David Owen Horace Cutler

Math 237: Midterm

March 15, 2024

Exercise 1. Let (X, ρ) be a compact metric space and $f : X \to X$ be a function such that

$$\rho(f(x), f(y)) < \rho(x, y) \quad \forall x \neq y.$$

Define $g : X \to \mathbb{R}$ by $g(x) = \rho(x, f(x))$.

(a) Prove that g is Lipschitz function, and that g has a minimum value that is achieved at a point $x \in X$. Conclude that there exists an $x \in X$ such that $g(x) = 0$.

Proof. We first want to show that g is Lipschitz, i.e. $\exists K \geq 0$ such that $\forall x, y \in X$ we have

$$|g(x) - g(y)| \leq K\rho(x, y).$$

As X is a metric space then, the reverse triangle inequality holds. In particular we note that

$$|\rho(x, f(x)) - \rho(x, f(y))| = |\rho(f(x), x) - \rho(f(y), x)| \leq \rho(f(x), f(y)),$$

and

$$|\rho(x, f(y)) - \rho(y, f(y))| \leq \rho(x, y).$$

We get then that

$$\begin{aligned}
|g(x) - g(y)| &= |\rho(x, f(x)) - \rho(x, f(y))| \\
&= |\rho(x, f(x)) - \rho(x, f(y)) + \rho(x, f(y)) - \rho(y, f(y))| && \text{(Introducing a term)} \\
&\leq |\rho(x, f(x)) - \rho(x, f(y))| + |\rho(x, f(y)) - \rho(y, f(y))| && \text{(Triangle inequality)} \\
&\leq \rho(f(x), f(y)) + \rho(x, y) && \text{(Reverse triangle inequality)} \\
&< 2\rho(x, y) && \text{(Definition of } f\text{)}
\end{aligned}$$

It follows that g is Lipschitz with Lipschitz constant 2. Note here we are implicitly using $x \neq y$ to conclude the last part, but this isn't a problem as when $x = y$ as regardless we have

$$0 = |g(x) - g(y)| \leq 2\rho(x, y) = 0$$

As g is Lipschitz then, it is continuous on X. It folows the **Extreme Value Theorem**[1] is applicable, as X is a compact metric space. In particular, this means g achieves a minimum, i.e. at some point x_0.

For the last part then, We claim then that $g(x_0) = 0$. To show this, we assume for the sake of contradiction that $g(x_0) \neq 0$.

By definition then, it must be $g(x_0) > 0$ then, i.e.

$$g(x_0) = \rho(x_0, f(x_0)) > 0.$$

[1] As stated on the Wikipedia article in the references.

By the properties of the metric then, it must be $x_0 \neq f(x_0)$. We get then

$$g(f(x_0)) = \rho(f(x_0), f(f(x_0))) < \rho(x_0, f(x_0)) = g(x_0),$$

which of course a contradiction as $g(x_0)$ should be the minimum value attained by g. Thus we must have $g(x_0) = 0$, which finishes off the proof. $\square$

(b) Show that f has a unique fixed point x_0.

Proof. By the previous part, there exists some x_0 such that $g(x_0) = 0$. Note then

$$\begin{aligned} g(x_0) &= 0 \\ \longrightarrow \rho(x_0, f(x_0)) &= 0 \\ \longrightarrow x_0 &= f(x_0) \end{aligned}$$

where the last deduction comes from properties of the metric. Thus f has a fixed point x_0.

We want to argue uniqueness then, so assume for the sake of contradiciton there is an $x_1 \in X$ such that $f(x_1) = x_1$, where $x_1 \neq x_0$.

Here we get

$$\rho(x_0, x_1) = \rho(f(x_0), f(x_1)) < \rho(x_0, x_1)$$

which is of course a contradiction. It follows f has unique fixed point x_0. $\square$

(c) Show that the assumption that X is compact cannot be ommited.

