ECE 345 / ME 380: Introduction to Control Systems
Problem Set #3

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Due Thursday, November 5, 2020 at 3:30pm

1. (+10 points) Consider a negative unity feedback system as in Figure 1 with $G(s) = \dfrac{(s+1)(s+2)}{s^2(s^2+2s+3)}$.

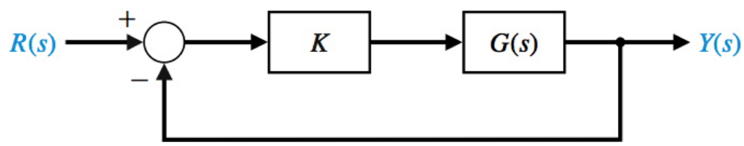


Figure 1: Negative unity feedback system.

(a) What is the type number of the closed-loop system $\frac{Y(s)}{R(s)}$?

Type 2

(b) What finite values of K, if any, will yield a steady-state error less than or equal to 0.1, in response to a unit step input?

$$e_{ss} = \frac{1}{1+K_p}$$

$$\begin{aligned} K_p &= \lim_{s\to 0} KG(s) \\ &= \lim_{s\to 0} K \cdot \frac{(s+1)(s+2)}{s^2(s^2+2s+3)} \\ &= \infty \end{aligned}$$

$$e_{ss} = 0$$

e_{ss} is 0, $\therefore$ all values of K will meet this constraint.

(c) What finite values of K, if any, will yield a steady-state error less than or equal to 0.1, in response to a unit ramp input?

$$e_{ss} = \frac{1}{K_v}$$

$$\begin{aligned} K_v &= \lim_{s\to 0} sKG(s) \\ &= \lim_{s\to 0} s \cdot K \cdot \frac{(s+1)(s+2)}{s^2(s^2+2s+3)} \\ &= \infty \end{aligned}$$

$$e_{ss} = 0$$

e_{ss} is 0, $\therefore$ all values of K will meet this constraint.

(d) What finite values of K, if any, will yield a steady-state error less than or equal to 0.1, in response to a unit parabolic input?

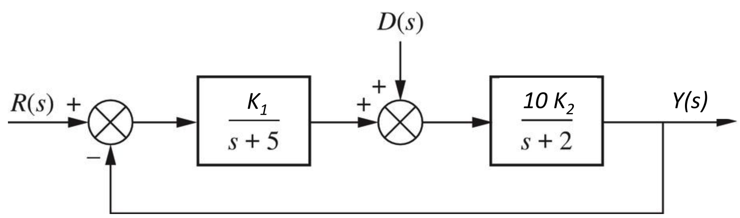
$$e_{ss} = \frac{1}{K_a}$$

$$\begin{aligned} K_a &= \lim_{s\to 0} s^2 KG(s) \\ &= \lim_{s\to 0} s^2 \cdot K \cdot \frac{(s+1)(s+2)}{s^2(s^2+2s+3)} \\ &= \frac{2K}{3} \end{aligned}$$

$$e_{ss} = \frac{3}{2K} \leq 0.1$$

$$K \geq 15$$

2. (+15 points) Consider the following system, with reference input $r(t)$ and disturbance input $d(t)$.



(a) Find the transfer function $G_R(s)$ with output $Y(s)$ and input $R(s)$, and the transfer function $G_D(s)$ with output $Y(s)$ and input $D(s)$, such that $Y(s) = G_R(s)R(s) + G_D(s)D(s)$.

Let $G_1(s) = \dfrac{K_1}{s+5}$ (our controller) and $G_2(s) = \dfrac{10K_2}{s+2}$ (our plant).

$$\begin{aligned} G_R(s) &= \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)} \\ &= \frac{10K_1K_2}{(s+5)(s+2)+10K_1K_2} \\ &= \frac{10K_1K_2}{s^2+7s+10+10K_1K_2} \end{aligned}$$

$$\begin{aligned} G_D(s) &= \frac{G_2(s)}{1+G_1(s)G_2(s)} \\ &= \frac{10K_2(s+5)}{(s+5)(s+2)+10K_1K_2} \\ &= \frac{10K_2(s+5)}{s^2+7s+10+10K_1K_2} \end{aligned}$$

(b) Describe the relationship between the characteristic equation of $G_R(s)$ and the characteristic equation of $G_D(s)$.

The characteristic equations are the same: $\Delta_R(s) = \Delta_D(s)$.

(c) Do $K_1 = 250$ and $K_2 = \frac{1}{10}$ meet the following specifications? Why or why not?

i. The steady-state *output response* due to a unit step disturbance input is 0.02.

