David Owen Horace Cutler **Math 237: Homework 2** April 3, 2024

Exercise 1. Show that the only function $f \in L^1(\mathbb{R})$ such that $f = f * f$ is $f = 0$ a.e.

Proof. Let $f \in L^1(\mathbb{R})$ such that $f = f * f$. Recall we have that by Heil **Exercise 9.2.6** that then

$$\hat{f} = (f * f)\hat{} = \hat{f}\hat{f},$$

so for any particular ξ we have

$$\hat{f}(\xi)^2 - \hat{f}(\xi) = 0,$$

so we consequently have

$$\hat{f}(\xi) = 0 \text{ or } 1, \quad \forall \xi \in \mathbb{R}.$$

Recall then that $\hat{f}$ is continuous on $\mathbb{R}$; it trivially follows that it thus must be either identically 0 or identically 1.

However, the **Riemann-Lebesgue Lemma** guarentees that $|\hat{f}|$ should decay to 0 as $|x| \to \infty$; so the only possibility is that $\hat{f}$ is constantly 0.

Note then clearly $\hat{0} = 0$; by the **Uniqueness Theorem** then, as we thus have $\hat{f} = \hat{0}$, it must be then that $f = 0$ a.e., as desired $\square$

Exercise 6. Suppose $f \in \mathrm{AC}(\mathbb{T})$, i.e., f is 1-periodic and absolutely continuous on $[0, 1]$.

6.1 Prove that $\hat{f'}(n) = 2\pi i n \hat{f}(n)$ for $n \in \mathbb{Z}$ and conclude $\lim_{|n|\to\infty} n\hat{f}(n) = 0$.

Proof. By assumption $f \in AC(\mathbb{T})$. We know of course that $e^{2\pi i n\xi}$ is furthermore absolutely continuous are we observe it is continuously differentiable.

Thus, we can apply **Integration by Parts**, which gets us the following (note $\frac{d}{d\xi}e^{-2\pi i n\xi} = -2\pi i n e^{-2\pi i n\xi}$):

$$\begin{aligned}\hat{f'}(n) &= \int_0^1 f'(\xi)e^{-2\pi i n\xi}d\xi \\ &= e^{-2\pi i n\xi}f(1) - f(0) - \int_0^1 -2\pi i n f(\xi)e^{-2\pi i n\xi}d\xi \\ &= e^{-2\pi i n\xi}f(1) - f(0) + 2\pi i n\hat{f}(n)\end{aligned}$$

We note then that for any value of $n \in \mathbb{Z}$, we have $e^{-2\pi i n\xi} = 1$. Moreover, as f is 1-periodic, we have $f(0) = f(1)$. Thus we can further reduce

$$\begin{aligned}&e^{-2\pi i n\xi}f(1) - f(0) + 2\pi i n\hat{f}(n) \\ &= 2\pi i n\hat{f}(n),\end{aligned}$$

as desired. Recall the by the **Riemann-Lebesgue Lemma** we know

$$\lim_{|n|\to\infty} |\hat{f'}(n)| = 0,$$

and thus

$$\lim_{|n|\to\infty} |\hat{f}'(n)| = \lim_{|n|\to\infty} |2\pi in\hat{f}(n)| = \lim_{|n|\to\infty} 2\pi n|\hat{f}(n)| = 0,$$

where from limit rules multiplying by $\frac{1}{2\pi}$ gets the limit of $n|\hat{f}(n)|$ as 0, which clearly implies the same for $n\hat{f}(n)$, as additionally desired. $\square$

6.2 Show that if $\int_0^1 f(x)dx = 0$, then

$$\int_0^1 |f(x)|^2 dx \le \frac{1}{4\pi^2} \int_0^1 |f'(x)|^2 dx$$