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Exercise 1. Show that the only function $f \in L^1(\mathbb{R})$ such that $f = f * f$ is $f = 0$ a.e.

Proof. Let $f \in L^1(\mathbb{R})$ such that $f = f * f$. Recall we have that by Heil **Exercise 9.2.6** that then

$$\hat{f} = (f * f)\hat{} = \hat{f}\hat{f},$$

so for any particular ξ we have

$$\hat{f}(\xi)^2 - \hat{f}(\xi) = 0,$$

so we consequently have

$$\hat{f}(\xi) = 0 \text{ or } 1, \quad \forall \xi \in \mathbb{R}.$$

Recall then that $\hat{f}$ is continuous on $\mathbb{R}$; it trivially follows that it thus must be either identically 0 or identically 1.

However, the **Riemann-Lebesgue Lemma** guarentees that $|\hat{f}|$ should decay to 0 as $|x| \to \infty$; so the only possibility is that $\hat{f}$ is constantly 0.

Note then clearly $\hat{0} = 0$; by the **uniqueness of Fourier transforms** then, as we thus have $\hat{f} = \hat{0}$, it must be then that $f = 0$ a.e., as desired □

Exercise 6. Suppose $f \in \mathrm{AC}(\mathbb{T})$, i.e., f is 1-periodic and absolutely continuous on $[0, 1]$.

6.1 Prove that $\hat{f'}(n) = 2\pi i n \hat{f}(n)$ for $n \in \mathbb{Z}$ and conclude $\lim\limits_{|n|\to\infty} n\hat{f}(n) = 0$.

Proof. By assumption $f \in AC(\mathbb{T})$. We know of course that $e^{2\pi i n\xi}$ is furthermore absolutely continuous are we observe it is continuously differentiable.

Thus, we can apply **integration by parts**, which gets us the following (note $\frac{d}{d\xi}e^{-2\pi i n\xi} = -2\pi i n e^{-2\pi i n\xi}$):

$$\begin{aligned} \hat{f'}(n) &= \int_0^1 f'(\xi)e^{-2\pi i n\xi}d\xi \\ &= e^{-2\pi i n\xi}f(1) - f(0) - \int_0^1 -2\pi i n f(\xi)e^{-2\pi i n\xi}d\xi \\ &= e^{-2\pi i n\xi}f(1) - f(0) + 2\pi i n\hat{f}(n) \end{aligned}$$

We note then that for any value of $n \in \mathbb{Z}$, we have $e^{-2\pi i n\xi} = 1$. Moreover, as f is 1-periodic, we have $f(0) = f(1)$. Thus we can further reduce

$$\begin{aligned} &e^{-2\pi i n\xi}f(1) - f(0) + 2\pi i n\hat{f}(n) \\ &= 2\pi i n\hat{f}(n), \end{aligned}$$

as desired. Recall the by the **Riemann-Lebesgue Lemma** we know

$$\lim_{|n|\to\infty} |\hat{f'}(n)| = 0,$$

and thus

$$\lim_{|n|\to\infty} |\hat{f}'(n)| = \lim_{|n|\to\infty} |2\pi i n \hat{f}(n)| = \lim_{|n|\to\infty} 2\pi n |\hat{f}(n)| = 0,$$

where from limit rules multiplying by $\frac{1}{2\pi}$ gets the limit of $n|\hat{f}(n)|$ as 0, which clearly implies the same for $n\hat{f}(n)$, as additionally desired. $\square$

6.2 Show that if $\int_0^1 f(x)dx = 0$, then

$$\int_0^1 |f(x)|^2 dx \leq \frac{1}{4\pi^2} \int_0^1 |f'(x)|^2 dx$$

Exercise 12. Fix $g \in L^2(\mathbb{R})$. Prove that $\{T_k g = g(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal sequence if and only if

$$\sum_{k\in\mathbb{Z}} |\hat{g}(\xi - k)|^2 = 1 \text{ a.e.}$$

Proof. Let $n, m \in \mathbb{Z}$. We can derive the following:

$$\begin{aligned} &\langle T_n g, T_m g\rangle \\ &= \int_{-\infty}^{\infty} T_n g(x) \overline{T_m g(x)} dx \\ &= \int_{\infty}^{\infty} (T_n g)\hat{}(\xi) \overline{(T_m g)\hat{}(\xi)} d\xi \qquad \text{(Parseval's Theorem)} \end{aligned}$$

We would like to say now that $(T_m g)\hat{}(\xi) = e^{-2\pi i m \xi}\hat{g}(\xi)$ for a.e. ξ, i.e. we have the translation identity for $L^2(\mathbb{R})$ functions.

