ECE 345 / ME 380: Introduction to Control Systems
Collaborative Quiz #1

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Due Thursday, September 10, 2020 at the end of class

Researchers have recently proposed a micro-gravity robot, the Hedgehog, for space exploration, that relies on internal flywheels to generate torques that facilitate movement [1]. The robot can hop, tumble, shuffle, and spin, depending on how the flywheels are manipulated (`https://www.youtube.com/watch?v=aVOOP8w_iVs`).

We consider a hopping maneuver (Figure 1), in which the flywheels are slowly spun up to a sufficiently high angular speed, then "impulsive" braking is applied to bring the flywheels to an abrupt stop. Due to conservation of energy, the angular momentum from the flywheels is transferred to the robot body, prompting a sudden increase in the robot's angular speed.

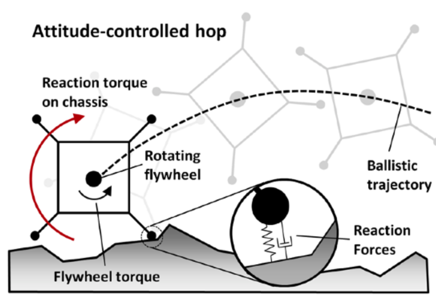

Figure 1: Hopping maneuver for the Hedgehog space robot. Images from [1].

In this scenario, the external torque $\tau(t)$ provided by the flywheels offsets the torque due to the gravitational force, so that the robot effectively balances on one spike at an angle $\theta(t)$. The linearized equations of motion for small values of θ are

$$I\ddot{\theta}(t) \quad = \quad -mgl\theta(t) - \tau(t) \tag{1}$$

with constant parameters that include robot inertia I, mass m, length l of the spike from the center of mass, and the gravitational constant g.

The main goal of this exercise is to model the robot dynamics during the hopping maneuver using both state-space and transfer function representations.

1 Questions to be completed before class

1. Take the Laplace transform of (1) to obtain the transfer function $G(s) = \dfrac{\theta(s)}{\tau(s)}$, with input $\tau(t)$ and output $\theta(t)$. Put the transfer function in proper form, i.e., ensure that the coefficient of s^n in the denominator is 1.

2. What is the denominator of the system $G(s)$?

3. Put the system (1) into state-space form, with state $x(t) = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$ and output $\theta(t)$. Show that the state-space matrices are

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -\frac{mgl}{I} & 0 \end{bmatrix}, & B &= \begin{bmatrix} 0 \\ -\frac{1}{I} \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 \end{bmatrix}, & D &= 0 \end{aligned} \tag{2}$$

2 Questions to be completed in class, in groups

1. Put the transfer function $G(s)$ from Pre-Class Question #1 into phase variable form, with matrices denoted A_P, B_P, C_P, D_P. Which of the following is correct?

   (a) $A_P = \begin{bmatrix} -mgl & 0 \\ 0 & 1 \end{bmatrix}$, $B_P = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $C_P = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D_P = \frac{1}{I}$

   (b) $A_P = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{mgl}{I} \end{bmatrix}$, $B_P = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $C_P = \begin{bmatrix} 0 & \frac{1}{I} \end{bmatrix}$, $D_P = 0$

   (c) $A_P = \begin{bmatrix} 0 & -\frac{mgl}{I} \\ 0 & 1 \end{bmatrix}$, $B_P = \begin{bmatrix} 0 \\ -\frac{1}{I} \end{bmatrix}$, $C_P = \begin{bmatrix} 0 & -1 \end{bmatrix}$, $D_P = 0$

   (d) $A_P = \begin{bmatrix} 0 & 1 \\ -\frac{mgl}{I} & 0 \end{bmatrix}$, $B_P = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_P = \begin{bmatrix} -\frac{1}{I} & 0 \end{bmatrix}$, $D_P = 0$

2. Compute the transfer function $H(s) = C(sI - A)^{-1}B + D$ from the state-space matrices in (2). Put $H(s)$ in proper form. What is the denominator of $H(s)$?

   (a) $s^2 + mgl/I$

   (b) $s^2 - mgl/I$

   (c) $-1/I$

   (d) $Is^2 + mgls$

3. Compare your result for $H(s)$ to that for $G(s)$, your answer to Pre-Class Question #2. Which of the following best describes the relationship between $G(s)$ and $H(s)$?

   (a) $G(s) = H(s)$, because a single state-space model can be represented as multiple transfer functions.

   (b) $G(s) = H(s)$, because a transfer function can be represented by many different state-space models.

   (c) $G(s) \neq H(s)$, because every state-space model has a unique transfer function associated with it.

   (d) $G(s) \neq H(s)$, because while their denominators are the same, the transfer functions have different numerators.