Proof. Simply consider $f : (0,1) \to (0,1)$ defined by

$$x \mapsto \frac{x}{2},$$

where $(0,1)$ is equipped with the standard metric. Then we get for $x, y \in (0,1)$ where $x \neq y$ that

$$|f(x) - f(y)| = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x - y| < |x - y|,$$

so f fulfills the given property. However, f trivially has no fixed point, as the only candidate is 0, which is not in the interval. $\square$

Exercise 2. Let X and Y be Banach spaces. Show that $T \in L(X, Y)$ is surjective if and only if range(T) is not meager in Y.

Proof. We will verify that $T \in L(X, Y)$ is surjective if and only if range(T) is not meager in Y.

(i) *(Surjectivity has range is not meager)* Assume for the sake of contradiction that range(T) is meager.

As T is surjective, clearly

$$\text{range}(T) = Y,$$

where Y is is Banach. By the **Baire Category Theorem** then, it should be Y is a nonmeager subset of itself.

But then clearly this is a contradiction, as we assume range$(T) = Y$ is meager. It follows range(T) must be not meager.

(ii) *(Range is not meager has surjectivity)* Let $B_n^X(0)$ denote the ball of radius n about the origin in X.

As the norm is finite then, of course we have

$$X = \bigcup_{n=1}^{\infty} B_n^X(0),$$

and as the image of a union is the union of images, we get

$$\text{range}(T) = T(X) = T\Big(\bigcup_{n=1}^{\infty} B_n^X(0)\Big) = \bigcup_{n=1}^{\infty} T(B_n^X(0)).$$

Linearity of T immediately has that $T(B_n^X(0)) = nT(B_1^X(0))$, where nT is the bounded, linear operator defined by $(nT)(n) = nT(n)$.

Thus we can additionally refine

$$\text{range}(T) = \bigcup_{n=1}^{\infty} T(B_n^X(0)) = \bigcup_{n=1}^{\infty} nT(B_1^X(0)),$$

and so as range(T) is nonmeager, it must be that some $nT(B_1^X(0))$ is *not* nowhere dense, i.e. for some $n \in N$ we have that $\overline{nT(B_1^X(0))}$ has nonempty interior.

As we noted nT is a bounded linear transformation then, we appeal to **Theorem 2.26** in Heil's *A Basis Theory Primer*, which has that there exists an $r > 0$ such that

$$B_r^Y(0) \subseteq nT(B_1^X 0).$$

Let $y \in Y$ then. We see that $\frac{ry}{2||y||} \in B_r^Y(0)$. Using then that $B_r^Y(0) \subseteq nT(B_1^X 0)$ then, it is that

$$\frac{ry}{2||y||} = nT(x),$$

for some x in the unit ball in X. We organize then to get

$$y = \frac{2||y||}{r} nT(x) = \frac{2||y||n}{r} T(x) = T\Big(\frac{2||y||n}{r} x\Big),$$

and so $y \in \text{range}(T)$. Thus T is surjective.

With both directions verified then, we get the desired result. $\square$

Exercise 3. Let $C_b(\mathbb{R})$ be the space of bounded, continuous functions, and let $C_b^1(\mathbb{R})$ be the space of functions such that $f, f' \in C_b(\mathbb{R})$. Equip both of these spaces with the uniform norm.

(a) Prove that $C_b(\mathbb{R})$ is complete but $C_b^1(\mathbb{R})$ is not.

Proof. We first move to prove that $C_b(\mathbb{R})$ is complete. To accomplish this, we reproduce the proof of **Theorem 1.3.3** in Heil's *Introduction to Real Analysis.*

(i) ($C_b(\mathbb{R})$ *is complete*) Let $\{f_n\}_{n=1}^\infty \subseteq C_b(\mathbb{R})$ be Cauchy in the uniform norm. Fixing some specific point x in $\mathbb{R}$, we note of course that for any given n and m we have

$$|f_n(x) - f_m(x)| \leq ||f_m - f_n||_u$$

and so $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence in our codomain, i.e. $\mathbb{R}$ or $\mathbb{C}$. However, both of these spaces are complete, so the sequence $\{f_n(x)\}_{n=1}^\infty$ must converge.

