Fall 2021 I609 Assignment 3

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Question

Background: Consider a Susceptible-Infected-Removed (SIR) system described by the following equations:

$$\frac{ds}{dt} = -\beta si \tag{1}$$

$$\frac{di}{dt} = \beta si - \gamma i \tag{2}$$

$$\frac{dr}{dt} = \gamma i \tag{3}$$

and the initial conditions $s(0) = 0.87, i(0) = 0.13$, and $r(0) = 0.00$. Assume that $\beta = 3$ and that $\gamma = 1.6$ (unless otherwise stated). Use the Euler Method with $dt = 0.001$ to integrate the equation up to time $T = 20$. Answer the following questions:

1. Plot $s(t), i(t), r(t)$ vs t in the same plot.
2. What is the fraction of infected at time $t = 5.2$ (i.e. what is $i(5.2)$)?
3. What fraction of the population will have had infected by the time the epidemics dies off? Note: $T = 20$ is sufficiently large to consider the epidemics finished.
4. The pandemics is considered over when the fraction of infected fall under 0.03. When does this happen?
5. At what time the number of infected reaches its maximum?
6. What is the fraction of susceptible when the maximum of infected is reached?
7. Can you express the fraction at point 6 in terms of parameters β and γ? Please offer a short explanation.

Consider the case $\beta = 1.5$ and $\gamma = 1.6$.

8. Plot $s(t), i(t), r(t)$ vs t in the same plot.
9. What fraction of the population will have had infected by the time the epidemics dies off? Note: $T = 20$ is sufficiently large to consider the epidemics finished.
10. Say in few lines what are the qualitative differences with the case of points 1-7.

Bonus question (2 points): Consider again the case $\beta = 3$, $\gamma = 1.6$. You have the possibility to enact a lockdown that will halve the infectivity rate β or, alternatively, to produce a medicine (to be given to people as soon as they become infected) that doubles the recovery rate γ. Which intervention would you choose if the goal is reducing the working hours lost to the epidemics? To compute the work time lost to the epidemics consider that a person that is infected in a given interval of time cannot work in that period. Justify your answer with some evidence.

Solutions

Please note that all supporting `R`-code is available at my Github repository here.
The first round of questions set $\beta = 3$ and $\gamma = 1.6$.

1. See Figure 1 below:

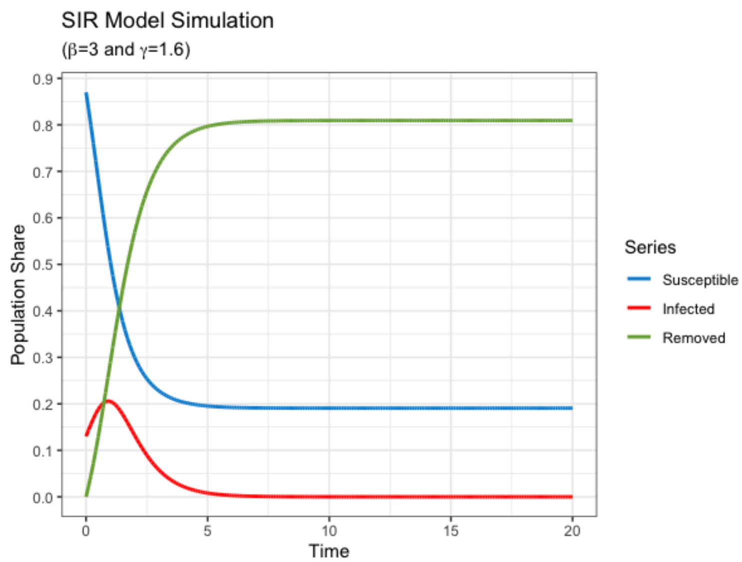


Figure 1: $s(t), i(t)$, and $r(t)$ time series with $\beta = 3$ and $\gamma = 1.6$.

