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1 Normed spaces

Let K denote either $\mathbb{R}$ or $\mathbb{C}$ and let $\mathcal{X}$ a vector space over K. A **subspace** of $\mathcal{X}$ is a vector a subspace. For $x \in X$, we denote Kx, the one-dimensional subspace spanned by x.

Definition (Seminorm and Norm). A **seminorm** on $\mathcal{X}$ is a function $||\cdot||_{\mathcal{X}} : \mathcal{X} \to [0, \infty)$ given by $x \mapsto ||x||_{\mathcal{X}}$ that satisfies the following two properties:

1. $\forall x, y \in \mathcal{X}, ||x + y||_{\mathcal{X}} \leq ||x||_{\mathcal{X}} + ||y||_{\mathcal{X}}$ **(Triangle inequality)**
2. $\forall x \in \mathcal{X}, \forall \lambda \in K, ||\lambda x||_{\mathcal{X}} = |\lambda| ||x||_{\mathcal{X}}$ **(Absolute homogeneity)**

If a seminorm also satisfies the property the following property:

$$||x||_{\mathcal{X}} \text{ iff } x = 0 \textbf{ (Positive definiteness)}$$

Then we say it is **norm**.

A vector space with a norm is called a **normed space**; on a normed space, the following definition:

$$d(x, y) = ||x - y||_{\mathcal{X}}$$

defines a metric; which furthermore induces a topology on $\mathcal{X}$, the "**norm topology**".

Definition T. wo norms $|| \cdot ||_1$ and $|| \cdot ||_2$ are said to be equivalent if $\exists C_1, C_2 > 0$ such that:

$$C_1 ||x||_1 \leq ||x||_2 \leq C_2 ||x||_1$$

A normed space which is complete with regard to the norm metric is called a **Banach space**.

Example: (Completion with respect to the norm metric) Every normed space can be embedded in a Banach space as a dense subspace; the prototypical example being $\mathbb{Q}$ in $\mathbb{R}$.

Definition (Convergent vs. absolutely convergent). If $\{x_n\}$ is a sequence in X, we say $\sum_{n=1}^{\infty} x_n$ **converges** to x if $\lim\limits_{N \to \infty} \sum_{n=1}^{N} x_n = x$, it is **absolutely convergent** if $\sum_{n=1}^{\infty} ||x_n||_{\mathcal{X}} < \infty$.

Theorem (Completeness characterization). A normed space $\mathcal{X}$ is complete if and only if every absolutely convergent series in $\mathcal{X}$ converges in $\mathcal{X}$.

Proof. $\Rightarrow$

If $\mathcal{X}$ is complete and $\sum_{n=1}^{\infty} ||x_n|| < \infty$, let $S_n = \sum_{n=1}^{N} x_n$. Take $N, k \in \mathbb{N}$, and consider the following:

$$\begin{aligned} ||S_N - S_{N+k}||_{\mathcal{X}} &= ||S_{N+k} - S_N|| \\ = \Big|\Big| \sum_{n=1}^{N+k} x_n - \sum_{n=1}^{N} x_n \Big|\Big|_{\mathcal{X}} &= \Big|\Big| \sum_{n=N}^{N+k} x_n \Big|\Big|_{\mathcal{X}} \\ \leq \sum_{n=N}^{N+k} ||x_n||_{\mathcal{X}} &\leq \sum_{n=N}^{\infty} \end{aligned} \tag{1}$$

But by Cauchy's criteria for series convergence, the "tail" of series, i.e. the last term in the given equation, goes to 0 as $N \to \infty$. It follows the sequence is Cauchy, and thus as $\mathcal{X}$ is complete it converges to some x.

$\Leftarrow$

Conversely, suppose that every absolutely convergent series converges and that $\{x_n\}$ is a Cauchy sequence.

We choose $n_1 < n_2 < ...$ such that:

$$||x_n - x_m||_{\mathcal{X}} < 2^{-j}, \forall m, n \geq n_j$$

With this, define $y_1 = x_{n_1}$ and $y_j = x_{n_j} - x_{n_{j-1}}$. Then we note that $\sum_{j=1}^{k} = x_{n_k}$ and $\sum_{j=1}^{\infty} ||y||_{\mathcal{X}} \leq ||y_1||_{\mathcal{X}} + \sum_{j=1}^{\infty} 2^{-j} = ||y_1||_{\mathcal{X}} + 1 < \infty$.

