Tufts University
Department of Mathematics

Math 237 | **Homework #1 : Due on 4/12**[1] | **Spring, 2024**

Instructions:

• **Problems** $1--3, 5--12$ **are from chapter 9 of Heil's Introduction to Real Analysis. I wrote the rest of the problems from notes I have accumulated over the years. But I'm sure there should be in some textbooks in some variations.**

• **Problems 1, 6, 12,14, 16 will be collected and graded.**

Problem 1. Show that the only function $f \in L^1(\mathbb{R})$ such that $f = f * f$ is $f = 0$ a.e.

2. Suppose that $f \in L^1(\mathbb{R})$ and there exist $\alpha \in (0,1)$ and $C > 0$ such that $|\hat{f}(\xi)| \leq C/|\xi|^{1+\alpha}$ foe $\alpha \neq 0$. Prove that f is Hölder continuous with exponent α.

3. Suppose that $f \in L^1(\mathbb{R})$ is such that $\hat{f} \in L^1(\mathbb{R})$. Prove the following statements:

   3.1 $f, \hat{f} \in C_0(\mathbb{R})$.

   3.2 $\hat{\hat{f}}(x) = f(-x)$ for all $x \in \mathbb{R}$.

   3.3 $\hat{\hat{\hat{\hat{f}}}}(x) = f(x)$ for all $x \in \mathbb{R}$

4. Let $h(x) = e^{-\pi x^2}$ for $x \in \mathbb{R}$. Justify why $\hat{h}(\xi)$ is the solution to the differential equation

$$\begin{cases} y'(\xi) + 2\pi\xi y(\xi) = 0 \ \ \forall \xi \in \mathbb{R} \\ y(0) = 1 \end{cases}$$

Conclude that $\hat{h}(\xi) = e^{-\pi\xi^2}$ for all $\xi \in \mathbb{R}$.

Problem 5. Suppose $\mathbb{T}$ is the one-dimensional torus. Suppose $f \in L^1(\mathbb{T})$ and $g \in C(\mathbb{T})$.

   5.1 Show that $f * g \in C(\mathbb{T})$.

   5.2 Assume now that $g \in C^1(\mathbb{T})$. Show that $f * g \in C^1(\mathbb{T})$ and $(f * g)' = f * g'$.

Problem 6. Suppose $f \in AC(\mathbb{T})$, i.e., f is 1 periodic and absolutely continuous on [01].

   6.1 Prove that $\hat{f}'(n) = 2\pi i n \hat{f}(n)$ for $n \in \mathbb{Z}$ and conclude that $\lim_{|n|\to\infty} n\hat{f}(n) = 0$.

   6.2 Show that if $\int_0^1 f(x)dx = 0$, then

$$\int_0^1 |f(x)|^2 dx \leq \tfrac{1}{4\pi^2} \int_0^1 |f'(x)|^2 dx.$$

Problem 7. Prove that if $f \in L^1(\mathbb{T})$ and $g \in L^\infty(\mathbb{T})$, then

$$\lim_{m\to\infty} \int_0^1 f(x)g(mx)dx = \hat{f}(0)\hat{g}(0) = \left(\int_0^1 f(x(dx\right)\left(\int_0^1 g(x(dx\right)$$

Problem 8. Let $f, g \in L^2(\mathbb{R})$. Prove the following statements.

   8.1 $\widehat{fg}$ is continuous, $\widehat{fg} = \hat{f} * \hat{g}$, and $f * g = (\hat{f}\hat{g})\check{}$.

   8.2 If in addition, $f*g \in L^2(\mathbb{R})$, then $\widehat{f * g} = \hat{f}\hat{g}$. In particular, this is the case when $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$.

8.3 $f \in AC(\mathbb{R})$ if and only if $f = g * h$ for some $g, h \in L^2(\mathbb{R})$. That is $AC(\mathbb{R}) = L^2(\mathbb{R}) * L^2(\mathbb{R})$.