For this, define $g_R = \mathbb{1}_{[-R,R]} g$. Then clearly each g_R is in $L^2(\mathbb{R})$, as as we have $L^2 \subseteq L^1$ on bounded domains we have $g_R \in L^1(\mathbb{R})$, i.e. the Fourier transform is defined.

Note then

$$\begin{aligned} &(T_m g_R)\hat{}(\xi) \\ &= \int_{-\infty}^{\infty} T_m g_R(x) e^{-2\pi i \xi x} dx \\ &= \int_{-\infty}^{\infty} g(x-m) e^{-2\pi i \xi x} dx \\ &= \int_{-\infty}^{\infty} g(u) e^{-2\pi i \xi (u+m)} dx \qquad (u - \text{substitution}) \\ &= \int_{-\infty}^{\infty} \left(g_R(u) e^{-2\pi i \xi u}\right) e^{-2\pi i \xi m} \\ &= e^{-2\pi i \xi m} \int_{-\infty}^{\infty} g_R(u) e^{-2\pi i \xi u} \\ &= e^{-2\pi i \xi m} \hat{g_R}(\xi), \end{aligned}$$

moreover, note we have $g_R \xrightarrow{L^2} g$. Thus, by definition, for $g \in L^2(\mathbb{R})$ it is precisely $\hat{g} = \lim\limits_{R\to\infty} \hat{g_R}$. Thus we get

$$\begin{aligned}
&(T_m g)\hat{} \\
&= \left(T_m\left(\lim_{R\to\infty} g_R\right)\right)\hat{} \\
&= \left(\lim_{R\to\infty} T_m g_R\right)\hat{} && \text{(Translation commutes with limit)} \\
&= \lim_{R\to\infty} (T_m g_R)\hat{} && \text{(Fourier transform operator is unitary)} \\
&= \lim_{R\to\infty} e^{-2\pi i m\xi}(g_R)\hat{} \\
&= e^{-2\pi i m\xi} \lim_{R\to\infty} \hat{g}_R \\
&= e^{-2\pi i m\xi}\hat{g},
\end{aligned}$$

where this equality is in the L^2 sense, i.e. precisely that $(T_m g)\hat{}(\xi) = e^{-2\pi i m\xi}\hat{g}(\xi)$ for a.e. ξ. Thus we can continue our derivation:

$$\begin{aligned}
&\int_{\infty}^{\infty} (T_n g)\hat{}(\xi)\overline{(T_m g)\hat{}(\xi)}d\xi \\
&= \int_{-\infty}^{\infty} e^{-2\pi i n\xi}\hat{g}(\xi)\overline{e^{-2\pi i m\xi}\hat{g}(\xi)}d\xi && \text{(Identity for transform of translation)} \\
&= \int_{-\infty}^{\infty} e^{-2\pi i n\xi}\overline{e^{-2\pi i m\xi}}\hat{g}(\xi)\overline{\hat{g}(\xi)}d\xi && \text{(Modulus is multiplicative)} \\
&= \int_{-\infty}^{\infty} e^{-2\pi i(n-m)\xi}|\hat{g}(\xi)|^2 d\xi && (z\overline{z} = |z|^2, \overline{e^{-2\pi i m\xi}} = e^{2\pi i m\xi}) \\
&= \sum_{k\in\mathbb{Z}} \int_{k}^{k+1} e^{-2\pi i(n-m)\xi}|\hat{g}(\xi)|^2 d\xi \\
&= \sum_{k\in\mathbb{Z}} \int_{0}^{1} e^{-2\pi i(n-m)(\xi+k)}|\hat{g}(\xi+k)|^2 d\xi && \text{(Translating the integral)}
\end{aligned}$$

We would like now to interchange the sum and integral. This is possible as a special case of Fubini's Theorem, with respect to the counting measure on $\mathbb{Z}$ and Lebesgue measure on $\mathbb{R}$.

In particular, refer temporarily to $f_{n,m}$ as the integrand. Then we have the following, as our exponenial always falls on the unit circle in the complex plane:

$$\int_0^1 \sum_{k\in\mathbb{Z}} |f_{n,m}| = \int_0^1 \sum_{k\in\mathbb{Z}} |\hat{g}(\xi+k)|^2 = \int_k^{k+1} \sum_{k\in\mathbb{Z}} |\hat{g}(\xi)|^2,$$

where monotone convergence, given the non-negativity of each $|\hat{g}(\xi)|^2$, has

$$\int_k^{k+1} \sum_{k\in\mathbb{Z}} |\hat{g}(\xi)|^2 = \sum_{k\in\mathbb{Z}} \int_k^{k+1} |\hat{g}(\xi)|^2 = \int_{\infty}^{\infty} |\hat{g}(\xi)|^2 < \infty,$$

where the finiteness of the last term follows as $g \in L^2(\mathbb{R}) \to \hat{g} \in L^2(\mathbb{R})$.

