Course ID: ECE 341 Communication Systems- Fall

Prof. Eirini Eleni Tsiropoulou
eirini@unm.edu / (505) – 277 – 5501
235D/ Office Hours: Mondays and Wednesdays 11:00am - 12:00pm
Lectures: Mondays and Wednesdays 9:30am-10:45 am, Room: EECE 118
Department of Electrical and Computer Engineering / University of New Mexico

Homework #2

Corresponding to Sections 2.4 – 2.9 of Principles of Communications, Rodger E. Zimmer and William H. Tranter, John Wiley, 7th Edition.

1. Find the Fourier transform of the signals: a) $x_1(t)=A\exp(-t/\tau)u(t)$, b) $x_2(t)=A\exp(t/\tau)u(-t)$, c) $x_3(t)= x_1(t) - x_2(t)$, d) $x_4(t)= x_1(t) + x_2(t)$, e) $x_5(t)= x_1(t-5)$, and f) $x_6(t)= x_1(t) - x_1(t-5)$

2. Obtain the time-average autocorrelation function of the signal $x(t)=3+ 6\cos(20\pi t)+3\sin(20\pi t)$ and obtain the power spectral density of the signal. What is its total average power?

3. Determine the autocorrelation function of the signal with power spectral density $S(f)=9\delta(f-20)+9\delta(f+20)$ and give the average power.

4. For the transfer function $H(f)=j2\pi f/(7+j2\pi f)$, determine the unit impulse response of the system.

5. Determine whether or not the filter with impulse response $h(t)=(1/t)\, u(t-1)$ is (BIBO) stable.

6. Given a filter with frequency response function $H(f)=5/(4+ j2\pi f)$ and input $x(t)=\exp(-3t)\, u(t)$, obtain the energy spectral densities of the input and output.

7. Prove that $x(t)=\sin(\omega_0 t)$ and $\hat{x}(t)$ are orthogonal signals.

To be delivered at instructor's office: 30 September 2019

Good Luck!

1. Find the Fourier transform of the signals: a) $x_1(t)=A\exp(-t/\tau)u(t)$, b) $x_2(t)=A\exp(t/\tau)u(-t)$, c) $x_3(t)= x_1(t) - x_2(t)$, d) $x_4(t)= x_1(t) + x_2(t)$, e) $x_5(t)= x_1(t-5)$, and f) $x_6(t)= x_1(t) - x_1(t-5)$

(a) $x_1(t) = A e^{-t/\tau} u(t)$

$$X_1(\omega) = A \int_0^\infty e^{-t/\tau} e^{-j\omega t} dt$$

$$= A \int_0^\infty e^{-t\left(\frac{1}{\tau}+j\omega\right)} dt$$

$$= A \left(-\frac{1}{\frac{1}{\tau}+j\omega}\right) e^{-t\left(\frac{1}{\tau}+j\omega\right)} \Big|_0^\infty$$

$$= \frac{-A}{\frac{1}{\tau}+j\omega}(0-1) \Big| \left(\frac{\tau}{\tau}\right)$$

$$\boxed{X_1(\omega) = \frac{A\tau}{1+j\omega\tau}}$$

(b) $x_2(t) = A e^{t/\tau} u(-t)$

$$X_2(\omega) = A \int_{-\infty}^{0} e^{t/\tau} e^{-j\omega t} dt$$

$$= A \int_{-\infty}^{0} e^{t\left(\frac{1}{\tau}-j\omega\right)} dt$$

$$\boxed{X_2(\omega) = \frac{A\tau}{1-j\omega\tau}}$$

(c) $x_3(t) = x_1(t) - x_2(t)$

$$X_3(\omega) = X_1(f) - X_2(f)$$

$$= \frac{A\tau}{1+j\omega\tau} - \frac{A\tau}{1-j\omega\tau}$$

$$= \frac{A\tau(1-j\omega\tau) - A\tau(1+j\omega\tau)}{(1+j\omega\tau)(1-j\omega\tau)}$$

$$\boxed{X_3(\omega) = \frac{-2A\tau^2 j\omega}{(\omega\tau)^2+1}}$$

(d) $x_4(t) = x_1(t) + x_2(t)$

$$X_4(\omega) = X_1(f) + X_2(f)$$

$$= \frac{A\tau}{1+j\omega\tau} + \frac{A\tau}{1-j\omega\tau}$$

$$= \frac{A\tau(1-j\omega\tau) + A\tau(1+j\omega\tau)}{(1+j\omega\tau)(1-j\omega\tau)}$$

