(A Very Brief) Introduction to Eigenvalues and Eigenvectors

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Preliminary Note

- This is a brief introduction to eigenvectors and eigenvalues from a practical and computer vision-oriented point of view.
- We will focus more on the intuitions behind than on the mathematical foundations.

Note (in Spanish):
Eigenvector = vector propio, vector característico o autovector
Eigenvalue = valor propio, valor característico o autovalor

Introduction

- Many “things” are best understood by breaking them down into their constituent parts.
  - Integers, for instance, can be decomposed into prime factors.
    - 12 = 2 x 2 x 3 → from this representation we can conclude useful properties, such as that 12 is not divisible by 5, or that any integer multiple of 12 will be divisible by 3.

Introduction

- Similarly, we can decompose matrices to extract information about their properties.
  - Information that is not obvious from the representation of a matrix as a mere array of elements.

Eigendecomposition

- One of the most common forms of decomposition is called **eigendecomposition**, where a matrix is decomposed into a set of **eigenvectors** and **eigenvalues**.
  - Given a square matrix **A**, we have

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

Matrix

Eigenvector

Eigenvalue

Calculation of eigenvalues

$$Av = \lambda v$$

Eigenvalue

Matrix

Eigenvector

Identity Matrix

$$Av = \lambda v \longrightarrow Av - \lambda v = 0 \longrightarrow \underbrace{|A - \lambda I|}_{\textbf{Determinant}} = 0$$

A **I**

$$\left| \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \longrightarrow \left| \begin{bmatrix} -6 - \lambda & 3 \\ 4 & 5 - \lambda \end{bmatrix} \right| = 0$$

Calculation of eigenvalues

$$A\mathbf{v} = \lambda \mathbf{v}$$

Matrix

Eigenvector

Eigenvalue

$$\left| \begin{bmatrix} -6-\lambda & 3 \\ 4 & 5-\lambda \end{bmatrix} \right| = 0$$

$(-6 - \lambda)(5 - \lambda) - 3 \cdot 4 = 0$

$\lambda^2 + \lambda - 42 = 0$

$\lambda = -7$ o $\lambda = 6$

In this case there are two possible eigenvalues

Calculation of eigenvectors

$$\mathbf{A}\mathbf{v} = \boldsymbol{\lambda}\mathbf{v}$$

Matrix

Eigenvector

Eigenvalue

Now that we know the eigenvalues (λ = -7 o λ = 6), **we must find the corresponding eigenvectors**. For instance, we proceed with λ = 6 (with λ = -7 would be the same procedure):

$$\underbrace{\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{v}} = \underbrace{6}_{\lambda} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{v}} \longrightarrow \begin{matrix} -6x + 3y = 6x \\ 4x + 5y = 6y \end{matrix} \longrightarrow y = 4x \longrightarrow \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Calculation of eigenvectors

We see that the expression (**Av** = **λv**) is fulfilled:

$$\begin{array}{cccc} \mathbf{A} & \mathbf{v} & = \boldsymbol{\lambda} & \mathbf{v} \\ \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \end{bmatrix} & = 6 & \begin{bmatrix} 1 \\ 4 \end{bmatrix} \end{array}$$

But… what is the intuition behind?

- An eigenvector is “a vector that does not change direction” (when multiplied by a matrix).

- The eigenvalue is the scale of transformation:

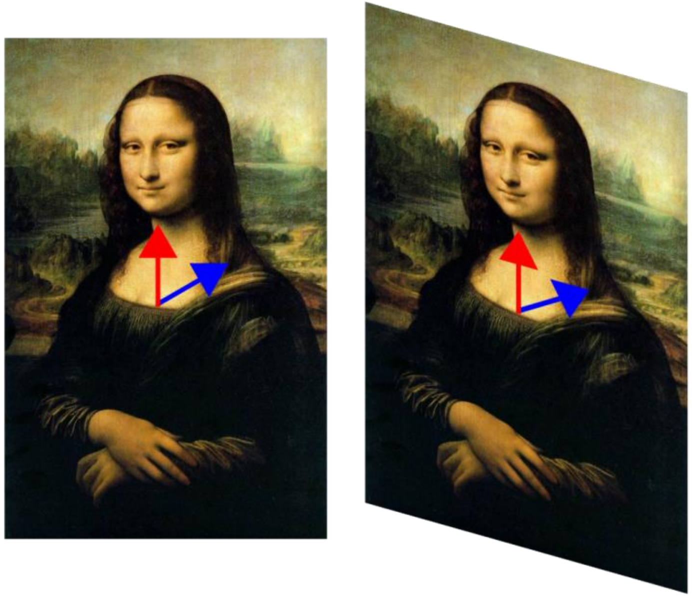
  1 → no change

  2 → double the magnitude/length

  -1 → reverse the vector sense

But… what is the intuition behind?

- An eigenvector does not change direction in a linear transformation

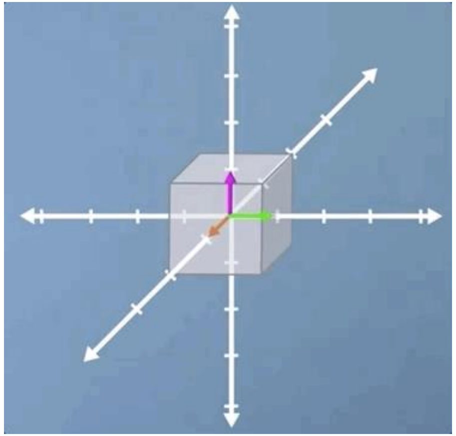


https://commons.wikimedia.org/wiki/File:Mona_Lisa_with_eigenvector.png

The image has been deformed in such a way that its vertical axis has not changed. The blue vector has changed direction, while the red vector has not changed. **The red one is an eigenvector** of the transformation, while the blue one is not. Since the red vector has not changed length, its eigenvalor is 1.

But... what is the intuition behind?

- An eigenvector does not change direction in a linear transformation



https://johngiorgi.github.io/mathematics-for-machine-learning/linear_algebra/week_5/

https://www.mathsisfun.com/algebra/images/eigen-transform.svg

- An *eigenvector* of a 3D rotation is a vector on the axis of rotation about which the rotation takes place. The corresponding *eigenvalue* is 1

More about eigenvectors and eigenvalues

- A matrix is singular (i.e. non-invertible) if, and only if, any of its eigenvalues is zero.

$$A \text{ singular} \iff \det(A) = 0 \iff \det(A - 0 \cdot I) = 0 \iff 0 \text{ is eigenvalue of } A.$$

- The determinant of a matrix can be calculated by multiplying its eigenvalues.

$$|A| = \lambda_1 \lambda_2 \ldots \lambda_n$$

- The trace would be the sum of its eigenvalues.

More about eigenvectors and eigenvalues

- The eigendecomposition of a real symmetric matrix can be used to optimize certain quadratic expressions.
- If we have a dataset and we want to project it onto a new space to reduce dimensionality.
  - In PCA, the direction of the first principal component is given by the eigenvector with the largest eigenvalue of the covariance matrix
    - https://stats.stackexchange.com/questions/217995/what-is-an-intuitive-explanation-for-how-pca-turns-from-a-geometric-problem-wit
    - https://stats.stackexchange.com/questions/2691/making-sense-of-principal-component-analysis-eigenvectors-eigenvalues

More about eigenvectors and eigenvalues

- They allow us to understand the asymptotic behaviour of linear dynamical systems
  - Section 5.6 (*Asymptotic Behavior of Discrete-Time Linear Dynamical Systems*) de Sayama, Hiroki. *Introduction to the modeling and analysis of complex systems*. Open SUNY Textbooks, 2015. See website.