2. The fraction of infected at time $t = 5.2$ is $i(5.2) = 0.006592962$.
3. The fraction of the population infected by the time the epidemic dies off is $r(20) = 0.8092948$.
4. The epidemic is over when less than 3% of the population is currently infected, e.g. $i(t_{\text{over}}) = 0.03$. Given the current model parameterization, $t_{\text{over}} = 3.686766$.
5. Define t_{imax} as the time that infections peak. In context, $t_{imax} = 0.9140407$ and $i(t_{imax}) = 0.2057529$. See Figure 2.

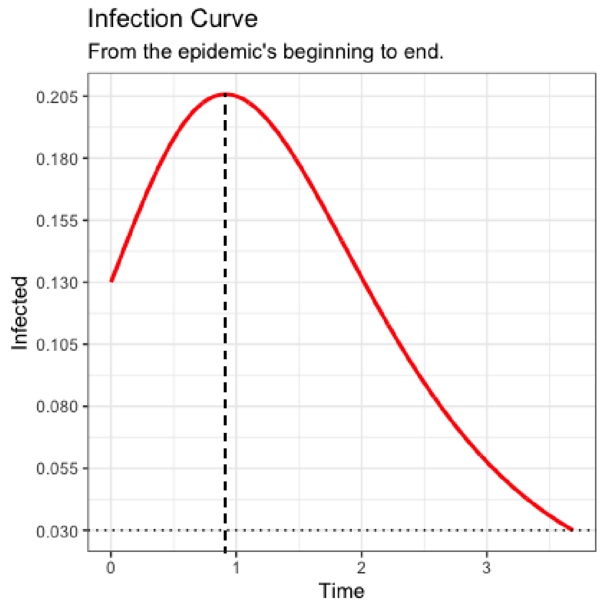


Figure 2: Infection curve w/ infections peaking at $t_{imax} \approx 0.91$

6. The susceptible population share at peak infection time is, $s(t_{imax}) = 0.5332579$.
7. One of the problem givens is that:

$$\frac{di}{dt} = \beta si - \gamma i$$

Infections peak when $\frac{di}{dt} = 0 \iff \beta si = \gamma i$. Assuming that $i \neq 0$, then $\beta s = \gamma \implies s = \frac{\gamma}{\beta}$. Assuming that $\beta = 3$ and $\gamma = 1.6$, then $s(t_{imax}) = \frac{\gamma}{\beta} = \frac{1.6}{3} = \frac{8}{5} \times \frac{1}{3} = \frac{8}{15} \approx 0.5\bar{3}3$. This answer means that our estimate of $s(t_{imax})$ in the preceding step is accurate up to the ten-thousands place.

Now assume that $\beta = 1.5$ and $\gamma = 1.6$.

8. The new plot of $s(t), i(t)$ and $r(t)$ is presented by Figure 3.

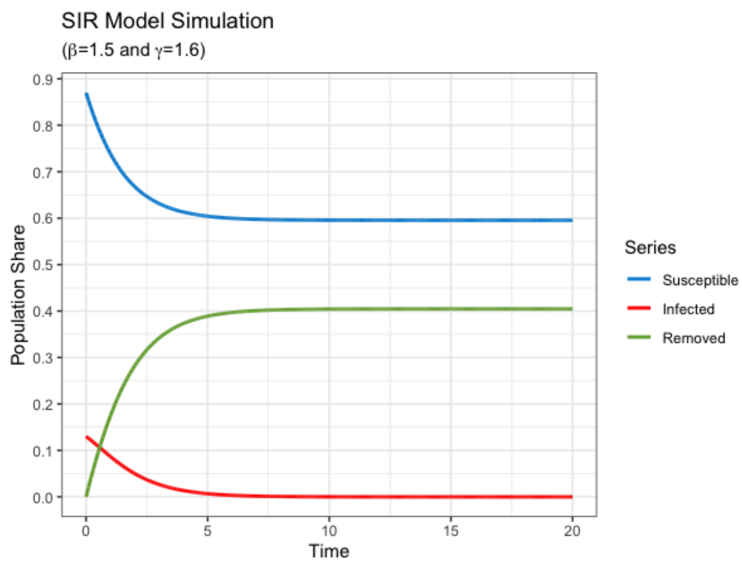


Figure 3: $s(t), i(t)$, and $r(t)$ time series with $\beta = 1.5$ and $\gamma = 1.6$.

