David Owen Horace Cutler **Math 237: Homework 2** March 8, 2024

Exercise 4. Let X be the space of C^1 functions on $[0,1]$ with $f(0)=0$ and let

$$\langle f,g\rangle = \int_0^1 f'(x)\overline{g'(x)}dx$$

(a) Prove that H, the completion of X, is a reproducing kernel Hilbert space.

Proof. We want to show H is a Hilbert space first. To show it is a Hilbert space then, we need to verify it is an inner product space. For this, we verify the inner product axioms.

(i) *(Linearity in the first Argument)* Let $f,g,h \in H$ and $a,b \in \mathbb{C}$. We note the following:

$$\begin{aligned}\langle af+bg,h\rangle &= \int_0^1 (af+bg)'(x)\overline{h'(x)}dx = \int_0^1 (af'(x)+bg'(x))\overline{h'(x)}dx \\ &= \int_0^1 af'(x)\overline{h'(x)} + bg'(x)\overline{h'(x)}dx = \int_0^1 af'(x)\overline{h'(x)}dx + \int_0^1 bg'(x)\overline{h'(x)}dx \\ &= a\int_0^1 f'(x)\overline{h'(x)}dx + b\int_0^1 g'(x)\overline{h'(x)}dx = a\langle f,h\rangle + b\langle g,h\rangle\end{aligned} \tag{1}$$

(ii) *(Conjugate symmetry)* Again, consider $f,g \in H$. We get the following:

$$\begin{aligned}\langle g,f\rangle &= \int_0^1 g'(x)\overline{f'(x)}dx = \int_0^1 (g_r'(x)+ig_i'(x))\overline{(f_r'(x)+if_i'(x))}dx \\ &= \int_0^1 (f_r'(x)g_r'(x)+f_i'(x)g_i'(x)) + i(g_i'(x)f_r'(x)-g_r'(x)f_i'(x))dx \\ &= \int_0^1 (f_r'(x)g_r'(x)+f_i'(x)g_i'(x))dx + i\int_0^1 (g_i'(x)f_r'(x)-g_r'(x)f_i'(x))dx \\ &= \overline{\int_0^1 (f_r'(x)g_r'(x)+f_i'(x)g_i'(x))dx - i\int_0^1 (g_i'(x)f_r'(x)-g_r'(x)f_i'(x))dx} \\ &= \overline{\int_0^1 (f_r'(x)g_r'(x)+f_i'(x)g_i'(x)) - i(g_i'(x)f_r'(x)-g_r'(x)f_i'(x))dx} \\ &= \overline{\int_0^1 (f_r'(x)+if_i'(x))\overline{(g_r'(x)+ig_i'(x))}dx} = \overline{\int_0^1 f'(x)\overline{g'(x)}} = \overline{\langle f,g\rangle}\end{aligned} \tag{2}$$

(iii) *(Positive-definiteness)* Let some $f \in H$. We note

$$\langle f,f\rangle = \int_0^1 f'(x)\overline{f'(x)}dx = \int_0^1 |f'(x)|^2 dx \geq 0 \tag{3}$$

as $|f'(x)|^2$ is nonnegative. We want then $\langle f,f\rangle = 0$ iff $f=0$. Obviously, the reverse direction holds.

For the forward direction, it is clear we need to mod out by an equivalence relation, in particular $f = g$ iff $f' = g'$ almost everywhere.

This should expected as this inner product is essentially the derivative version of the L^2 inner product.

With this in mind, we note that by (3), it is clear that if $\langle f, f \rangle = 0$, we have $|f'(x)|^2 = 0$ almost everywhere. Thus $|f'(x)| = 0$ a.e. and so finally $f'(x) = 0$ a.e.

In particular then, $f' = 0$ a.e., which has that $f = g$.

From this, it follows H is an inner product space. Moreover, as H is the completion of X under the norm induced by this inner product, it is that H is a Banach space.

It follows H is a Hilbert space. To show it is a reproducing kernel Hilbert space then, we need to show the evaluation functional $L_x : H \to \mathbb{C}$ is bounded for each $x \in [0, 1]$.