Define the the function f by stipulating

$$f(x) = \lim_{n\to\infty} f_n(x),$$

we claim that $f_n \xrightarrow{u} f$, i.e. f_n converges to f *uniformly.*

For this sake, let $\epsilon > 0$. Using the Cauchyness of $\{f_n\}_{n=1}^\infty$ in the uniform norm, we know there exists an N such that

$$||f_n - f_m||_u < \epsilon \text{ for } n, m \geq N,$$

and thus if $m \geq N$ we have for every $x \in \mathbb{R}$ that

$$|f(x) - f_m(x)| = \lim_{n\to\infty} |f_n(x) - f_m(x)| \leq \limsup_{m\to\infty} ||f_m - f_n||_u \leq \epsilon.$$

We can take the supremum over all $x \in \mathbb{R}$ to get $||f - f_m||_u \leq \epsilon$ for $m \geq N$, and so it follows $f_n \to f$ uniformly.

In addition to this, the **Uniform Limit Theorem** tells us that f is continuous on $\mathbb{R}$, as all the f_n are. Moreover, f is bounded as all the f_n are, given that

$$||f||_u \leq ||f - f_n + f_n||_u \leq ||f - f_n||_u + ||f_n||_u,$$

and so taking $||f - f_n||_u$ small shows that $||f||_u$ bounded. It follows $f \in C_b(\mathbb{R})$, where $f_n \xrightarrow{u} f$. Thus $C_b(\mathbb{R})$ is complete.

(ii) ($C_b^1(\mathbb{R})$ *is not complete*) We define a sequence of functions $\{f_n\}_{n=1}^{\infty}$ in the following way:

$$f_n(x) = \sum_{k=0}^{n} 2^{-k}\cos(13^k \pi x)$$

We can establish $\{f_n\}_{n=1}^{\infty} \subseteq C_b^1(\mathbb{R})$ by showing f_n and f_n' are bounded for each n. Let $n \in N$ then. For f then we note

$$\begin{aligned}|f_n(x)| = \Big|\sum_{k=0}^{n} 2^{-k}\cos(13^k\pi x)\Big| &\leq \sum_{k=0}^{n} |2^{-k}\cos(13^k\pi x)| \\ &\leq \sum_{k=0}^{n} 2^{-k} < \sum_{k=0}^{\infty} 2^{-k} = 2,\end{aligned}$$

and for f' we have

$$\begin{aligned}|f_n'(x)| = \Big|\sum_{k=0}^{n} -\pi\Big(\frac{13}{2}\Big)^k \sin(13^k\pi x)\Big| &\leq \sum_{k=0}^{n} \Big|\pi\Big(\frac{13}{2}\Big)^k \sin(13^k\pi x)\Big| \\ &\leq \sum_{k=0}^{n} \pi\Big(\frac{13}{2}\Big)^k < \infty,\end{aligned}$$

so both f_n and f_n' are bounded. Additionally, it is evident both f_n and f_n' are continuous. It follows $\{f_n\}_{n=1}^{\infty} \subseteq C_b^1(\mathbb{R})$.

Moreover, we can verify this sequence is uniformly Cauchy, taking N where $n, m \geq N$ and with $n > m$ without loss of generality. Then for any $x \in \mathbb{R}$ we get

$$\begin{aligned}|f_n(x) - f_m(x)| &= \Big|\sum_{k=0}^{n} 2^{-k}\cos(13^k\pi x) - \sum_{k=0}^{m} 2^{-k}\cos(13^k\pi x)\Big| \\ &\leq \Big|\sum_{k=m+1}^{n} 2^{-k}\cos(13^k\pi x)\Big| \leq \sum_{k=m+1}^{n} 2^{-k} \leq \sum_{k=N}^{n} 2^{-k},\end{aligned}$$

where taking $N \to 0$ thus shows Cauchyness, and the last sum gets arbitrarily small. As this bound is not dependent on x, taking the supremum over all x has the sequence is uniformly Cauchy.