Problem 9. $AC(\mathbb{T}) = L^2(\mathbb{T}) * L^2(\mathbb{T})$ that is $f \in AC(\mathbb{T})$ if and only if $f = g * h$ for some $g, h \in L^2(\mathbb{T})$

Problem 10. Find a nontrivial $f \in L^2(\mathbb{R})$ such that $f = f * f$ a.e.

Problem 11. Fix $g \in L^2(\mathbb{R})$. Prove that $\{T_a g := g(\cdot - a)\}_{a\in\mathbb{R}}$ is complete in $L^2(\mathbb{R})$ if and only if $\hat{g}(\xi) \neq 0$ a.e.

Problem 12. Fix $g \in L^2(\mathbb{R})$. Prove that $\{T_k g := g(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal sequence if and only if

$$\sum_{k\in\mathbb{Z}} |\hat{g}(\xi - k)|^2 = 1 \ a.e.$$

Problem 13. Let $H = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \, \forall |\xi| > 1/2\}$. Any function in H is called a badlimited function with bandwidth 1.

13.1 Prove that H is a closed subspace of $L^2(\mathbb{R})$.

13.2 Prove that $f \in H$, then the inversion formula $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi = \int_{-1/2}^{1/2} \hat{f}(\xi) e^{2\pi i x \xi}\, d\xi$ holds for all $x \in \mathbb{R}$, and $f \in C^\infty(\mathbb{R})$. That is, $H \subset L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$.

13.3 For each $k \in \mathbb{Z}$ define $\varphi_k(x) = \operatorname{sinc}(x-k) = \frac{\sin \pi(x-k)}{\pi(x-k)}$. Prove that $\{\varphi_k\}$ is an orthonormal sequence in H.

13.4 Suppose that $f \in H$ is such that $\langle f, \varphi_k \rangle = 0$ for all $k \in \mathbb{Z}$. Prove that $\hat{f}(\xi) = 0$ a.e., and conclude that $\{\varphi_k\}$ is an orthonormal basis for H.

13.5 Prove that for each $f \in H$

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\varphi_k(x) = \sum_{k\in\mathbb{Z}} f(k) \frac{\sin \pi(x-k)}{\pi(x-k)}$$

where the series converges in L^2 and uniformly. This formula is the Shannon Sampling Theorem.

Problem 14. Let $p(x) = \chi_{[0,1)}(x)$ and $h(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x)$. For $j, k \in \mathbb{Z}$, set $I_{jk} = [2^{-j}k, 2^{-j}(k+1))$ and define

$$\begin{cases} p_{jk}(x) = 2^{j/2} p(2^j x - k) \\ h_{jk}(x) = 2^{j/2} h(2^j x - k) \end{cases}$$

14.1 Prove that $\{h_{jk}\}_{j,k\in\mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$.

14.2 For $j \in \mathbb{Z}$, show that $\{p_{jk}\}_{k\in\mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$.

14.3 Fix$j \in \mathbb{Z}$ and let $g_j(x)$ be a step function, constant on all (dyadic intervals) $I_{j,k}$ $k \in \mathbb{Z}$. Show that $g_j(x) = g_{j-1}(x) + r_{j-1}(x)$ where

$$r_{j-1}(x) = \sum_{k\in\mathbb{Z}} a_{j-1}(k) h_{j-1,k}(x)$$

for some coefficients $\{a_{j-1}(k)\}_{k\in\mathbb{Z}}$ and $g_{j-1}(x)$ is a step function constant on all intervals $I_{j-1,k}$ $k \in \mathbb{Z}$

14.4 Now fix $J \geq 0$ and consider

$$\{p_{Jk} : 0 \leq k \leq 2^J - 1\} \cup \{h_{j,k} : j \geq J \,\&\, 0 \leq k \leq 2^j - 1\}$$

Prove that $\{p_{Jk} : 0 \leq k \leq 2^J - 1\} \cup \{h_{j,k} : j \geq J \,\&\, 0 \leq k \leq 2^j - 1\}$ is an orthonormal sequence in $L^2[0, 1]$.

