Sampling Theory

09/16/2019 1

Instantaneous Sampled Waveform

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(nT_s)\cdot\delta(t-nT_s)$$

T_s → sampling interval

Uniform Sampling Theorem for lowpass signals

$x(t)$: does not contain any component for frequency above $f = W$.

I can describe COMPLETELY its form by instantaneous samples with $T_s < \frac{1}{2W}$

$2W$ → Nyquist frequency

Input: $X_\delta(f) = f_s \sum_{n=-\infty}^{+\infty} X(f-nf_s)$

Ideal Lowpass Filter

$$H(f) = H_0 \Pi\left(\frac{f}{2B}\right)\cdot e^{-j2\pi f t_0}$$

B → $W \leq B \leq f_s - W$ (f_s: sampling frequency)

Output $\| \quad \underbrace{Y(f) = f_s H_0 \cdot X(f) \cdot e^{-j2\pi f t_0}}$

$$\updownarrow$$

$$y(t) = f_s \cdot H_0 \cdot x(t - t_0)$$

Bandbass Sampling Theorem

$x(t) \to$ spectrum bandwidth W & upper frequency limit f_u }

$\Rightarrow$ sampling rate f_s is $\dfrac{2f_u}{m}$

m: largest integer that does not exceed $\dfrac{f_u}{W}$

** if sample shifted from 5-10 Hz to 95-100HZ, need more samples because higher freq. = higher data

Hilbert Transform

$$\underset{X(f)}{x(t)} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \hat{x}(t)$$

$$H(f) = -j \cdot \text{sgn}(f)$$

$$\mathrm{sgn}(f) = \begin{cases} 1 & , f > 0 \\ 0 & , f = 0 \\ -1 & , f < 0 \end{cases}$$

$$\hat{x}(t) = x(t) * h(t)$$

$$= \mathcal{F}^{-1}\Big[\underbrace{-j \; X(f) \; \mathrm{sgn}(f)}_{H(f)}\Big]$$

We can prove: $\dfrac{j}{\pi \cdot t} \longleftrightarrow \mathrm{sgn}(f)$

Hilbert Transform $\Big\|$ $\hat{x}(t) = \displaystyle\int_{-\infty}^{+\infty} \frac{x(\lambda)}{\pi \cdot (\cancel{t}\,\tau - \lambda)} \, d\lambda$

$$\boxed{\hat{\hat{x}}(t) = -x(t)}$$

Explanation

$$X(f) \to \boxed{H(f)} \xrightarrow{Y_1(f)} \boxed{H(f)} \to$$

$Y_1(f) = -j\,\mathrm{sgn}(f) \cdot X(f)$

$$\hat{\hat{X}}(f) = Y_1(f) \cdot H(f)$$
$$= -j\,\mathrm{sgn}(f) \cdot X(f) \cdot (-j\,\mathrm{sgn}(f))$$
$$= -[\mathrm{sgn}(f)]^2 X(f) = -X(f)$$

$$\boxed{\hat{X}(f) = -j\,\mathrm{sgn}(f)\cdot X(f)}$$

Example:

$$x(t) = \cos(2\pi f_0 t) \longleftrightarrow$$

$$X(f) = \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$

Hilbert transform

$$\hat{X}(f) = \frac{1}{2}\delta(f-f_0)\cdot e^{-j\frac{\pi}{2}} + \frac{1}{2}\delta(f+f_0)\cdot e^{j\frac{\pi}{2}}$$

$$\updownarrow$$

$$\hat{x}(t) = \frac{1}{2}e^{j2\pi f_0 t}\cdot e^{-j\frac{\pi}{2}} + \frac{1}{2}e^{-j2\pi f_0 t}\cdot e^{j\frac{\pi}{2}}$$

$$= \frac{1}{2}e^{j(2\pi f_0 t - \frac{\pi}{2})} + \frac{1}{2}e^{-j(2\pi f_0 t - \frac{\pi}{2})}$$

$$= \cos\left(2\pi f_0 t - \frac{\pi}{2}\right) = \sin(2\pi f_0 t)$$

$$\boxed{\widehat{\cos(2\pi f_0 t)} = \sin(2\pi f_0 t)}$$

$$\boxed{\widehat{\sin(2\pi f_0 t)} = -\cos(2\pi f_0 t)}$$

$$\boxed{\widehat{e^{j2\pi f_0 t}} = -j\,\mathrm{sgn}(f_0)\cdot e^{j2\pi f_0 t}}$$

Properties of Hilbert Transform:

1) The energy (power) in a signal $x(t)$ and $\hat{x}(t)$ are equal.

$$|\hat{X}(f)|^2 = |-j\,\mathrm{sgn}(f)\cdot X(f)|^2 = |-j\,\mathrm{sgn}(f)|^2\cdot|X(f)|^2$$
$$= 1\cdot|X(f)|^2 = |X(f)|^2$$

2) $x(t), \hat{x}(t)$: orthogonal signals

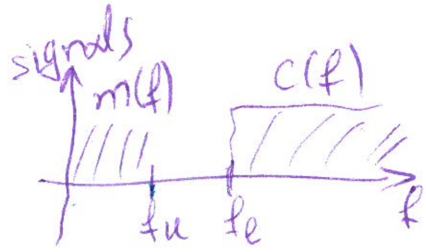
$$\int_{-\infty}^{+\infty} x(t)\,\hat{x}(t)\,dt = 0 \quad \text{(energy signals)}$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\cdot\hat{x}(t)\,dt = 0 \quad \text{(power signals)}$$