However, this is an issue as this sequence of functions is precisely the definition of a **Weierstrass Function**[2], as $\frac{1}{2}(13) > \frac{3\pi}{2}$. Thus its limit is not in $C_1^b(\mathbb{R})$, as it is nowhere differentiable.

Appealing to uniqueness of the uniform limit then, it follows $C_b^1(\mathbb{R})$ is not complete as $\{f_n\}_{n=1}^{\infty} \subseteq C_b^1(\mathbb{R})$ is a Cauchy sequence with no limit in $C_b^1(\mathbb{R})$.

[2]See Brent Nelson's online notes titled "The Weierstrass Function" for proof of uniform convergence to a nonwhere differentiable function. Listed in references.

□

(b) Show that the differentiation operator $D : C_b^1(\mathbb{R}) \to C_b(\mathbb{R})$ given by $Df = f'$ is unbounded, but has a closed graph.

Proof. We need to show D is unbounded but that it has a closed graph.

(D is unbounded) To show D is unbounded, it is sufficient to show there is a bounded sequence $\{f_k\}_{k=1}^{\infty}$ that has $\{Df_k\}_{k=1}^{\infty}$ unbounded.

For this, consider the sequence such that $f_k = \sin(kx)$. Of course for each k we have

$$||f_k||_u = 1,$$

but we note $Df_k = k\cos(kx)$. But then $Df_k(0) = k\cos(k \cdot 0) = k$. Of course

$$k = |Df_k(0)| \leq ||Df_k||_u,$$

which clearly has that $\{Df_k\}_{k=1}^{\infty}$ is unbounded, taking $k \to \infty$.

(ii) *(D has a closed graph)* Let $f_k \xrightarrow{u} f \in C_b^1(\mathbb{R})$ and let $Df_k \xrightarrow{u} g \in C_b(\mathbb{R})$, i.e. $f_k' \xrightarrow{u} g \in C_b(\mathbb{R})$. To show D has a closed graph, we need to show $Df = f' = g$.

For this, consider an arbitrary compact interval $[a, b]$. As we have $f_k' \xrightarrow{u} g$ uniformly on $\mathbb{R}$, we have uniform convergence on $[a, b]$, and consequently L^1 convergence.

In particular we note, considering the indefinite integral on $[a, b]$ that

$$\lim_{k\to\infty} \int_a^x f_k' = \int_a^x g$$

As f is C^1 then, the **Fundamental Theorem of Calculus** is applicable, i.e.

$$\lim_{k\to\infty} f_k(x) - f_k(a) = \lim_{k\to\infty} \int_a^x f_k' = \int_a^x g,$$

where $f_k \xrightarrow{u} f$ additionally has pointwise convergence, so get we finally

$$f(x) - f(a) = \int_a^x g.$$

We can apply then the second statement of the **FTC**, given the continuity of g, to get

$$f'(x) = D(f(x) - f(a)) = D\Big(\int_a^x g\Big) = g(x),$$

and as this holds over all compact intervals, we can extrapolate to get $f' = g$ in general. Thus $Df = f' = g$, and so we have a closed graph.

□

Exercise 4. Let $H = L^2([0,1], \mathbb{R})$ be the space of all real-valued Lebesgue measurable and square-integrable functions on $[0,1]$. Let K be a nonempty closed convex subset of H, and $P = P_K$ be the orthogonal projection of H onto K.

(a) Let $x \in H$. Prove that the following two statements are equivalent.

(i) There exists a unique $z \in K$ such that $||x - z|| = \min_{y \in K} ||x - y||$.