Terminology

- **Positive-definite** matrix: all eigenvalues are positive.
- **Positive semi-definite** matrix: all eigenvalues are positive or zero.
- **Negative-definite** matrix: all eigenvalues are negative.
- **Negative semi-definite** matrix: all eigenvalues are negative or zero.
- **Indefinite** matrix: There are positive and negative eigenvalues.

Can they always be calculated?

- Not all matrices allow decomposition into eigenvalues and eigenvectors.
- Sometimes the decomposition exists, but it involves complex instead of real numbers.
- Any symmetric real matrix ($\mathbf{A}^T = \mathbf{A}$) has an eigendecomposition, but this may not be unique.

Singular Value Decomposition (SVD)

- Another way to factorize/decompose a matrix. In this case into singular values and vectors.
- It is a more general method.
  - For example, if a matrix in not squared, the eigendecomposition is not defined, and SVD must be used instead.

Singular Value Decomposition (SVD)

- In this case, we will write matrix **A** as

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^\top . \tag{2.43}$$

Suppose that $\boldsymbol{A}$ is an $m \times n$ matrix. Then $\boldsymbol{U}$ is defined to be an $m \times m$ matrix, $\boldsymbol{D}$ to be an $m \times n$ matrix, and $\boldsymbol{V}$ to be an $n \times n$ matrix.

- The elements on the diagonal of **D** are the singular values.
- The columns of **U** are the eigenvectors of $\mathbf{A}\mathbf{A}^T$.
- The columns of **V** are the eigenvectors of $\mathbf{A}^T\mathbf{A}$.

How are singular values and eigenvalues related?

- The singular values of matrix **A** M×N are the square roots of the eigenvalues of an $\mathbf{A}^T\mathbf{A}$ N×N matrix.
- If **A** is a real, symmetric N×N matrix with non-negative eigenvalues, then the eigenvalues and singular values are the same.

What can SVD be useful for?

- Matrix inversion is not defined for non-square matrices.
  - In such cases, the generalization called Moore-Penrose Pseudoinverse can be used.
  - For this, a computationally simple and accurate way to compute the pseudoinverse is via SVD
- And many other applications:
  - https://en.wikipedia.org/wiki/Singular_value_decomposition#Applications_of_the_SVD
  - Including the separability of convolutional filters:
    - any 2D-mask admits a SVD decomposition (*low-rank approximations*): sum of several separable kernels. See website.

What do eigenvalues have to do with Harris detector?

- As we saw in theory, E(u,v) is approximated by a quadratic form

$$E(u,v) \approx Au^2 + 2Buv + Cv^2$$

$$\approx \begin{bmatrix} u & v \end{bmatrix} \underbrace{\begin{bmatrix} A & B \\ B & C \end{bmatrix}}_{H} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$A = \sum_{(x,y)\in W} I_x^2$$

$$B = \sum_{(x,y)\in W} I_x I_y$$

$$C = \sum_{(x,y)\in W} I_y^2$$

What do eigenvalues have to do with Harris detector?

- **H** is a symmetric matrix and has a decomposition of the form **H** = **UΛU**^{T}, where **Λ** is a diagonal matrix with the eigenvalues of **H**

$$E(u,v) \approx \begin{bmatrix} u & v \end{bmatrix} \underbrace{\begin{bmatrix} A & B \\ B & C \end{bmatrix}}_{H} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$E(u,v) \approx [u,v] \overbrace{\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}^T}^{H} \begin{bmatrix} u \\ v \end{bmatrix} = [u',v'] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}$$

What do eigenvalues have to do with Harris detector?

