Course ID: ECE 341 Communication Systems- Fall
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235D/ Office Hours: Mondays and Wednesdays 11:00am - 12:00pm
Lectures: Mondays and Wednesdays 9:30am-10:45 am, Room: EECE 118
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Homework #5

Corresponding to Chapter 6 of Principles of Communications, Rodger E. Zimmer and William H. Tranter, John Wiley, 7th Edition.

1. A circle is divided in 21 equal parts. A pointer is spun until it stops on one of the parts, which are numbered from 1 through 21. Describe the sample space and assuming equally likely outcomes, find: (a) P(an even event), (b) P(the number 21), (c) P(the numbers 4,5, or 9), and P(a number greater than 10).

2. What equations must be satisfied in order for three events A, B, and C to be independent? Think that they must be independent by pairs, but this is not sufficient.

3. Given the table of joint probabilities, (a) find the probabilities omitted from the table, and (b) find the probabilities $P(A_3|B_3)$, $P(B_2|A_1)$ and $P(B_3|A_2)$.

	B_1	B_2	B_3	$P(A_i)$
A_1	0.05		0.45	0.55
A_2		0.15	0.10	
A_3	0.05	0.05		0.15
$P(B_j)$				1.0

4. A certain continuous random variable has the cumulative-distribution function.

$$F_X(x) = \begin{cases} 0, x < 0 \\ Ax^4, 0 \le x \le 12 \\ B, x > 12 \end{cases}$$

(a) Find the proper values for A and B. (b) Obtain the pdf $f_X(x)$. (c) Compute $P(X>5)$. and (d) Compute $P(4 \le X < 6)$

5. The joint pdf of two random variables is

$$f_{XY}(x, y) = \begin{cases} C(1+xy), 0 \le x \le 4, 0 \le y \le 2 \\ 0, otherwise \end{cases}$$

Find the following: (a) The constant C, (b) $f_{XY}(1,1.5)$, (c) $f_{XY}(x,3)$, and (d) $f_{X|Y}(x|3)$.

6. The joint pdf of the random variables X and Y is $f_{XY}(x,y)=Axye^{-(x+y)}$, $x \ge 0$ and $y \ge 0$. (a) Find the constant A. (b) Find the marginal pdfs of X and Y, $f_X(x)$ and $f_Y(y)$. (c) Are X and Y statistically independent? Justify your answer.

7. Let $f_X(x)=A\exp(-bx)u(x-2)$ for all x where A and b are positive constants. (a) Find the relationship between A and b such that this function is a pdf. (b) Calculate E(X) for this random variable. (c) Calculate $E(X^2)$ for this random variable. (d) What is the variance of this random variable?

8. The random variable has pdf $f_X(x)=(1/2)*\delta(x-5)+(1/8)*[u(x-4)-u(x-8)]$, where u(x) is the unit step. Determine the mean and the variance of the random variable.

9. Two random variables X and Y have means and variances given below: $m_x=1$, $\sigma_x^2=4$, $m_y=3$, $\sigma_y^2=7$. A new random variable Z is defined as Z=3X-4Y. Determine the mean and variance of Z of the following case of correlation between the random variables X and Y: $\rho_{xy}=0$.

10. A random variable X is defined by $f_X(x)=4\exp(-8|x|)$. The random variable Y is related to X by $Y=4+5X$. (a) Determine $E[X]$, $E[X^2]$, and σ_x^2. (b) Determine $f_y(y)$. (c) Determine $E[Y]$, $E[Y^2]$, and σ_y^2.

To be delivered at instructor's office: 18 November 2019

Good Luck!

1. A circle is divided in 21 equal parts. A pointer is spun until it stops on one of the parts, which are numbered from 1 through 21. Describe the sample space and assuming equally likely outcomes, find:

The sample space is all possible, mutually exclusive outcomes, so in this case, 21

(a) P(an even event) $P(x = \text{even}) = \boxed{\frac{10}{21}}$

(b) P(the number 21) $P(x = 21) = \boxed{\frac{1}{21}}$

(c) P(the numbers 4, 5, or 9) $P(x = 4, 5, \text{or } 9) = P(4) + P(5) + P(9) = \frac{3}{21} = \boxed{\frac{1}{7}}$

and P(a number greater than 10). $P(x > 10) = \boxed{\frac{11}{21}}$

2. What equations must be satisfied in order for three events A, B, and C to be independent?
Think that they must be independent by pairs, but this is not sufficient.

