Hilbert spaces

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Hilbert spaces

- A vector space X over the reals $\mathbb{R}$ is an inner product space if there exists a real-valued symmetric bilinear map $< \cdot, \cdot >$ that satisfies

$$\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$$

  This map is an inner or dot product. It is strict if the equality is true only for $\mathbf{x} = \mathbf{0}$.

- If the dot product is strict, then we can define a norm $||\mathbf{x}||_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ and a metric $||\mathbf{x} - \mathbf{z}||_2$.
- A Hilbert space $\mathcal{F}$ is a strict inner product space that is *separable* and *complete*.

- *Completeness* means that every Cauchy sequence h_n converges to an element $h \in \mathcal{F}$. A Cauchy sequence is one for which

$$\lim_{n\rightarrow\infty} \sup_{m>n} ||h_n - h_m|| \rightarrow 0$$

- *Separability* means that a countable set of elements $h_1 \cdots h_n$ in $\mathcal{F}$ exists such that for all $h \in \mathcal{F}$ and for $\epsilon > 0$

$$||h_i - h|| < \epsilon$$

- ℓ^2 space. This is the space of all countable sequences of real numbers $\mathbf{x} = x_1, x_2, \cdots, x_n, \cdots$ such that $\sum_{i=1}^{\infty} x_i^2 < \infty$. Its dot product is

$$\langle \mathbf{x}, \mathbf{z} \rangle = \sum_{i=0}^{\infty} x_i z_i$$

Examples of dot products

- Let $X = \mathcal{R}^n$. A suitable dot product is

$$\langle \mathbf{x}, \mathbf{z} \rangle = \sum_{i=1}^{n} \lambda_i x_i z_i = \mathbf{x}^\top \mathbf{\Lambda} \mathbf{z}$$

  Note that this is equivalent to the "standard" dot product over the transformed vectors $\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{x}$ and $\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{z}$.

- The scalars λ_i must be positive. Otherwise, a vector can be found for which the corresponding metric is negative. If one or more values of λ_i are zero, then the dot product is not strict.

Examples of dot products

- Let $\mathcal{F} = L_2(X)$ be the space of square integrable functions on a compact subset of X. For $f, g \in \mathcal{F}$, the dot product is

$$\langle f, g \rangle = \int_X f(x)g(x)dx$$

- When functions are defined in $\mathbb{C}$, an equavalent expression exists where one of the functions is conjugated.

The Cauchy-Schwartz inequality

- In an inner product space

$$\langle \mathbf{x}, \mathbf{z} \rangle^2 \leq ||\mathbf{x}||^2 ||\mathbf{z}||^2$$

The angle between two vectors in an inner product space is defined as

$$\cos\theta = \frac{\langle \mathbf{x}, \mathbf{z} \rangle}{||\mathbf{x}|| ||\mathbf{z}||}$$

Obviously, if the cosine is 1, then both vectors are parallel. If the cosine is 0, then both vectors are orthogonal.

Positive semi-definite matrices

- A symmetric matrix $\mathbf{A}$ is positive semi-definite if its eigenvalues are all non-negative.
- Indeed, the Courant-Fisher theorem says that the minimal eigenvalue of a symmetric matrix is

$$\lambda_m(\mathbf{A}) = \min_{\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^{\mathbf{n}}} = \frac{\mathbf{v}^\top \mathbf{A} \mathbf{v}}{\mathbf{v}^\top \mathbf{v}}$$

- Then, a symmetric matrix $\mathbf{A}$ is positive semi-definite if and only if

$$\mathbf{v}^\top \mathbf{A} \mathbf{v} \geq 0$$

Positive semi-definite matrices

- Gram and Kernel matrices are positive semi-definite matrices.
- First, we formally define a kernel as a function $k(\cdot, \cdot)$ such that

$$k(\mathbf{x}, \mathbf{z}) = \langle \boldsymbol{\varphi}(\mathbf{x}), \boldsymbol{\varphi}(\mathbf{z}) \rangle$$

where $\boldsymbol{\varphi}(\cdot)$ is a mapping from a space X to a Hilbert space $\mathcal{F}$

$$\boldsymbol{\varphi} : \mathbf{x} \rightarrow \boldsymbol{\varphi}(\mathbf{x}) \in \mathcal{F}$$

- Them we define the kernel matrix as

$$K_{ij} = k(\mathbf{x}_i, \mathbf{x}_j)$$

- With that, we can prove tha the product $\mathbf{v}^\top \mathbf{K} \mathbf{v}$ is nonnegative $\forall \mathbf{v}$.

$$
\begin{aligned}
\mathbf{v}^\top \mathbf{K} \mathbf{v} &= \sum_i \sum_j v_i v_j k(\mathbf{x}_i, \mathbf{x}_j) \\
&= \sum_i \sum_j v_i v_j \langle \boldsymbol{\varphi}(\mathbf{x}_i), \boldsymbol{\varphi}(\mathbf{x}_j) \rangle \\
&= \left\langle \sum_i v_i \boldsymbol{\varphi}(\mathbf{x}_i), \sum_i v_i \boldsymbol{\varphi}(\mathbf{x}_i) \right\rangle \\
&= \left\| \sum_i v_i \boldsymbol{\varphi}(\mathbf{x}_i) \right\|^2
\end{aligned}
$$

This expression is nonnegative for any sequence v_i.

Kernel matrices are positive semi-definite

- A matrix $\mathbf{A}$ is positive semi-definite if and only if $\mathbf{A} = \mathbf{B}^\top \mathbf{B}$.
- The proof is straightforward if we assume that $\mathbf{A} = \mathbf{B}^\top \mathbf{B}$ and we compute $\mathbf{v}^\top \mathbf{A} \mathbf{v}$:

$$\mathbf{v}^\top \mathbf{A} \mathbf{v} = \mathbf{v}^\top \mathbf{B}^\top \mathbf{B} \mathbf{v} = \|\mathbf{B}\mathbf{v}\|^2$$

  Here we identify in $\mathbf{B}^\top \mathbf{B}$ a matrix of dot product in a space where coordinates are given. Since there is a isometric isomorphism between $\mathcal{F}$ and ℓ^2 or $\mathbb{R}^n$, this and the previous result become equivalent.

Outcomes of this lesson

In this lesson we have seen the definition of a Hilbert space and some important aspects of Hilbert spaces:

- Completeness, separability and ℓ^2 space.
- Examples of dot products.
- Positive semi definite property of kernel matrices.