(ii) The vector z in (i) is characterized by

$$\begin{cases} z \in K \\ \langle x - z, y - z \rangle \leq 0 \quad \forall y \in K \end{cases}$$

Proof. We start with the forward direction, assuming (i) and hoping to show (ii). With this in mind, let $y \in K$. We define y_t by stipulating

$$y_t = (1-t)z + ty \text{ for } t \in (0,1)$$

we note $y_t \in K$ for $t \in (0,1)$, as $z, y \in K$ and K is convex. Thus by assumption

$$||x - z|| \leq ||x - y_t||.$$

We get the following derivation then, making extensive use of the properties of the inner product:

$$\begin{aligned} ||x - z||^2 \leq ||x - y_t||^2 = ||x - (1-t)z - ty||^2 \\ = ||x - z + tz - ty||^2 = \langle x - z + tz - ty, x - z + tz - ty \rangle \\ = \langle x - z, x - z + tz - ty \rangle + \langle tz - ty, x - z + tz - ty \rangle \\ = \langle x - z, x - z \rangle + \langle x - z, tz - ty \rangle + \langle tz - ty, x - z \rangle + \langle tz - ty, tz - ty \rangle \\ = ||x - z||^2 + ||tz - ty||^2 + 2t\langle x - z, z - y \rangle \end{aligned}$$

We can rewrite this even more to get

$$||x - z||^2 \leq ||x - z||^2 + t^2||z - y||^2 - 2t\langle x - z, y - z \rangle,$$

i.e.

$$\langle x - z, y - z \rangle \leq \frac{t}{2}||z - y||^2.$$

Taking $t \to 0$ then gets that

$$\langle x - z, y - z \rangle \leq 0,$$

where then this holds for all $y \in K$ given our arbitrary choice of $y \in K$.

Assume then (ii) (i.e. consider z with the outlined properties), we want to show (i). Let $y \in K$; we get the following:

$$\begin{aligned} \langle x - z, y - z \rangle = \langle x - z, y - x + x - z \rangle \\ = \langle x - z, y - x \rangle + \langle x - z, x - z \rangle \\ = ||x - z||^2 + \langle x - z, y - z \rangle \\ = ||x - z||^2 - \langle x - z, x - y \rangle \end{aligned}$$

By our assumption then we have

$$||x-z||^2 - \langle x-z, x-y\rangle \leq 0,$$

and so we can rearrange to get

$$||x-z||^2 \leq \langle x-z, x-y\rangle.$$

Applying Cauchy-Schwarz thus gets

$$\begin{aligned} ||x-z||^2 \leq \langle x-z, x-y\rangle \leq |\langle x-z, x-y\rangle| \leq ||x-z||||x-y|| \\ \longrightarrow ||x-z|| \leq ||x-y||, \end{aligned}$$

and so as $y \in K$ was arbitrary we have $||x-z|| \leq \min_{y\in K}||x-y||$, with $z \in K$ by assumption.

We want to argue uniqueness of this z then. For this, suppose we have some $z' \in K$ fulfilling the same inequality.

Clearly then, we have

$$||x-z|| \leq ||x-z'|| \text{ and } ||x-z'|| \leq ||x-z||,$$

so $||x-z|| = ||x-z'||$. In addition we of course have

$$\frac{x-z}{||x-z||} = \frac{x-z'}{||x-z'||} = 1,$$

so the strict convexity of the unit ball with respect to the L^2 norm (as proven in the first homework) has for $t \in (0,1)$ that

$$\left|\left|t\frac{x-z}{||x-z||} + (1-t)\frac{x-z'}{||x-z'||}\right|\right| < 1.$$

As $||x-z|| = ||x-z'||$ then, we can multiply by $||x-z||$ to get the following:

$$||t(x-z) + (1-t)(x-z')|| < ||x-z||$$

But this presents an issue, as we note

$$\begin{aligned} t(x-z) + (1-t)(x-z') \\ = x + tz' - tz - z' \\ = x - (tz + z' - tz') \\ = x - (tz + (1-t)z') \end{aligned}$$

So we have

$$||x - (tz + (1-t)z')|| < ||x-z||,$$

but this is a problem as $z, z' \in K$, so $(tz+(1-t)z') \in K$ for $t \in (0,1)$ by convexity.

