin are negative!

Gain and Phase Margin

Clicker question

Is the closed-loop system with Bode diagram below stable?

A. Yes, $G_M \approx 5$ dB, $\Phi_M \approx 20°$

B. Yes, $G_M \approx -5$ dB, $\Phi_M \approx 20°$

C. No, $G_M \approx -5$ dB, $\Phi_M \approx -160°$

D. No, $G_M \approx 5$ dB, $\Phi_M \approx -20°$