ECE 345 / ME 380
Introduction to Control Systems
Lecture Notes 8

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Learning Objectives

- State the phase and gain properties of a root locus
- Sketch a root locus, by identifying
  - Real-axis segments
  - Start and end points of the locus
  - Number of asymptotes, and location of the asymptote center
- Refine a root locus sketch, by identifying
  - Real-axis breakaway and break-in points
  - Angles of departure and arrival
  - Imaginary axis crossings
- Find the gain associated with a point on the root locus
- Use root locus to meet a transient response specification

References:

- Nise, Chapter 8.1–8.7

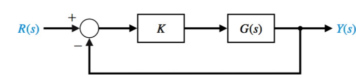
Outline

- What is a root locus?
- Review: Vector representations of complex numbers
- Phase and gain property of the root locus
- Sketching the root locus
- Refining the root locus sketch
- Control design via root locus

What is the root locus?

For the negative unity feedback system with

$$G(s) = \frac{1}{s(s+10)}$$



the characteristic equation of $\frac{Y(s)}{R(s)}$ is

$$\Delta(s) = 1 + KG(s) = s^2 + 10s + K$$

The poles of the closed-loop transfer function

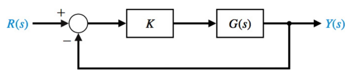
$$s = -5 \pm \sqrt{5^2 - K}$$

will be real-valued for $K \leq 25$ and complex-valued for $K > 25$.

What is the root locus?

For the negative unity feedback system with $\Delta(s) = 1 + KG(s) = s^2 + 10s + K$

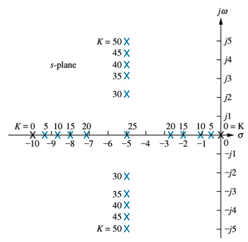
$$G(s) = \frac{1}{s(s+10)}$$



Plot the poles for various values of $K > 0$.

Also plot the poles of the open-loop system $KG(s)$.

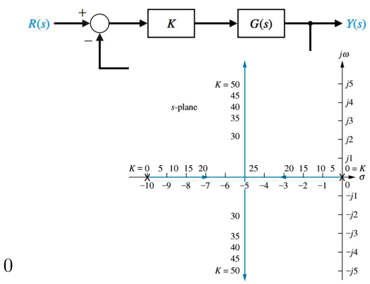
As $K \to 0$, the poles of $1 + KG(s)$ move towards the poles of $KG(s)$.



What is the root locus?

For the negative unity feedback system with $\Delta(s) = 1 + KG(s) = s^2 + 10s + K$

$$G(s) = \frac{1}{s(s+10)}$$

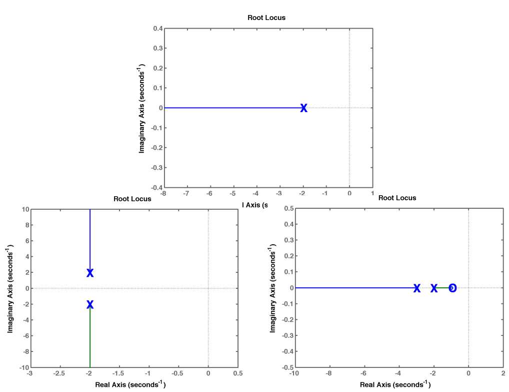


The *root locus* is a plot of the roots of the closed-loop system as K varies from 0_+ to ∞.

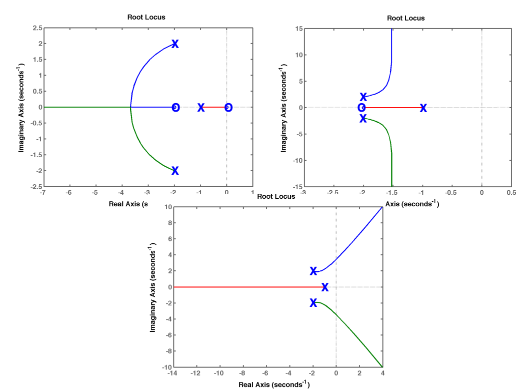
The plot starts at the poles of $KG(s)$ with $K = 0$.

$$\Delta(s) = 1 + KG(s) = D(s) + KN(s) = 0$$

What is the root locus?