In particular, we let some $x \in [0, 1]$. For $f \in X$, we get the following argument, appealing to Hölder's inequality and FTC (given the functions are C^1 and have $f(0) = 0$):

$$\begin{aligned} |L_x(f)| = |f(x)| = \Big| \int_0^x f'(x) dx \Big| \leq \int_0^x |f'(x)| \\ \leq \sqrt{\int_0^x 1^2 dx} \sqrt{\int_0^x |f'(x)|^2 dx} \leq \sqrt{x} ||f'||_2 = \sqrt{x} ||f||_H \end{aligned} \tag{4}$$

In general for $f \in H$ we let $f_n \xrightarrow{H} f$ where $\{f_n\}_{n=1}^\infty \subseteq X$. Substituting $|f_n(x) - f_m(x)|$ into (4) and taking the supremum over $x \in [0, 1]$ gets us that

$$||f_n - f_m||_\infty \leq ||f_n - f_m||_H \tag{5}$$

As the sequence is convergent in the H norm, it is Cauchy in the H norm, and so it follows then that $\{f_n\}_{n=1}^\infty$ is Cauchy in the L^∞ norm by (5).

Recall $C[0, 1]$ is complete with the L^∞ norm, so as $\{f_n\}_{n=1}^\infty \subseteq X \subseteq C[0, 1]$ then, we have that for some $g \in C[0, 1]$ that $f_n \xrightarrow{\infty} g$. Appealing to uniqueness then it should be that $g = f$ everywhere.

As uniform convergence has pointwise convergence then, clearly

$$\lim_{n \to \infty} f_n(x) = g(x) = f(x),$$

which is that then

$$\lim_{n \to \infty} |L_x(f_n)| = \lim_{n \to \infty} |f_n(x)| = |f(x)| = |L_x(f)|,$$

where we can substitute into (4), given that each $f_n \in X$, to get

$$|L_x(f)| = \lim_{n\to\infty} |L_x(f_n)| \leq \lim_{n\to\infty} \sqrt{x}||f_n||_H = \sqrt{x}||f||_H,$$

which shows the evaluation functional is bounded in general. $\square$

(b) Prove that $K(x, y) = \min(x, y)$.

Proof. Let some $x \in [0, 1]$. We first compute the derivative of $\min(x, \cdot)$ as a function of y, which intuitively is

$$\min{}'(x, y) = \begin{cases} 1 & \text{if } y < x \\ 0 & \text{if } y > x \end{cases} = \mathbb{1}_{[0,x]}(y) \text{ a.e.} \tag{6}$$

where the derivative is not necessarily defined at $y = x$. Recall then that $K(x, y) = k_x(y)$, where $k_x(y)$ is the unique element in H such that for all $f \in H$ we have

$$f(x) = \langle f, k_x \rangle$$

As this element is unique, per the **Riesz Representation Theorem**, to prove $K(x, y) = \min(x, y)$, i.e. proving $k_x = \min(x, \cdot)$ as functions in y, we can show $f(x) = \langle f, \min(x, \cdot)\rangle$ for any given $f \in H$.

Consider $f \in H$. Let $\{f_n\}_{n=1}^{\infty} \subseteq X$ where $f_n \xrightarrow{H} f$. Using our observation in (6), we get the following:

$$\begin{aligned}
\langle f, \min(x, \cdot)\rangle &= \langle \lim_{n\to\infty} f_n, \min(x, \cdot)\rangle \\
&= \lim_{n\to\infty} \langle f_n, \min(x, \cdot)\rangle && \text{(Continuity of the inner product)} \\
&= \lim_{n\to\infty} \int_0^x f_n'(y)\, dy && \text{(Applying our computation in (6))} \\
&= \lim_{n\to\infty} f_n(x) && \text{(FTC)} \\
= \lim_{n\to\infty} L_x(f_n) &= L_x(f) && \text{(Continuity of the evaluation functional)} \\
&= f(x)
\end{aligned}$$

Note the last part follows are continuous $\Leftrightarrow$ bounded. It follows then by the earlier argument that $\min(x, \cdot) = k_x$ as a function in y, and so

$$K(x, y) = k_x(y) = \min(x, y),$$

for all x, y, which is the desired result.

$\square$

Exercise 6. Let $Y = \ell^1(\mathbb{N})$ and $X = \{f \in Y \mid \sum_{n=1}^{\infty} n|f(n)| < \infty\}$ equipped with the ℓ^1 norm.

(a) Prove that X is a proper dense subspace of Y, hence X is not complete.

Proof. We need to verify that X is a subspace, it is a proper one, and that it is dense.

(i) *(Subspace)* We need to verify that X contains the zero vector and that it is closed under addition and scalar multiplication.