Thus $\int_0^1 \sum_{k\in\mathbb{Z}} |f_{n,m}|$ is finite; by Tonelli's theorem then, again with respect to the counting and Lebesgue measures, $\sum_{k\in\mathbb{Z}} \int_0^1 |f_{n,m}|$ is as well. Thus the hypotheses for Fubini's theorem are fulfilled, and we interchange the sum. We continue then:

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} \int_0^1 e^{-2\pi i(n-m)(\xi+k)} |\hat{g}(\xi+k)|^2 d\xi \\ &= \int_0^1 \sum_{k\in\mathbb{Z}} e^{-2\pi i(n-m)(\xi+k)} |\hat{g}(\xi+k)|^2 d\xi \qquad \text{(Fubini's Theorem)} \\ &= \int_0^1 \sum_{k\in\mathbb{Z}} e^{-2\pi i(n-m)\xi} |\hat{g}(\xi+k)|^2 d\xi \\ &= \int_0^1 \Big(\sum_{k\in\mathbb{Z}} |\hat{g}(\xi+k)|^2 \Big) e^{-2\pi i(n-m)\xi} d\xi \end{aligned}$$

Define $H(\xi) = \sum_{k\in\mathbb{Z}} |\hat{g}(\xi+k)|^2$. Note then our earlier work done to show we can apply Fubini's theorem has $H \in L^1(\mathbb{R})$.

We also observe the 1-periodicity of $\sum_{k\in\mathbb{Z}} |\hat{g}(\xi+k)|^2$, which is obvious as k varies over all $\mathbb{Z}$.

Thus $H(\xi) = \sum_{k\in\mathbb{Z}} |\hat{g}(\xi+k)|^2 \in L^1(\mathbb{T})$; so its Fourier transform is defined. We can continue then:

$$\begin{aligned} &\int_0^1 \Big(\sum_{k\in\mathbb{Z}} |\hat{g}(\xi+k)|^2 \Big) e^{-2\pi i(n-m)\xi} d\xi \\ &= \int_0^1 H(\xi) e^{-2\pi i(n-m)\xi} d\xi \\ &= \hat{H}(n-m) \end{aligned}$$

Which is essentially the final fact we need. Assume then $\{T_k g\}_{k\in\mathbb{Z}}$ is an orthonormal sequence; then by our work we get the Fourier coefficients of H as

$$\hat{H}(i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases} = \delta(i),$$

where δ is the Dirac delta function. Using the **uniqueness of Fourier coefficients** then, we must have $H = 1$ a.e., as the Dirac delta function gives precisely the Fourier coefficients of 1.

Thus we have $H = 1$ a.e., which is precisely $\sum_{k\in\mathbb{Z}} |\hat{g}(\xi+k)|^2 = 1$ a.e.

Assume then instead we have $\sum_{k\in\mathbb{Z}} |\hat{g}(\xi+k)|^2 = 1$ a.e.; we recall in the process of our derivation we also established

$$\langle T_n g, T_m g\rangle = \int_0^1 \Big(\sum_{k\in\mathbb{Z}} |\hat{g}(\xi+k)|^2\Big) e^{-2\pi i(n-m)\xi} d\xi,$$

so under our assumption this clearly just becomes

$$\langle T_n g, T_m g\rangle = \int_0^1 e^{-2\pi i(n-m)\xi}.$$

When $n = m$, the integrand is 1, so the integral reduces to 1. When $n \neq m$, we can apply the **FTC** (note the integrand is continuous) to get

$$\begin{aligned}\int_0^1 e^{-2\pi i(n-m)\xi} &= \frac{-e^{-2\pi i(n-m)}}{2\pi i(n-m)} - \left(-\frac{1}{2\pi i(n-m)}\right)\\ &-\frac{1}{2\pi i(n-m)} - \left(-\frac{1}{2\pi i(n-m)}\right) = 0,\end{aligned}$$

which in tandem with the previous fact clearly shows that $\{T_k g\}_{k\in\mathbb{Z}}$ is an orthonormal sequence. Thus we have shown both directions, and we are done. $\square$