ECE 345 / ME 380
Introduction to Control Systems
Lecture Notes 3

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Learning Objectives

- Find a state-space representation for a linear, time-invariant system
- Model electrical and mechanical systems in state-space representations
- Convert a transfer function to a state-space representation
- Convert a state-space representation to a transfer function
- Linearize a state-space representation of a system

References:

- Nise, Chapter 3

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Outline

- State-space representations
  - Common terminology
  - First-order ODEs
  - Independent state elements
  - Examples: electrical and mechanical systems
- Frequency domain vs. time-domain
  - State-space to transfer function
  - Transfer function to state-space
  - Uniqueness of state-space representations
- Linearization

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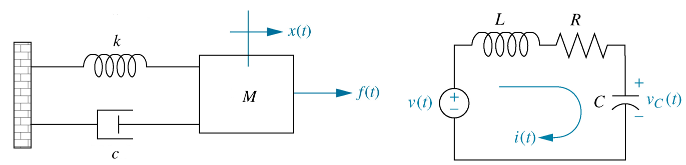
State-space models



- What information is required to predict the system's behavior at the next instant in time?

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State-space models

Time-domain modeling

- More compact form than integro-differential or n^{th} order differential equations (that would arise from Kirchhoff's laws and Newton's laws)
- Complementary to transfer function representation
- Most multivariable design methods are based on state equations
- Routine to move between state-space and transfer function representations

What is the *state*?

- A complete set of variables required to account for storage of mass, momentum, and energy in a system

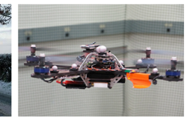
These variables are grouped together in a vector, and referred to as the *state vector* or simply the *state*.

State-space models

Examples of state variables:

- Position and velocity of a point-mass model of a car
- Voltage across the capacitor of a series RC circuit
- Room temperature for a thermostat
- Angle and rotational speed of a rigid, fixed-length pendulum
- Kinetic energy and potential energy of a spring-mass-damper system
- Speed, flightpath angle, and altitude for longitudinal movement of a civil jet aircraft

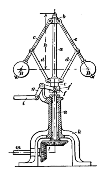
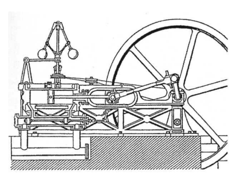




State-space models

Clicker question

Consider the flyball governor, a control mechanism that helped power the Industrial Revolution.

Which of the following is *least likely* to be a feasible set of state variables?

A. Weight of the spheres and vertical speed of the spheres
B. Height of the spheres and vertical speed of the spheres
C. Height of the spheres and rotational speed of the shaft
D. Vertical speed of the sphere and rotational speed of the shaft

State-space models

A single-input, single-output (SISO) state-space model is defined by four constant matrices A, B, C, D:

- State equation $\dot{x}(t) = Ax(t) + Bu(t)$
- Output equation $y(t) = Cx(t) + Du(t)$

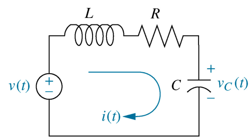
with input $u(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$, and state $x(t) \in \mathbb{R}^n$.

Matrix	Name	Dimension	Role
A	State matrix	$n \times n$	Propagate state-based changes in x
B	Input matrix	$n \times 1$	Capture effect of actuation on x
C	Output matrix	$1 \times n$	Sensed quantities
D	Direct term	1×1	Direct feedthrough from input to output

State-space models

Clicker question

Recall the RLC series circuit.



Which one of the following is a feasible state vector?

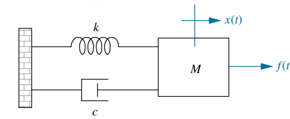
A. $x(t) = \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$

B. $x(t) = \begin{bmatrix} v_c(t) \\ \int_0^\tau i(\tau) d\tau \end{bmatrix}$

C. $x(t) = \begin{bmatrix} v_c(t) \\ v(t) \end{bmatrix}$

D. $x(t) = \begin{bmatrix} i(0) \\ \int_0^\tau i(\tau) d\tau \end{bmatrix}$

Example: Spring-Mass-Damper System



We previously obtained one, 2^{nd} order equation of motion with input $f(t)$ and output $x(t)$.