What is the root locus?



What is the root locus?

The $root\ locus$ is a plot of the roots of the closed-loop system as K varies from 0_+ to ∞.

- Developed by Walter Evans (1920–1999) as a graduate student at UCLA.
- Generic technique to sketch roots of a polynomial as a function of *one* parameter in the polynomial.
- Simple graphical representation of *all* potential locations for poles of the closed-loop system as the gain K increases.
- Particularly convenient for negative unitary feedback systems, because analysis of *open-loop* system $KG(s)$ allows quick assessment of behavior of *closed-loop* system $\frac{KG(s)}{1+KG(s)}$.
- Back-of-the envelope sketching techniques are important.
- Numerical tools in Matlab (`rlocus`, `rlocfind`).

Phase and Gain Criteria of The Root Locus

All plots on the locus satisfy $$\Delta(s) = 1 + KG(s) = 0$$

Phase criterion

- *All* points $s \in \mathbb{C}$ on the locus satisfy $\angle G(s) = 180° \pm 360k, k \in \mathbb{Z}$.
  (Recall that $G(s)$ is simply a complex number. Then for all $K > 0$,
  $$-\frac{1}{K} = G(s)$$

Gain criterion

- The gain required to place one of the poles of the closed-loop system at a desired location s^* is
  $$K = \left|\frac{1}{G(s^*)}\right|$$

Sketching the Root Locus

Main idea

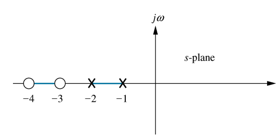
- Back-of-the-envelope sketch
- Identify basic features of the root locus
- Based on gain and phase properties of the root locus
- Can refine later, or plot numerically

Key features

- Symmetry
- Number of branches
- Real-axis segments
- Start and end points
- Behavior at infinity

Sketching the Root Locus

1. Symmetry
   - The root locus always has symmetry about the real axis
2. Number of branches
   - A *branch* is the path a single pole traverses as K increases
   - The number of branches of the root locus is equal to the number of poles.
3. Real-axis segments
   - The root locus exists on the real line to the *left* of an odd number of poles and zeros

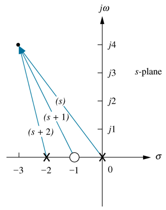


Review: Vector representation of complex numbers

For a transfer function

$$G(s) = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- The vector $s+p$ is the vector drawn from pole at $-p$ to a point s
- The vector $s+z$ is the vector drawn from zero at $-z$ to point s



Review: Vector representation of complex numbers

The *magnitude* of G(s) is

$$|G(s)| = \frac{|s+z_1|\cdot|s+z_2|\cdots|s+z_m|}{|s+p_1|\cdot|s+p_2|\cdots|s+p_n|}$$

$$= \frac{\text{Product of magnitudes of vectors drawn from zeros to } s}{\text{Product of magnitudes of vectors drawn from poles to } s}$$

The *phase* of G(s) is

$$\begin{aligned}\angle G(s) &= (\angle(s+z_1)+\angle(s+z_2)+\cdots+\angle(s+z_m)) \\ &\quad -(\angle(s+p_1)+\angle(s+p_2)+\cdots+\angle(s+p_n)) \\ &= (\text{Sum of angles of vectors drawn from zeros to } s) \\ &\quad -(\text{Sum of angles of vectors drawn from poles to } s)\end{aligned}$$

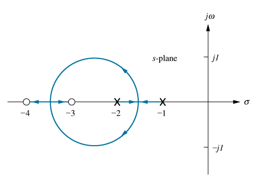
Sketching the Root Locus

4. Start and end points
   - The root locus starts at the poles of $G(s)$ and ends at the zeros of $G(s)$

$$\Delta(s) = 1 + KG(s) = D(s) + KN(s) = \frac{1}{K} + \frac{N(s)}{D(s)} = 0$$

For K small, $0 = D(s) + KN(s) \approx D(s)$.
For K large, $0 = \frac{1}{K} + \frac{N(s)}{D(s)} \Rightarrow N(s) \approx 0$.

