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Math 237: Lecture 2

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Definition (Banach Space). Recall: X is a **Banach space** if X is a linear normed space which is complete with respect to the metric norm.

Theorem (Completeness Characterization). Recall: A normed space is complete if and only if if every absolutely convergent series in X converges in converges in X.

Definition (Linearity and Boundedness). For X, Y normed spaces, then $T : X \to Y$ is said to be **linear** if $\forall a, b \in \mathbb{C}, \forall f, g \in X$ we have:

$$T(af + bg) = aT(f) + bT(g)$$

Furthermore, T is said to be **bounded** if there is $c > 0$ such that $\forall f \in X$:

$$||Tf||_Y \leq c||f||_X$$

Theorem (Continuity). $T : X \to Y$ between normed spaces is bounded iff T is continuous at 0 iff T is continuous on X

Definition (Norm of a Transformation). $L(X, Y)$ is the set of all bounded linear mappings from X to Y.

The function $T \in L(X, Y) \mapsto ||T|| = \sup\{||Tx||_Y, ||x|| = 1\}$ defines a norm on $L(X, Y)$ and:

$$||T|| = \sup\left\{\frac{||Tx||_Y}{||x||_X}, x \neq 0\right\} = \inf\{c > 0 : ||Tx||_Y \leq c||x||_X\}$$

Theorem (Lp Spaces are Normed Spaces). *Proof.* Let $1 \leq p \leq \infty$. Recall $L^p(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C}, f$ Lebesgue measurable and $\int_{\mathbb{R}^d} |f|^p < \infty\}$.

$f \mapsto ||f||_p = \left(\int_{\mathbb{R}^d} |f|^p\right)^{\frac{1}{p}}$ is a norm on $L^p(\mathbb{R}^d)$.

1. $f, g \in L^p \to f + g \in L^p$
2. $\forall c \in \mathbb{C}, f \in L^p \to cf \in L^p$

For (1), note using convexity:

$$|f+g|^p \leq (|f|+|g|)^p = 2^p\left(\frac{|f|}{2}+\frac{|g|}{2}\right)^p \leq 2^p\left(\frac{|f|}{2}+\frac{|g|}{2}\right)^p \leq 2^p\left(\frac{1}{2}|f|^p+\frac{1}{2}|g|^p\right) = 2^{p-1}(|f|^p+|g|^p)$$

So we get:

$$\int_{\mathbb{R}^d} |f+g|^2 \leq 2^{p-1}\left(\int_{\mathbb{R}^d} |f|^p + \int_{\mathbb{R}^d} |g|^p\right) < \infty$$

This proves (1.), where (2.) is trivial. $\square$

Theorem (Minkowski's Inequality). For all $f, g \in L^p(\mathbb{R}^d)$, we have the following:

$$||f + g||_p \leq ||f||_p + ||g||_p$$

Definition (Dual). For $p \in [1, \infty]$, the unique p for which $\frac{1}{p} + \frac{1}{q}$ is called the **dual** of p.

Theorem (Hölder's Inequality). If $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, then $fg \in L^1(\mathbb{R}^d)$ and $||fg||_1 \leq ||f||_p||g||_q$.

Proof. I will prove this for $p, q \geq 1$, as these are generally the only cases considered in functional analysis.

Consider first the special case $p = \infty, q = 1$ (or $q = \infty, p = 1$, which is symmetric).

Then we have $|fg| \leq ||f||_\infty |g|$ almost everywhere, and so we get the desired result from monotonicity of the integral.

Consider the general case $p, q > 1$ then. We normalize f and g by $||f||_p$ and $||g||_q$ respectively, so we get:

$$||f||_p = ||g||_q = 1$$

Note here we are substituing $\frac{f}{||f||_p}$ for f and $\frac{g}{||g||_q}$ for g. We then apply Young's inequality to get the following (this only holds for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$):

$$|fg| \leq \frac{|f|^p}{p} + \frac{|g|^q}{q}$$

Integrating both sides gets:

$$||fg||_1 \leq \frac{||f||_p^p}{p} + \frac{||g||_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1$$

So we get $||fg||_1 \leq 1$, where *denormalizing* gets $||fg||_1 \leq ||f||_p||g||_q$. □

Theorem (Lp Spaces are Complete). Consider some L^p space for $p \geq 1$. Then $L^p(\mathbb{R}^d)$ is complete.