We therefore get a contradiction, as it should then of course be (as z minimizes the distance) that we have

$$||x - z|| \leq ||x - (tz + (1-t)z')||.$$

It follows the point z must be unique then, so we have (i). As we have shown both directions then, the two statements are equivalent. $\square$

(b) Let A be a continuous bilinear mapping from $H \times H$ into $\mathbb{R}$ such that for some $\alpha > 0$ we have

$$A(f,f) \geq \alpha ||f||_2^2 \, \forall f \in H$$

(1) Fix $u \in H$ and prove that there exists a unique $Tu \in H$ such that $A(u,v) = \langle Tu, v\rangle$ for all $v \in H$. Prove that T is a bounded linear mapping on H.

Proof. Fix some $u \in H$, as A is continuous on $H \times H$ and bilinear, it follows $A(u, \cdot)$ is continuous on H and linear.

More specifically, $A(u, \cdot)$ is a bounded linear functional. It follows by the **Riesz Representation Theorem** that for some unique $z \in H$, we have

$$A(u,v) = \langle v, z\rangle \, \forall v \in H.$$

Define then T by stipulating $Tu = z$; the **RRT** also thus has T is well-defined and Tu is unique and the given z must be unique.

We thus have for the unique $Tu \in H$ that

$$A(u,v) = \langle v, Tu\rangle = \langle Tu, v\rangle \quad \forall v \in H.$$

We want then to show in addition that T is linear and bounded. We start by showing T is linear, i.e. for $a, b \in \mathbb{R}$ and $x, y \in H$ we have

$$T(ax+by) = aT(x) + bT(y).$$

Using the linearity of A in the first argument we get that

$$A(ax+by, v) = aA(x,v) + bA(y,v) \quad \forall v \in H$$

so it by construction we have

$$\begin{aligned} \langle T(ax+by), v\rangle &= a\langle T(x), v\rangle + b\langle T(y), v\rangle \quad \forall v \in H \\ \longrightarrow \langle T(ax+by), v\rangle &= \langle aT(x) + bT(y), v\rangle \quad \forall v \in H \\ \longrightarrow \langle T(ax+by) - aT(x) - bT(y), v\rangle &= 0 \quad \forall v \in H, \end{aligned}$$

but the only vector orthogonal to all other vectors is the zero vector, i.e. $T(ax + by) - aT(x) - bT(y) = 0$, so

$$T(ax + by) = aT(x) + bT(y).$$

The last thing to show then is that T is bounded; for this, we can show it has finite operator norm.

By our earlier working with **RRT**, we know for each u that $||A(u,\cdot)||_{B(H,\mathbb{R})} = ||Tu||_H$. Thus we get

$$||T||_{H^*} = \sup_{||u||_H=1} ||Tu||_H = \sup_{||u||_H=1} ||A(u,\cdot)||_{B(H,\mathbb{R})},$$

where each $||A(u,\cdot)||_{B(H,\mathbb{R})}$ is finite, given each is bounded/continuous on H. We want now to apply the **Uniform Boundedness Principle** (note H and $\mathbb{R}$ are Banach spaces).

Let some $v \in H$, then

$$\sup_{||u||_H=1} |A(u,\cdot)(v)| = \sup_{||u||_H=1} |A(u,v)| = ||A(\cdot,v)||_{B(H,\mathbb{R})}$$

by definition. Of course, the latter quantity is finite as fixing A is one variable leads to a continuous/bounded linear function on H, and so it has finite operator norm.