A, B, and C must be independent pairs as stated, but must also be mutually independent. (i.e.:

$$P(A \cap B) = P(A) \times P(B)$$

$$P(B \cap C) = P(B) \times P(C)$$

$$P(A \cap C) = P(A) \times P(C)$$

& $P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$

3. Given the table of joint probabilities,

	B_1	B_2	B_3	$P(A_i)$
A_1	0.05	0.05	0.45	0.55
A_2	0.05	0.15	0.10	0.30
A_3	0.05	0.05	0.05	0.15
$P(B_j)$	0.15	0.25	0.60	1.0

(a) find the probabilities omitted from the table please see table

(b) find the probabilities

$$P(A_3|B_3) = \frac{P(A_3 \cap B_3)}{P(B_3)} = \frac{0.05}{0.60} = \frac{1}{12} \approx \boxed{0.0833}$$

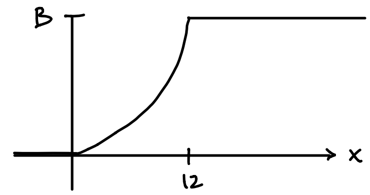
$$P(B_2|A_1) = \frac{P(B_2 \cap A_1)}{P(A_1)} = \frac{0.05}{0.55} = \frac{1}{11} \approx \boxed{0.09091}$$

$$P(B_3|A_2) = \frac{P(B_3 \cap A_2)}{P(A_2)} = \frac{0.10}{0.30} = \frac{1}{3} \approx \boxed{0.333}$$

4. A certain continuous random variable has the cumulative-distribution function

$$F_X(x) = \begin{cases} 0, & x < 0 \\ Ax^4, & 0 \le x \le 12 \\ B, & x > 12 \end{cases}$$