$$\boxed{X_4(\omega) = \frac{2A\tau}{(\omega\tau)^2+1}}$$

(e) $x_5(t) = x_1(t-5)$

$$X_5(\omega) = X_1(\omega)\, e^{-j5\omega}$$

$$\boxed{X_5(\omega) = \frac{A\tau}{1+j\omega\tau} e^{-j5\omega}}$$

(f) $x_6(t) = x_1(t) - x_1(t-5)$

$$X_6(\omega) = X_1(\omega) - X_5(\omega)$$

$$= \frac{A\tau}{1+j\omega\tau} - \frac{A\tau}{1+j\omega\tau} e^{-j5\omega}$$

$$\boxed{X_6(\omega) = \frac{A\tau}{1+j\omega\tau}\left(1 - e^{-j5\omega}\right)}$$

Superposition Theorem

$$a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 X_1(f) + a_2 X_2(f)$$

Time-Delay Theorem

$$x\left(t - t_0\right) \longleftrightarrow X(f)\, e^{-j2\pi f t_0}$$

2. Obtain the time-average autocorrelation function of the signal x(t)=3+ 6cos(20πt)+3sin(20πt) and obtain the power spectral density of the signal. What is its total average power?

$$R(\tau) = \frac{1}{T_0}\int_{T_0} x(t)x(t+\tau)\,dt$$

$$x(t) = 3 + 6\cos(20\pi t) + 3\sin(20\pi t)$$
$$= 3 + 3\sqrt{5}\cos\left[20\pi t + \tan^{-1}\left(\tfrac{3}{6}\right)\right]$$

$$R(\tau) = \frac{1}{0.1}\int_0^{0.1}\left\{3+3\sqrt{5}\cos\left[20\pi t + \tan^{-1}\left(\tfrac{1}{2}\right)\right]\right\}\left\{3+3\sqrt{5}\cos\left[20\pi(t+\tau) + \tan^{-1}\left(\tfrac{1}{2}\right)\right]\right\}dt$$

$$= 10\int_0^{0.1}\left\{\begin{array}{l}9 + 9\sqrt{5}\cos\left[20\pi(t+\tau)+\tan^{-1}\left(\tfrac{1}{2}\right)\right] + 9\sqrt{5}\cos\left[20\pi t + \tan^{-1}\left(\tfrac{1}{2}\right)\right] \\ +45\cos\left[20\pi(t+\tau)+\tan^{-1}\left(\tfrac{1}{2}\right)\right]\cos\left[20\pi t + \tan^{-1}\left(\tfrac{1}{2}\right)\right]\end{array}\right\}$$

$$= 10\int_0^{0.1} 9\,dt$$
$$+ 90\sqrt{5}\int_0^{0.1}\cancelto{0}{\cos\left[20\pi(t+\tau)+\tan^{-1}\left(\tfrac{1}{2}\right)\right]dt}$$
$$+ 90\sqrt{5}\int_0^{0.1}\cancelto{0}{\cos\left[20\pi t+\tan^{-1}\left(\tfrac{1}{2}\right)\right]dt}$$
$$+ \frac{450}{2}\int_0^{0.1}\cos(20\pi\tau)\,dt \longrightarrow \frac{450}{20}\cos(20\pi\tau)$$
$$+ \cancelto{0}{\frac{450}{2}\int_0^{0.1}\cos\left[40\pi t + 20\pi\tau + 2\tan^{-1}\left(\tfrac{1}{2}\right)\right]dt}$$

$$\boxed{R(\tau) = 9 + \frac{45}{2}\cos(20\pi\tau)}$$

$$S(f) = \mathfrak{F}[R(\tau)]$$

$$\cos(2\pi f_0 t) \leftrightarrow \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$

$$S(f) = F\left[9 + \frac{45}{2}\cos(20\pi\tau)\right]$$
$$= 9F[1] + \frac{45}{2}F\left[\cos(20\pi\tau)\right]$$

$$\boxed{S(f) = 9\delta(f) + \frac{45}{4}\delta(f-10) + \frac{45}{4}\delta(f+10)}$$

Integrating over all of f gives:

$$R(0) = \langle x^2(t)\rangle = \int_{-\infty}^{\infty} S(f)\,df$$

$$\boxed{P = 9 + \frac{45}{2} = 31.5\,\frac{\text{Watts}}{\text{ohm}}}$$

$$P = \int_{-\infty}^{\infty} S(f)\,df = \langle x^2(t)\rangle$$

3. Determine the autocorrelation function of the signal with power spectral density S(f)=9δ(f-20)+9δ(f+20) and give the average power.

