ECE 345
Introduction to Control Systems
Lecture Notes 7

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Learning Objectives

- Find the steady-state error for a unity feedback system
- Specify a system's type number and error constant
- Design the gain of a closed-loop system to achieve a desired steady-state error specification
- Find the steady-state error for a non-unity feedback system
- Find the steady-state error for systems represented in state-space
- Compute steady-state error due to a disturbance input

References:

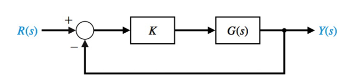
- Nise, Chapter 7.1-7.6, 7.8

Outline

- What is steady-state error?
- Steady-state error in unity feedback systems
- Type number
- Steady-state error in non-unity feedback systems
- Steady-state error due to disturbance inputs
- Steady-state error in state-space

What is steady-state error?

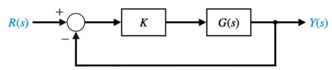
System 1: $G_1(s) = \dfrac{1}{s(s+2)}$



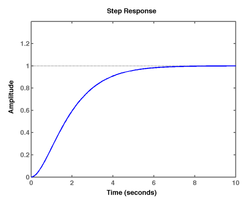
- Closed-loop system
$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + 2s + K}$$
- Asymptotically stable
  $\Rightarrow$ Natural response is $x(t) \to 0$ as $t \to \infty$
- BIBO stable
  $\Rightarrow$ Forced response $y(t)$ is *bounded* whenever forcing function $r(t)$ is also *bounded*

What is steady-state error?

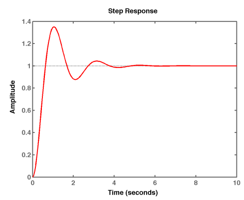
System 1: $G_1(s) = \dfrac{1}{s(s+2)}$



- What happens when we apply a step input?



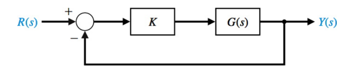
$K = 1$



$K = 10$

What is steady-state error?

System 2: $G_2(s) = \dfrac{1}{(s+1)(s+2)}$



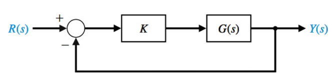
- Closed-loop system

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + 3s + (2+K)}$$

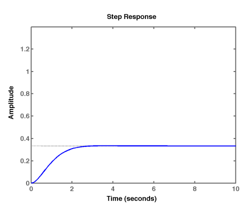
- Asymptotically stable
- BIBO stable

What is steady-state error?

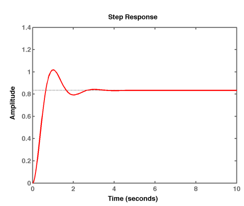
System 2: $G_2(s) = \dfrac{1}{(s+1)(s+2)}$



- What happens when we apply a step input?



$K = 1$



$K = 10$

What is steady-state error?

Both systems have the same stability properties, but their steady-state responses to a step input are *qualitatively different.*

- Ideally, a system that is supposed to track a step input should have a steady-state value of 1 in response to a step input
- The *error* $E(s)$ in response to a desired input $R(s)$ is defined as

$$E(s) = R(s) - Y(s)$$

Compare the Matlab plots on slides 4 and 6.

- When $K = 1$, what value does the error of System 1 approach in steady-state?
- When $K = 10$, what value does the error of System 2 approach in steady-state?

What is steady-state error?

Why does the error of System 1 converge to 0 in steady-state, while the error of System 2 does not?

We can compute the error as a function of the gain K.

$$\begin{aligned} E(s) &= R(s) - \frac{KG(s)}{1+KG(s)}R(s) \\ &= \left(\frac{1}{1+KG(s)}\right) R(s) \end{aligned}$$

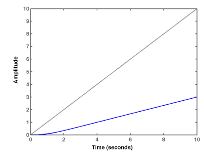
- System 1: $e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\dfrac{s(s+2)}{s^2+2s+K} \cdot \dfrac{1}{s} = 0$
- System 2: $e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\dfrac{(s+1)(s+2)}{s^2+3s+(2+K)} \cdot \dfrac{1}{s} = \dfrac{2}{2+K}$

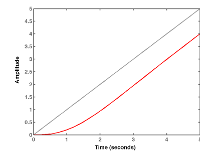
Using the Final Value Theorem, we can see that for any finite gain K, System 2 will have a non-zero steady-state error $e_{ss} \triangleq \lim_{t\to\infty} e(t)$.

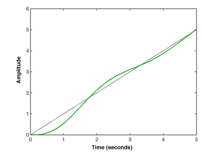
What is steady-state error?

Clicker question

Which of the following shows a system with steady-state error $e_{ss} = 0$ in response to a unit ramp input?


A.

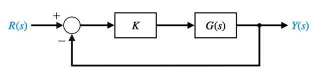

B.