Note trivially the zero vector (the zero sequence) is in X. Consider then $g, h \in X$, i.e. $\sum_{n=1}^{\infty} n|f(n)| < \infty$ and $\sum_{n=1}^{\infty} n|g(n)| < \infty$. Thus

$$\sum_{n=1}^{\infty} n|(f+g)(n)| = \sum_{n=1}^{\infty} n|f(n)+g(n)| \leq \sum_{n=1}^{\infty} n|f(n)| + n|g(n)|$$
$$= \sum_{n=1}^{\infty} n|f(n)| + \sum_{n=1}^{\infty} n|g(n)| < \infty,$$

which has $f + g \in X$. Moreover for a scalar in our field c, we get

$$\sum_{n=1}^{\infty} n|cf(n)| = \sum_{n=1}^{\infty} n|c||f(n)| = |c| \sum_{n=1}^{\infty} n|f(n)| < \infty,$$

which has $cf \in X$, and so we get X is a subspace.

(ii) *(Properness)* Consider $f(n) = \frac{1}{n^2}$. By the p-series test, we note immediately $f \in \ell^1(\mathbb{N}) = Y$. However we note

$$\sum_{n=1}^{\infty} n|f(n)| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

so $f \notin X$.

(iii) *(Density)* Let $g \in Y \setminus X$. Then clearly we have $\sum_{n=1}^{\infty} |g(n)| < \infty$. In particular then, by the **Cauchy Criterion for Series Convergence**, we have for every $\epsilon > 0$ that there exists an M such that for all $m \geq M$ we have

$$\sum_{n=m}^{\infty} |g(n)| < \epsilon$$

We construct f then such that $f(n) = g(n)$ for $n < M$ and $f(n) = 0$ for $n \geq M$. We note immediately $f \in X$ as it only has finitely many nonzero terms. Note then

$$||f - g||_1 = \sum_{n=1}^{\infty} |f(n) - g(n)| = \sum_{n=M}^{\infty} |g(n)| < \epsilon,$$

and so as we can take ϵ arbitrarily small, we can always construct f in a way that it is ϵ-close to g. As our method of constructing f always leads to a term in X then, X is dense.

*Note specifically if $g \in X$ we could just take the term to be g itself.

$\square$

(b) Define T from X to Y by $Tf(n) = nf(n)$. Then T is closed but not bounded.

Proof. We need to show that T is a closed mapping, but it is not bounded.

(i) *(Mapping is Closed)* Let K be a closed set in X.

That is by definiton that if we have $\{f_k\}_{k=1}^\infty \subseteq K$ such that $f_k \xrightarrow{\ell^1} f$, then $f \in K$.

We want to show that $T(K)$ is closed then. For this let $\{g_k\}_{n=1}^\infty \subseteq T(K)$ be a convergent sequence in $T(K)$, i.e. $g_k \xrightarrow{\ell^1} g \in Y$. Then by the definition of $T(K)$, we get a corresponding sequence $\{f_k\}_{k=1}^\infty \subseteq K$ where

$$g_k(n) = Tf_k(n) = nf_k(n)$$

for all $n \in \mathbb{N}$. Note this has that $nf_k \xrightarrow{\ell^1} g$, i.e.

$$\lim_{k\to\infty} \sum_{n=1}^\infty |nf_k(n) - g(n)| = 0,$$

so we can appeal to limit rules to note then

$$\lim_{k\to\infty} \sum_{n=1}^\infty \left| f_k(n) - \frac{g(n)}{n} \right| = 0,$$

i.e. if we define $h \in X$ but $h(n) = \frac{g(n)}{n}$, then $f_k \xrightarrow{\ell^1} h$. But $\{f_k\}_{k=1}^\infty$ is a sequence in K, so as we assumed K to be closed it is that $h \in K$. But then of course

$$Th(n) = nh(n) = n\frac{g(n)}{n} = g(n),$$

which is that $Th = g$. As $h \in K$ then, $g \in T(K)$. This shows closedness.

(ii) *(Unboundedness)* To show that T is not bounded, we will produce a bounded sequence $\{f_k\}_{k=1}^\infty \subseteq X$ where $\{Tf_k\}_{k=1}^\infty \subseteq Y$ is unbounded.