(a) Find the proper values for A and B



Has to be continuous @ $x = 12$, $x = 0$

$F(12) = A(12)^4 = B \qquad B = 20736A$

but $F_x(x) \le 1$ so $\boxed{B = 1 \qquad A = \frac{1}{20736}}$

(b) Obtain and plot the pdf $fx(x)$

$$f_x(x) = \frac{d\,F_x(x)}{dx} = \begin{cases} 0, & x < 0 \\ 4Ax^3, & 0 \le x \le 12 \\ 0, & x > 12 \end{cases} = \begin{cases} \frac{1}{5184} x^3, & 0 \le x \le 12 \\ 0, & \text{otherwise} \end{cases}$$

```
figure(1); clf
f = piecewise(x<0, 0, 0<=x<=12, 1/5184*x^3, x>12, 0);
fplot(f, 'linewidth',2)
axis([0 14 0 inf])
xlabel('x');
ylabel('f_x(x)');
title('Probability Density Function');
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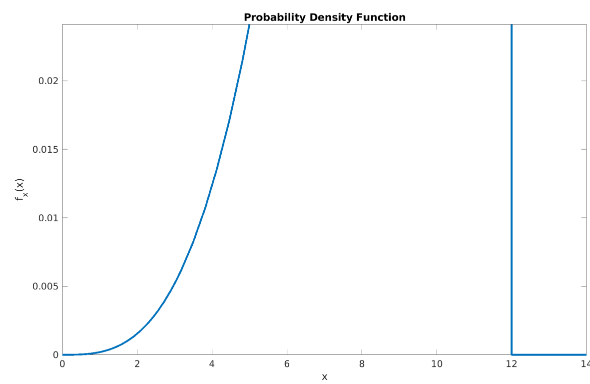


(c) Compute $P(X>5)$

$$P(x>5) = \frac{1}{5184}\int_5^{12} x^3\,dx = \boxed{0.969}$$

(d) Compute $P(4 \le X < 6)$

$$P(4 \le x < 6) = \frac{1}{5184}\int_4^{6} x^3\,dx = \boxed{0.0502}$$

5. The joint pdf of two random variables is

$$f_{XY}(x,y) = \begin{cases} C(1+xy), & 0 \le x \le 4,\ 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the following:

(a) The constant C

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) = 1 \longrightarrow C\int_{x=0}^{x=4}\int_{y=0}^{y=2} (1+xy)\,dy\,dx = 1$$

$$C\int_{x=0}^{x=4} y + \frac{xy^2}{2}\Big|_{y=0}^{y=2} dx = 1 \longrightarrow C\int_{x=0}^{x=4}\left(2+2x\right)dx = 1$$

$$C\left(2x + x^2\right)\Big|_0^4 = 1 \longrightarrow C\left(8+16\right) = 1 \qquad \boxed{C = \frac{1}{24}}$$

(b) $f_{XY}(1, 1.5)$

$$f_{XY}(1,1.5) = \frac{1}{24}\left(1+(1)(1.5)\right) = \left(\frac{1}{24}\right)\left(\frac{5}{2}\right) = \boxed{\frac{5}{48}}$$

(c) $f_{XY}(x,3)$

$$f_{XY}(x,3) = \frac{1}{24}\left(1+3x\right) = \begin{cases} \frac{1}{24} + \frac{x}{8}, & 0 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

(d) $f_{X|Y}(x|3)$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \begin{cases} \frac{1}{24}\int_0^4 (1+xy)\,dx, & 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{6}(1+2y), & 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\frac{1}{24}(1+xy)}{\frac{1}{6}(1+2y)} = \frac{1+xy}{8+4y} \qquad \begin{matrix} 0 \le x \le 4 \\ 0 \le y \le 2 \end{matrix}$$

$$f_{X|3}(x|3) = \frac{1+3x}{8+4(3)} = \begin{cases} \frac{1+3x}{28}, & 0 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

6. The joint pdf of the random variables X and Y is

$f_{XY}(x, y) = Axye^{-(x+y)},\ x \geq 0$ and $y \geq 0$

(a) Find the constant A.

$$A\int_{y=0}^{y=\infty}\int_{x=0}^{x=\infty} xye^{-(x+y)}\,dx\,dy = 1$$

$$A\cancelto{1}{\int_{y=0}^{y=\infty} ye^{-y}\,dy}\ \cancelto{1}{\int_{x=0}^{x=\infty} xe^{-x}\,dx} = 1 \qquad \boxed{A = 1}$$

(b) Find the marginal pdfs of X and Y, $f_X(x) f_Y(y)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^{\infty} xe^{-x}ye^{-y}\,dy = xe^{-x}\cancelto{1}{\int_0^{\infty} ye^{-y}\,dy} = \boxed{xe^{-x}}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^{\infty} xe^{-x}ye^{-y}\,dx = ye^{-y}\cancelto{1}{\int_0^{\infty} xe^{-x}\,dx} = \boxed{ye^{-y}}$$

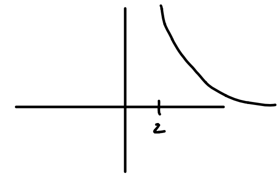
(c) Are X and Y statistically independent? Justify your answer.

$$f_{XY}(x,y) \stackrel{?}{=} f_X(x)\, f_Y(y)$$

$$xye^{-(x+y)} \stackrel{\checkmark}{=} (xe^{-x})(ye^{-y})$$

Because the pdf factors into a product of the marginals, X and Y are statistically independent.

7. Let $f_X(x) = A\exp(-bx)u(x-2)$ for all x where A and b are positive constants.



(a) Find the relationship between A and b such that this function is a pdf.