Now consider a more complex model of the hopping robot, which takes into account the dynamics of the flywheel needed to generate a desired torque. We presume that the flywheel angular velocity is $z(t)$, and that the input $u(t)$ is the flywheel angular acceleration. The flywheel inertia is J, and the thrust $\tau(t) = \frac{J}{\epsilon} z(t)$ is generated by bringing the flywheel to rest quickly, over ϵ seconds. That is, we presume the following equations of motion in lieu of (1):

$$\begin{aligned} I\ddot{\theta}(t) &= mgl \cdot \theta(t) - \tfrac{J}{\epsilon} z(t) \\ \dot{z}(t) &= u(t) \end{aligned} \tag{3}$$

With state $\tilde{x}(t) = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \\ z(t) \end{bmatrix}$, we obtain the state-space model

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{mgl}{I} & 0 & -\frac{J}{I\epsilon} \\ 0 & 0 & 0 \end{bmatrix}, & \tilde{B} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ \tilde{C} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, & \tilde{D} &= 0 \end{aligned} \tag{4}$$

4. Consider the state equation, governed by matrices $\tilde{A}, \tilde{B}$ from (4). Which of the following most correctly describes the role of the input in this system?

   (a) The input directly affects the flywheel angular velocity $z(t)$, and indirectly affects the robot angle $\theta(t)$.

   (b) The input directly affects the robot angular velocity $\dot{\theta}(t)$, and indirectly affects the robot angle $\theta(t)$.

   (c) The input indirectly affects the flywheel angular velocity $z(t)$, the robot angle $\theta(t)$, and the robot angular rate $\dot{\theta}(t)$.

   (d) The input indirectly affects the generated torque $\tau(t)$ and the flywheel angular velocity $\dot{\theta}(t)$.

We now consider the problem of control design. When braking is applied to the flywheels at time t_{hop}, the angular velocity of the flywheel drops from $z(t_{\text{hop}})$ to 0 in ϵ seconds. The input trajectory

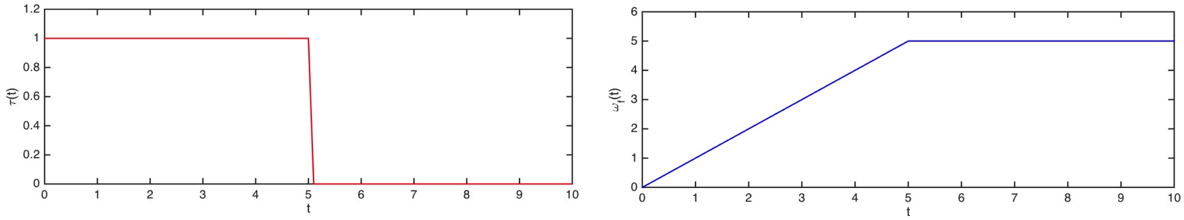

Figure 2: Prototypical flywheel acceleration $u(t)$ (left) and flywheel velocity $z(t)$ (right), with $t_{\text{hop}} = 5$ seconds and $\epsilon = 0.1$ seconds.

to create this effect is of the form $u(t) = \mathbf{1}(t) - \mathbf{1}(t - t_{\text{hop}})$, and is shown in Figure 2. The robot is imparted with an additional angular velocity

$$\dot{\theta}_{\text{hop}} = \frac{Jz(t_{\text{hop}})}{I}, \tag{5}$$

which, if sufficiently high, will overcome the gravitational force, and result in a 'hop'. A rough estimate for this threshhold is $\dot{\theta}_{\text{hop}} \geq \sqrt{\frac{g}{l}}$.

5. On space missions, energy conservation is paramount. Consider the case in which the primary energy expenditure is proportional to the duration that the flywheel needs to spin up, in order to hop. As a design engineer, would you prefer a flywheel with low inertia or high inertia? Why? That is, how does your choice help reduce energy expenditure?

3 If your group finishes early...

Other points to consider (not necessary to hand in):

- Find the transfer function for the system (4) and compare it to $G(s)$. Which poles are the same? Can you identify them in $\tilde{A}$?
- Balancing on a single spike occurs when the torque is above a minimum value $\tau_{\min}$, below which the robot has two spikes in contact with the ground, and when the torque is below a maximum value $\tau_{\max}$, above which the robot will tumble onto another spike.

  For a robot starting from rest, with flywheels initially at rest, describe the three phases of motion of the robot that result as the flywheels are slowly spun up, then abruptly braked to a full stop.

References

[1] B. Hockman, A. Frick, I. A. D. Nesnas, and M. Pavone, "Design, control, and experimentation of internally-actuated rovers for the exploration of low-gravity planetary bodies," in *Field and Service Robotics: Results of the 10th International Conference* (S. D. Wettergreen and D. T. Barfoot, eds.), (Cham), pp. 283–298, Springer International Publishing, 2016.