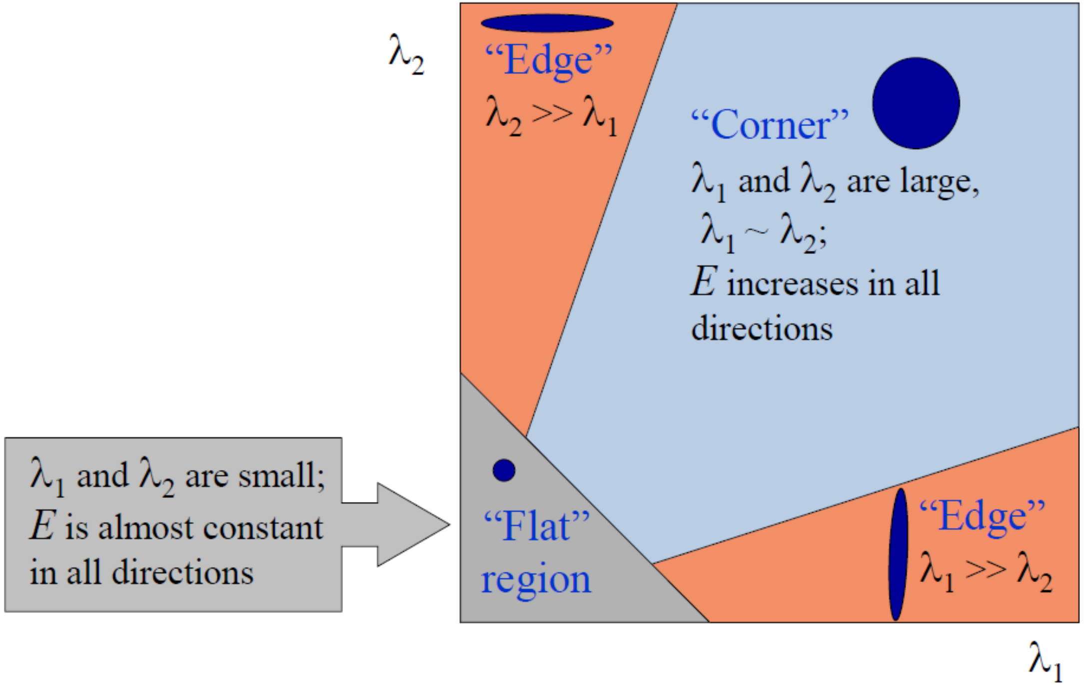
- These eigenvalues will always be non-negative, so **H** is positive semidefinite.

$$[u\ v]\begin{bmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = u^2I_x^2 + 2uvI_xI_y + v^2I_y^2 = (uI_x + vI_y)^2 \geq 0$$

- The eigenvector of the matrix corresponding to the largest eigenvalue gives the direction of fastest change.
- Since **H** is symmetric (**H**^{T}=**H**), its eigenvectors (corresponding to different eigenvalues) are orthogonal.

What do eigenvalues have to do with Harris detector?

- Finally, we use these eigenvalues to characterize a point in the image:



Some References

- Chapter I.2 of the book Goodfellow, I., Bengio, Y. & Courville, A. (2016). *Deep learning*. https://www.deeplearningbook.org/contents/linear_algebra.html
- Chapter 6. of the book Strang, G. (2016). *Introduction to Linear Algebra.* https://math.mit.edu/~gs/linearalgebra/ila5/linearalgebra5_6-1.pdf
- Eigenvalues vs. Singular Values by Suraj Rampure (Univ. of Berkeley): https://rampure.org/resources/data100/notes/eigen-singular.html
- Linear Algebra 36: Eigenapplications, 2. Ellipses (Jonathan Evans)
- Interesting discussion on the relationship between eigenvectors/eigenvalues with PCA: https://stats.stackexchange.com/questions/2691/making-sense-of-principal-component-analysis-eigenvectors-eigenvalues
- Interesting discussion on the algebraic aspects of corner detection: https://dsp.stackexchange.com/questions/69349/detecting-corners-using-structure-tensor-matrix
- Notes on singular values (in Spanish): http://www.ehu.eus/izaballa/Cursos/valores_singulares.pdf
- https://www.mathsisfun.com/algebra/eigenvalue.html

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