$$A\int_2^\infty e^{-bx}\,dx = -\frac{1}{b}e^{-bx}\Big|_2^\infty = \frac{A}{b}e^{-2b} = 1$$

$$\boxed{A = be^{2b}}$$

(b) Calculate $E(X)$ for this random variable.

$$E(x) = \int_{-\infty}^{\infty} x\,f_x(x)\,dx = A\int_2^\infty xe^{-bx}\,dx = A\left(\frac{(2b+1)e^{-2b}}{b^2}\right) = \boxed{\frac{2b+1}{b}}$$

(c) Calculate $E(X^2)$ for this random variable.

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx = A\int_2^\infty x^2e^{-bx}\,dx = A\left(\frac{(4b^2+4b+2)e^{-2b}}{b^3}\right)$$

$$= \boxed{\frac{4b^2+4b+2}{b^2}}$$

(d) What is the variance of this random variable?

$$\text{var}(x) = E(x^2) - E(x)^2 = \frac{4b^2+4b+2}{b^2} - \left(\frac{2b+1}{b}\right)^2 = \boxed{\frac{1}{b^2}}$$

8. The random variable has pdf, where $u(x)$ is the unit step. Determine the mean and the variance of the random variable. $f_X(x) = \frac{1}{2}\delta(x-5) + \frac{1}{8}[u(x-4) - u(x-8)]$

$$E(x) = \int_{-\infty}^{\infty} x\, f_x(x)\,dx = \int_{-\infty}^{\infty} x \left\{ \frac{1}{2} f(x-5) + \frac{1}{8}\left[u(x-4) - u(x-8)\right] \right\} dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x f(x-5)\,dx + \frac{1}{8}\int_{4}^{8} x\,dx = \frac{5}{2} + 3 = \boxed{\frac{11}{2}}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx = \int_{-\infty}^{\infty} x^2 \left\{ \frac{1}{2} f(x-5) + \frac{1}{8}\left[u(x-4) - u(x-8)\right] \right\} dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x^2 f(x-5)\,dx + \frac{1}{8}\int_{4}^{8} x^2\,dx = \frac{25}{2} + \frac{56}{3} = \frac{187}{6}$$

$$\mathrm{var}(x) = E(x^2) - E(x)^2 = \frac{187}{6} - \left(\frac{11}{2}\right)^2 = \boxed{\frac{11}{12}}$$

9. Two random variables X and Y have means and variances given below:

$m_x = 1 \quad \sigma_x^2 = 4 \quad m_y = 3 \quad \sigma_y^2 = 7$

A new random variable Z is defined as $Z = 3X - 4Y$

Determine the mean and variance of Z for the following case of correlation between the random variables X and Y: $\rho_{XY} = 0$.

$$E(z) = E(3X - 4Y) = 3E(X) - 4E(Y) = 3(1) - 4(3) = \boxed{-9}$$

$$\begin{aligned}
var(z) &= E(z^2) - E(z)^2 = E[(3X-4Y)^2] - E^2(3X - 4Y) \\
&= E[3X - 4Y - 3E(x) + 4E(Y)]^2 \\
&= E[3(x - E(x)) - 4(Y - E(Y))]^2 \\
&= E[9(x - E(x))^2 - 24(x - E(x))(Y - E(Y)) + 16(Y - E(Y))^2] \\
&= 9(\sigma_x^2) - 24(\sigma_x \sigma_Y \sigma_{xY}) + 16(\sigma_Y^2)
\end{aligned}$$

$$\rho_{xY} = 0 \rightarrow var(z) = 9(4) - 0 + 16(7) = \boxed{148}$$

10. A random variable X is defined by $f_X(x) = 4e^{-8|x|}$

The random variable Y is related to X by $Y = 4 + 5X$

(a) Determine $E[X]$, $E[X^2]$, and σ_x^2.

$$E(x) = \int_{-\infty}^{\infty} x f_x(x)\,dx = 2\int_0^{\infty} x(4e^{-8x})\,dx = \boxed{\frac{1}{8}}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx = 2\int_0^{\infty} x^2(4e^{-8x})\,dx = \boxed{\frac{1}{32}}$$

$$var(x) = E(x^2) - E(x)^2 = \frac{1}{32} - \left(\frac{1}{8}\right)^2 = \boxed{\frac{1}{64}}$$

(b) Determine $f_Y(y)$.

$$f_Y(y) = f_X(x)\frac{dx}{dy} \rightarrow \boxed{\frac{4}{5}e^{-8|x|}}$$

(c) Determine $E[Y]$, $E[Y^2]$, and σ_y^2.

$$E(Y) = E(4+5X) = E(4) + 5E(x) = 4 + 5\left(\frac{1}{8}\right) = \boxed{\frac{37}{8}} \approx 4.6250$$

$$E(Y^2) = E\left[(4+5X)^2\right] = E(16 + 40X + 25X^2) = 16 + 40E(x) + 25E(x^2)$$

$$= 16 + 40\left(\frac{1}{8}\right) + 25\left(\frac{1}{32}\right) = \boxed{\frac{697}{32}} \approx 21.7813$$

$$var(Y) = E(Y^2) - E(Y)^2 = \frac{697}{32} - \left(\frac{37}{8}\right)^2 = \boxed{\frac{25}{64}} \approx 0.3906$$