$$M\ddot{x}(t) = -kx(t) - c\dot{x}(t) + f(t) \tag{1}$$

Now we reorganize (1) as two, 1^{st} order ODEs

$$\begin{aligned} \dot{x}(t) &= v(t) \\ \dot{v}(t) &= -\tfrac{k}{M}x(t) - \tfrac{c}{M}v(t) + \tfrac{1}{M}f(t) \end{aligned} \tag{2}$$

Example: Spring-Mass-Damper System

More formally,

$$\begin{aligned} \dot{\underline{x}}_1 &= 0 \cdot \underline{x}_1 + 1 \cdot \underline{x}_2 + 0 \cdot u(t) \\ \dot{\underline{x}}_2 &= -\tfrac{k}{M}\underline{x}_1 - \tfrac{c}{M}\underline{x}_2 + \tfrac{1}{M}u(t) \end{aligned} \tag{3}$$

gives us the state equation

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{c}{M} \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t) \tag{4}$$

And writing the output as

$$\begin{aligned} y(t) &= x(t) \\ &= 1 \cdot \underline{x}_1(t) + 0 \cdot \underline{x}_2(t) + 0 \cdot u(t) \end{aligned} \tag{5}$$

gives us the output equation

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \end{bmatrix} \cdot u(t) \tag{6}$$

State-space models

Clicker question

Consider the second-order system represented by

$$\ddot{x} = -2\dot{x} - 2x \tag{7}$$

Which of the following is a *first-order* system that is equivalent to (7)?

A. $\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 2\dot{x}_2 \end{aligned}$

B. $\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_2 + 2x_1 \end{aligned}$

C. $\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_2 - 2x_1 \end{aligned}$

D. $\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2x_2 - 2x_1 \end{aligned}$

State-space models

Why state-space models?

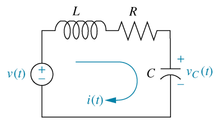
- **Closed-form solution** for $x(t)$ (and hence $y(t)$)
  An aside (for now... you will be responsible for this later in the course):

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- Easily extends to multiple-input, multiple-output systems
- Represents a wide variety of physical systems
- Easy to compute with because of efficient handling of matrices (for very large systems)

State-space models

State-space representations are NOT unique!



$$u(t) = v(t),\ y(t) = i(t)$$

What are A, B, C, D? What are F, G, H, J?

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \\ x &= \begin{bmatrix} \int_0^t i(\tau)d\tau \\ i(t) \end{bmatrix} \end{aligned} \qquad \begin{aligned} \dot{z} &= Fz + Gu \\ y &= Hz + Ju, \\ z &= \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix} \end{aligned}$$

While the input and output are the same in both systems, the *state* is chosen differently in each case.

Frequency domain vs. time-domain

Time-domain $\longrightarrow$ frequency-domain

- Take the Laplace transform of the state and output equations, presuming initial conditions are zero ($x(0) = 0$).

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

- With algebraic manipulation, the transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

- where I is a $n \times n$ identity matrix.

Exercise: Show that the systems (A, B, C, D) and (F, G, H, J) on slide 9 result in *the same* transfer function $\frac{Y(s)}{U(s)}$.

Frequency domain vs. time-domain

Linear algebra review

For this class, you should be able to

- Easily manipulate matrices of up to (and including) 3D by hand.
  - Matrix addition
  - Matrix multiplication
  - Matrix inversion

If you need to review, you may want to check out:

- Matlab resources
- Textbook resources

Frequency domain vs. time-domain

Clicker question: For the system

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} -5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned} \tag{8}
$$

with input $u(t)$ and output $y(t)$, what is the transfer function $\frac{Y(s)}{U(s)}$?

A. $\frac{Y(s)}{U(s)} = \frac{1}{s+2}$
B. $\frac{Y(s)}{U(s)} = \frac{3}{(s+2)(s+5)}$
C. $\frac{Y(s)}{U(s)} = \frac{2s+7}{(s+2)(s+5)}$
D. $\frac{Y(s)}{U(s)} = \frac{7}{(s-2)(s-5)}$
E. None of the above