Proof. Let $\{f_n\}_{n=1}^\infty$ in $L^p(\mathbb{R}^d)$ be Cauchy in the L^p norm, specifically so that:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n, m, \geq N, ||f_n - f_m||_p < \epsilon$$

Let $\epsilon > 0$ and take N. Then by Chebyshev's inequality for L^p spaces:

$$|\{|f_n - f_m| > \epsilon\}| \leq \frac{1}{\epsilon^p}||f_n - f_m||_p^p \tag{1}$$

Taking $N \to \infty$ makes the latter quality small, so we observe $\{f_n\}_{k=1}^\infty$ is Cauchy in measure.

Recall convergence in measure is a "complete" characteristic, so we have $f_n \xrightarrow{m} f$ where f is a measurable function, i.e. f_n converges to some f in measure.

Moreover, this implies there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ such that $f_{n_k} \xrightarrow{\text{a.e}} f$.

With this in mind, note for $j, j' \geq M$, we have:

$$||f_{n_j} - f_{n_{j'}}||_p < \epsilon$$

Fix $j \geq M$ then, and let $j' \to \infty$. Then we get the following using our earlier convergence of a subsequence and Fatou's lemma:

$$\begin{aligned} \int_{\mathbb{R}^d} |f - f_{n_j}|^p &= \int_{\mathbb{R}^d} \liminf_{j \to \infty} |f_{n_j} - f_{n_{j'}}|^p \\ &\leq \liminf_{j \to \infty} \int_{\mathbb{R}^d} |f_{n_j} - f_{n_{j'}}| < \epsilon^p \end{aligned} \tag{2}$$

Recall then that $\{f_n\}_{n=1}^{\infty}$ is Cauchy in the L^p norm. As it has a subsequence that converges in L^p, it follows it must converge to the same limit f. Thus we have completeness. □

Theorem (Closed Subspaces). (a) A subspace E of X **closed** if and only if E contains all of its limit points.

(b) If $E \subseteq X$ is a subspace, the set:

$$\overline{E} = E \cup \{\text{limit points of } E\}$$

is a closed subspace called the **closure** of E.

Furthermore, E is closed if and only if $E = \overline{E}$.

(c) A subspace $E \subseteq X$ is **dense** if every point in X is the limit of a sequence in E.

(d) If X is a Banach space and E is a subspace of X then E is a Banach space if and only if E is closed.

Proof. No proof is given. Perhaps I will give one in the future! :) □

Definition (Seperability). A normed space is **seperable** if a contains a countable dense subset.

Definition (Inner Product Space)). A complex vector space is called an **inner product space** if there exists a mapping $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ that satisfies the following properties:

(a) $\forall x, y, z \in H, \forall a, b \in \mathbb{C}$:

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle \textbf{ (Linearity in the First Argument)}$$

(b) $\forall x, y \in H$:

$$\langle x, y \rangle = \overline{\langle x, y \rangle} \textbf{ (Conjugate Symmetry)}$$

(c) $\forall x \in H$,

$$\langle x, x \rangle \geq 0 \text{ and } \langle x, x \rangle = 0 \text{ iff } x = 0 \textbf{ (Positive Definiteness)}$$

Definition (Pre-Hilbert Space). If is H a vector space and the candidate norm $|| \cdot ||_H : H \to [0, \infty)$ defined by $x \mapsto ||x||_H = \sqrt{\langle x, x \rangle}$ is a norm, then H is a **pre-Hilbert space**.

Definition (Hilbert Space). If H is a complete pre-Hilbert space, then it is called a **Hilbert space**.