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# Math 237: Lecture 6

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**Example:** If $X, Y$ are normed spaces, the $L(X, Y)$ is a normed space. If $Y$ is complete, then $L(X, Y)$ is complete. If $X$ is a normed space, then the normed space $X^* = L(X, F)$ is the dual of $X$:

$$X* = \{L : X \to F : L \text{ is continuous and linear}\}$$

**Example:** Let $X, Y, Z$ be normed spaces. Furthermore, take $T \in L(X, Y), S \in L(Y, Z)$. We have then $S \circ T \in L(X, Z)$. It is easy to to check linearity, and boundedness follows from the submultiplicative property of the operator norm, in particular we have:

$$\begin{aligned}\sup_{||x||_X=1} ||S \circ T(x)||_Z = \sup_{||x||_X=1} ||S(T(x))||_Z \leq \sup_{||x||_X=1} ||S||_{L(Y,Z)}||Tx||_Y = ||S||_{L(Y,Z)} \sup_{||x||_X=1} ||Tx||_Y \\ \leq ||S||_{L(Y,Z)}||T||_{L(X,Y)} \sup_{||x||_X=1} ||x||_X = ||S||_{L(Y,Z)}||T||_{L(X,Y)}\end{aligned} \tag{1}$$

Thus in general we have something like:

$$||S \circ T||_{L(X,Z)} \leq ||T||_{L(X,Y)}||S||_{L(Y,Z)}$$

Under this, we note that the bounded linear operators on a normed space form a normed algebra.

We get some additional definitions/facts on $T \in L(X, Y)$:

(a) $T$ is injective if $\ker(T) = \{x \in X, Tx = 0\} = \{0\}$.

(b) $T$ is surjective if range$(T)\{Tx, x \in X\} = Y$.

(c) $T$ is an isomorphism of normed spaces if $T$ is bijective.

(d) $T$ is isometric if $||Tx||_Y = ||x||_X, \forall x \in X$.

**Theorem (Riesz Representation Theorem for Hilbert Space).** Let $H$ be a Hilbert space. Any bounded linear functional $T$ on $H$ has the form $Tx = \langle x, z\rangle$ for a unique $z \in H$. and $||T|| = ||z||$.