C.

D. Both A. and B.

E. Both B. and C.

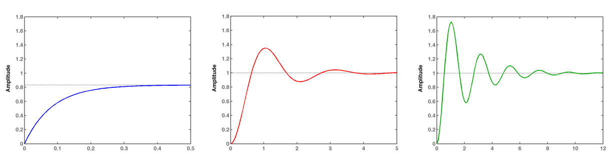
Steady-state error of unity feedback systems



- For the closed-loop system $\frac{Y(s)}{R(s)}$ with open-loop transfer function

$$KG_0(s) = \frac{1}{s+2} \qquad KG_1(s) = \frac{1}{s(s+2)} \qquad KG_2(s) = \frac{s+1}{s^2(s+2)}$$

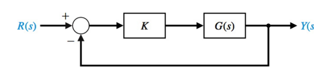
- Output response to a **step** input



- Steady-state error e_{ss} is

Finite and non-zero | **0** | **0**

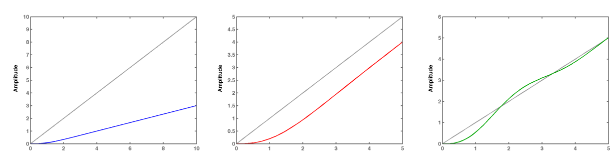
Steady-state error of unity feedback systems



- For the closed-loop system $\frac{Y(s)}{R(s)}$ with open-loop transfer function

$$KG_0(s) = \frac{1}{s+2} \qquad KG_1(s) = \frac{1}{s(s+2)} \qquad KG_2(s) = \frac{s+1}{s^2(s+2)}$$

- Output response to a **ramp** input

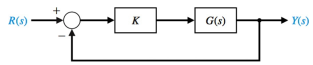


- Steady-state error e_{ss} is

∞ | Finite and non-zero | **0**

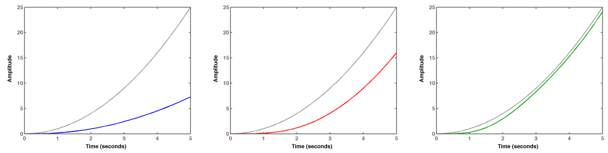
Steady-state error of unity feedback systems

- For the closed-loop system $\frac{Y(s)}{R(s)}$ with open-loop transfer function



$$KG_0(s) = \frac{1}{s+2} \qquad KG_1(s) = \frac{1}{s(s+2)} \qquad KG_2(s) = \frac{s+1}{s^2(s+2)}$$

- Output response to a **parabolic** input



- Steady-state error e_{ss} is

∞ $\qquad$ ∞ $\qquad$ Finite and non-zero

Steady-state error of unity feedback systems

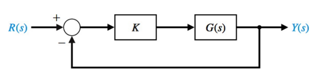
Type number

- Property of the *closed-loop* system $\frac{Y(s)}{R(s)}$
- Defined as the number of poles of the *open-loop* transfer function $KG(s)$ **at 0**
- Allows easy determination of steady-state error e_{ss} of the *closed-loop* transfer function $\frac{KG(s)}{1+KG(s)}$

e_{ss}	Type 0	Type 1	Type 2	Error Constant
Step Input	$\frac{1}{1+K_p}$	0	0	$K_p =$
Ramp Input	∞	$\frac{1}{K_v}$		$K_v =$
Parabolic Input		∞	$\frac{1}{K_a}$	$K_a =$

Steady-state error of unity feedback systems

Clicker question

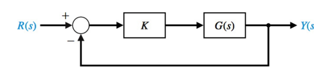


A system with which of the following type number(s) will be able to track a unit ramp with steady-state error *less than 0.05*?

A. Type 0
B. Type 1
C. Type 2
D. Types 0 and 1
E. Types 1 and 2

Steady-state error of unity feedback systems

Clicker question

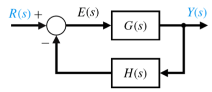


Consider a system for which $G(s) = \frac{1}{s+5}$. What positive value of K, if any, is required to make the closed-loop system $\frac{Y(s)}{R(s)}$ track a step input with steady-state error less than or equal to 0.1?

A. $K \geq 25$
B. $K = 55$
C. $K \geq 45$
D. $K < 15$
E. No finite value of K will achieve the desired response

Exercise: What is the type number and error constant for this system?

Steady-state error of non-unity feedback systems



With error

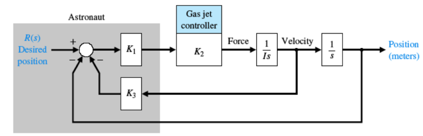
$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= R(s) - \frac{G(s)}{1+G(s)H(s)}R(s) \\ &= \frac{1+G(s)H(s)-G(s)}{1+G(s)H(s)}R(s) \end{aligned}$$

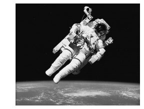
Steady-state error is

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1+G(s)H(s)-G(s)}{1+G(s)H(s)}R(s)$$

Steady-state error of non-unity feedback systems

Consider an astronaut in space, controlling position with gas jet propulsion.