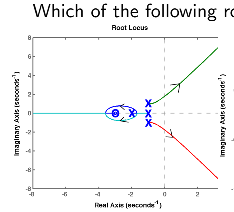
5. Behavior at infinity
   - For a system with n poles and m zeros, $n-m$ branches *do not end*.
   - "Infinite zeros"

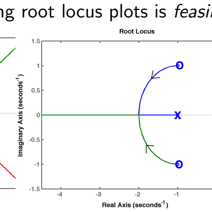


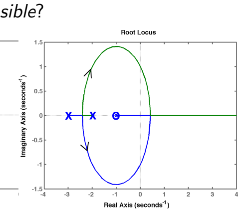
Sketching the Root Locus

Clicker question

Which of the following root locus plots is *feasible*?







A. B. C.

D. Both A. and C.

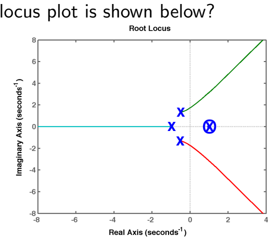
E. Both B. and C.

Sketching the Root Locus

Clicker question

Which of the following is correct, based on the system whose root locus plot is shown below?

A. The system is unstable for low gains.
B. The system is unstable for high gains.
C. The system is stable for any gain.
D. The system is unstable for any gain.



Sketching the Root Locus

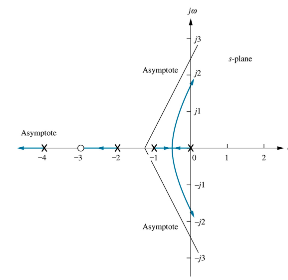
5. Behavior at infinity (cont'd)
   - $n - m$ branches converge to *asymptotes* that approach infinity as $K \to \infty$
   - The asymptotes have $(n-m)$ angles

$$\begin{aligned} \theta_k &= \tfrac{(2k+1)\pi}{n-m} \text{ [rad]}, \\ k &= 0, \cdots, n-m-1 \end{aligned}$$

   - The asymptotes have 1 centroid

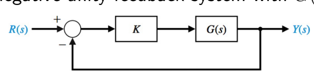
$$\sigma = \frac{\sum_{k=1}^{n}(-p_k) - \sum_{k=1}^{m}(-z_k)}{n-m}$$

for $G(s) = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$



Sketching the Root Locus Sketch

Sketch the root locus for the negative unity feedback system with $G(s) = \frac{s+1}{s(s+2)(s+3)}$.

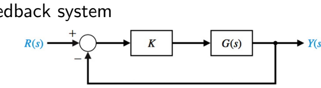


1. How many poles and zeros, and where are they located?
2. Which parts of the real line are on the root locus?
3. How many asymptotes, and at what angles θ_k?
4. Where is the centroid σ of the asymptotes?

Sketching the Root Locus

Clicker question

Consider a negative unity feedback system

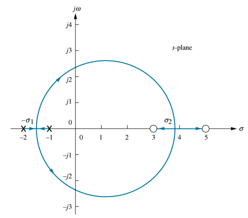


for which $G(s)$ has no zeros. Use a root locus sketch to determine under which of the following scenarios the closed-loop system $\frac{Y(s)}{R(s)}$ will remain stable for *all* values of $K > 0$.

A. $G(s)$ has one pole in the LHP.
B. $G(s)$ has two poles, both in the LHP.
C. $G(s)$ has three poles, all in the LHP.
D. A. and B.
E. A. and B. and C.

Refining the Root Locus Sketch

6. Real-axis breakaway and break-in points
   - Occurs when branches intersect
   - A breakaway occurs when the locus leaves the real axis
   - A break-in occurs when the locus returns to the real axis
   - Angle of the breakaway / break-in with respect to the real axis is $180°/n$ and *maintains symmetry about the real axis*
     - 2 poles $\Rightarrow \pm 90°$
     - 3 poles $\Rightarrow 60°, 180°, 240°$
     - 4 poles $\Rightarrow 45°, 135°, 225°, 315°$
     - $\vdots$



