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Math 237: Lecture 6

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Theorem (Riesz Representation Theorem for Hilbert Space). Let H be a Hilbert space. Any bounded linear functional T on H has the form $Tx = \langle x, z\rangle$ for a unique $z \in H$. and $||T|| = \sup -||x||_H = 1|Tx|||z||_H$.

Proof. 1. If $Tx = 0, \forall x \in H$, then $z = 0$.

2. $T \neq 0 \Leftrightarrow x_0 \in H : Tx_0 \neq 0 \Leftrightarrow \ker T =\neq H$.

   But $\ker(T)$ is a closed linear subspace of H, so we have $H = \ker(T) \oplus \ker(T)^{\perp}$.

   Thus we have that $x_0 = z_1 + z_0$ for $z_1 \in \ker(T), z_0 \in \ker(T)^{\perp}$. Let $x \in H$ and consider then the following for $v = T(z_0)x - T(x)z_0$:

$$Tv = T(z_0)T(x) - T(x)T(z_0) = 0 \tag{1}$$

   Thus $v \in \ker(T)$. Then as $z_0 \in \ker(T)^{\perp}$, we get the following:

$$\begin{aligned} \langle v, z_0\rangle &= 0 \\ \langle T(z_0)x - T(x)z_0, z_0\rangle &= 0 \\ T(z_0)\langle x, z_0\rangle - T(x)||z_0||^2 &= 0 \\ T(x) &= \langle x, \frac{\overline{T(z_0)z_0}}{||z_0||^2}\rangle \end{aligned} \tag{2}$$

   Thus we get the desired result with $z = \frac{\overline{T(z_0)z_0}}{||z_0||^2}$. To show uniqueness, we just say we have $z_1, z_2 \in H_2$ with the given properties.

   Then $T(x) = \langle x, z_1\rangle - \langle x, z_2\rangle$, which has $\langle x, z_1 - z_2\rangle = 0, \forall x \in H$. Thus taking $x = z_1 - z_2$ has that $||z_1 - z_2||^2 = 0$.

□

Note then we get the following as a corollary:

Theorem Riesz Corollary. Let H be a Hilbert space. We consider a mapping $L : H \to H^*$ given by $z \mapsto L_z$, where $L_z(x) = \langle x, z\rangle$.

Then L is antilinear, and L is an isomorphism.

Theorem L. et $1 \leq p \leq \infty$, and $q : \frac{1}{p} + \frac{1}{q} = 1$. Then for $\forall g \in L^q(E)$ for a measurable set E, then we have the following:

$$||g||_q = \sup_{||f||_p = 1} \left| \int_E f\overline{g} \right| \tag{3}$$

Also, if $1 \leq p < \infty$, then $(L^p)^* = L^q$ with the mapping given $\forall g \in L^q$ by $L_g : L^p \to \mathbb{C}$ with $f \mapsto \int f\overline{g}$.

Proof. Consider the case $1 < p < \infty$. By Holder's inequality, $\forall g \in L^q$ with $\square$

Theorem L. et X be a linear space and M be a subspace of X. Let ρ be a seminorm on X and $\lambda : M \to F$ be a linear functional on M such that $|\lambda(x)| \le \rho(x), \forall x \in M$. Then there exists a linear functional Λ on X such that $\Lambda|_M = \lambda$ and $|\Lambda(x)| \le \rho(x), \forall x \in X$.