In particular, define $\{f_k\}_{k=1}^\infty$ by

$$f_k(n) = \frac{1}{n^{2+\frac{1}{k}}};$$

we note clearly for each k that $f_k \in X$ as

$$\sum_{n=1}^\infty n|f_k(n)| = \sum_{n=1}^\infty \frac{n}{n^{2+\frac{1}{k}}} = \sum_{n=1}^\infty \frac{1}{n^{1+\frac{1}{k}}} < \infty$$

where convergence of the last series follows from the p-series test with $p = 1 + \frac{1}{k} > 1$. Moreover, we establish the sequence is bounded as for each k we have

$$||f_k||_1 = \sum_{n=1}^{\infty} \frac{1}{n^{2+\frac{1}{k}}} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

where convergence of the last series again follows from the p-series test. We consider now the ℓ^1 norm of the image sequence, i.e.

$$||Tf_k||_1 = \sum_{n=1}^{\infty} \frac{n}{n^{2+\frac{1}{k}}} = \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{k}}},$$

which has

$$\liminf_{k\to\infty} ||Tf_k||_1 = \liminf_{k\to\infty} \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{k}}}$$

by simply applying the lim inf to both sides. As $\forall n, k \in N$ we have $\frac{1}{n^{1+\frac{1}{k}}} > 0$, we can apply **Fatou's Lemma for Series**, which gets

$$\infty = \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \liminf_{k\to\infty} \frac{1}{n^{1+\frac{1}{k}}} \leq \liminf_{k\to\infty} \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{k}}} = \liminf_{k\to\infty} ||Tf_k||_1,$$

i.e. $\{Tf_k\}_{k=1}^{\infty}$ diverges off to ∞. As we have produced a bounded sequence that is unbounded under the image, this consequently shows T is unbounded.

□

(c) Let $S = T^{-1}$. Prove that $S : Y \to X$ is bounded and surjective but not open.

Proof. To verify the existence of S we show T is injective and surjective.

(i) *(Injectivity)* Let $f, g \in X$ such that $Tf = Tg$, we want to show $f = g$. Note we have

$$\begin{aligned} Tf &= Tg \\ \to Tf(n) &= Tg(n), \forall n \in \mathbb{N} \\ \to nf(n) &= ng(n), \forall n \in N \\ \to f(n) &= g(n), \forall n \in N && \text{(Dividing by } n\text{)} \\ \to f &= g \end{aligned}$$

And so T is an injection.

(ii) *(Surjectivity)* Let $g \in Y$. We want $f \in X$ such that $Tf = g$.

As $g \in Y$, we have that $\sum_{n=1}^{\infty} |g(n)| < \infty$. We define $h \in X$ then by

$$h(n) = \frac{g(n)}{n}$$

In particular, we note $h \in X$ as

$$\sum_{n=1}^{\infty} n|h(n)| = \sum_{n=1}^{\infty} n\Big|\frac{g(n)}{n}\Big| = \sum_{n=1}^{\infty} |g(n)| < \infty,$$

and so we finally note $Th = g$, as $Th(n) = nh(n) = n\frac{g(n)}{n} = g(n)$. Thus $h \mapsto g$ under T, and so we have surjectivity.

It follows T is a bijection, so it has a set-theoretic inverse $S = T^{-1}$, which is assuredly also a surjection as the inverse is also a bijection.

We want to show then S is bounded but not open.

(i) *(Boundedness)* Recall boundedness of S on Y is equivalent to continuity of S on Y. In particular, we remember S is continuous on Y if for every closed set $F \subseteq X$ we have

$$S^{-1}(F) \text{ is a closed set in } Y,$$

but of course

$$S^{-1}(F) = (T^{-1})^{-1}(F) = T(F),$$

where we showed T is a closed mapping, i.e. $T(F)$ is closed. It follows S is continuous on Y, and thus bounded.

(ii) *(Mapping is not Open)* Assume for the sake of contradiction S is an open mapping, i.e. for every open set $\mathcal{O} \subseteq Y$ we have

$$S(\mathcal{O}) \text{ is an open set in } X,$$

but this generates a contradiction as we recall T is bounded iff it is continuous, and this implies continuity of T as we have for every open set in $\mathcal{O} \subseteq Y$ that

$$T^{-1}(\mathcal{O}) = S(\mathcal{O}) \text{ is an open set in } X,$$

so here T is bounded, but we showed earlier this in not the case. It follows S must not be an open mapping.

It follows S is a surjection that is bounded but not open. □

Exercise 16. Let $\mathrm{Lip}[0,1] = \{f \in C[0,1] \mid f \text{ is Lipschitz}\}$, and for each $n \geq 1$ let $F_n = \{f \in C[0,1] \ : \ |f(x) - f(y)| \leq n|x-y|, \ \forall x, y \in [0,1]\}$.

(a) Prove for each $n \geq 1$, F_n is a closed, nowhere dense subset of $\mathrm{Lip}[0,1]$.

