Theorem (Parallelogram Law). If H is an inner product space, then for every $x, y \in H$:

$$||x+y||^2 + ||x-y||^2 + 2(||x||^2 + ||y||^2) \tag{1}$$

Example: (lp Norm) We define the l_p norm for $\mathbb{C}^d$ for $\forall p \in (0, \infty)$ by the following convention:

$$||x||_p = \left(\sum_{k=1}^{d} |x_k|^p\right)^{\frac{1}{p}} \tag{2}$$

Interestingly, $p = 2$ uniquely generates the norm that satisifes the Parallelogram law.

Theorem (Unique Closest Point). Let H be a Hilbert space and M be a closed subspace of H.

For every $x \in H$, there exists a unique $p \in M$ that is closest to x. That is that for some $p \in M$:

$$||x-p|| = \inf\{||x-m||, m \in M\} = \operatorname{dist}(x, M) \tag{3}$$

Proof. Let $d = \operatorname{dist}(x, M)$. By the definition of the infinum, $\{y_n\}_{n=1}^{\infty} \subseteq M$ such that $d = \lim\limits_{n\to\infty} ||x - y_n||$.

Take then $d^2 = \lim\limits_{n\to\infty} ||x-y_n||^2$. Let $\epsilon > 0$. Using the definition of the limit, there $\exists N \in \mathbb{N}$ such that $\forall n \geq N$:

$$d^2 \leq ||x-y_n||^2 < d^2 + \epsilon$$

Take then $n, m \geq N$, this yields the following by applying the Parallelogram law:

$$||(x-y_n) - (x-y_m)||^2 + ||(x-y_n) + (x+y_m)||^2 = 2(||x-y_n||^2 + ||x-y_m||^2) \tag{4}$$

However, we also get the following by using homogeneity:

$$||(x-y_n) - (x-y_m)||^2 + ||(x-y_n) + (x+y_m)||^2 = ||y_n - y_m||^2 + 4||x - \frac{(y_n+y_n)}{4}||^2 \tag{5}$$

Which gets us the following (as $\frac{(y_n+y_m)}{2} \in M$):

$$\begin{gathered} 4d^2 + ||y_n - y_m||^2 \leq 4d^2 + 4\left|\left|x - \frac{(y_n - y_m)}{2}\right|\right|^2 \\ = 2(||x-y_n||^2 + ||x-y_m||^2) = 2||x-y_n||^2 + 2||x-y_m||^2 < 4d^2 + 4\epsilon \end{gathered} \tag{6}$$

Which of course finally has:

$$||y_n - y_m||^2 < \epsilon \tag{7}$$

So $\{y_n\}_{n=1}^{\infty}$ is Cauchy, and so it converges to some $y \in M$, where we get the final idea:

$$d = \lim_{n\to\infty} ||x - y_n|| = ||x - \lim_{n\to\infty} y_n|| = ||x - y|| \tag{8}$$

Uniqueness is left as exercise to the reader. □