Math 146: Homework 2

Due: Friday, February 9, by 11:59pm

Please submit your solutions as a single PDF on Canvas

Reminders:

- You are encouraged to work together on homework. If you work with other students, please indicate who you worked with on this assignment.
- The usual extension policy applies to this assignment: if you would like an extension on this homework over the weekend, simply email the professor to ask for it. No justification is required.

1. Let $F = \mathbb{Q}(i)$, which you proved on your previous homework is a field; in particular, you may assume that F is a field throughout this problem. This problem will have you explore properties of non-zero (ring) homomorphisms from F to itself. Thus, let $\phi\colon F \to F$ be a non-zero homomorphism.

(a) Prove that ϕ must be an isomorphism.

(b) Prove that $\phi(a) = a$ for any $a \in \mathbb{Q}$.

(c) Suppose $\alpha \in F$ is such that $\phi(\alpha) = i$. Find a complete list of possibilities for α.

(d) Determine all non-zero homomorphisms $F \to F$. You should describe each homomorphism as explicitly as possible (e.g., by determining the value $\phi(a + bi)$).

2. Recall that $\mathbb{Z}_p$ is a field for primes p. To emphasize the fact that this is a field, going forward we'll typically write it as $\mathbb{F}_p$ instead of $\mathbb{Z}_p$ (so $\mathbb{F}_p = \mathbb{Z}_p$). This problem aims to have you become more familiar with that convention and to notice parallels between $\mathbb{F}_p$ and $\mathbb{Q}$. For concreteness, we'll focus on the field $\mathbb{F}_5$. You'll ultimately construct a field with 25 elements. This will be pretty explicit, but we'll soon learn how to do this more abstractly and more generally.

(a) Prove there is no element $\alpha \in \mathbb{F}_5$ such that $\alpha^2 = 2$.

(b) Now, let $R = \{a + bx : a, b \in \mathbb{F}_5\}$ be the set of (at most) linear polynomials over $\mathbb{F}_5$. For $f, g \in R$, define $f + g$ normally; define $f \cdot g$ almost normally, except subject to the "extra" rule $x^2 = 2$.[1] As two examples, we have:

- $x \cdot (x + 1) = x^2 + x = x + 2 \in R$; and
- $(3x + 1) \cdot (2x + 4) = 6x^2 + 14x + 4 = 14x + 16 = 4x + 1 \in R$ (remember, we are working in $\mathbb{F}_5$).

Prove that R is a ring under these operations.

(c) Prove that $(a + bx)(a - bx) \neq 0$ unless $a = b = 0$.

(d) Prove that R is a field.

3. Let R be a commutative ring and let $I, J \subseteq R$ be ideals. We define a set $I + J$ as the set of all possible sums of elements in I and J, i.e. $I + J := \{i + j : i \in I, j \in J\}$.

(a) Prove that $I + J$ is an ideal.

(b) Suppose $K \subseteq R$ is any ideal containing both I and J (i.e. $I \subseteq K$ and $J \subseteq K$). Prove that $I + J \subseteq K$. (In other words, you are proving that $I + J$ is the smallest ideal containing both I and J.)

(c) Prove that $I \cap J$ is an ideal.

[1] Very shortly, we'll be talking about quotients instead of "extra rules," but some of the things we'll do for quotients will be more easily understood initially in terms of this more ad hoc notion.