Spring 2022 I609 Final Project
Vicesk Model: A Replication Effort

Github Repository

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1 Introduction

The goal of this final project is to replicate/simulate the Vicesk model of collective motion/system of self-driven particles (or birds) developed by Vicsek et al. (1995). In particular, I will reproduce both panels of Figure 2 (Vicsek et al., 1995, p.1128).

1.1 Vicesk Model Setup

Consider a collection of $N \in \mathbb{N}$ birds placed within an $L \times L \subset \mathbb{R}^2$ grid with periodic boundaries (also note that $L \in R_+$). Each bird $i = 1, \ldots, N$ has an initial placement $\big(x_{1,i}(0), x_{2,i}(0)\big)$ in the grid such that $x_{1,i}(0), x_{2,i}(0) \sim \mathcal{U}(0, L)$. The density of the system is defined as $\rho = N/L^2$ The model is run for a total of T periods. Denote the position vector of all bird i at time t as $\mathbf{x}_i(t) = \langle x_{1,i}(t), x_{2,i}(t) \rangle$. Given a marginal change in time, namely Δt, the position of bird i is:

$$\mathbf{x}_i(t + \Delta t) = \mathbf{x}_i(t) + \Delta t \mathbf{v}_i(t), \text{ given } \mathbf{v}_i(t) = \begin{bmatrix} v_0 \cos \theta_i(t) \\ v_0 \sin \theta_i(t) \end{bmatrix} \tag{1}$$

where $\mathbf{v}_i(t)$ is bird i's velocity, v_0 is speed, and $\theta_i(t)$ is defined by equation (2.2.1):

$$\theta_i(t + \Delta t) = \frac{1}{N} \sum_{|\mathbf{x}_i - \mathbf{x}_k| < r} \theta_k(t) + \mathcal{U}\Big(-\frac{\eta}{2}, \frac{\eta}{2} \Big) \tag{2}$$

Equation (2.2.1) says that bird i's orientation for the time step is a function of the average angle of all birds within radius r and a uniformly distributed noise term.
The order parameter of this model is the absolute value of the average normalized velocity, namely v_a :

$$v_a = \frac{1}{N v_0} \Big| \sum_{i=1}^{N} \mathbf{v}_i \Big| \tag{3}$$

Changes in v_a reflect a phase transition. When $v_a \simeq 0$, the motion of individual particles is randomly distributed whereas $v_a > 0$ represents ordered motion such that $v_a \to 1$, all particles/birds will have the same orientation.

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1.2 Periodic Boundary Notes

One of the key challenges when simulating the Viseck model is to determine whether birds i and j are within radius r of each other. Consider Figure 1. Birds i and j have (an Euclidean) distance of $d_{ij} = \sqrt{(x_{1,j} - x_{1,i})^2 + (x_{2,j} - x_{2,i})^2}$. In this example, $r < d_{ij}$, so it might be tempting for bird i to ignore j when i is computing its neighbors' average angle. However, the Vicesk model relies on periodic boundaries. Figure 2 copies and pastes the $L \times L$ grid around the original grid. Figure 2 shows that i's periodic distance with j is:

$$\delta_{ij} = \sqrt{\Big((x_{1,j} + L) - x_{1,i}\Big)^2 + \Big((x_{2,j} + L) - x_{2,i}\Big)^2} < r < d_{ij}$$

Thus, bird i *should not* ignore j when i computes its neighbors' average angle. In practice, I compute all pairwise distances (per period) between bird i and bird $j = 1, \ldots N$ translated across the $3L \times 3L$ grid (similar to Figure 2) and save the smallest of the nine (9) distances.

Moreover, if bird i moves outside the original grid, then I map i back into the original grid by having it approach from the cell opposite to the cell i would have entered after leaving the original grid. For example— and appealing to Figure 2— if bird i left the original grid (cell V) and flew into cell I, then in the next period i would fly in from cell IX to the bottom right hand corner of cell V.