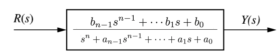
Frequency domain vs. time-domain

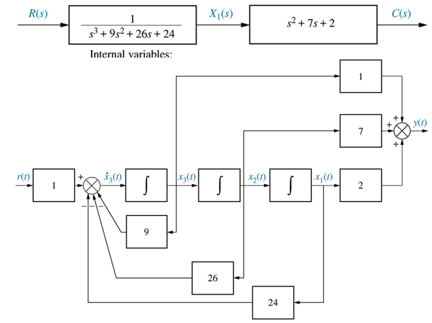
Frequency-domain $\longrightarrow$ time-domain
- Many (infinite) possibilities
- 'Canonical' forms
- *Phase variable* form

$$
A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}
$$

$$
C = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \end{bmatrix}, \quad D = 0
$$

for

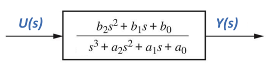


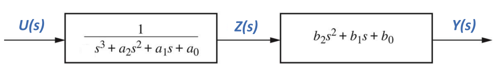


Frequency domain vs. time-domain

Phase variable form

- Represent denominator and numerator as successive systems (in series)
- Define $x_1(t)$ as the intermediary signal $z(t)$; remaining states are higher derivatives
- Dimension of state vector = order of polynomial in denominator
- Denominator effects in feedback terms
- Numerator effects in feedforward terms



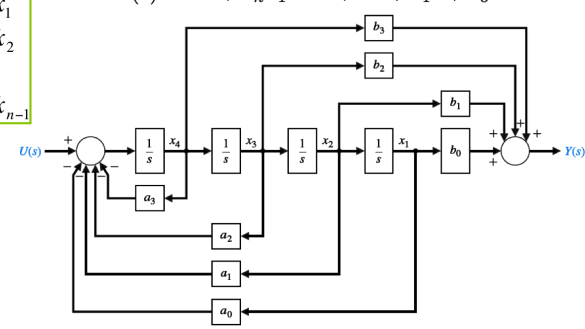


Frequency domain vs. time-domain

Phase variable form

$$
\boxed{\begin{aligned} x_1 &= z \\ x_2 &= \dot{x}_1 \\ x_3 &= \dot{x}_2 \\ &\vdots \\ x_n &= \dot{x}_{n-1} \end{aligned}}
$$

$$
\frac{Y(s)}{U(s)} = \frac{b_m \cdot s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}
$$



Frequency domain vs. time-domain

Clicker question

For the system with transfer function $\frac{Y(s)}{U(s)} = \frac{s+2}{s^2+3s+2}$, which of the following systems produce exactly the same input-output behavior?

A. $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 2 & 1 \end{bmatrix} x$

B. $\frac{Y(s)}{U(s)} = \frac{1}{s+1}$

C. $\dot{x} = -x + u$, $y = x$

D. A and B, but not C

E. A, B, and C

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Linearization

Generalization of scalar linearization from Chapter 2

- Linearize $y = f(x) \in \mathbb{R}$ for $x \in \mathbb{R}^n$ near $x_0 \in \mathbb{R}^n$
- Taylor's series approximation
- Works for $(x - x_0)$ "small enough"

$$f(x) - f(x_0) \approx \left.\frac{\partial f}{\partial x}\right|_{x=x_0} \cdot (x - x_0)$$

Note that the Jacobian is a matrix, not a scalar!

$$\frac{\partial f}{\partial x} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x=x_0} \in \mathbb{R}^{n \times n} \tag{9}$$

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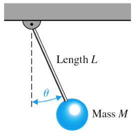
Linearization

Frictionless rigid pendulum

- Linearize the dynamics $\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ MgL\sin\theta \end{bmatrix}$ around $\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.



$$\begin{aligned} \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} &\approx \left.\begin{bmatrix} 0 & 1 \\ MgL\frac{\partial \sin\theta}{\partial\theta} & 0 \end{bmatrix}\right|_{\theta=0,\omega=0} \left(\begin{bmatrix} \theta \\ \omega \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \\ \begin{bmatrix} \delta\dot{\theta} \\ \delta\dot{\omega} \end{bmatrix} &\approx \begin{bmatrix} 0 & 1 \\ MgL & 0 \end{bmatrix} \cdot \begin{bmatrix} \delta\theta \\ \delta\omega \end{bmatrix} \end{aligned}$$

- with $\delta\theta = \theta$, $\delta\omega = \omega$

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Key Concepts

1. State-space form: (A, B, C, D)
2. Converting transfer function to state-space
3. Converting state-space to transfer function
4. Linearization of scalar functions of vector variables

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