Applying the **UBP** thus has

$$\sup_{||u||_H=1} ||A(u,\cdot)||_{B(H,\mathbb{R})} < \infty,$$

and so consequently $||T||_{H^*} < \infty$, which is sufficient to determine that T is bounded as we always have

$$||Tu||_H \leq ||T||_{H^*}||u||_H$$

for $u \in H$. □

(2) Let $\rho > 0$ and define a mapping S_ρ on K given by $S_\rho v = P(\rho f, pTv + v)$ for $v \in K$. Prove that $\rho > 0$ can be chosen such that there exists $0 < k < 1$ with the property that

$$||S_\rho v_1 - S_\rho v_1|| \leq k||v_1 - v_2|| \quad \forall v_1, v_2 \in K$$

Proof. We need first to prove that the projection onto K is Lipschitz with constant 1.

For this, consider $x, y \in H$. We want to show that

$$||Px - Py|| \leq ||x - y||.$$

Assume then $x \neq y$, as $x = y$ is trivial. By our earlier work in (a) then, we thus have

$$\langle x - Px, Py - Px \rangle \leq 0$$
$$\langle y - Py, Px - Py \rangle \leq 0,$$

as $Px, Py \in K$. Working through this then, we get the following derivation:

$$\begin{aligned} \langle x - Px, Py - Px \rangle + \langle y - Py, Px - Py \rangle &\leq 0 \\ \longrightarrow \langle y - Py, Px - Py \rangle - \langle x - Px, Px - Py \rangle &\leq 0 \\ \longrightarrow \langle y - Py, Px - Py \rangle + \langle Px - x, Px - Py \rangle &\leq 0 \\ \longrightarrow \langle y - Py - x + Px, Px - Py \rangle &\leq 0 \\ \longrightarrow \langle y - x + (Px - Py), Px - Py \rangle &\leq 0 \\ \longrightarrow -\langle x - y, Px - Py \rangle + ||Px - Py||^2 &\leq 0 \\ \longrightarrow ||Px - Py||^2 &\leq \langle x - y, Px - Py \rangle \end{aligned}$$

Thus, applying Cauchy-Schwarz we get

$$||Px - Py||^2 \leq \langle x - y, Px - Py \rangle \leq ||x - y||||Px - Py||,$$

and so

$$||Px - Py|| \leq ||x - y||.$$

With this in mind then, we move to prove the desired result. We note for $v_1, v_2 \in K$ then that

$$\begin{aligned} &||S_\rho v_1 - S_\rho v_2||^2 \\ &= ||P(\rho f - \rho T v_1 + v1) - P(\rho f - \rho T v_2 + v2)||^2 \\ &\leq ||(\rho f - \rho T v_1 + v_1) - (\rho f - \rho T v_2 + v_2)||^2 && \text{(Lipschitz with 1)} \\ &= ||(v_1 - v_2) - \rho T(v_1 - v_2)||^2 \\ &= ||v_1 - v_2||^2 - 2\rho \langle T(v_1 - v_2), v_1 - v_2 \rangle + ||\rho T(v_1 - v_2)||^2 && \text{(Expanding inner product)} \\ &\leq ||v_1 - v_2||^2 - 2\rho \langle T(v_1 - v_2), v_1 - v_2 \rangle + \rho^2 ||T||^2 ||v_1 - v_2||^2 \\ &= ||v_1 - v_2||^2 - 2\rho A(v_1 - v_2, v_1, v_2) + \rho^2 ||T||^2 ||v_1 - v_2||^2 \\ &\leq ||v_1 - v_2||^2 - 2\rho\alpha ||v_1 - v_2||^2 + \rho^2 ||T||^2 ||v_1 - v_2||^2 && \text{(Inequality on } A\text{)} \\ &= (1 - 2\rho\alpha + \rho^2 ||T||^2) ||v_1 - v_2||^2 \end{aligned}$$