3) $c(t)$, $m(t)$: non-overlapping spectra
with

$c(t)$ ↓ highpass signal

$m(t)$ ↓ lowpass signal



$$\widehat{c(t)\cdot m(t)} = m(t)\cdot \hat{c}(t)$$

Example: $m(t)$: lowpass signal

$$\widehat{m(t)\cdot \cos(2\pi f_0 t)} = m(t)\cdot \sin(2\pi f_0 t)$$

$$\widehat{m(t)\cdot \sin(2\pi f_0 t)} = -m(t)\cdot \cos(2\pi f_0 t)$$

Analytic Signals

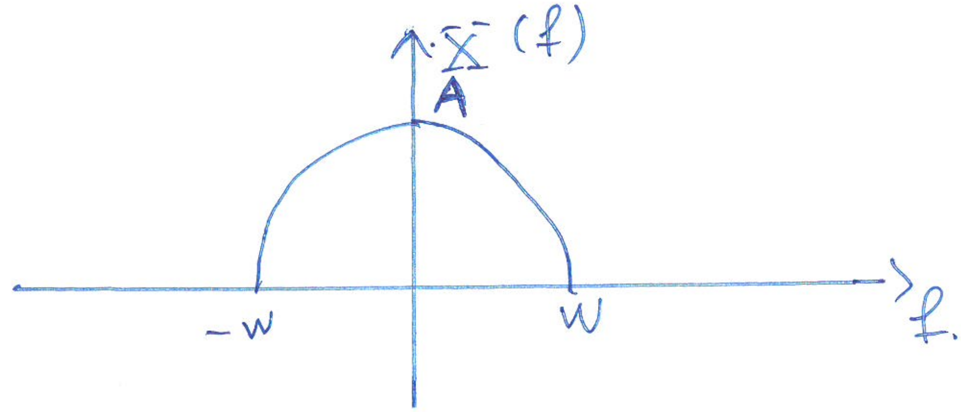
$x(t)$: real signal. Positive frequency

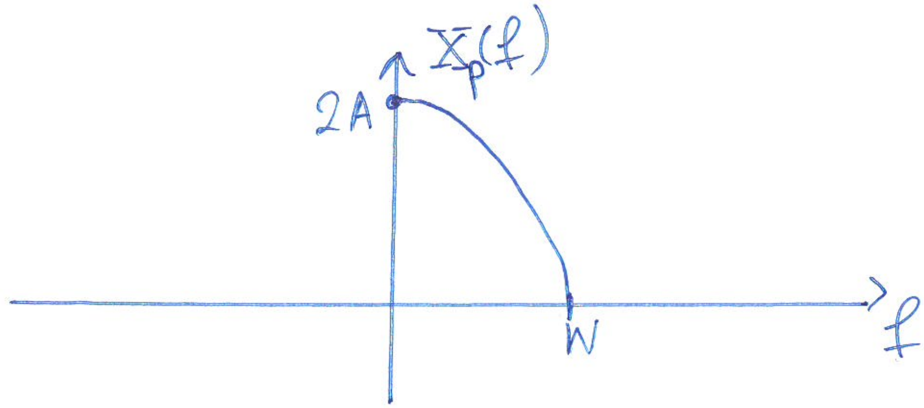
$$\boxed{X_p(t) = x(t) + j\hat{x}(t)}$$

$$X_p(f) = X(f) + j\left[-j\,\mathrm{sgn}(f)\cdot X(f)\right]$$

$$= X(f) + \mathrm{sgn}(f)\cdot X(f)$$

$$= \begin{cases} 2X(f), & f>0 \\ 0, & f<0 \end{cases}$$





Negative frequency

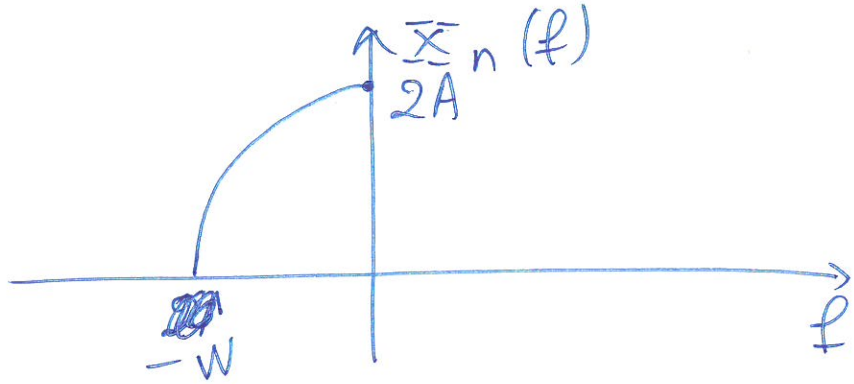
$$x_n(t) = x(t) - j\hat{x}(t)$$

$$\updownarrow$$

$$X_n(f) = X(f) - j\left[-j\,\mathrm{sgn}(f)\cdot X(f)\right]$$

$$= X(f) - \mathrm{sgn}(f)\,X(f)$$

$$= \begin{cases} 0, & f>0 \\ 2X(f), & f<0 \end{cases}$$



$\underline{X}_n(f)$
2A
-W
f