

Exercise 1. Show that the only function $f \in L^1(\mathbb{R})$ such that $f = f * f$ is $f = 0$ a.e.

Proof. Let $f \in L^1(\mathbb{R})$ such that $f = f * f$. Recall we have that by Heil **Exercise 9.2.6** that then

$$\hat{f} = (f * f)^\wedge = \hat{f}\hat{f},$$

so for any particular ξ we have

$$\hat{f}(\xi)^2 - \hat{f}(\xi) = 0,$$

so we consequently have

$$\hat{f}(\xi) = 0 \text{ or } 1, \quad \forall \xi \in \mathbb{R}.$$

Recall then that \hat{f} is continuous on \mathbb{R} ; it trivially follows that it thus must be either identically 0 or identically 1.

However, the **Riemann-Lebesgue Lemma** guarantees that $|\hat{f}|$ should decay to 0 as $|x| \rightarrow \infty$; so the only possibility is that \hat{f} is constantly 0.

Note then clearly $\hat{0} = 0$; by the **Uniqueness Theorem** then, as we thus have $\hat{f} = \hat{0}$, it must be then that $f = 0$ a.e., as desired \square

Exercise 6. Suppose $f \in AC(\mathbb{T})$, i.e., f is 1-periodic and absolutely continuous on $[0, 1]$.

6.1 Prove that $\hat{f}'(n) = 2\pi i n \hat{f}(n)$ for $n \in \mathbb{Z}$ and conclude $\lim_{|n| \rightarrow \infty} n \hat{f}(n) = 0$.

Proof. By assumption $f \in AC(\mathbb{T})$. We know of course that $e^{2\pi i n \xi}$ is furthermore absolutely continuous and we observe it is continuously differentiable.

Thus, we can apply **Integration by Parts**, which gets us the following (note $\frac{d}{d\xi} e^{-2\pi i n \xi} = -2\pi i n e^{-2\pi i n \xi}$):

$$\begin{aligned} \hat{f}'(n) &= \int_0^1 f'(\xi) e^{-2\pi i n \xi} d\xi \\ &= e^{-2\pi i n \xi} f(1) - f(0) - \int_0^1 -2\pi i n f(\xi) e^{-2\pi i n \xi} d\xi \\ &= e^{-2\pi i n \xi} f(1) - f(0) + 2\pi i n \hat{f}(n) \end{aligned}$$

We note then that for any value of $n \in \mathbb{Z}$, we have $e^{-2\pi i n \xi} = 1$. Moreover, as f is 1-periodic, we have $f(0) = f(1)$. Thus we can further reduce

$$\begin{aligned} e^{-2\pi i n \xi} f(1) - f(0) + 2\pi i n \hat{f}(n) \\ = 2\pi i n \hat{f}(n), \end{aligned}$$

as desired. Recall then by the **Riemann-Lebesgue Lemma** we know

$$\lim_{|n| \rightarrow \infty} |\hat{f}'(n)| = 0,$$

and thus

$$\lim_{|n| \rightarrow \infty} |\hat{f}'(n)| = \lim_{|n| \rightarrow \infty} |2\pi i n \hat{f}(n)| = \lim_{|n| \rightarrow \infty} 2\pi n |\hat{f}(n)| = 0,$$

where from limit rules multiplying by $\frac{1}{2\pi}$ gets the limit of $n|\hat{f}(n)|$ as 0, which clearly implies the same for $n\hat{f}(n)$, as additionally desired. \square

6.2 Show that if $\int_0^1 f(x)dx = 0$, then

$$\int_0^1 |f(x)|^2 dx \leq \frac{1}{4\pi^2} \int_0^1 |f'(x)|^2 dx$$

Exercise 12. Fix $g \in L^2(\mathbb{R})$. Prove that $\{T_k g = g(\cdot - k)\}_{k \in \mathbb{Z}}$ is an orthonormal sequence if and only if

$$\sum_{k \in \mathbb{Z}} |\hat{g}(\xi - k)|^2 = 1 \text{ a.e.}$$

Proof. Let $n, m \in \mathbb{Z}$. We can derive the following:

$$\begin{aligned} & \langle T_n g, T_m g \rangle \\ &= \int_{-\infty}^{\infty} T_n g(x) \overline{T_m g(x)} dx \\ &= \int_{-\infty}^{\infty} (T_n g)^\wedge(\xi) \overline{(T_m g)^\wedge(\xi)} d\xi && \text{(Parseval's Theorem)} \\ &= \int_{-\infty}^{\infty} e^{-2\pi i n \xi} \hat{g}(\xi) \overline{e^{-2\pi i m \xi} \hat{g}(\xi)} d\xi && \text{(Identity for transform of translation)} \\ &= \int_{-\infty}^{\infty} e^{-2\pi i n \xi} e^{-2\pi i m \xi} \hat{g}(\xi) \overline{\hat{g}(\xi)} d\xi && \text{(Modulus is multiplicative)} \\ &= \int_{-\infty}^{\infty} e^{-2\pi i (n+m) \xi} |\hat{g}(\xi)|^2 d\xi && (z\bar{z} = |z|^2) \\ &= \sum_{k \in \mathbb{Z}} \int_k^{k+1} e^{-2\pi i (n+m) \xi} |\hat{g}(\xi)|^2 d\xi \\ &= \sum_{k \in \mathbb{Z}} \int_0^1 e^{-2\pi i (n+m)(\xi+k)} |\hat{g}(\xi+k)|^2 d\xi && \text{(Translating the integral)} \\ &= \int_0^1 \sum_{k \in \mathbb{Z}} e^{-2\pi i (n+m)(\xi+k)} |\hat{g}(\xi+k)|^2 d\xi && \text{(Fubini's Theorem)} \\ &= \int_0^1 \sum_{k \in \mathbb{Z}} e^{-2\pi i (n+m)\xi} |\hat{g}(\xi+k)|^2 d\xi \\ &= \int_0^1 \left(\sum_{k \in \mathbb{Z}} |\hat{g}(\xi+k)|^2 \right) e^{-2\pi i (n+m)\xi} d\xi \end{aligned}$$

Say then the given sequence is orthonormal. \square