Properties of Autocorrelation Function

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1) $|R(\tau)| \leq R(0)$, $\overset{\text{foreach}}{\forall \tau}$

2) $R(-\tau) = R(\tau)$

3) If $\{X(t)\}$ does not have any periodic component $\Rightarrow \lim_{|\tau| \to +\infty} R(\tau) = \overline{X(t)}^2$

4) If $\{X(t)\}$: periodic $\Leftrightarrow R(\tau)$: periodic

5) $F\{R(\tau)\}$ non-negative

Example

Power Spectral Density $\quad S(f) = \begin{cases} \frac{1}{2} N_0, & |f| \leq B \\ 0, & \text{otherwise} \end{cases}$

$\downarrow F^{-1}$

Autocorrelation Function $\quad R(\tau) = \int_{-B}^{B} \frac{1}{2} N_0 \cdot e^{(j2\pi f \cdot \tau)} df$

$$= \frac{N_0}{2} \left[\frac{e^{j2\pi f \tau}}{j2\pi\tau} \right]_{-B}^{B} = BN_0 \frac{\sin(2\pi B\tau)}{2\pi B\tau}$$

$= BN_0 \, \mathrm{sinc}(2\pi B \tau)$

Autocorrelation Function for Random Pulse Trains:

Random Process $\| \; X(t) = \sum_{k=-\infty}^{+\infty} a_k \, p(t - kT - \Delta)$

a_k → sequence of random variables

$$E[a_k \cdot a_{k+m}] = R_m$$

Autocorrelation Function:

$$R_X(\tau) = E[X(t) \cdot X(t+\tau)] =$$

$$= E\left\{ \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} a_k \cdot a_{k+m} \, p(t - kT - \Delta) \cdot p(t + \tau - (k+m)T - \Delta) \right\}$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} E[a_k \cdot a_{k+m}] \cdot E[p(t-kT-\Delta) \cdot p(t+\tau-(k+m)T-\Delta)]$$

$$= \sum_{m=-\infty}^{+\infty} R_m \sum_{k=-\infty}^{+\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{p(t-kT-\Delta) \cdot p[t+\tau-(k+m)T-\Delta]}{T} \, d\Delta$$

Set $u = t - kT - \Delta$

$$= \sum_{m=-\infty}^{+\infty} R_m \sum_{k=-\infty}^{+\infty} \int_{t-\frac{(k+1)}{2}T}^{t-\left(\frac{k-1}{2}\right)T} \frac{p(u)\cdot p(u+\tau-mT)\,du}{T}$$

$$= \sum_{m=-\infty}^{+\infty} \frac{R_m}{T} \left[\int_{-\infty}^{+\infty} p(u+\tau-mT)\,p(u)\,du \right]$$

$$\boxed{R_X(\tau) = \sum_{m=-\infty}^{+\infty} R_m\, r(\tau - mT)}$$

Autocorrelation Function

$$r(\tau) \triangleq \frac{\int_{-\infty}^{+\infty} p(t+\tau)\cdot p(t)\,dt}{T}$$

Cross-correlation Function

Random Process $\|$ $n(t) = X(t) + Y(t)$

Power in the sum:

$$E[n^2(t)] = E[(X(t)+Y(t))^2] = \underbrace{E[X^2(t)]}_{P_X} + \overbrace{2E(X(t)\cdot Y(t))}^{\substack{\text{cross-power} \\ P_{XY}}} + \underbrace{E[Y^2(t)]}_{P_Y}$$

cross-correlation function:

$$R_{XY}(\tau) = E[X(t) \cdot Y(t+\tau)]$$

$$\boxed{P_{XY} = R_{XY}(\tau = 0)}$$ Cross-power

if 2 random processes:

a) statistically independent

b) at least one of them has zeromean

$\Rightarrow$ $\boxed{\text{ORTHOGONAL}}$

* Orthogonal Random Processes $\Rightarrow$ $R_{XY}(\tau=0) = 0$

$$\boxed{P_{XY} = 0}$$

* orthogonal random processes $\not\Rightarrow$ statistically independent

not necessarily

$$\boxed{R_{XY}(\tau) = R_{XY}(-\tau)}$$

X, Y: statistically independent

Cross-power spectral density :

$$S_{XY}(f) = F\{R_{XY}(\tau)\}$$

Linear Systems & Random Processes

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Output Spectral Density || $S_Y(f) = |H(f)|^2 \cdot S_X(f)$

Autocorrelation function of output || $R_Y(\tau) = F^{-1}(S_Y(f))$

$$= \int_{-\infty}^{+\infty} |H(f)|^2 S_X(f) \cdot e^{j2\pi f\tau} df$$

$$\boxed{R_{XY}(\tau) =} \; E[x(t) \cdot y(t+\tau)]$$

$$y(t) = \int_{-\infty}^{+\infty} h(u) \cdot x(t-u) du$$

$$= E\left[x(t) \cdot \int_{-\infty}^{+\infty} h(u) \cdot x(t+\tau-u) du\right\}$$

$$= \int_{-\infty}^{+\infty} h(u) \cdot \underbrace{E[x(t) \cdot x(t+\tau-u)]}_{\to R_x(\tau-u)} du$$

$$= \int_{-\infty}^{+\infty} h(u) \cdot R_x(\tau-u) du$$

$$\boxed{= h(\tau) * R_x(\tau-u)}$$ cross-correlation function

$\downarrow$

$$\boxed{S_{xy}(f) = H(f) \cdot S_x(f)}$$ Cross-power spectral density

$$\boxed{S_{yx}(f) = \mathcal{F}\{R_{yx}(\tau)\} = \mathcal{F}[R_{xy}(-\tau)] = S_{xy}^*(f)}$$

$$H^*(f) = H(-f)$$

$$S_x^*(f) = S_x(f)$$

$$S_{yx}(f) = S_{xy}^*(f) = H(-f) \cdot S_x(f) = H^*(f) \cdot S_x(f)$$

$$R_{yx}(\tau) = \mathcal{F}^{-1}[S_{yx}(f)] = h(-\tau) * R_x(\tau)$$

$$R_{xy}(\tau) = E[x(t) \cdot y(t+\tau)]$$
$$= E[x(t) \cdot h(t) * x(t+\tau)]$$
$$= h(\tau) * R_x(\tau)$$

$$R_{yx}(\tau) = E[y(t) \cdot x(t+\tau)]$$
$$= E[h(t) * x(t) \cdot x(t+\tau)]$$
$$= h(-\tau) * R_x(\tau)$$

$$\boxed{\begin{aligned} R_{xy}(\tau) &= h(\tau) * R_x(\tau) \\ R_{yx}(\tau) &= h(-\tau) * R_x(\tau) \end{aligned}}$$

Autocorrelation function of the output

$$R_y(\tau) = E[y(t) \cdot y(t+\tau)]$$
$$= E[y(t) \cdot h(t) * x(t+\tau)]$$
$$= h(\tau) * \underbrace{E[y(t) \cdot x(t+\tau)]}_{R_{yx}}$$
$$= h(\tau) * R_{yx}(\tau)$$

$$= h(\tau) * \{ h(-\tau) * R_X(\tau) \}$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(u) \cdot h(v) \cdot R_X(\tau + v - u) dv du$$

$$S_Y(f) = \mathcal{F}[R_Y(\tau)] = \mathcal{F}[h(\tau) * R_{YX}(\tau)]$$

$$= H(f) \cdot S_{YX}(f)$$

$$= |H(f)|^2 \cdot S_X(f)$$