Math primer

0.1 Distributions and expected values (2 points)

Let X be a random variable and $p : \mathbb{R} \to \mathbb{R}$ with:

$$p(x) = \begin{cases} c \cdot \sin(x), & x \in [0, \pi] \\ 0, & \text{elsewhere} \end{cases}$$

(a) Determine the parameter value $c \in \mathbb{R}$ so that $p(x)$ is a probability density.

(b) For such c, determine the expected value $\langle X \rangle_p$

(c) Determine the variance $\left\langle X^2 \right\rangle_p - \langle X \rangle_p^2$.

0.2 Marginal densities (2 points)

Assume that the probability density function of a two-dimensional random vector $\boldsymbol{z} = (x, y)^T$ is

$$p_{\boldsymbol{z}}(\boldsymbol{z}) = p_{x,y}(x, y) = \begin{cases} \frac{3}{7}(2 - x)(x + y), & x \in [0, 2], y \in [0, 1] \\ 0, & \text{elsewhere} \end{cases}$$

(a) Write down the marginal densities $p_x(x)$ and $p_y(y)$ of the variables x and y.

(b) Determine if the two variables are independent.

0.3 Taylor expansion (1 point)

For the function $\sqrt{1 + x}$, write down the Taylor series expansion around $x_0 = 0$ up to 3rd order.

0.4 Determinant of a matrix (1 point)

Consider the 3×3 matrix

$$A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$$

Calculate the determinant and the trace of A (directly, not via Eigenvalues).

0.5 Critical points (2 points)

Consider the two functions

$$f(x, y) := c + x^2 + y^2$$
$$g(x, y) := c + x^2 - y^2,$$

where c is a constant.

(a) Show that $\mathbf{a} = (0, 0)$ is a critical point of both functions.

(b) Check for f and for g whether $\mathbf{a}$ is a minimum, maximum, or no extremum by calculating the Hessian matrix. Make use of the fact that a matrix is positive (negative) definite if and only if all its eigenvalues are positive (negative).

0.6 Bayes rule (1 point)

Assume it is known that 1% of the population suffer from a certain disease. A company has developed a test for diagnosing the disease, which comes up either positive ("+", disease found) or negative ("-", disease not found). People suffering from the disease (D) are diagnosed positive with probability 0.95, and healthy people ($\bar{D}$) are diagnosed negative with probability 0.999.

Apply Bayes' rule to find

- the probabilities that a person for which the test yielded a positive result is indeed suffering from the disease $P(D|+)$, respectively is healthy $P(\bar{D}|+)$.
- the probabilities that a person for which the test yielded a negative result is indeed healthy $P(\bar{D}|-)$, respectively is suffering from the disease $P(D|-)$.

0.7 Learning paradigms (1 point)

(a) Describe the difference between *supervised, unsupervised,* and *reinforcement learning*.

(b) Which of the above learning techniques would be most appropriate in the following cases and what would be the corresponding *observations*, *labels* and/or *rewards*?

- To teach a dog to catch a ball
- To read hand written addresses from letters
- To identify groups of users with the same taste of music

total: 10 points