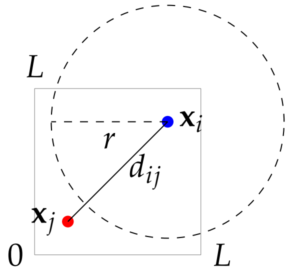


Figure 1: Simple Grid

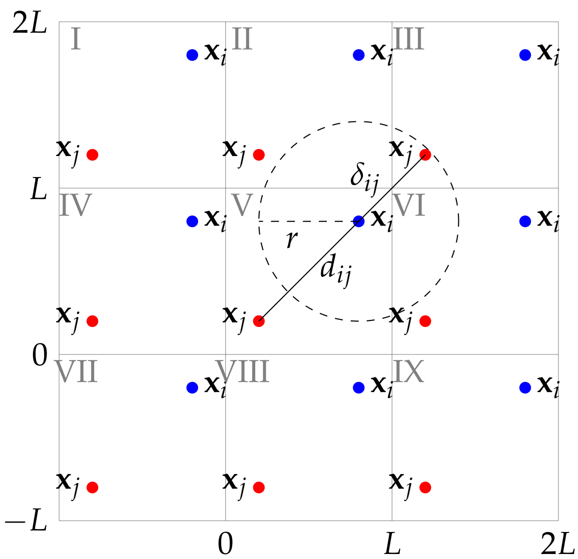


Figure 2: Full Grid

2 Simulation Exercise

Vicsek et al. (1995)'s Figure 2 features two (2) panels. Fig. 2(a) shows the relationship between η and v_a whereas Fig. 2(b) shows the relationship between ρ and v_a. Both Fig. 2(a) and Fig. 2(b) are created using $r = 1, v_0 = 0.03$. Fig. 2(a) uses different number of birds, namely $N \in \{40, 100, 400, 4,000, 10,000\}$ but holds density constant (e.g. $L = \sqrt{N/4}$). Fig. 2(b) holds noise η constant,[1] but increases ρ by increasing the number of birds (e.g. $L = 20$ for all trials).

2.1 Practical Judgement Calls

I had to make a number of operational decisions to replicate Figure 2. The first choice was to run the model for a 100 iterations with a step size of $\Delta t = 1/1000$. Other decisions are a touch more subjective. These decisions include: measuring/defining v_a; choosing which parameterization to run; and, setting the number of trials per parameterization. Per parameterization, I run the simulation fifty (50) times. I then define v_a by computing (3) using the last twenty (20) observations across all simulation trials; this choice assumes that the model has run long enough for v_a to be "time-invariant." In terms of parameter selection, I replicated Fig. (1a) by choosing 25 linearly spaced values for η between $[0, 5]$ and by choosing $N = \{40, 100, 400\}$.[2] I replicated Fig. (2b) by first setting $\eta = 1/100$ and $N = 40$, and then choosing forty (40) exponentially gridded values for L between $[2, 20]$ so that ρ will vary between $[0.1, 10]$.[3]

2.2 Result Preview

2.2.1 Fig. 2(a)

My replication effort of Figure 3a matches the Vicsek et al. (1995)'s Fig. 2(a). As the noise parameter, η grows, the variance of the random noise term also increases (e.g. variance of noise term is $\frac{1}{3}\eta^2$). The noisier the system, the harder it becomes for the particles/birds to move uniformly. Thus $\partial v_a/\partial\eta < 0$. Figure 3a does show some non-monotonicity, especially for $N = 400$. My guess is that for larger values of N and lower values of η that a larger number of trials are need for sufficient v_a stability.

2.2.2 Fig. 2(b)

As a general matter, one should expect that $\partial v_a/\partial\rho > 0$: as the density of particles/birds increases, uniform motion should be easier/quicker to achieve. Roughly speaking, this relationship between ρ and v_a is presented by my replication efforts in Figure 4a and by the original Figure 4b. Interestingly enough, my replication efforts suggest that for $0 < \rho \leq 0.46$, that $\partial v_a/\partial\rho < 0$ but that once $\rho > 0.46$ then $\partial v_a/\partial\rho > 0$. As with the previous section, I conjecture that some combination of: (i) numerical instability/small number of simulation runs; (ii) not knowing all the parameters used to generate the original figure (e.g. what is η?); and, (iii) varying L as opposed to N to affect density all contributed to the unusual behavior of $v_a(\rho; \rho < 0.46)$.

