Math 146: Homework 4
Due: Friday, March 1, by 11:59pm
Please submit your solutions as a single PDF on Canvas

Reminders:

- You are encouraged to work together on homework. If you work with other students, please indicate who you worked with on this assignment.
- The usual extension policy applies to this assignment: if you would like an extension on this homework over the weekend, simply email the professor to ask for it. No justification is required.

1. Let F be a field and let $f \in F[x]$ be a monic polynomial. Prove that f is irreducible if and only if f cannot be factored as a product of two *monic* polynomials of smaller degree.

2. Let p be a prime and let $\mathbb{F}_p$ be the field with p elements.
a) List all possible ways that a monic polynomial $f \in \mathbb{F}_p[x]$ with $\deg f = 3$ could factor.
b) Prove that there are exactly $\frac{p^3-p}{3}$ monic, irreducible polynomials $f \in \mathbb{F}_p[x]$ with degree 3. (*Hint:* You may use that there are exactly $\frac{p^2-p}{2}$ monic irreducible degree 2 polynomials.)
c) Conclude that there is a field of order p^3.

3. Let F and K be field and suppose that K is an extension of F. Let $\alpha \in K$ be such that α^2 is algebraic over F. Prove that α is algebraic over F.

4. Let F and K be fields and suppose that K is an extension of F. An *automorphism* of K over F is a bijective homomorphism $\phi\colon K \to K$ such that $\phi(a) = a$ for every $a \in F$. Let $\mathrm{Aut}_F(K)$ be the set of automorphisms of K over F.
a) Prove that $\mathrm{Aut}_F(K)$ is a group under composition of functions.
b) If $\alpha \in K$ is algebraic over F, prove that $\phi(\alpha)$ is also algebraic over F for every $\phi \in \mathrm{Aut}_F(K)$.
c) Find $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(i))$, $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}))$, and $\mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$.