So $\lim\limits_{k\to\infty} x_{n_k} = \sum_{j=1}^{\infty} y_j$ exists, so x_n is Cauchy and has a convergent subsequence, so x_n converges to some $x \in \mathcal{X}$. $\square$

If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces $\mathcal{X} \times \mathcal{Y}$ becomes a normed space equipped with the product norm $||(x, y)||_{\mathcal{X}\times\mathcal{Y}} = \max\{||x||_X, ||y||_Y\}$.

A linear map $T : \mathcal{X} \to \mathcal{Y}$ between two normed space is called **bounded** if:

$$\exists C > 0 \text{ s.t. }, ||T(x)||_Y \leq C||x||_X, \forall x \in \mathcal{X}$$

Theorem (Continuity characterization). If $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces and $T : X \to Y$ is a linear map, then the following statements are equivalent:

1. T is continuous.
2. T is continuous at 0.
3. T is bounded.

Proof. We see (1) has (2) automatically, so we assume T is continuous at $0 \in \mathcal{X}$ and work to show it is bounded. For this, note by definition of continuity that we have for some $\delta > 0$ that $||x||_{\mathcal{X}} \leq \delta \to ||T(x)||_{\mathcal{Y}} \leq 1$. It follows:

$$||T(x)||_Y = \Big|\Big|\frac{||x||_{\mathcal{X}}}{\delta} T\Big(\frac{\delta x}{||x||_{\mathcal{X}}}\Big)\Big|\Big| \leq \frac{||x||_{\mathcal{X}}}{\delta}$$

So $C = \frac{1}{\delta}$ bounds T.

Finally, if $||T(x)||_Y \leq C||x||_{\mathcal{X}}$, i.e. bounded, then $\forall x \in \mathcal{X}$ we have taking $||x-y||_X < \frac{\epsilon}{C}$:

$$||T(x) - T(y)||_{\mathcal{Y}} = ||T(x-y)||_{\mathcal{Y}} \leq C||x-y||_X < C\frac{\epsilon}{C} = C$$

So all statements are equivalent. One notes this also has continuous $\Leftrightarrow$ uniformly continuous. $\square$

If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces, we denote the space of all bounded linear maps from $\mathcal{X}$ to $\mathcal{Y}$ by $L(\mathcal{X}, \mathcal{Y})$. This trivially forms a vector space, which becomes a normed space under the following definition:

$$||T||_{L(\mathcal{X},\mathcal{Y})} = \sup\{||T(x)||_Y : ||x||_X = 1\} = \sup\Big\{\frac{||T(x)||_Y}{||x||_X} : x \neq 0\Big\}$$

We quickly verify the two given definitions are equivalent:

Proof. Let $A = \{||T(x)||_Y : ||x||_X = 1\}$ and $B = \Big\{\frac{||T(x)||_Y}{||x||_X} : x \neq 0\Big\}$. Let $a \in A$, i.e. for some $x \in \mathcal{X}$, $a = ||T(x)||_{\mathcal{Y}}$ where $||x||_{\mathcal{X}} = 1$.

By positive definiteness, we know $x \neq 0$, so $a = \frac{||T(x)||_{\mathcal{Y}}}{||x||_{\mathcal{X}}}$ for $x \neq 0$, for $a = b \in B$. And thus $\forall a \in A, a \leq \sup B$, so $\sup A \leq \sup B$.

For the other directions, consider then $b \in B$, i.e. $b = \frac{||T(x)||_{\mathcal{Y}}}{||x||_{\mathcal{X}}}$ for $x \neq 0$. In this case, $\frac{x}{||x||}$ exists, and its norm is just 1. But then:

$$b = \frac{||T(x)||_{\mathcal{Y}}}{||x||_{\mathcal{X}}} = \Big|\Big|T(\frac{x}{||x||_{\mathcal{X}}})\Big|\Big|_{\mathcal{Y}} \in A$$

And the same argument yields then that $\sup A = \sup B$. $\square$

Theorem I. f $\mathcal{Y}$ is complete, so is $L(\mathcal{X}, \mathcal{Y})$.