Proof. Let $n \in N$ and consider the corresponding F_n. We need to verify F_n is closed and nowhere dense (clearly it is a subset, given the definition of Lipschitz).

(i) *(Closedness)* Let $\{f_k\}_{k=1}^{\infty} \subseteq F_n$ such that $f_k \to f$ uniformly. We want to show $f \in F_n$. For this, let $\epsilon > 0$ and take K such that $||f - f_k||_\infty < \epsilon$ for $k \geq K$.

Moreover, let $x, y \in [0, 1]$. We note the following:

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_k(x) + f_k(x) - f_k(y) + f_k(y) - f(y)| \\ &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \\ &\leq 2||f - f_k||_\infty + n|x - y| < 2\epsilon + n|x - y| \end{aligned}$$

Taking $\epsilon \to 0$ then, we note $|f(x) - f(y)| \leq n|x - y|$. Thus $f \in F_n$.

(ii) *(Nowhere Dense)* We want to show the closure of each F_n, i.e. $\bar{F}_n$, has empty interior. As F_n is closed, $\bar{F}_n = F_n$, so we are just showing F_n has empty interior.

Assume for the sake of contradiction that F_n does not have empty interior, i.e. there some $f \in F_n$ such that for some $r > 0$ we have $B_r(f) \subseteq F_n$. To get a contradiction, we first prove a small lemma.

Lemma 1. Let $f \in \text{Lip}[0, 1]$. Then for $c > 0$, $f + c\sqrt{x} \notin \text{Lip}[0, 1]$.

Proof. Let some $f \in \text{Lip}[0, 1]$ with Lipschitz constant $K \geq 0$. Recall that Lipschitz functions are differentiable a.e., and their derivatives are bounded by their Lipschitz constants, i.e. here

$$|f'(x)| \leq K$$

To show $f + c\sqrt{x} \notin \text{Lip}[0, 1]$ then, it suffices to show its derivative is unbounded, i.e. $\forall M \geq 0$, we have some $x \in [0, 1]$ for which

$$\left| f'(x) + \frac{c}{2\sqrt{x}} \right| > M$$

Let $M \geq 0$. Take $\{x_n\}_{n=1}^{\infty} \subseteq [0, 1]$ such that $x_n \to 0$ then. As $c > 0$, we know

$$\lim_{n \to \infty} \frac{c}{2\sqrt{x_n}} = \infty,$$

and moreover we have by use of the reverse triangle inequality that

$$\frac{c}{2\sqrt{x}} - |f'(x)| \leq \left| \frac{c}{2\sqrt{x}} - |f'(x)| \right| \leq \left| f'(x) + \frac{c}{2\sqrt{x}} \right|$$

We take $N \in \mathbb{N}$ then such that $\frac{c}{2\sqrt{x_N}} \geq M + K$ (which exists as the limit diverges to ∞), which substituing into the previous inequality gets

$$M \leq \frac{c}{2\sqrt{x_N}} - K \leq \frac{c}{2\sqrt{x_N}} - |f'(x_N)| \leq \left| \frac{c}{2\sqrt{x_N}} - |f'(x_N)| \right| \leq \left| f'(x_N) + \frac{c}{2\sqrt{x_N}} \right|$$

and so the derivative is unbounded. It follows $f + c\sqrt{x} \notin \text{Lip}[0, 1]$. $\square$

Return then to our f such that $B_r(f) \subseteq F_n$. Consider the function g defined by

$$g(x) = f(x) + \frac{r}{2}\sqrt{x}$$

As f is Lipschitz, **Lemma 1** shows that g is not. But this is problematic as we see

$$||f - g||_\infty = \sup_{x \in [0,1]} |f(x) - g(x)| = \sup_{x \in [0,1]} \left| \frac{r}{2}\sqrt{x} \right| = \frac{r}{2},$$

which renders a contradiction as then $g \in B_r(f) \subseteq F_n$, i.e. g should be Lipschitz. It follows F_n must have empty interior, and so it is nowhere dense.

It follows for each $n \in \mathbb{N}$ that F_n is closed and nowhere dense. □

(b) Conclude that $\mathrm{Lip}[0,1]$ is a countable union of nowhere dense subsets of $C[0,1]$.

Proof. Note the definition of each F_n is precisely the set of Lipschitz functions with Lipschitz constant at most n. It follows then immediately that

$$\bigcup_{n=1}^{\infty} F_n \subseteq \mathrm{Lip}[0,1]$$

Moreover, letting an arbitrary $f \in C[0,1]$ with Lipschitz constant $K \geq 0$, we note there is some $N \in \mathbb{N}$ such that $N \geq K$ by the Archimedean principle.

