David Owen Horace Cutler **Math 237: Lecture Notes** February 27, 2024

Theorem (Hahn-Banach). Let X be a linear space, ρ a seminorm on X, and M be a subspace of X. If $\lambda : M \to F$ is a linear functional such that

$$|\lambda(x)| \leq \rho(x),\ \forall x \in M$$

, then there exists a linear functional $\Lambda X \to F$ such that $\Lambda|_M = \lambda$ and:

$$|\Lambda(x)| \leq \rho(x),\ \forall x \in X$$

Theorem (Hahn-Banach Corollary 1). Let X be a normed linear space and $M \subseteq X$ be a subspace. If $\lambda \in M^\star$, then $\exists \Lambda \in X^\star$ such that:

$$\Lambda|_M = \lambda \text{ and } ||\Lambda||_{X^\star} = ||\lambda||_{M^\star}$$

Theorem (Hahn-Banach Corollary 2). Let X be a Banach space. Then for every $x \in X$, $||x||_X = \sup\limits_{f \in X^\star, ||f||_{X^\star}=1} |f(x)| = \alpha$.

Proof. Fix $x \in X$, $\forall f \in X^s$
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Theorem (Banach-Steinhaus). Let X and Y be Banach spaces, if $\{A_n\} \subseteq B(X,Y)$ and $Ax = \lim\limits_{n\to\infty} A_n x$ exists for all $x \in X$, then $A \in B(X,Y)$ and $||A|| \leq \sup\limits_n ||A_n||$.

Theorem (The Open Mapping Theorem). Definition (Open Mapping). Let X, Y be normed spaces. A mapping $T : X \to Y$ if for $T(U)$ is open in Y (under the norm topology), whenver U is open in X.

Let X and Y be Banach spaces, and $A : X \to Y$ a bounded, linear surjection. Then A is an open mapping.

Proof. Remark $X = \bigcup_{n=1}^\infty B_n(0)$. Moreover, $Y = AX = \bigcup_{n=1}^\infty A(B_n(0)) \subseteq \bigcup_{n=1}^\infty \overline{A(B_n(0))} \subseteq Y$.

Thus $Y = \bigcup_{n=1}^\infty A(B_n(0))$. By Baire Category theorem, $\exists n_0$ such that $\overline{A(B_{n_0}(0))}$ contains an open ball. □