QR Factorization by Householder Reflectors (3×3 Worked Example, No sign)

We factor $A = QR$ with Q orthogonal and R upper triangular using Householder reflectors

$$H = I - 2uu^\top, \qquad \|u\|_2 = 1,$$

$$A = \begin{bmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{bmatrix}.$$

Step 1: Zero out entries 2 and 3 of the first column

Let x be the first column of A, $x = (12, 6, -4)^\top$. Take

$$\|x\|_2 = \sqrt{12^2 + 6^2 + (-4)^2} = 14.$$

Form the unit vector

$$u_1 = \frac{x - \|x\|_2 e_1}{\|x - \|x\|_2 e_1\|_2} = \frac{(12, 6, -4) - (14, 0, 0)}{\|(-2, 6, -4)\|_2} = \frac{(-1,\ 3,\ -2)}{\sqrt{14}}.$$

Thus

$$H_1 = I - 2u_1 u_1^\top = \begin{bmatrix} \frac{6}{7} & \frac{3}{7} & -\frac{2}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{6}{7} \\ -\frac{2}{7} & \frac{6}{7} & \frac{3}{7} \end{bmatrix}.$$

Apply it:

$$A_1 = H_1 A = \begin{bmatrix} 14 & 21 & -14 \\ 0 & -49 & -14 \\ 0 & 168 & -77 \end{bmatrix}.$$

Step 2: Act on the 2×2 trailing principal minor

Consider the bottom-right principal minor $A_1(2{:}3,\ 2{:}3)$. Take its first column

$$x = \begin{bmatrix} -49 \\ 168 \end{bmatrix}, \qquad \|x\|_2 = \sqrt{(-49)^2 + 168^2} = 175.$$

Compute the unit vector directly:

$$u_2 = \frac{x - \|x\|_2 e_1}{\|x - \|x\|_2 e_1\|_2} = \frac{(-49, 168) - (175, 0)}{\|(-224, 168)\|_2} = \frac{(-224, 168)}{280} = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}.$$

The 2×2 reflector on that block is

$$\widetilde{H}_2 = I - 2u_2u_2^\top = \begin{bmatrix} -\frac{7}{25} & \frac{24}{25} \\ \frac{24}{25} & \frac{7}{25} \end{bmatrix}.$$

Embed it block-diagonally to act on rows/cols 2:3:

$$H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{7}{25} & \frac{24}{25} \\ 0 & \frac{24}{25} & \frac{7}{25} \end{bmatrix}.$$

Apply it:

$$A_2 = H_2A_1 = \begin{bmatrix} 14 & 21 & -14 \\ 0 & 175 & -70 \\ 0 & 0 & -35 \end{bmatrix} =: R,$$

which is upper triangular.

Assemble Q and R

Since $R = H_2H_1A$ and each H_k is symmetric and orthogonal,

$$Q = H_1H_2 = \begin{bmatrix} \frac{6}{7} & -\frac{69}{175} & \frac{58}{175} \\ \frac{3}{7} & \frac{158}{175} & -\frac{6}{175} \\ -\frac{2}{7} & \frac{6}{35} & \frac{33}{35} \end{bmatrix}, \qquad R = \begin{bmatrix} 14 & 21 & -14 \\ 0 & 175 & -70 \\ 0 & 0 & -35 \end{bmatrix}.$$

One may verify $Q^\top Q = I$, $Q^\top A = R$, and hence $A = QR$.