Límites sucesiones

80. Calcula los límites de las sucesiones $\{x_n\}$ definidas por:

a) $x_n = \dfrac{1^\alpha + 2^\alpha + 3^\alpha + \cdots + n^\alpha}{n^{\alpha+1}}$, donde $\alpha > -1$.

b) $x_n = \sqrt[k]{(n+a_1)(n+a_2)\cdots(n+a_k)} - n$, donde $k \in \mathbb{N}$, $a_j \in \mathbb{R}$, $1 \leqslant j \leqslant k$.

c) $x_n = \left(\dfrac{\alpha\sqrt[n]{a} + \beta\sqrt[n]{b}}{\alpha+\beta}\right)^n$ donde $a > 0$, $b > 0$ y $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta \neq 0$.

d) $x_n = \left(\dfrac{1 + 2^{p/n} + 3^{p/n} + \cdots + p^{p/n}}{p}\right)^n$, donde $p \in \mathbb{N}$.

e) $x_n = n\left(\dfrac{1 + 2^k + 3^k + \cdots + n^k}{n^{k+1}} - \dfrac{1}{k+1}\right)$, donde $k \in \mathbb{N}$.

f) $x_n = \left(\dfrac{3}{4}\dfrac{1 + 3^2 + 5^2 + \cdots + (2n-1)^2}{n^3}\right)^{n^2}$

g) $x_n = n\left[\left(1 + \dfrac{1}{n^3\log(1+1/n)}\right)^n - 1\right]$

h) $x_n = \dfrac{1}{n}\left(n + \dfrac{n-1}{2} + \dfrac{n-2}{3} + \cdots + \dfrac{2}{n-1} + \dfrac{1}{n} - \log(n!)\right)$

i) $x_n = \left(\dfrac{\log(n+2)}{\log(n+1)}\right)^{n\log n}$

j) $x_n = \sqrt[n]{\dfrac{(pn)!}{(qn)^{pn}}}$ $(p, q \in \mathbb{N})$

a) $x_n = \dfrac{1^\alpha + 2^\alpha + 3^\alpha + \cdots + n^\alpha}{n^{\alpha+1}}$, $\alpha > -1$

Aplico el Criterio de Stolz.

$$\frac{x_{n+1}-x_n}{y_{n+1}-y_n} = \frac{(n+1)^\alpha}{(n+1)^{\alpha+1} - n^{\alpha+1}} = \frac{(n+1)^\alpha}{n^{\alpha+1}\left(1+\frac{1}{n}\right)^{\alpha+1} - n^\alpha \cdot n} =$$

$$= \frac{(n+1)^\alpha}{n^\alpha} \cdot \frac{1}{n\left(1+\frac{1}{n}\right)^{\alpha+1} - n} = \underset{\downarrow 1}{\left(\frac{n+1}{n}\right)^\alpha} \cdot \underset{\downarrow 1/\alpha+1}{\frac{1}{n\left[\left(1+\frac{1}{n}\right)^{\alpha+1} - 1\right]}} \to \frac{1}{\alpha+1}$$

Si tomamos $\mu_n = n\left[\overset{\to 1}{\left(1+\frac{1}{n}\right)^{\alpha+1}} - 1\right] \sim n\left(\log\left(1+\frac{1}{n}\right)^{\alpha+1}\right) =$

$$= n(\alpha+1)\log\Big(1+\overset{\to 0}{\frac{1}{n}}\Big) \sim \frac{n}{n}\,\alpha+1 \longrightarrow \alpha+1$$

c) $x_n = \left(\dfrac{\alpha \sqrt[n]{a} + \beta \sqrt[n]{e}}{\alpha+\beta} \right)^n$

$\downarrow$

1 claramente

Indeterminación 1^∞, procedemos como sigue:

$$n\left(\frac{\alpha\sqrt[n]{a} + \beta\sqrt[n]{e}}{\alpha+\beta} - 1 \right) = n\left(\frac{\alpha\sqrt[n]{a} + \beta\sqrt[n]{e} - \alpha - \beta}{\alpha+\beta} \right) =$$

$$= n\left(\frac{\alpha(\sqrt[n]{a}-1) + \beta(\sqrt[n]{e}-1)}{\alpha+\beta} \right) = \cancel{n}\left(\frac{\alpha(\cancel{\tfrac{1}{n}}\log a) + \beta(\cancel{\tfrac{1}{n}}\log e)}{\alpha+\beta} \right) =$$

$$= \frac{\alpha}{\alpha+\beta}\log a + \frac{\beta}{\alpha+\beta}\log e$$

j) $x_n = \sqrt[n]{\dfrac{(pn)!}{(qn)^{pn}}}$ $\quad \sqrt[n]{x_n} \to L \Rightarrow \dfrac{x_{n+1}}{x_n} \to L$

$$\frac{x_{n+1}}{x_n} = \frac{(pn+p)!}{(qn+q)^{pn+p}} \cdot \frac{qn^{pn}}{(pn)!} = \frac{(pn+1)(pn+2)\cdots(pn+p)\ qn^{pn}}{(qn+q)^{pn}(qn+q)^{p}} =$$

$$= \left(\frac{qn}{qn+q}\right)^{pn} \cdot \frac{(pn+1)(pn+2)\cdots(pn+p)}{(qn+q)^p}$$

$\downarrow \qquad\qquad \downarrow$

$\dfrac{1}{e^p} \qquad\qquad \left(\dfrac{p}{q}\right)^p$

$$pn\left(\frac{n}{n+1}-1\right) = pn\left(\frac{-1}{n+1}\right) \longrightarrow -p$$

La sucesión converge a $\left(\dfrac{p}{qe}\right)^p$