[1] Paper does not specify which η is used. Nonetheless, I think that the qualitative results of Fig. 2(b) will still be straightforward to replicate.

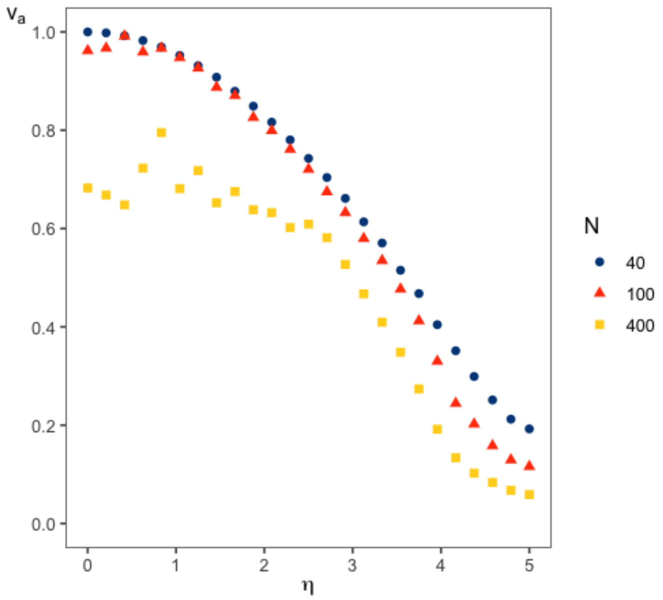
[2] My code—and for that my matter, my computer—is not optimized to run this model for sufficiently large models. For this reason, I ignore the paper's run of $N = \{4,000, 10,000\}$.

[3] The original paper held L constant at $L = 20$ and then varied N from 40 to 4,000. Computing pairwise distance is $\mathcal{O}(N^2)$ whereas varying the grid density by changing the grid size shouldn't affect computational complexity.

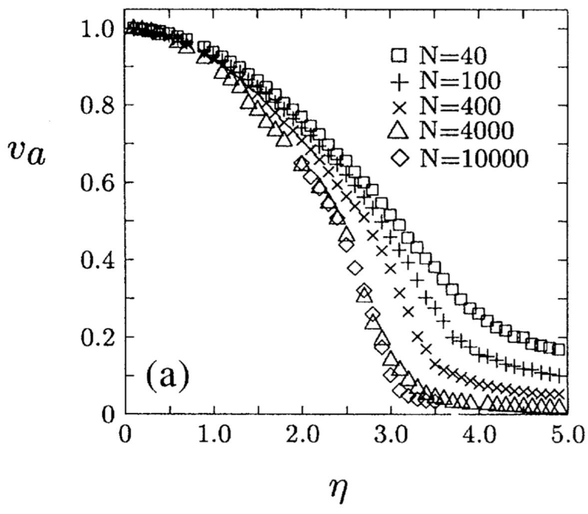
2.2.3 Birds in motion

I created a GIF of birds flying in Figure 5. Enjoy!