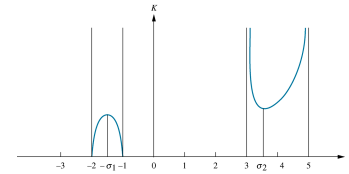
Refining the Root Locus Sketch

6. Real-axis breakaway and break-in points
   - A breakaway / break-in occurs for the value of s in the appropriate range that solves

$$\frac{dK}{ds} = \frac{d}{ds}\left(-\frac{1}{G(s)}\right) = 0$$

   - Or equivalently, for s that solves

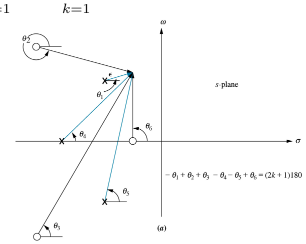
$$\sum_{k=1}^{m} \frac{1}{s+z_k} = \sum_{k=1}^{n} \frac{1}{s+p_k}$$

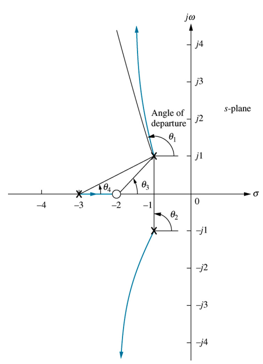


Refining the Root Locus Sketch

7. Angles of departure and arrival
   - Angle of departure *from each pole*
   - Angle of arrival *to each zero*

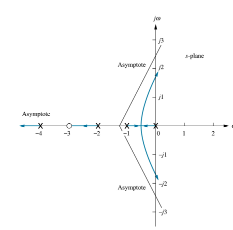
$$\sum_{k=1}^{m} \theta_{z_k} - \sum_{k=1}^{n} \theta_{p_k} = 180°$$





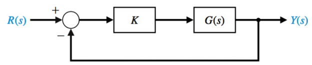
Refining the Root Locus Sketch

8. Imaginary axis crossings
   - Routh table
     - Choose K such that there is a row of zeros
     - Solve the roots of the polynomial formed from row *above* row of zeros to find the location of the imaginary axis crossing
     - See Nise Example 8.5
   - Hurwitz criterion
     - Choose K such that constraints on coefficients *equal 0* instead of being greater than zero



Refining the Root Locus Sketch

Refine the root locus sketch for the negative unity feedback system with $G(s) = \frac{s+1}{s(s+2)(s+3)}$.

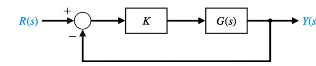


1. How many poles and zeros, and where are they located?
2. Which parts of the real line are on the root locus?
3. How many asymptotes, and at what angles θ_k?
4. Where is the centroid σ of the asymptotes?
5. Where is the real-axis breakaway point?
6. For poles and zeros not on the real line, what are the departure / arrival angles?

Refining the Root Locus Sketch

Clicker question

$$G(s) = \frac{1}{s(s+4)(s^2+8s+32)}$$



Consider the above system, with open-loop poles at $s = 0, -4, -4 \pm 4j$. Which of the following is the most correct?

A. With four asymptotes at $\theta_k = 0°, \pm 90°, 180°$, with high enough gain, $\frac{Y(s)}{R(s)}$ will become unstable with *one* pole in the RHP.

B. With four asymptotes at $\theta_k = \pm 45°, \pm 135°$, with high enough gain, $\frac{Y(s)}{R(s)}$ will become unstable with *two* poles in the RHP.

C. $\frac{Y(s)}{R(s)}$ is unstable at low gains with *one* pole at the origin.

D. $\frac{Y(s)}{R(s)}$ is unstable for all values of K, with *either one or two* poles in the RHP depending on the value of K.

Summary

- Root locus is a plot of the poles of the *closed-loop system* as K varies from 0 to ∞
- Sketching the root locus allows for quick assessment of what is needed to stabilize a system
- Precise numerical tools available in Matlab (`rlocus`)
- Root locus *starts at the poles* of the open-loop system and *ends at the zeros* of the open-loop system
- Important landmarks: Location of open-loop poles and zeros, root locus on the real axis, asymptote angles and centroid
- Additional landmarks: Real axis breakaway / break-in points, departure / arrival angles, imaginary axis crossings