9. The fraction of the population infected by the time the epidemic dies off is $r(20) = 0.4046989$.

10. Qualitatively, halving β from 3 to 1.5 fundamentally changes the trajectory/behavior of the epidemic. Recall that $\mathcal{R}_0$ is the expected number of secondary infections produced by a single (typical) infection. If we define $\mathcal{R}_0$ as

$$\mathcal{R}_0 = \frac{\beta}{\gamma}$$

then we see that $\{\beta_0, \gamma_0\} = \{3, 1.6\} \implies \mathcal{R}_0(\beta_0, \gamma_0) > 1$. Likewise, we also know that $\{\beta_1, \gamma_1\} = \{1.5, 1.6\} \implies \mathcal{R}_0(\beta_1, \gamma_1) < 1$.

$\mathcal{R}_0$ is meaningful with respect to how the *rate of infections evolves*. At $t = 0$, assume that $s \approx 1$ and that $i > 0$. Then,

$$\frac{di}{dt} = (\beta s - \gamma)i \approx (\beta - \gamma)i \implies \frac{di}{dt} \gtrless 0 \iff \mathcal{R}_0 \gtrless 1$$

When $\mathcal{R}_0 < 1$, this means that the number of infections will converge towards zero right from the outset of the epidemic as represented by Figure 3. Likewise, $\mathcal{R}_0 > 1$ leads to the exponential growth of infections as seen in Figure 2. Practically speaking, lowering $\mathcal{R}_0$ from 1.875 (e.g. Questions 1-7) to 0.9375 (e.g. Questions 8-9) roughly halves the number of total infections from 0.809 to 0.404.

Bonus Question

Answer up front

If the grader is uninterested in reading through all of this section's details, then the medicine-based approach is the best policy intervention because it effectively halves the lifespan of the epidemic (relative to the lockdown-based approach). Even though the same number of people are infected by the end of the epidemic, the average duration of illness (e.g. $1/\gamma$) is cut in half); hence, more working hours are saved by improving the recovery rate.

Reducing β from $3 \rightarrow 1.5$ (Option 1/Lockdown) and increasing γ from $1.6 \rightarrow 3.2$ (Option 2/Medicine) has the same effect on the *total number of people infected*. To see this, note that $\mathcal{R}_0$ doesn't change:

$$\mathcal{R}_0 = \underbrace{\frac{\frac{1}{2}\beta}{\gamma}}_{=\text{Option 1}} = \underbrace{\frac{\beta}{2\gamma}}_{=\text{Option 2}}$$

What's more, knowing what $\mathcal{R}_0$ allows us to characterize the total number of people infected by the end epidemic. First, integrate equation (3) with respect to time:

$$\int_0^\infty \frac{dr}{dt}dt = \int_0^\infty \gamma i dt \iff r(\infty) - \underbrace{r(0)}_{=0 \text{ by construction}} = \int_0^\infty \gamma i dt \implies r(\infty) = \gamma \int_0^\infty i dt$$

Next, re-arrange equation (1):

$$\frac{ds}{dt} = -\beta s i \iff i = -\frac{1}{\beta s}\frac{ds}{dt}$$

Back substitute into the previous equation:

$$r(\infty) = -\frac{\gamma}{\beta}\int_0^\infty \frac{1}{s}\frac{ds}{dt}dt \iff -\underbrace{\frac{\beta}{\gamma}}_{=\mathcal{R}_0} r_\infty = \ln\left(s(\infty)\right) - \ln\left(\underbrace{s(0)}_{=1 \text{ by construction}}\right) \implies$$

$$-\mathcal{R}_0 r(\infty) = \ln\left(s(\infty)\right)$$

Since $i(\infty) = 0$ and $i(t) + r(t) + s(t) = 1 \forall t$, we can re-express the last result as

$$\ln\Big(1 - r(\infty)\Big) = -\mathcal{R}_0 r(\infty) \tag{4}$$

Equation (4) shows that the total number of people that are infected by the end of the epidemic is (implicitly) a function of $\mathcal{R}_0$. Since, $\mathcal{R}_0$ doesn't change across either policy options/interventions, we can conclude that the total number of people infected won't change.