14.5 For $f \in L^2[0, 1]$ find g_j $(j \geq J)$ a step function constant on all $I_{j,k}$ $k \in \mathbb{Z}$ that approximate f in the L^2 norm. Use this and the previous question to show $\{p_{Jk} : 0 \leq k \leq 2^J - 1\} \cup \{h_{j,k} : j \geq J \,\&\, 0 \leq k \leq 2^j - 1\}$ is an orthonormal basis for $L^2[0, 1]$

Problem 15. Let $Df = f'$ and for $k \in \mathbb{N}$ let $D^k f = f^{(k)}$.

15.1 Show that if f is $n-$times differentiable and $x^j f^{(k)}(x) \in L^1(\mathbb{R})$ for $j = 0, 1, 2, \ldots, m$ and $k = 0, 1, 2, \ldots, n$, then

$$(D^n(\widehat{(-2\pi ix)^m} f(x)))(\xi) = (2\pi i\xi)^n D^m \hat{f}(\xi), \; \forall \xi \in \mathbb{R}$$

15.2 The Schwartz space is

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R}) \text{ for } \quad \text{all } m, n \geq 0\}$$

Exhibit a nonzero function in $\mathcal{S}(\mathbb{R})$, and show that if $f \in \mathcal{S}(\mathbb{R})$, then $f^{(n)} \in L^1(\mathbb{R})$ for every $n \geq 0$. Prove that $\mathcal{S}(\mathbb{R})$ is dense in $L^1(\mathbb{R})$.

15.3 Show that if $f \in \mathcal{S}(\mathbb{R})$, then $\hat{f} \in \mathcal{S}(\mathbb{R})$.

15.4 Prove that the Fourier transform maps $\mathcal{S}(\mathbb{R})$ bijectively onto itself.

Problem 16. Let $0 \neq \varphi \in L^2(\mathbb{R})$ and for each $f \in L^2(\mathbb{R})$ let $V_\varphi f$ be the function defined on $\mathbb{R}^2$ by

$$V_\varphi f(x, \xi) = \int_{\mathbb{R}} f(t)\overline{\varphi(t-x)} e^{-2\pi it\xi}\, dt.$$

For $a, b \in \mathbb{R}$, let T_a be the translation operator by a ($T_a f(x) = f(x-a)$), and M_b the modulation operator by b ($M_b f(x) = e^{2\pi bx} f(x)$).

16.1 Prove that for each $f \in L^2(\mathbb{R})$, $V_\varphi f$ is uniformly continuous on $\mathbb{R}^2$ and that $\lim_{|(x,\xi)| \to \infty} V_\varphi f(x, \xi) = 0$.

16.2 Prove that if $f, \varphi \in \mathcal{S}(\mathbb{R})$, then $V_\varphi f \in \mathcal{S}(\mathbb{R}^2)$; that is $V_\varphi f \in C^\infty(\mathbb{R}^2)$ and for all $n, m, k, \ell \geq 0$

$$x^n y^m \tfrac{\partial^{k+l} V_\varphi f(x,y)}{\partial^k x \partial^\ell y} \in L^\infty(\mathbb{R}^2)$$

16.3 Prove that V_φ is an isometry from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^2)$ and that $\|V_\varphi f\|_{L^2(\mathbb{R}^2)} = \|\varphi\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}$ for each $f \in L^2(\mathbb{R})$.

16.4 Show that the operator V_φ^* given by

$$V_\varphi^* F(t) = \|\varphi\|_2^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, \xi) e^{2\pi i\xi t} \varphi(t-x) dx d\xi$$

maps $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R})$ and the for each $f \in L^2(\mathbb{R})$ the following inversion formula holds in a sense you should specify:

$$f(t) = \|\varphi\|_2^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V_\varphi f(x, \xi) e^{2\pi i\xi t} \varphi(t-x) dx d\xi$$