For $x, y \in [0,1]$ then we have

$$|f(x) - f(y)| \leq K|x - y| \leq N|x - y|,$$

i.e. $f \in F_N \subseteq \bigcup_{n=1}^{\infty} F_n$, and so

$$\mathrm{Lip}[0,1] \subseteq \bigcup_{n=1}^{\infty} F_n,$$

and so of course

$$\mathrm{Lip}[0,1] = \bigcup_{n=1}^{\infty} F_n,$$

i.e. $\mathrm{Lip}[0,1]$ is a countable union of nowhere dense sets, as the F_n are nowhere dense as proven in (a). □

Exercise 17. Let f be a nonnegative Lebesgue measurable function defined on $\mathbb{R}$. Assume that for all $g \in L^2(\mathbb{R})$ we have $fg \in L^1(\mathbb{R})$. Prove that $T_f(g) = \int_{\mathbb{R}} gf$ is a bounded linear functional on $L^2(\mathbb{R})$ and conclude that $f \in L^2(\mathbb{R})$.

Proof. Note first that $T_f(g)$ is linear given distributivity of multiplication and the linearity of the integral.

We want to show $T_f(g)$ is a bounded linear functional on $L^2(\mathbb{R})$, i.e. for all $g \in L^2(\mathbb{R})$ we have

$$|T_f(g)| = \Big| \int_{\mathbb{R}} gf \Big| \leq C||g||_2,$$

for some $C \geq 0$ not dependent on g. Consider then the family of sets $\{E_k\}_{k=1}^{\infty}$ where

$$E_k = [-k, k];$$

for any given E_k then, we define $f_k : \mathbb{R} \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} \min(f(x), k) & \text{if } x \in E_k \\ 0 & \text{if } x \notin E_k \end{cases} \tag{7}$$

which gives a family of functions the nonnegative functions $\{f_k\}_{k=1}^{\infty}$. We note that each f_k is measurable relatively easily, as the minimum of two measurable functions is measurable.

In particular, we know $\min(f, k)$ is measurable. For $a > 0$ then, the set $\{f_k \geq a\}$ is just $\{\min(f, k) \geq a\} \cap E_k$, which is measurable as it is the intersection of measurable sets.

If instead $a \leq 0$, then clearly the set $\{f_k \geq a\}$ is just $\mathbb{R}$, which is also measurable. Thus each function in $\{f_k\}_{k=1}^{\infty}$ is measurable. With this, we can quickly establish each is in $L^2(\mathbb{R})$, as

$$||f_k||_2 = \sqrt{\int_{\mathbb{R}} |f_k|^2} = \sqrt{\int_{E_k} |f_k|^2} \leq \sqrt{\int_{E_k} k^2} = \sqrt{2k^3} < \infty,$$

i.e. each f_k has finite 2-norm. Consider then the corresponding family of functionals on $L^2(\mathbb{R})$ given by

$$T_k(g) = \int_{\mathbb{R}} gf_k,$$

i.e. consider $\{T_k\}_{k=1}^{\infty}$. Linearity of each T_k follows trivially from distributivity and linearity of the integral. Moreover, we note for $g \in L^2(\mathbb{R})$ has

$$|T_k(g)| = \Big| \int_{\mathbb{R}} gf_k \Big| \leq \int_{\mathbb{R}} |gf_k| = ||gf_k||_1 \leq ||g||_2||f_k||_2 \leq \sqrt{2k^3}||g||_2,$$

given our work in the previous part. It follows every functional in the family $\{T_k\}_{k=1}^{\infty}$ is bounded and linear.

Our goal now is to apply the **Uniform Boundedness Principle**. For this, consider some fixed $g \in L^2(\mathbb{R})$. Moreover, let $k \in \mathbb{N}$. As $f_k \leq f$ by definition, we get

$$|T_k(g)| = \Big| \int_{\mathbb{R}} f_k g \Big| \leq \int_{\mathbb{R}} |f_k g| \leq \int_{\mathbb{R}} |fg| = ||fg||_1 < \infty,$$

where we recall $||fg||_1 < \infty \Leftrightarrow fg \in L^1(\mathbb{R})$ by assumption. Thus we have shown $|T_k(g)| \leq ||fg||_1$, and so

$$\sup_{k\in\mathbb{N}} |T_k(g)| \leq ||fg||_1 < \infty$$

for all $g \in L^2(\mathbb{R})$. Thus the **Uniform Boundedness Principle** applies, in particular we get

$$\sup_{k\in\mathbb{N}} ||T_k|| < \infty,$$

i.e. the supremum over the operator norms is finite. We want then to conclude that

$$||T_f|| \leq \sup_{k\in\mathbb{N}} ||T_k||,$$

i.e. $||T_f||$ is finite $\Leftrightarrow T_f$ is bounded. For this, first note it must be that f is finite a.e., as if it were infinite on a set A of positive measure, we could choose $g \in L^2(\mathbb{R})$ to be an indicator function of a measurable subset of A with finite measure.

