Math League Contest Problem Set 12213

Team Round Problem 9

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Let $a_0 = \frac{1}{4}$ and $a_n = \frac{1+a_{n-1}}{2}$ for all $n \geq 1$. Let S be the sum of the numerator and denominator of a_{2021} when expressed in simplest form. What is the remainder when S is divided by 100?

Trying some small cases, we find that

$$a_0 = \frac{1}{4}, a_1 = \frac{5}{8}, a_2 = \frac{13}{16}, a_3 = \frac{29}{32}.$$

We can rewrite these as

$$a_0 = \frac{2^{0+2} - 3}{2^{0+2}}, a_1 = \frac{2^{1+2} - 3}{2^{1+2}}, a_2 = \frac{2^{2+2} - 3}{2^{2+2}}, a_3 = \frac{2^{3+2} - 3}{2^{3+2}}.$$

$$a_n = \frac{2^{n+2} - 3}{2^{n+2}}.$$

$$a_{2021} = \frac{2^{2023} - 3}{2^{2023}}$$

Since S is the sum of a_{2021}'s numerator and denominator, we have

$$S = 2^{2023} + 2^{2023} - 3 = 2^{2024} - 3$$

$$S \equiv ? \mod 100$$

Find the remainder when S is divided by 100.

$$S = 2^{2024} - 3$$

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$$2^{2024} \equiv ? \bmod 100$$

$$2^{2024} \equiv ? \bmod 4, \quad 2^{2024} \equiv ? \bmod 25$$

Find the remainder when S is divided by 100.

$$S = 2^{2024} - 3$$

$$S \equiv ? \bmod 100$$

$$2^{2024} \equiv ? \bmod 100$$

$$2^{2024} \equiv ? \bmod 4, \quad 2^{2024} \equiv ? \bmod 25$$

$$2^{2024} = (2^2)^{1012} = 4^{1012} \equiv 0 \bmod 4$$

Find the remainder when S is divided by 100.

$$S = 2^{2024} - 3$$

$$S \equiv ? \bmod 100$$

$$2^{2024} \equiv ? \bmod 100$$

$$2^{2024} \equiv 0 \bmod 4, \quad 2^{2024} \equiv ? \bmod 25$$

Find the remainder when S is divided by 100.

$$S = 2^{2024} - 3$$

$$S \equiv ? \bmod 100$$

$$2^{2024} \equiv ? \bmod 100$$

$$2^{2024} \equiv 0 \bmod 4, \quad 2^{2024} \equiv ? \bmod 25$$

$$2^{10} = 1024 \equiv 24 \equiv -1 \bmod 25$$

Find the remainder when S is divided by 100.

$$S = 2^{2024} - 3$$

$$S \equiv ? \bmod 100$$

$$2^{2024} \equiv ? \bmod 100$$

$$2^{2024} \equiv 0 \bmod 4, \quad 2^{2024} \equiv ? \bmod 25$$

$$2^{10} = 1024 \equiv 24 \equiv -1 \bmod 25$$

$$2^{2024} = (2^{10})^{202} \cdot 2^4 \equiv (-1)^{202} \cdot 16 \equiv 16 \bmod 25$$

Find the remainder when S is divided by 100.

$$S = 2^{2024} - 3$$

$$S \equiv ? \bmod 100$$

$$2^{2024} \equiv ? \bmod 100$$

$$2^{2024} \equiv 0 \bmod 4, \quad 2^{2024} \equiv 16 \bmod 25$$

Find the remainder when S is divided by 100.

$$S = 2^{2024} - 3 \equiv ? \bmod 100$$

$$2^{2024} \equiv ? \bmod 100$$

$$2^{2024} \equiv 0 \bmod 4, \quad 2^{2024} \equiv 16 \bmod 25$$

Find the remainder when S is divided by 100.

$$S = 2^{2024} - 3 \equiv ? \bmod 100$$

$$2^{2024} \equiv ? \bmod 100$$

$$2^{2024} \equiv 0 \bmod 4, \quad 2^{2024} \equiv 16 \bmod 25$$

$$2^{2024} = 16 + 25k, \text{ for some integer } k$$

$$S = 2^{2024} - 3 \equiv \text{?} \mod 100$$

$$2^{2024} \equiv \text{?} \mod 100$$

$$2^{2024} \equiv 0 \mod 4, \quad 2^{2024} \equiv 16 \mod 25$$

$$2^{2024} = 16 + 25k, \text{ for some integer } k$$

For all integers k,

$$2^{2024} \equiv 0 \mod 4 \iff 16 + 25k \equiv 0 \mod 4$$

Find the remainder when S is divided by 100.

$$S = 2^{2024} - 3 \equiv ? \bmod 100$$

$$2^{2024} \equiv ? \bmod 100$$

$$2^{2024} \equiv 0 \bmod 4, \quad 2^{2024} \equiv 16 \bmod 25$$

$$2^{2024} = 16 + 25k, \text{ for some integer } k$$

For all integers k,

$$\begin{aligned} 2^{2024} \equiv 0 \bmod 4 &\iff 16 + 25k \equiv 0 \bmod 4 \\ &\iff 25k \equiv -16 \bmod 4 \end{aligned}$$

Find the remainder when S is divided by 100.