Proof. Let $\{T_n\}$ be a Cauchy sequence in $L(\mathcal{X}, \mathcal{Y})$. If $x \in \mathcal{X}$, then $\{T_n(x)\}$ is Cauchy in Y as we have the following lemma:

Lemma (Operator norm). Note that $||T(x)||_{\mathcal{Y}} \leq ||T||_{L(\mathcal{X},\mathcal{Y})}||x||_X$. In particular this is trivially shown for $x = 0$, and for $x \neq 0$ this is equivalent to:

$$\frac{||T(x)||_{\mathcal{Y}}}{||x||_{\mathcal{X}}} \leq ||T||_{L(\mathcal{X},\mathcal{Y})}$$

But this follows from definition.

So $||T_n(x) - T_m(x)||_{\mathcal{Y}} = ||(T_n - T_m)(x)||_{\mathcal{Y}} \leq ||T_n - T_m||_{L(\mathcal{X},\mathcal{Y})}||x||_{\mathcal{X}}$. So $\{T_n\}$ has $\{T_n(x)\}$ Cauchy in Y for every $x \in \mathcal{X}$, so as $\mathcal{Y}$ is complete each sequence $\{T_n(x)\}$ converges in $\mathcal{Y}$.

So we define $T(x) = \lim_{n\to\infty} T_n(x)$. This is clearly linear via the linearity of the limit, and we can show $\lim_{n\to\infty} ||T_n - T||_{L(\mathcal{X},\mathcal{Y})}$ by considering the set of $||(T_n - T)(x)||_Y$ for $||x||_{\mathcal{X}} = 1$. First though, we prove a lemma:

Lemma (Operator norm is actually a max). Let $T : X \to Y$ be a bounded linear operator. Then the function defined by $||T(x)||_{\mathcal{Y}}$ is continuous on X.

Proof. For this, just note that we have the following:

$$\begin{aligned} |||T(x)||_{\mathcal{Y}} - ||T(y)||_{\mathcal{Y}}| &\leq ||T(x) - T(y)||_{\mathcal{Y}} \\ &= ||T(x-y)||_{\mathcal{Y}} \leq C||x-y||_X \end{aligned} \tag{2}$$

And so it is continuous. ☐

It follows then that the restriction of this function to the unit circle in $\mathcal{X}$ is continuous, and so the supremum in the operator norm definition is actually a max by the extreme value theorem, i.e. $||T||_{L(\mathcal{X},\mathcal{Y})} = ||T(x)||_{\mathcal{Y}}$ for some $x \in X$ with $||x||_{\mathcal{X}} = 1$.

We return then to looking at $\lim_{n\to\infty} ||T_n - T||_{L(\mathcal{X},\mathcal{Y})}$. Under our lemma, this reduces to $\lim_{n\to\infty} \max\{||T_n(x) - T(x)||_Y : ||x||_{\mathcal{X}} = 1||\}$. **This is an incomplete proof, see https://math.stackexcha y-is-complete-then-bx-y-is-complete**. ☐

2 Linear functionals

Let $\mathcal{X}$ be a vector space over K, where $K = \mathbb{R} or \mathbb{C}$. A linear map from $\mathcal{X}$ to K is called a linear functional on $\mathcal{X}$. If $\mathcal{X}$ is a normed space, the space $L(\mathcal{X}, K)$ of bounded linear functionals on $\mathcal{X}$ is called the **dual space** of $\mathcal{X}$ and is denoted by $\mathcal{X}^*$. According to the previous theorem, $\mathcal{X}^*$ is a Banach space with the operator norm.

Note also if $\mathcal{X}$ a vector space over $\mathbb{C}$, it also is over $\mathbb{R}$. We can construct a relationship between complex and real functionals in this way then.

Theorem L. et $\mathcal{X}$ be a vector space over $\mathbb{C}$. If f is a complex linear functional and $u = \text{Re} f$, then u is a real linear functional, and $f(x) = u(x) - iu(ix)$ for all $x \in X$. Conversely, if u is a real linear functional and f is defined by $f(x) = u(x) - iu(ix)$, then f is complex linear. In this case, if $\mathcal{X}$ is normed, we have $||u|| = ||f||$.

Proof. I will prove this later, I'm tired. ☐

Definition (Minkowski functional). If $\mathcal{X}$ is a real vector space, a **Minkowski functional** on $\mathcal{X}$ is a map $p : \mathcal{X} \to \mathbb{R}$ such that $p(x+y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for all $x, y \in \mathcal{X}$ and $\lambda > 0$.