The resulting product fg would then of course not be finite a.e., but this would contradict that it should be $fg \in L^1(\mathbb{R})$.

As f is finite a.e. then, it is clear that for each $g \in L^2(\mathbb{R})$ where $||g||_2 = 1$ (also finite a.e.) $gf_k \to gf$ pointwise a.e. by construction (and these functions are measurable). Moreover we recall

$$|gf_k| \leq |gf|$$

where of course $|fg| \in L^1(\mathbb{R})$. It follows we can apply the **Dominated Convergence Theorem**, i.e.

$$|T_f(g)| = \Big|\int_{\mathbb{R}} gf\Big| = \Big|\lim_{k\to\infty}\int_{\mathbb{R}} g_f k\Big| = |\lim_{k\to\infty} T_k(g)| = \lim_{k\to\infty}|T_k(g)| \leq \liminf_{k\to\infty}||T_k|| \leq \sup_{k\in\mathbb{N}}||T_k||,$$

where this then holds for all $g \in L^2(\mathbb{R})$ with unit norm, so taking the supremum over $g \in L^2(\mathbb{R})$ with unit norm gets

$$||T_f|| = \sup_{||g||_2=1} |T_f(g)| \leq \sup_{k\in\mathbb{N}}||T_k||,$$

and so we get the desired inequality. In particular this shows $||T_f|| = \sup_{g\neq 0} \frac{|T_f(g)|}{||g||_2}$ is finite, and so for $g \neq 0$ we have

$$\frac{|T_f(g)|}{||g||_2} \leq ||T_f|| \longrightarrow |T_f(g)| \leq ||T_f||||g||_2,$$

where this trivally holds for $g = 0$. Thus we $T_f(g)$ is bounded with $C = ||T_f||$.

We want now to conclude then $f \in L^2(\mathbb{R})$. For this, note by the **Riesz Representation Theorem** that there is a unique $h \in L^2(\mathbb{R})$ such that for all g

$$T_f(g) = \langle g, h\rangle,$$

i.e. we have

$$\int_{\mathbb{R}} gf = \int_{\mathbb{R}} gh$$

for all $g \in L^2(\mathbb{R})$. It follows in particular that

$$0 = \int_{\mathbb{R}} gf - \int_{\mathbb{R}} gh = \int_{\mathbb{R}} gf - gh = \int_{\mathbb{R}} g(f-h) = 0$$

for all $g \in L^2(\mathbb{R})$. Let then g an indicator function of a finite interval I then. Clearly $g \in L^2(\mathbb{R})$, so this has

$$\int_{\mathbb{R}} g(f-h) = \int_I f - h = 0,$$

but we know that if the integral of a function is zero over every interval the only possibility is that it equals zero almost everywhere (this was proven in *Math 235*).

Thus it must be $f = h$ a.e., as so as h is in $L^2(\mathbb{R})$ it follows $f \in L^2(\mathbb{R})$ as desired. □

Exercise 19. Let $\mathcal{B}$ a Banach space, and S a closed proper subspace. Fix some $f_0 \notin S$. Show that there is a continuous linear functional γ such that $\gamma(f) = 0$ for all $f \in S$ and $\gamma(f_0) = 1$.

Moreover, show that we may choose the linear functional such that $||\gamma|| = \frac{1}{d}$, where d is the distance from f_0 to S.

Proof. We first want to establish that $d > 0$. In particular, we denote this must be the case because if the distance were 0, we could find a sequence $\{s_n\} \subseteq S$ such that

$$\lim_{n\to\infty} ||f_0 - s_n|| = 0,$$

which would have that $f_0 \in S$ appealing to the closure of S, but this is a contradiction.

