Trigonometric Form of. Fourier Series

08/26/2019 (1)

x(t) real signal

$$\bar{X}_n \cdot e^{jn\omega_0 t} + \bar{X}_{-n} \cdot e^{-jn\omega_0 t} = |\bar{X}_n| e^{j(n\omega_0 t + \angle X_n)} + |\bar{X}_n| \cdot e^{-j(n\cdot\omega_0 t + \angle X_n)}$$

Euler's Theorem

$$= 2|\bar{X}_n| \cdot \cos(n\omega_0 t + \angle X_n)$$

$$\boxed{x(t) = \bar{X}_0 + \sum_{n=1}^{+\infty} 2|\bar{X}_n| \cdot \cos(n\omega_0 t + \angle X_n)}$$

$$x(t) = \sum_{n=-\infty}^{+\infty} \bar{X}_n \cdot e^{jn\omega_0 t}$$

$|\bar{X}_n| \cdot e^{j\angle X_n}$

Euler's Theorem

$$= \bar{X}_0 + \sum_{n=1}^{+\infty} A_n \cos(n\omega_0 t) + \sum_{n=1}^{+\infty} B_n \sin(n\cdot\omega_0 t)$$

$A_n = 2|\bar{X}_n| \cos\angle X_n$

$B_n = -2|\bar{X}_n| \sin\angle X_n$

Parseval's Theorem

(Periodic Signals)

Average Power of a periodic signal

$$P = \underbrace{\bar{X}_0^2}_{DC} + \underbrace{\sum_{n=1}^{+\infty} 2|\bar{X}_n|^2}_{AC}$$

$$\left(\operatorname{sinc} z = \frac{\sin(\pi z)}{\pi \cdot z} \qquad \text{sinc function} \right)$$

Example

Pulse train : $x(t) = A \cdot \Pi\left(\frac{t - t_0}{\tau}\right)$, $\tau < T_0$

$\tau = \frac{1}{2} T_0$

$t_0 = 0$

$$X_n = \frac{1}{2} A \operatorname{sinc}\left(\frac{1}{2} n\right)$$

(Fourier coeffic)

$$\operatorname{sinc}\left(\frac{n}{2}\right) = \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)} = \begin{cases} 1 & , n=0 \\ 0 & , n: \text{even} \\ \left|\frac{2}{n\pi}\right| & , n = \pm 1, \pm 5, \\ -\left|\frac{2}{n\pi}\right| & , n = \pm 3, \pm 7, \end{cases}$$

$$x(t) = \overbrace{\cdots + \frac{A}{5\pi} e^{-j5\omega_0 t} - \frac{A}{3\pi} e^{-j3\omega_0 t} + \frac{A}{\pi} e^{-j\omega_0 t}}^{n<0}$$

$$\underbrace{+ \frac{A}{2}}_{n=0} \underbrace{+ \frac{A}{\pi} e^{j\omega_0 t} - \frac{A}{3\pi} \cdot e^{j3\omega_0 t} + \frac{A}{5\pi} e^{j5\omega_0 t} + \cdots}_{n>0}$$

Euler's Theorem

Exponential Form

$$= \frac{A}{2} + \frac{2A}{\pi}\left[\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t) - \cdots\right]$$

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Trigonometric Form

Line Spectra → Amplitude, Phase

{ Single-side
{ double-side

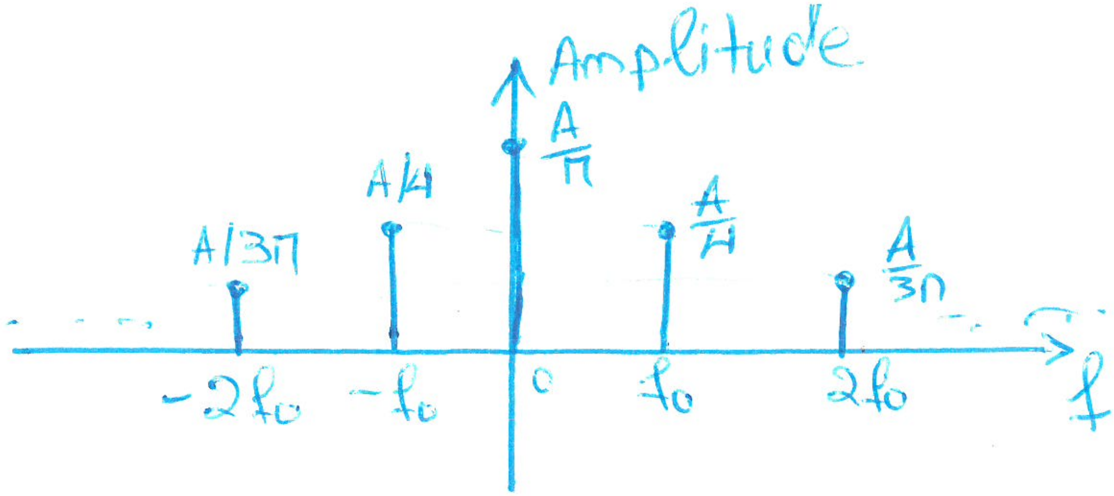
Example

Half-rectified sinewave:

$$x(t) = \begin{cases} A\sin(\omega_0 t), & 0 \le t \le \frac{T_0}{2} \\ 0, & -\frac{T_0}{2} \le t \le 0 \end{cases}$$

$$X_n = \begin{cases} \frac{A}{\pi(1-n^2)}, & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n = \pm 3, \pm 5, \ldots \\ -\frac{1}{4} jnA, & n = \pm 1 \end{cases}$$

$$|X_n| = \begin{cases} \left|\frac{A}{\pi(1-n^2)}\right|, & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n = \pm 3, \pm 5, \ldots \\ \frac{A}{4}, & n = \pm 1 \end{cases}$$



$$\angle X_n = \begin{cases} 0 & , n = 0 \\ -\dfrac{\pi}{2} & , n = 1 \\ \dfrac{\pi}{2} & , n = -1 \\ 0 & , n = \pm 3, \pm 5 \ldots \\ \pi & , n = 2, 4, \ldots \\ -\pi & , n = -2, -4, \ldots \end{cases}$$

$n = -2, -4, \ldots$

$$X_n = -\left|\frac{A}{\pi \cdot (1-n^2)}\right| = \left|\frac{A}{\pi \cdot (1-n^2)}\right| e^{-j\pi}$$

$n = 2, 4, \ldots$

$$X_n = -\left|\frac{A}{\pi (1-n^2)}\right| = \left|\frac{A}{\pi \cdot (1-n^2)}\right| e^{j\pi}$$

$e^{j\pi} = -1$

$e^{-j\pi} = -1$

$X_n = |X_n| \cdot e^{j\angle X_n}$

Fourier Transform

$$t \longleftrightarrow f$$

$$x(t) = \sum_{n=-\infty}^{+\infty} \underbrace{\boxed{\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} x(\lambda)\, e^{-j2\pi f_0 \lambda}\, d\lambda}}_{X_n} \cdot e^{j2\pi f_0 n t}$$

$$|t| < \frac{T_0}{2}$$

$T_0 \rightarrow \infty$

$$x(t) = \int_{-\infty}^{+\infty} \underbrace{\boxed{\int_{-\infty}^{+\infty} x(\lambda) \cdot e^{-j2\pi f \lambda}\, d\lambda}}_{= X(f)} \cdot e^{j2\pi f t}\, df$$

Fourier Transform

$$x(t) = \int_{-\infty}^{+\infty} X(f) \cdot e^{j2\pi f t}\, df.$$

Amplitude/Phase Spectra:

$$X(f) = |X(f)| \cdot e^{j\theta(f)} \qquad \theta(f) = \angle X(f)$$

Real signal: $|\bar{X}(f)| = |\bar{X}(-f)|$ even symmetry

$\theta(f) = -\theta(-f)$ odd symmetry

$\left(\angle \bar{X}(f)\right)$

$$Re\{X(f)\} = \int_{-\infty}^{+\infty} x(t) \cdot \cos(2\pi f t)\, dt$$

$$Im\{\bar{X}(f)\} = \int_{-\infty}^{+\infty} x(t) \sin(2\pi f t)\, dt.$$

Symmetry Properties

Even signal

$x(t) = x(-t)$

$Im(X(f)) = 0$

$Re\{\bar{X}(f)\}$ even w.r.t. f

Odd signal

$x(t) = -x(-t)$

$Re\{X(f)\} = 0$

$Im\{\bar{X}(f)\}$ odd w.r.t. f

Rayleigh's Energy Theorem

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df.$$

Energy Spectral Density of a Signal

$$G(f) \triangleq |X(f)|^2$$

Example

$f = 10\,Hz$

$$x(t) = 40\,\text{sinc}(20\pi t) \longleftrightarrow X(f) = 2\Pi\left(\frac{f}{20}\right)$$

Energy Spectral Density

$$G(f) = |X(f)|^2 = \left|2\Pi\left(\frac{f}{20}\right)\right|^2$$

$$= 4\Pi\left(\frac{f}{20}\right)$$

Total Energy

$$E = \int_{-\infty}^{+\infty} G(f)\,df = \int_{-10}^{10} G(f)\,df$$

$$= \int_{-10}^{10} 4\Pi\left(\frac{f}{20}\right) df = 80\,J$$

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Energy in (g, w) $\|$ $E = \int_{-w}^{w} G(f)\,df = 2\int_{0}^{w} G(f)\,df =$

$$= 2\int_{0}^{w} 4\Pi\left(\frac{f}{20}\right) df = \begin{cases} 8w, & w \le 10 \\ 80, & w > 10 \end{cases}$$