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Math 146: Lecture 1

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Example: Why is S_3 "built from abelian groups"?

Recall: the order of S_3 is 6. S_3 has a normal subgroup $C_3 \subseteq S_3$, where $|C_3| = 3$.

Because it is normal, we can construct the quotient group S_3/C_3 which has order 2. This is one group up to isomorphism of order 2, so it is abelian given we must have $S_3/C_3 \cong C_2$.

So S_3 has a normal, abelian subgroup that has the property that the quotient of S_3 with it is also abelian; we appeal to this for intuition as to what "built from abelians means".

Continuing this example on S_4, we note that A_4 does not work, as it isn't abelian. Considering over the Klein group however, we get interestingly that $S_4/V_4 \cong S_3$.

So we extend our definition that S_4 is "built from abelians" in the sense that it has an abelian normal subgroup which determines a quotient that itself is "built from abelians".

Definition (Solvable). A group which is "built from abelians" is also called **solvable**.

Example: This is why the quintic formula fails. For $n \geq 5$, the only normal subgroups of S_n are the trivial subgroup and A_n, but A_n for $n \geq 5$ is simple and thus has no meaningful quotient.

Definition (Ring). A **ring** is an abelian group R (with operation $+$) with an additional operation (denoted $\cdot$) such that:

(a) Multiplication is associative.

(b) Multiplication distributes over addition both ways.

(c) There is a multiplicative identity.

Lemma e. = ee' = e'

0 multiply by anything