Thus to get what we desired, we just need $(1 - 2\rho\alpha + \rho^2||T||^2) < 1$. Note then

$$\begin{aligned} (1 - 2\rho\alpha + \rho^2||T||^2) &< 1 \\ \leftrightarrow \rho^2 ||T||^2 &< 2\rho\alpha \\ \leftrightarrow \rho &< \frac{2\alpha}{||T||^2} \end{aligned}$$

Thus choosing $\rho < \frac{2\alpha}{||T||^2}$ gets us a $k = (1 - 2\rho\alpha + \rho^2||T||^2) < 1$ for which we have

$$||S_\rho v_1 - S_\rho v_2||^2 \leq k||v_1 - v_2|| \quad \forall v_1, v_2 \in K$$

as desired. $\square$

(3) Conclude that for the value of ρ given in (2), that S_p is a contraction, and thus has a unique fixed point $u \in K$.

Proof. As our given k in the last part is such that $k < 1$, S_ρ is by definition a contraction. As K is a closed subset of H then and H is complete, K is a complete metric space.

Moreover, S_ρ is a function $K \to K$.

By the **Banach Fixed Point Theorem**[3] then, S_ρ admits a unique fixed point $u \in K$. □

(4) By writing $\rho f - \rho Tu = \rho f - \rho Tu + u - u$ and using part (a), show that

$$\langle \rho f - \rho Tu, v - u \rangle \leq 0 \quad \forall v \in K.$$

Proof. Let $v \in K$. Considering $w = \rho f + \rho Tu + u$ (with u as produced in the last part) we get

$$Pw = P(\rho f + \rho Tu + u) = S_\rho(u) = u.$$

As $u \in K$, we get by (1) that

$$\begin{aligned} \langle w - u, v - u \rangle &\leq 0 \\ \longrightarrow \langle \rho f + \rho Tu, v - u \rangle &\leq 0 \end{aligned}$$

and so

$$\langle \rho f + \rho Tu, v - u \rangle \leq 0 \quad \forall v \in K,$$

as desired. □

(5) Conclude that given any $f \in H$, there exists a unique $u \in K$ such that

$$A(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in K.$$

Proof. Let $f \in H$ and consider some $v \in K$. Using the previous part then, we get the following:

$$\begin{aligned} \langle \rho f - \rho Tu, v - u \rangle &\leq 0 \\ \longrightarrow \rho (f - Tu, v - u) &\leq 0 \\ \longrightarrow \langle f - Tu, v - u \rangle &\leq 0 \\ \longrightarrow \langle f, v - u \rangle - \langle Tu, v - u \rangle &\leq 0 \\ \longrightarrow \langle f, v - u \rangle &\leq \langle Tu, v - u \rangle \end{aligned}$$

But note of course

$$\langle Tu, v - u \rangle = A(u, v - u),$$

so we get

$$\langle f, v - u \rangle \leq A(u, v - u).$$

As $v \in K$ was arbitrary then, this holds for all $k \in K$, as desired.

Additionally, the uniqueness of u is inherited from the previous part. □

[3] As stated on the Wikipedia article in the references.

References

[1] C. Heil, *Introduction to Real Analysis*, 2018.

[2] C. Heil, *A Basis Theory Primer*, 2011.

[3] B. Nelson, *The Weierstrass Function*. Available: https://math.berkeley.edu/~brent/files/104_weierstrass.pdf

[4] Mar 2024. [Online] Available: https://en.wikipedia.org/wiki/Banach_fixed-point_theorem

[5] Mar 2024. [Online] Available: https://en.wikipedia.org/wiki/Extreme_value_theorem

[6] I collobarated extensively with multiple classmates; seen here:

My main collobarators were Scott, Eric, Josh, Ishaan, Vievie, Satchel, and Danny, however problems were likely mentioned and discussed to some degree with essentially all individuals in the picture.