Example

Two random variables X, Y

$$f_{XY}(x,y) = \begin{cases} A e^{-(2x+y)}, & x,y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Solution:

1) Find A: $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) = 1 \Leftrightarrow$

$$\int_{\cancel{-\infty}}^{\infty}\int_{\cancel{-\infty}}^{\infty} A e^{-(2x+y)}\,dx\,dy = 1 \Leftrightarrow \int_{\cancel{-\infty}}^{\infty}\int_{0}^{\infty} A e^{-2x} \cdot e^{-y}\,dx\,dy = 1 \Leftrightarrow$$

$$A\int_{0}^{\infty}\int_{0}^{\infty} e^{-2x} \cdot e^{-y}\,dx\,dy = 1 \Leftrightarrow A\int_{0}^{\infty} e^{-2x}\left[-e^{-y}\right]_0^{\infty} dx = 1 \Leftrightarrow$$

$$A\int_{0}^{\infty} e^{-2x}\left[-\frac{1}{e^{y}}\right]_0^{\infty} dx = 1 \Leftrightarrow A\int_{0}^{\infty} e^{-2x}\,dx = 1 \Leftrightarrow$$

$$A\left[-\frac{1}{2}e^{-2x}\right]_0^{\infty} = 1 \Leftrightarrow \boxed{A = 2}$$

2) Find the marginal pdfs

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_{0}^{\infty} 2e^{-2x} \cdot e^{-y}\,dy \Leftrightarrow$$

$$f_X(x) = 2e^{-2x}$$

$$\underline{f_X(x)} = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_{0}^{\infty} 2e^{-2x} \cdot e^{-y}\,dx \Leftrightarrow$$

$$f_Y(y) = e^{-y}$$

$$\underline{f_Y(y)} = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

3) Find the joint cdf:

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x',y')\,dx'\,dy' \Leftrightarrow$$

$$F_{XY}(x,y) = \int_{0}^{x}\int_{0}^{y} 2e^{-2x'} \cdot e^{-y'}\,dx'\,dy' \Leftrightarrow$$

$$F_{XY}(x,y) = \int_{0}^{x} 2e^{-2x'} \int_{0}^{y} e^{-y'}\,dy'\,dx' \Leftrightarrow$$

$$F_{XY}(x,y) = \int_{0}^{x} 2e^{-2x'} \left[-e^{-y'}\right]_{0}^{y} dx' \Leftrightarrow$$

$$F_{XY}(x,y) = \int_{0}^{x} 2e^{-2x'}(1-e^{-y})\,dx' \Leftrightarrow F_{XY}(x,y) = (1-e^{-y})\int_{0}^{x} 2e^{-2x}d$$

$$\Leftrightarrow F_{XY}(x,y) = 2(1-e^{-y})\int_{0}^{x} e^{-2x'}dx = 2(1-e^{-y})\left[-\frac{1}{2}e^{-2x'}\right]_{0}^{x}$$

$$F_{XY}(x,y) = (1-e^{-y})(1-e^{-2x}).$$

$$F_{XY}(x,y) = \begin{cases} (1-e^{-y})(1-e^{-2x}), & x,y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

4) Find the marginal cdfs :

$$F_X(x) = F_{XY}(x,\infty) = \begin{cases} 1-e^{-2x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$F_Y(y) = F_{XY}(\infty,y) = \begin{cases} 1-e^{-y}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

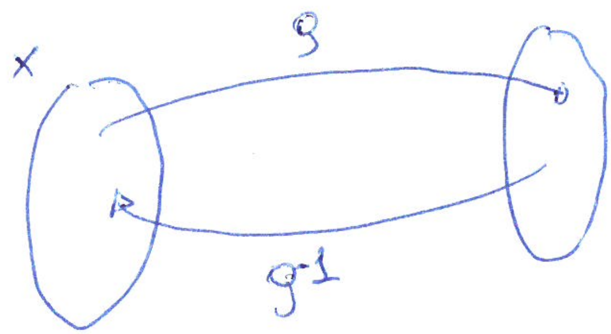
Transformation of Random Variables :

⇒ We may know the pdf (or the cdf) of a random variable X, and to need find the corresponding functions of a random variable, which is given as a function of Y:

$$Y = g(X).$$

→ If g is a monotonic function $\begin{cases} \text{non-increasing} \\ \text{non-decreasing} \end{cases}$

then $$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|_{x = g^{-1}(y)}.$$



Example:

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise.} \end{cases}$$

$Y = -\frac{1}{\pi}\Theta + 1$ (1)   pdf of Y?

Solution:

$$f_Y(y) = f_\Theta(\theta) \left| \frac{d\theta}{dy} \right|.$$

(1) $\Rightarrow$ $\boxed{\Theta = -\pi Y + \pi}$ $\Rightarrow$ $\frac{d\theta}{dy} = -\pi$

$$f_Y(y) = \begin{cases} 1/2, & -1 \le y \le 1. \\ 0, & \text{otherwise} \end{cases}$$

$0 \le \theta \le 2\pi \iff 0 \le -\pi Y + \pi \le 2\pi \iff$

$\cancel{1 \le -\pi Y \le 2\pi \iff -2 \le Y \le -1/\pi}$

$-1 \le y \le 1$

If g is non monotonic :

$$f_Y(y) = \sum_{i=1}^{N} f_X(x_i) \left| \frac{dx_i}{dy} \right|$$

where N is the number of disjoint intervals where the g is monotonic.

Example : $Y = X^2$, $f_X(x) = 0.5\, e^{-|x|}$

$$X^2 = Y \iff X = \begin{cases} x_1 = +\sqrt{y} \\ x_2 = -\sqrt{y} \end{cases} , \quad N = 2.$$

$$\frac{dx_1}{dy} = \frac{1}{2\sqrt{y}} , \quad \frac{dx_2}{dy} = -\frac{1}{2\sqrt{y}}$$

$$f_Y(y) = \sum_{i=1}^{2} f_X(x_i) \left| \frac{dx_i}{dy} \right|$$

$$= \frac{1}{2} \frac{e^{-\sqrt{y}}}{\sqrt{y}} , \quad y > 0$$

Statistical Averages:

Most of times, we are not able to find the cdfs or pdfs. In that case, a partial description of a random variable is given in terms of various statistical averages.

Statistical Average:

We have a random variable X, that take the values $x_1, x_2, \dots, x_M$, with probabilities $P_1, P_2, \dots, P_M$.

$$\bar{X} = E[X] = \sum_{i=1}^{M} x_i P_i \quad \text{(Discrete R.V)}$$

$$\bar{X} = E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx.$$