Vector Lattices

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Introduction

The goal of this project is to formalize the main theorems in the theory of vector and Banach lattices. Currently we are working towards a formalization of Kakutani's theorem.

Chapter 1

The basics

In this chapter we introduce vector lattices and prove some basic facts such as the Riesz decomposition theorem.

1.1 Vector lattices

Definition 1. A *vector lattice* is a real vector space X together with a lattice order $\leq$ (i.e., a partial order for which any pair of elements has a supremum and an infimum) satisfying:

1. if $x \leq y$ and $z \in X$, then $x + z \leq y + z$;
2. if $x \leq y$ and $\lambda \in \mathbb{R}_+$, then $\lambda x \leq \lambda y$.

In this chapter, X will always denote a vector lattice.

Definition 2. For every $x \in X$, we define its *positive part* as $x_+ = x \vee 0$, its *negative part* as $x_- = (-x) \vee 0$, and its *absolute value* as $|x| = x \vee (-x)$.

Proposition 3. *For every $x, y \in X$ and $a \in \mathbb{R}_+$ the following hold:*

$$a(x \vee y) = (ax) \vee (ay) \text{ and } a(x \wedge y) = (ax) \wedge (ay).$$

Proof. For $a = 0$ the result is direct, so assume $a > 0$. Since $x \leq x \vee y$, we have $ax \leq a(x \vee y)$. Similarly, $ay \leq a(x \vee y)$. Therefore $(ax) \vee (ay) \leq a(x \vee y)$. To prove the reverse inequality, note that $x = a^{-1}ax \leq a^{-1}[(ax) \vee (ay)]$, and since the same is true of y, it follows that

$$x \vee y \leq a^{-1}[(ax) \vee (ay)].$$

Multiplying both sides by a, $a(x \vee y) \leq (ax) \vee (ay)$. The equality for the infimum follows from the identity $x \vee y = -(-x) \wedge (-y)$ that is true in every lattice ordered group. □

Now we can explore some properties of the positive and negative parts.

Proposition 4. *For every $x \in X$:*

$$x = x_+ - x_- \text{ and } |x| = x_+ + x_-.$$

Proof. For every a and b in a lattice ordered group, $a + b = a \vee b + a \wedge b$. Putting $a = x$ and $b = 0$:

$$x = x \vee 0 + x \wedge 0 = x_+ - (-x) \vee 0 = x_+ - x_-.$$

For the absolute value, compute:

$$\begin{aligned} x_+ + x_- &= 2x_+ - x \\ &= (2x) \vee 0 - x \\ &= x \vee (-x) & = |x|, \end{aligned}$$

where we are using that $c + a \vee b = (c + a) \vee (c + b)$ holds for every a, b, c in a lattice ordered group. □

Lemma 5. *For every $x, y \in X$, $x \leq y$ if and only if both $x_+ \leq y_+$ and $y_- \leq x_-$.*

Lemma 6. *Let $x, y \in X$ be such that $x \wedge y = 0$. Then $x + y = x \vee y$.*

The positive part (and therefore the negative part) is characterized by the following property.

Proposition 7. *Let $x, u, v \in X$ be such that $x = u - v$ and $u \wedge v = 0$. Then $u = x_+$.*

Next we provide some properties of the absolute value. All of them are already in `mathlib` but, for some reason, the first and the second are only proved under the assumption that the space is totally ordered.

Lemma 8. *For every $x, y \in X$ and $a \in \mathbb{R}$:*

1. $|x| = 0$ *if and only if* $x = 0$*;*
2. $|ax| = |a||x|$*;*
3. $|x + y| \leq |x| + |y|$.

To prove the Riesz decomposition theorem, we will need the following fact.

Lemma 9. *For every $x, y \in X$,*

$$x - x \wedge y = (x - y)_+.$$

Theorem 10 (Riesz decomposition)**.** *Let $x, y, z \in X$ be positive elements satisfying $x \leq y + z$. Then there exist $0 \leq x_1 \leq y$ and $0 \leq x_2 \leq z$ such that $x = x_1 + x_2$.*

From this point on, we will mostly deal with Archimedean vector lattices.

Definition 11. A vector lattice X is said to be *Archimedean* if for every $x, y \in X$ with $y > 0$ there exists a natural number $n \in \mathbb{N}$ such that $x \leq ny$.

Lemma 12. *Let X be an Archimedean vector lattice. If $x, y \in X$ are such that $nx \leq y$ for all $n \in \mathbb{N}$, then $x \leq 0$.*