2.3 Results

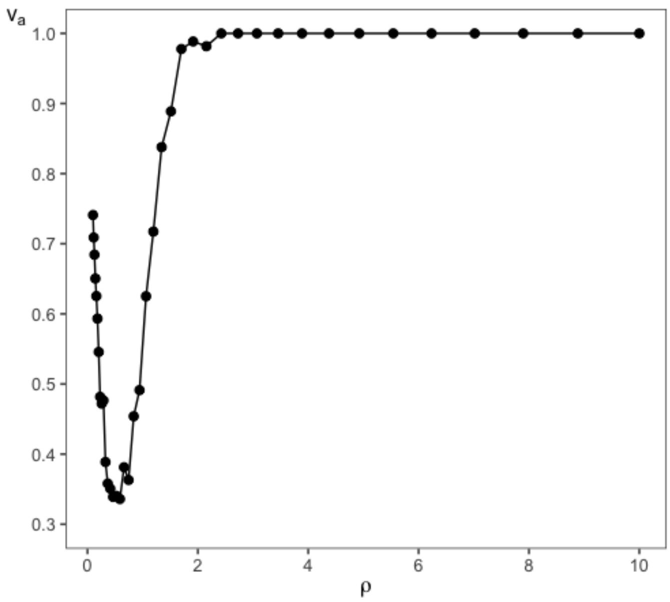


(a) Replication

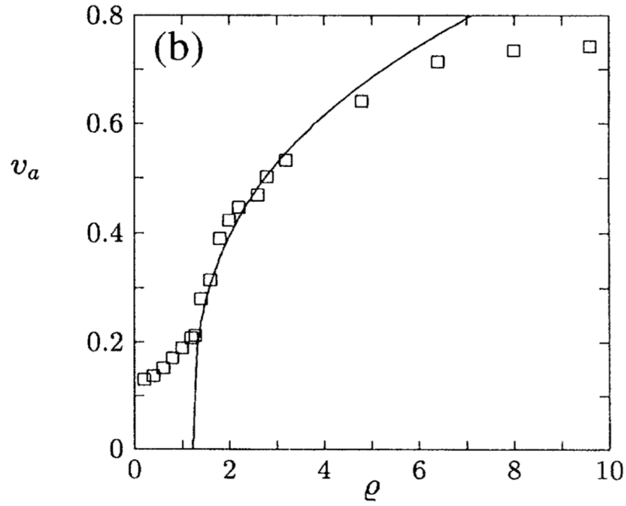


(b) Original

Figure 3: Noise η vs order parameter v_a (Paper Fig. 2(a))



(a) Replication



(b) Original

Figure 4: Density ρ vs order parameter v_a (Paper Fig. 2(b))

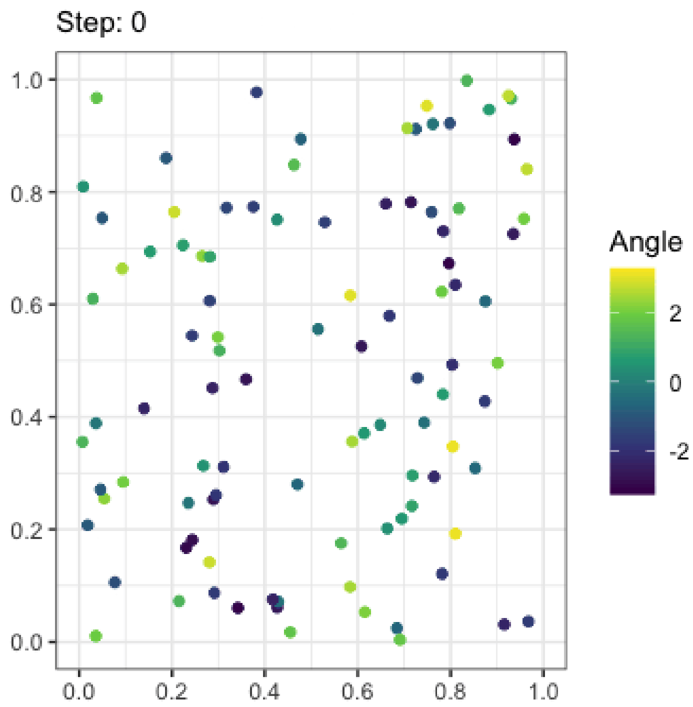


Figure 5: Vicsek Model in Motion

Note: This figure is a GIF which can played by clicking on the figure in Adobe Reader, KDE Ocular, or any other PDF reader with JavaScript support. The figure was generated using a radius of $r = 1/10$, $N = 100$ birds, absolute velocity(speed) of $v_0 = 10$. and $L = 1$. The simulation was run for 50 iterations.

References

Tamás Vicsek, András Czirók, Eshel Ben-Jacob, Inon Cohen, and Ofer Shochet. Novel type of phase transition in a system of self-driven particles. *Physical review letters*, 75(6):1226, 1995.