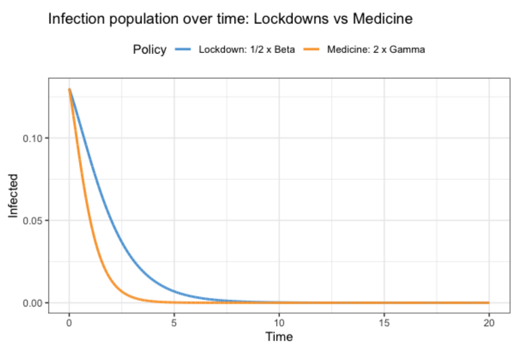
However, the policy objective in effect is to *minimize the total sick time* in our population. If each individual is endowed with h working hours and infection precludes work, then policy makers want to minimize:

$$h\int_{t=0}^\infty i(t)dt = h\int_{t=0}^\infty \Big(\int_{t=0}^\infty (\beta s(i)i(t) - \gamma i(t))dt\Big)dt \tag{5}$$

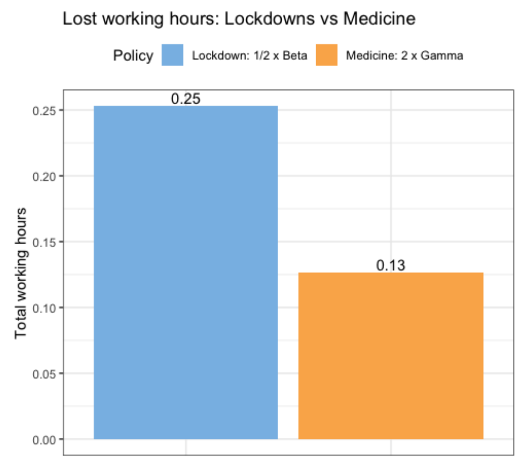
After normalizing $h = 1$, I evaluate equation (5) by: first simulating both SIR models; and, second evaluating the "outer" integral using the trapezoid rule of integration (with $n = 1,000$ partitions). Thus, equation (5) is approximated by:

$$\int_{t=0}^{20} i(t)dt \approx \frac{20}{n}\left(\frac{i(0)}{2} + \sum_{k=1}^{n-1}\Big[i(20k/n)\Big] + \frac{i(20)}{2}\right)$$

Figure 4 shows that the "medicine" based intervention *is the most effective at reducing lost working hours*. Figure 4a shows how the epidemic is effectively over (e.g. $i \approx 3\%$) for the medicine-based intervention at time $t = 1.41$ compared to the lockdown-based intervention's time of $t = 2.82$. The fact that the epidemic lasts nearly twice as long for the lockdown-based intervention translates into nearly double the amount of lost working hours as shown in Figure 4b.



(a) Infections over time



(b) Lost hours

Figure 4: Infections and lost working hours from two policy interventions

The medicine-based intervention is more effective at hastening the end of the pandemic (at a mechanical level) because targeting γ impacts di/dt more than targeting β. Note that:

$$\left|\frac{\partial}{\partial\gamma}\left(\frac{di}{dt}\right)\right| = i \text{ and } \left|\frac{\partial}{\partial\beta}\left(\frac{di}{dt}\right)\right| = si \implies \frac{\left|\frac{\partial}{\partial\gamma}\left(\frac{di}{dt}\right)\right|}{\left|\frac{\partial}{\partial\beta}\left(\frac{di}{dt}\right)\right|} = \frac{i}{si} = \frac{1}{s} > 1 \tag{6}$$

since $s \in (0,1)$. The expressions in (6) signify that that if γ and β are proportionally scaled, then modifying γ yields a bigger marginal impact on the rate of infection growth. The full impact of decreasing β is mitigated by the s; the "s" offset is not present for changing γ.