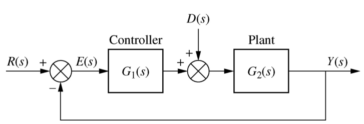


For fast and accurate performance, the system should have

- Settling time of 0.3 seconds
- Overshoot less than 5%
- Steady-state error to a ramp input of less than 0.5 m

What values of K_1, K_2, K_3 will provide an acceptable response?

Steady-state response to disturbance inputs



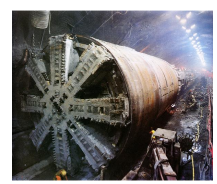
- Output $Y(s)$ due to reference input $R(s)$ *and* disturbance input $D(s)$

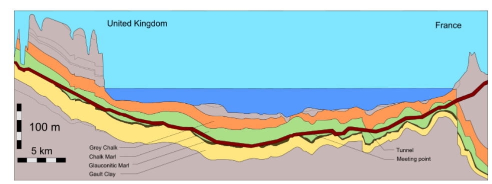
$$\begin{aligned} Y(s) &= G_2(s)\left(D(s) + G_1(s)(R(s) - Y(s))\right) \\ &= \frac{G_2(s)}{1+G_1(s)G_2(s)}D(s) + \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)}R(s) \end{aligned}$$

- Consider the effect of each input separately
  - When $D(s) = 0$, and $R(s)$ is test signal (e.g., step, ramp, parabola)
  - When $D(s)$ is a test signal, and $R(s) = 0$
- Controller design to reduce steady-state error due to $R(s)$
- Reduce effect of $D(s)$ by making steady-state output due to $D(s)$ small

Steady-state response to disturbance inputs

Eurotunnel construction

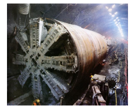


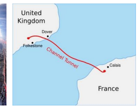


- Between France and Great Britian, 1987–1992
- $14 Billion USD
- 23.5 miles long, bored 200 m below sea level

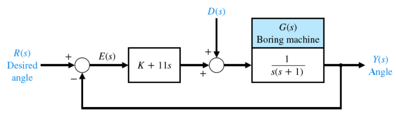
Steady-state response to disturbance inputs

Eurotunnel construction





Consider the following simplified model of the longitudinal dynamics of the boring machine with feedback controller $G_c(s) = K + 11s$.



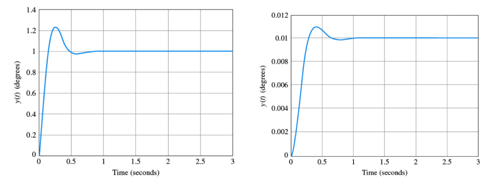
We wish to select K to achieve good transient performance while also rejecting disturbances.

Steady-state response to disturbance inputs

Eurotunnel construction

$$Y(s) = \frac{1}{s^2 + 12s + K} D(s) + \frac{K + 11s}{s^2 + 12s + K} R(s)$$

Evaluate the step responses with $K = 100$

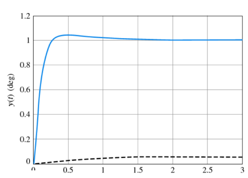


Exercise: For what value(s) of K will the steady-state response due to a unit step disturbance be less than equal to 0.05?

Steady-state response to disturbance inputs

Eurotunnel construction

For $K = 20$



- While $K = 100$ provides good disturbance rejection, performance is poor due to excessive overshoot
- Reducing to $K = 20$ improves performance by reducing overshoot, but also worsens disturbance rejection, as measured by *steady-state response* to a step in the disturbance input

Steady-state error in state-space form

Recall that the transfer function of a system

$$\dot{x} = Ax + Br, \quad y = Cx$$

is given by $G(s) = C(sI - A)^{-1}B$

Hence the error for a SISO system is

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= \left(1 - C(sI - A)^{-1}B\right) \cdot R(s) \end{aligned}$$

The steady-state error is

$$e_{ss} = \lim_{s \to 0} sE(s) = s\left(1 - C(sI - A)^{-1}B\right) \cdot R(s)$$

Summary

- Stable systems *cannot* always accomplish tracking with error that converges to 0
- Steady-state error is only relevant for *stable* systems
- Type number provides a quick way to assess
  - The class of signals that a given system can track
  - The steady-state error, if any, that results from tracking

  in unity feedback systems
- Steady-state response provides an indirect measure of the overall responsiveness of the system to
  - Reference inputs (which the system should track well)
  - Disturbance inputs (which the system should reject)