$y_{ss} \leq 0.02$ due to $D(s) = \dfrac{1}{s}$ and pressure $R(s) = 0$.

$$\begin{aligned} Y(s) &= \frac{10K_2(s+5)}{s^2+7s+10+10K_1K_2} \cdot D(s) \\ &= \frac{s+5}{s^2+7s+260} \cdot \frac{1}{s} \end{aligned}$$

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$$
\begin{aligned}
y_{ss} &= \lim_{s\to 0} sY(s) \\
&= \lim_{s\to 0} s\left(\frac{s+5}{s^2+7s+260}\cdot\frac{1}{s}\right) \\
&= \frac{1}{52} = 0.01923 \le 0.02 \checkmark
\end{aligned}
$$

ii. The steady-state *error* due to a unit step reference input is 0.05.

$$e_{ss} \le 0.05 \text{ for } R(s) = \frac{1}{s} \text{ and pressure } D(s) = 0.$$

$$\frac{Y(s)}{R(s)} = \frac{10K_1K_2}{(s+5)(s+2)+10K_1K_2}$$

Type 0 system

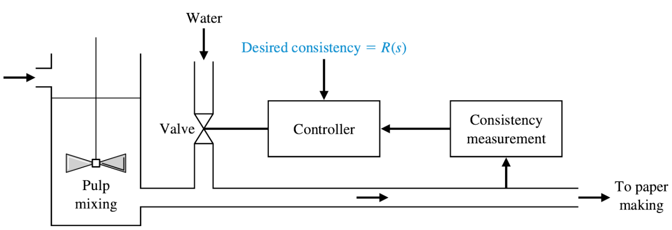
$$e_{ss} = \frac{1}{1+K_p}$$

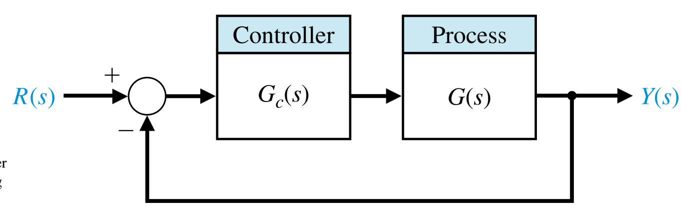
$$
\begin{aligned}
K_p &= \lim_{s\to 0}\ KG(s) \\
&= \lim_{s\to 0}\ G_1(s)G_2(s) \\
&= \lim_{s\to 0}\ \left(\frac{K_1}{s+5}\right)\left(\frac{10K_2}{s+2}\right) \\
&= K_1K_2 \\
&= 25
\end{aligned}
$$

$$e_{ss} = \frac{1}{26} = 0.03846 \le 0.05 \checkmark$$

Yes, these values of K_1 & K_2 meet the specifications.

3. (+15 points) Pulp dilution is an important part of the paper-making process. We model the dynamics of pulp dilution with plant $G(s) = \dfrac{s+2}{s^2+2s+3}$.





(a) Consider the controller $G_c(s) = \dfrac{K}{s+1}$. What is the type number of the closed-loop system $\frac{Y(s)}{R(s)}$?

$$= K\cdot\frac{s+2}{(s+1)(s^2+2s+3)}$$

Type 0

(b) What value of $K > 0$ will ensure that the steady-state error in response to a unit step input is at most 0.01?

$$e_{ss} = \frac{1}{1 + K_p}$$

$$\begin{aligned} K_p &= \lim_{s \to 0} \; KG(s) \\ &= \lim_{s \to 0} \; K \cdot G_c(s)G(s) \\ &= \lim_{s \to 0} \; K \cdot \frac{s+2}{(s+1)(s^2+2s+3)} \\ &= \frac{2K}{3} \end{aligned}$$

$$e_{ss} = \frac{1}{1 + \frac{2K}{3}} \leq 0.01$$

$$K \geq 148.5$$

(c) Now consider the controller $G_c(s) = \dfrac{K}{s(s+1)}$. What is the type number of the closed-loop system $\frac{Y(s)}{R(s)}$?

$$\boxed{K} \longrightarrow \boxed{G_c(s)G(s)} = K \cdot \frac{s+2}{s(s+1)(s^2+2s+3)}$$

Type 1

(d) What value of $K > 0$ will ensure that the steady-state error in response to a unit step input is at most 0.01?

$$e_{ss} = \frac{1}{1 + K_p}$$

$$\begin{aligned} K_p &= \lim_{s \to 0} \; KG(s) \\ &= \lim_{s \to 0} \; K \cdot G_c(s)G(s) \\ &= \lim_{s \to 0} \; K \cdot \frac{s+2}{s(s+1)(s^2+2s+3)} \\ &= \infty \end{aligned}$$

$$e_{ss} = 0$$

e_{ss} is 0, $\therefore$ all values of K will meet this constraint.

(e) Consider the gain $K = 150$. Which controller has the best steady-state performance? Why?

The controller with the extra s term (Type 1) will have the better steady-state performance. Even though $K = 150$ satisfies the constraint for the Type 0 system, there will still be slight steady-state error. The steady-state error for the Type 1 controller however will always be 0.