Define then the vector subspace T by

$$T = \text{span}\{S, f_0\},$$

which is indeed trivially a subspace of $\mathcal{B}$ given the definition of span. Moreover, by the definition of span we know we can represent each element $t \in T$ like

$$t = s_t + \lambda_t f_0,$$

where $s_t \in S$ and λ_t a scalar. We claim this representation is unique. On the contrary, suppose we had $t = s_t + \lambda_t f_0 = s_t' + \lambda_t' f_0$. It follows

$$(s_t - s_t') + (\lambda_t - \lambda_t') f_0 = 0,$$

but as $f_0 \in S$ then the only possibility is that $(\lambda_t - \lambda_t') f_0 = 0$ and so $(s_t - s_t') = 0$. In particular, otherwise we would have

$$-f_0 = \frac{(s_t - s_t')}{(\lambda_t - \lambda_t')},$$

which immediately has $f_0 \in S$ using subspace properties. It follows $s_t = s_t'$ and $a_t - a_t' = 0$, i.e. $a_t = a_t'$. Thus the representation is unique.

With this in mind, define the functional $\delta : T \to K$, where K is the field $\mathcal{B}$ is vector space over. In particular, define δ for $t \in T$ by

$$\delta(t) = \lambda_t,$$

where λ_t is defined as above. Note this mapping is well-defined per the uniqueness we proved. Consider $t, t' \in T$ then, we show linearity of δ in a straightforward way:

$$\delta(t + t') = \delta(s_t + \lambda_t f_0 + s_{t'} + \lambda_{t'} f_0) = \delta((s_t + s_{t'}) + (\lambda_t + \lambda_{t'}) f_0) = \lambda_t + \lambda_{t'} = \delta(t) + \delta(t')$$

Where of course $s_t + s_{t'} \in S$. Moreover, for additional constant $k \in K$ we get

$$\delta(kt) = \delta(k(s_t + \lambda_t f_0)) = \delta(ks_t + k\lambda_t f_0) = k\lambda_t = k\delta(t),$$

and so δ is assuredly linear.

Note then by construction we have $\delta|_S = 0$, each element $s \in S$ has $\lambda_S = 0$, as they have the representation $s = s + 0 \cdot f_0$. Similarly, $\delta(f_0) = 1$.

We need to show then that δ is bounded and that $||\delta|| = \frac{1}{d}$.

Consider then some vector t in T. We assume without loss of generality that $\lambda_t \neq 0$, as as otherwise $\delta(t) = 0$ which is trivially bounded.

It follows λ_t has an inverse, so we write

$$t = s_t + \lambda_t f_0 = \lambda_t(\lambda_t^{-1} s_t + f_0),$$

where we furthermore get that

$$||t|| = ||\lambda_t(\lambda_t^{-1} s_t + f_0)|| = |\lambda_t|||\lambda_t^{-1} s_t + f_0||$$

Note then we have $\lambda_t^{-1} s_t \in S$, as S is a subspace. It follows then that $||\lambda_t^{-1} s_t + f_0|| \geq d$ by the definition d.

In particular, we recall

$$d = \inf_{s \in S} ||f_0 - s||,$$

so taking $s = -\lambda_t^{-1} s_t$ suffices to show we have a lower bound by d. Thus

$$\begin{aligned} ||t|| = |\lambda_t|||\lambda_t^{-1} s_t + f_0|| &\geq |\delta(t)|d \\ &\longrightarrow |\delta(t)| \leq \frac{1}{d}||t||, \end{aligned}$$

and so taking the supremum over unit norm t has $||\delta|| \leq \frac{1}{d}$, and of course that $||d||$ exists.

To show $\frac{1}{d} \leq ||\delta||$ then, consider the sequence that minimizes the distance, i.e. the sequence $\{s_n\}_{n=1}^{\infty} \subseteq S$ such that $||f_0 - s_n|| \to d$ as $n \to \infty$. Then we have

$$1 = |\delta(f_0)| = |\delta(f_0 - s_n)| \leq ||\delta||||f_0 - s_n||,$$

where taking the limit thus gets $1 \leq ||\delta||d$. It follows $\frac{1}{d} \leq ||\delta||$, and so $||\delta|| = \frac{1}{\delta}$.

Using **Corollary 2.2** from Heil then, as $\delta \in T^*$, we have there exists a $\gamma \in \mathcal{B}^*$ such that

$$\gamma|_T = \delta \text{ and } ||\gamma|| = ||\delta|| = \frac{1}{d},$$

so in particular as we have $\gamma|_T = \delta$ we have $\gamma(f) = 0$ for $f \in S$ and $\gamma(f_0) = 1$. This concludes the problem. $\square$