$$S = 2^{2024} - 3 \equiv ? \bmod 100$$

$$2^{2024} \equiv ? \bmod 100$$

$$2^{2024} \equiv 0 \bmod 4, \quad 2^{2024} \equiv 16 \bmod 25$$

$$2^{2024} = 16 + 25k, \text{ for some integer } k$$

For all integers k,

$$\begin{aligned} 2^{2024} \equiv 0 \bmod 4 &\iff 16 + 25k \equiv 0 \bmod 4 \\ &\iff 25k \equiv -16 \bmod 4 \\ &\iff k \equiv 0 \bmod 4 \end{aligned}$$

$$S = 2^{2024} - 3 \equiv \, ? \mod 100$$

$$2^{2024} \equiv \, ? \mod 100$$

$$2^{2024} \equiv 0 \mod 4, \quad 2^{2024} \equiv 16 \mod 25$$

$$2^{2024} = 16 + 25k, \text{ for some integer } k$$

For all integers k,

$$\begin{aligned} 2^{2024} \equiv 0 \mod 4 &\iff 16 + 25k \equiv 0 \mod 4 \\ &\iff 25k \equiv -16 \mod 4 \\ &\iff k \equiv 0 \mod 4 \\ \implies 16 + 25k \equiv 0 \mod 4 &\iff k = 0 + 4\ell, \text{ for some integer } \ell \end{aligned}$$

$$S = 2^{2024} - 3 \equiv ? \mod 100$$

$$2^{2024} \equiv ? \mod 100$$

$$2^{2024} \equiv 0 \mod 4, \quad 2^{2024} \equiv 16 \mod 25$$

$$2^{2024} = 16 + 25k, \text{ for some integer } k$$

For all integers k,

$$2^{2024} \equiv 0 \mod 4 \iff 16 + 25k \equiv 0 \mod 4$$

$$\implies 16 + 25k \equiv 0 \mod 4 \iff k = 0 + 4\ell, \text{ for some integer } \ell$$

$$S = 2^{2024} - 3 \equiv ? \bmod 100$$

$$2^{2024} \equiv ? \bmod 100$$

$$2^{2024} \equiv 0 \bmod 4, \quad 2^{2024} \equiv 16 \bmod 25$$

$$2^{2024} = 16 + 25k, \text{ for some integer } k$$

For all integers k,

$$2^{2024} \equiv 0 \bmod 4 \iff 16 + 25k \equiv 0 \bmod 4$$

$$\implies 16 + 25k \equiv 0 \bmod 4 \iff k = 0 + 4\ell, \text{ for some integer } \ell$$

Thus, $2^{2024} \equiv 0 \bmod 4$ and $2^{2024} \equiv 16 \bmod 25$ if and only if $2^{2024} = 16 + 25 \cdot (0 + 4\ell) = 16 + 100\ell$, for some integer ℓ.

$$S = 2^{2024} - 3 \equiv ? \bmod 100$$

$$2^{2024} \equiv ? \bmod 100$$

$$2^{2024} \equiv 0 \bmod 4, \quad 2^{2024} \equiv 16 \bmod 25$$

$$2^{2024} = 16 + 25k, \text{ for some integer } k$$

For all integers k,

$$2^{2024} \equiv 0 \bmod 4 \iff 16 + 25k \equiv 0 \bmod 4$$
$$\implies 16 + 25k \equiv 0 \bmod 4 \iff k = 0 + 4\ell, \text{ for some integer } \ell$$

Thus, $2^{2024} \equiv 0 \bmod 4$ and $2^{2024} \equiv 16 \bmod 25$ if and only if $2^{2024} = 16 + 25 \cdot (0 + 4\ell) = 16 + 100\ell$, for some integer ℓ.

$$\implies 2^{2024} \equiv 16 \bmod 100$$

$$S = 2^{2024} - 3 \equiv ? \bmod 100$$

$$2^{2024} \equiv 16 \bmod 100$$

$$2^{2024} \equiv 0 \bmod 4, \quad 2^{2024} \equiv 16 \bmod 25$$

$$2^{2024} = 16 + 25k, \text{ for some integer } k$$

For all integers k,

$$2^{2024} \equiv 0 \bmod 4 \iff 16 + 25k \equiv 0 \bmod 4$$
$$\implies 16 + 25k \equiv 0 \bmod 4 \iff k = 0 + 4\ell, \text{ for some integer } \ell$$

Thus, $2^{2024} \equiv 0 \bmod 4$ and $2^{2024} \equiv 16 \bmod 25$ if and only if $2^{2024} = 16 + 25 \cdot (0 + 4\ell) = 16 + 100\ell$, for some integer ℓ.

$$\implies 2^{2024} \equiv 16 \bmod 100$$

$$S = 2^{2024} - 3 \equiv \boxed{13} \bmod 100$$

$$2^{2024} \equiv 16 \bmod 100$$

$$2^{2024} \equiv 0 \bmod 4, \quad 2^{2024} \equiv 16 \bmod 25$$

$$2^{2024} = 16 + 25k, \text{ for some integer } k$$

For all integers k,

$$2^{2024} \equiv 0 \bmod 4 \iff 16 + 25k \equiv 0 \bmod 4$$
$$\implies 16 + 25k \equiv 0 \bmod 4 \iff k = 0 + 4\ell, \text{ for some integer } \ell$$

Thus, $2^{2024} \equiv 0 \bmod 4$ and $2^{2024} \equiv 16 \bmod 25$ if and only if $2^{2024} = 16 + 25 \cdot (0 + 4\ell) = 16 + 100\ell$, for some integer ℓ.

$$\implies 2^{2024} \equiv 16 \bmod 100$$

Key Concepts

- Modular Arithmetic
- Chinese Remainder Theorem