$$R(\tau) = F^{-1}\left[9\delta(f-20) + 9\delta(f+20)\right]$$

$$\boxed{R(\tau) = 18\cos(40\pi t)}$$

$$P = R(0) = 18\,\cancel{\cos(0)}^{\,1}$$

$$\boxed{P = 18\ \frac{\text{Watts}}{\text{ohm}}}$$

$$R(\tau) = \mathfrak{F}^{-1}[S(f)]$$

$$\cos(2\pi f_0 t) \leftrightarrow \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

$$P = \int_{-\infty}^{\infty} S(f)\,df = \langle x^2(t)\rangle$$

$$R(0) = \langle x^2(t)\rangle = \int_{-\infty}^{\infty} S(f)\,df$$

4. For the transfer function H(f)=j2πf/(7+j2πf), determine the unit impulse response of the system.

$(-jt)^n f(t)$	$\dfrac{d^n F(\omega)}{d\omega^n}$
$u(t)e^{-\alpha t}$	$\dfrac{1}{\alpha + j\omega}$

$$H(f) = \frac{j2\pi f}{7 + j2\pi f} \rightarrow H(\omega) = \frac{j\omega}{7 + j\omega}$$

$$\boxed{h(t) = \frac{d}{dt} e^{-7t} u(t)}$$

$$= 1 - 7e^{-7t}$$

5. Determine whether or not the filter with impulse response h(t)=(1/t) u(t-1) is (BIBO stable.

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

$$\int_{-\infty}^{\infty} \frac{1}{t}\, u(t-1)\, dt$$

$$\int_{1}^{\infty} \frac{1}{t}\, dt = \ln(t) \Big|_{1}^{\infty} \longrightarrow \infty$$

The filter is $\boxed{\text{unstable}}$

6. Given a filter with frequency response function H(f)=5/(4+ j2πf) and input x(t)=exp(-3t) u(t), obtain the energy spectral densities of the input and output.

input

$$G_x(\omega) = |X(\omega)|^2 \qquad X(\omega) = \frac{1}{3+j\omega}$$

$$G(f) = |X(f)|^2$$

$$= \left|\frac{1}{3+j\omega}\right|^2$$

$$\boxed{\cancel{G_x(\omega) = \frac{1}{(3+j\omega)^2}}} \qquad \frac{1}{9+\omega^2}$$

output

$$G_y(f) = |H(f)|^2 G_x(f)$$

$$G_y(\omega) = |H(\omega)|^2 G_x(\omega)$$

$$= \left|\frac{5}{4+j\omega}\right|^2 \cancel{\frac{1}{(3+j\omega)^2}} \qquad \left(\frac{25}{16+\omega^2}\right)\left(\frac{1}{9+\omega^2}\right)$$

$$= \left[\frac{25}{(4+j\omega)^2}\right]\left[\frac{1}{(3+j\omega)^2}\right]$$

$$\boxed{G_y(\omega) = \frac{25}{(4+j\omega)^2(3+j\omega)^2}}$$

7. Prove that x(t)=sin(ω_0t) and $\hat{x}(t)$ are orthogonal signals.

$$\sin(2\pi f_0 t) \longleftrightarrow \frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$$

$$x(t) = \sin(\omega_0 t) = \sin(2\pi f_0 t)$$

$$X(f) = \frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$$

$$\hat{X}(f) = \frac{1}{2j}\delta(f - f_0)e^{-j\pi/2} - \frac{1}{2j}\delta(f + f_0)e^{j\pi/2}$$

$$\hat{x}(t) = \sin\left(2\pi f_0 t - \frac{\pi}{2}\right)$$
$$= -\cos(2\pi f_0 t)$$

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\hat{x}(t)\,dt$$

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\hat{x}(t)\,dt = 0 \text{ (power signals)}$$

$$= \lim_{T\to\infty} -\frac{1}{2T}\int_{-T}^{T} \sin(\omega_0 t)\cos(\omega_0 t)\,dt$$

$$= \lim_{T\to\infty} -\frac{1}{2T}\int_{-2\pi}^{2\pi} \frac{1}{2}\sin(2\omega_0 t)\,dt$$

$$= \lim_{T\to\infty} -\frac{1}{4T}\int_{-2\pi}^{2\pi} \sin(2\omega_0 t)\,dt$$

$$= \lim_{T\to\infty} +\frac{1}{4T}\left(\frac{1}{2\omega_0}\cos(2\omega_0 t)\right)\Big|_{-T}^{T} = 0$$

since the limit goes to 0, they are orthogonal