David Owen Horace Cutler

Math 237: Homework 2

April 4, 2024

Exercise 1. Show that the only function $f \in L^1(\mathbb{R})$ such that $f = f * f$ is $f = 0$ a.e.

Proof. Let $f \in L^1(\mathbb{R})$ such that $f = f * f$. Recall we have that by Heil **Exercise 9.2.6** that then

$$\hat{f} = (f * f)\hat{} = \hat{f}\hat{f},$$

so for any particular ξ we have

$$\hat{f}(\xi)^2 - \hat{f}(\xi) = 0,$$

so we consequently have

$$\hat{f}(\xi) = 0 \text{ or } 1, \quad \forall \xi \in \mathbb{R}.$$

Recall then that $\hat{f}$ is continuous on $\mathbb{R}$; it trivially follows that it thus must be either identically 0 or identically 1.

However, the **Riemann-Lebesgue Lemma** guarentees that $|\hat{f}|$ should decay to 0 as $|x| \to \infty$; so the only possibility is that $\hat{f}$ is constantly 0.

Note then clearly $\hat{0} = 0$; by the **Uniqueness Theorem** then, as we thus have $\hat{f} = \hat{0}$, it must be then that $f = 0$ a.e., as desired □

Exercise 6. Suppose $f \in \mathrm{AC}(\mathbb{T})$, i.e., f is 1-periodic and absolutely continuous on $[0, 1]$.

6.1 Prove that $\hat{f'}(n) = 2\pi i n \hat{f}(n)$ for $n \in \mathbb{Z}$ and conclude $\lim_{|n|\to\infty} n\hat{f}(n) = 0$.

Proof. By assumption $f \in AC(\mathbb{T})$. We know of course that $e^{2\pi i n \xi}$ is furthermore absolutely continuous are we observe it is continuously differentiable.

Thus, we can apply **Integration by Parts**, which gets us the following (note $\frac{d}{d\xi}e^{-2\pi i n \xi} = -2\pi i n e^{-2\pi i n \xi}$):

$$\hat{f'}(n) = \int_0^1 f'(\xi)e^{-2\pi i n \xi} d\xi$$
$$= e^{-2\pi i n \xi} f(1) - f(0) - \int_0^1 -2\pi i n f(\xi) e^{-2\pi i n \xi} d\xi$$
$$= e^{-2\pi i n \xi} f(1) - f(0) + 2\pi i n \hat{f}(n)$$

We note then that for any value of $n \in \mathbb{Z}$, we have $e^{-2\pi i n \xi} = 1$. Moreover, as f is 1-periodic, we have $f(0) = f(1)$. Thus we can further reduce

$$e^{-2\pi i n \xi} f(1) - f(0) + 2\pi i n \hat{f}(n)$$
$$= 2\pi i n \hat{f}(n),$$

as desired. Recall the by the **Riemann-Lebesgue Lemma** we know

$$\lim_{|n|\to\infty} |\hat{f'}(n)| = 0,$$

and thus

$$\lim_{|n|\to\infty} |\hat{f}'(n)| = \lim_{|n|\to\infty} |2\pi in\hat{f}(n)| = \lim_{|n|\to\infty} 2\pi n|\hat{f}(n)| = 0,$$

where from limit rules multiplying by $\frac{1}{2\pi}$ gets the limit of $n|\hat{f}(n)|$ as 0, which clearly implies the same for $n\hat{f}(n)$, as additionally desired. □

6.2 Show that if $\int_0^1 f(x)dx = 0$, then

$$\int_0^1 |f(x)|^2 dx \leq \frac{1}{4\pi^2}\int_0^1 |f'(x)|^2 dx$$

Exercise 12. Fix $g \in L^2(\mathbb{R})$. Prove that $\{T_k g = g(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal sequence if and only if

$$\sum_{k\in\mathbb{Z}} |\hat{g}(\xi - k)|^2 = 1 \text{ a.e.}$$

Proof. Let $n, m \in \mathbb{Z}$. We can derive the following:

$$\begin{aligned}
&\langle T_n g, T_m g\rangle \\
&= \int_{-\infty}^{\infty} T_n g(x)\overline{T_m g(x)}dx \\
&= \int_{\infty}^{\infty} (T_n g)\hat{}(\xi)\overline{(T_m g)\hat{}(\xi)}d\xi && \text{(Parseval's Theorem)} \\
&= \int_{-\infty}^{\infty} e^{-2\pi in\xi}\hat{g}(\xi)\overline{e^{-2\pi im\xi}\hat{g}(\xi)}d\xi && \text{(Identity for transform of translation)} \\
&= \int_{-\infty}^{\infty} e^{-2\pi in\xi}\overline{e^{-2\pi im\xi}}\hat{g}(\xi)\overline{\hat{g}(\xi)}d\xi && \text{(Modulus is multiplicative)} \\
&= \int_{-\infty}^{\infty} e^{-2\pi i(n-m)\xi}|\hat{g}(\xi)|^2 d\xi && (z\overline{z} = |z|^2, \overline{e^{-2\pi im\xi}} = e^{2\pi im\xi}) \\
&= \sum_{k\in\mathbb{Z}}\int_k^{k+1} e^{-2\pi i(n-m)\xi}|\hat{g}(\xi)|^2 d\xi \\
&= \sum_{k\in\mathbb{Z}}\int_0^1 e^{-2\pi i(n-m)(\xi+k)}|\hat{g}(\xi+k)|^2 d\xi && \text{(Translating the integral)} \\
&= \int_0^1 \sum_{k\in\mathbb{Z}} e^{-2\pi i(n-m)(\xi+k)}|\hat{g}(\xi+k)|^2 d\xi && \text{(Fubini's Theorem)} \\
&= \int_0^1 \sum_{k\in\mathbb{Z}} e^{-2\pi i(n-m)\xi}|\hat{g}(\xi+k)|^2 d\xi \\
&= \int_0^1 \Big(\sum_{k\in\mathbb{Z}} |\hat{g}(\xi+k)|^2\Big) e^{-2\pi i(n-m)\xi} d\xi
\end{aligned}$$

Define $H(\xi) = \sum_{k\in\mathbb{Z}} |\hat{g}(\xi + k)|^2$. Note here then that (by translating the integral back)

$$\int_0^1 \sum_{k\in\mathbb{Z}} |\hat{g}(\xi + k)|^2 d\xi = \int_{-\infty}^{\infty} |\hat{g}(\xi)|^2 d\xi < \infty,$$

where the last part follows as $g \in L^2(\mathbb{R}) \to \hat{g} \in L^2(\mathbb{R})$.

We also observe the 1-periodicity of $\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + k)|^2$, which is obvious as k varies over all $\mathbb{Z}$ (redoing the last equation with the bounds $n, n+1$ shows existence a.e. of this sum).

Thus $H(\xi) = \sum_{k\in\mathbb{Z}} |\hat{g}(\xi + k)|^2 \in L^1(\mathbb{T})$; so its Fourier transform is defined. We can continue then:

$$\begin{aligned} \int_0^1 \Big(\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + k)|^2\Big) e^{-2\pi i(n-m)\xi} d\xi \\ = \int_0^1 H(\xi) e^{-2\pi i(n-m)\xi} d\xi \\ = \hat{H}(n-m) \end{aligned}$$

it's pretty obv from here and the earlier ending of the first set of equations, use uniqueness and boom $\square$