Theorem (Parallelogram Law). If H is an inner product space, then for every $x, y \in H$:

$$||x+y||^2 + ||x-y||^2 + 2(||x||^2 + ||y||^2) \tag{1}$$

Example: (lp Norm) We define the l_p norm for $\mathbb{C}^d$ for $\forall p \in (0, \infty)$ by the following convention:

$$||x||_p = \left(\sum_{k=1}^{d} |x_k|^p \right)^{\frac{1}{p}} \tag{2}$$

Interestingly, $p = 2$ uniquely generates the norm that satisifes the Parallelogram law.

Theorem (Unique Closest Point). Let H be a Hilbert space and M be a closed subspace of H.

For every $x \in H$, there exists a unique $p \in M$ that is closest to x. That is that for some $p \in M$:

$$||x-p|| = \inf\{||x-m||, m \in M\} = \text{dist}(x, M) \tag{3}$$

Proof. Let $d = \text{dist}(x, M)$. By the definition of the infinum, $\{y_n\}_{n=1}^{\infty} \subseteq M$ such that $d = \lim\limits_{n\to\infty} ||x - y_n||$.

Take then $d^2 = \lim\limits_{n\to\infty} ||x-y_n||^2$. Let $\epsilon > 0$. Using the definition of the limit, there $\exists N \in \mathbb{N}$ such that $\forall n \geq N$:

$$d^2 \leq ||x-y_n||^2 < d^2 + \epsilon$$

Take then $n, m \geq N$, this yields the following by applying the Parallelogram law:

$$||(x-y_n) - (x-y_m)||^2 + ||(x-y_n) + (x+y_m)||^2 = 2(||x-y_n||^2 + ||x-y_m||^2) \tag{4}$$

However, we also get the following by using homogeneity:

$$||(x-y_n) - (x-y_m)||^2 + ||(x-y_n) + (x+y_m)||^2 = ||y_n - y_m||^2 + 4||x - \frac{(y_n+y_n)}{4}||^2 \tag{5}$$

Which gets us the following (as $\frac{(y_n+y_m)}{2} \in M$):

$$\begin{gathered} 4d^2 + ||y_n - y_m||^2 \leq 4d^2 + 4\left|\left|x - \frac{(y_n - y_m)}{2}\right|\right|^2 \\ = 2(||x-y_n||^2 + ||x-y_m||^2) = 2||x-y_n||^2 + 2||x-y_m||^2 < 4d^2 + 4\epsilon \end{gathered} \tag{6}$$

Which of course finally has:

$$||y_n - y_m||^2 < \epsilon \tag{7}$$

So $\{y_n\}_{n=1}^{\infty}$ is Cauchy, and so it converges to some $y \in M$, where we get the final idea:

$$d = \lim_{n\to\infty} ||x-y_n|| = ||x - \lim_{n\to\infty} y_n|| = ||x-y|| \tag{8}$$

Uniqueness is left as exercise to the reader. □

Definition (Orthogonal Projection). If M is a closed subspace of a Hilbert space H, for every $x \in H$, the unique $p \in M : ||x-p|| = \text{dist}(x, M)$ is called the **orthogonal projection** of x onot M denoted $P_M x$.

Turning this into a function P_M, we note it is a surjection. Moreover $P_M \in B(H)$.

Definition (Orthogonal Complement). Let A be a subset of an inner product space H, we define the following to be the **orthogonal complement of A**:

$$A^\perp = \{x \in H : \langle x, a \rangle = 0, \forall a \in A\} \tag{9}$$

If $A \subseteq H$ is a subset and H is an inner product space, then $A^\perp$ is a subspace of H.

Definition (Spans and Completeness). Let S be a subset of a normed space H. The **span** of S is the subspace of H that contains all finite linear combinations of vectors in S.

The **closed linear span** of S is the closure of the linear span of S, denoted $\overline{\text{span}(S)}$.

S is said to be **complete** if $\overline{\text{span}(S)} = H$.

Theorem (Orthogonal Decomposition). Let H be a Hilbert space, M a closed subspace, and $x \in H$. Then, the following statements are equivalent:

(a) $x = p + e$, where $p = P_M x$

(b) $x = p + e$ where $p \in M$ and $e \in M^\perp$

(c) $x = p + e$, where $e = P_{M^\perp} x$

Theorem (Spans and Orthogonal Complements). Let H be a Hilbert space, and A a subset of H.

(a) If M is a closed subspace of H:
$$(M^\perp)^\perp = M$$

(b) If $A \subseteq H$:
$$A^\perp = (\overline{\text{span}(A)})^\perp$$
and
$$(A^\perp)^\perp = ((\overline{\text{span}(A)})^\perp)^\perp = \overline{\text{span}(A)}$$

(c) If $A = \{y_n\}_{n=1}^\infty$ is a sequence, A is complete if and only $\forall x \in H$, $\langle x, y_n \rangle = 0, \forall n \in \mathbb{N}$.

Definition (Basis). Let H be a Hilbert space. A sequence $\{x_n\}_{n=1}^\infty$ is a **basis** for H if $\forall x \in H$:

$$x = \sum_{k=1}^\infty c_k x_n$$

For some unique constants $\{c_k\}_{k=1}^\infty$.

Definition (Orthogonal Sequence). A sequence $\{x_n\}_{n=1}^{\infty}$ is called **orthogonal** if $\langle x_n, x_m \rangle = 0$ when $n \neq m$.

Definition (Orthonormal Sequence). An **orthonormal** sequence is is an orthogonal sequence with the added behavior that:

$$\langle x_n, x_m \rangle = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \tag{10}$$

Definition (Orthonormal Basis). A sequence $\{x_n\}_{n=1}^{\infty}$ is an **orthonormal basis** if it is both orthonormal and a basis.

Theorem (Big Theorem on Orthnormal Sequences). If $\{x_n\}_{n=1}^{\infty}$ is an orthonormal sequence in a Hilbert space H, then:

(a) **(Bessel's Inequality)** $\forall x \in H, \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq ||x||^2$

(b) If $x = \sum_{n=1}^{\infty} cx_n$ for a sequence $\{c_n\}_{n=1}^{\infty} \subseteq F$, then $c_n = \langle x, x_n \rangle$ for all $n \geq 1$.

(c) $\sum_{n=1}^{\infty} c_n x_n$ converges $\Leftrightarrow \sum_{n=1}^{\infty} |c_n|^2 < \infty$

(d) $x \in \overline{\text{span}\{x_n\}_{n=1}^{\infty}} \Leftrightarrow x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$

(e) For $x \in H, p = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$ is the orthogonal projection of x onto $\overline{\text{span}\{x_n\}_{n=1}^{\infty}}$.

Proof. Proof in next lecture's notes. □