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Definition (Semi-Norm and Norm). Let X be a vector space. Then if the mapping $||\cdot||_X : X \to [0,\infty)$ given by $x \to ||x||$ has:

1. $\forall x, y \in X, ||x+y|| \leq ||x|| + ||y||$ **(Triangle Inequality)**
2. $\forall \lambda \in K, \forall x \in X, ||\lambda x|| = |\lambda| ||x||$ **(Absolute Homogeneity)**
3. $||0|| = 0$ and if $||x|| = 0 \to x = 0$ **(Positive Definiteness)**

It is called a **norm** on X. Fulfilling just the first two yields a **semi-norm**.

Definition (Normed Vector Space). A **normed vector space** is a vector space X on a field K equipped with a norm.

Definition (Metric Norm). If $(X, ||\cdot||)$ is a normed space, then $d(x,y) = ||x-y||$ defines a distance (or metric) on X called the **norm metric**.

Definition (Completeness in Norm Metric). A normed space is a said to be **complete** with respect to the norm metric if every Cauchy sequence in X converges in X.

Definition (Banach Space). A normed vector space which is complete is the norm metric is a **Banach space**.

Theorem (Norm Equivalence on Finite-Dimensional Normed Vector Spaces). If V is a finite-dimensional vector space, then all norms on V are **equivalent**. That is, in $||\cdot||_1$ and $||\cdot||_2$ are any 2 norms on V,

$$\exists c_1, c_2 > 0 \text{ such that } c_1||x||_1 \leq ||x||_2 \leq c_2||x||_1, \forall x \in V$$

Definition (Equivalent Norms). If X is a vector space, we say two norms $||\cdot||_1$ and $||\cdot||_2$ are **equivalent** if there are

$$c_1, c_2 > 0 \text{ such that } \forall x \in X, c_1||x||_1 \leq ||x||_2 \leq c_2||x||_1$$

Definition (Linear Mapping and Its Boundedness). Let X, Y be two vector spaces. A map $T : X \to Y$ is **linear** if:

$$\begin{gathered} \forall x_1, x_2 \in X, T(x_1 + x_2) = T(x_1) + T(x_2) \\ \forall \lambda \in K, \forall x \in X, T(\lambda x) = \lambda T(x) \end{gathered} \tag{1}$$

If X and Y are normed spaces then the linear map $T : X \to Y$ is said to be **bounded** if:

$$\exists c > 0 \text{ such that } ||T(x)||_Y \leq c||x||_X, \forall x \in X$$

Theorem (Continuity Conditions on Normed Spaces). Let $T : X \to Y$ be a linear map between two normed spaces. Then the following statements are equivalent:

(a) T is continuous on X

(b) T is continuous at 0

(c) T is bounded

Proof. Remark $(a) \to (b)$ is trivial, so we proceed with $(b) \to (c)$. Let $\epsilon > 0$ then. Using continuity at 0, we know that $\exists \delta > 0$ such that $||x|| < \delta$ has $||T(x)|| < \epsilon$. Consider some $x \in V$ then, rewriting yields the following:

$$||T(x)|| = \left|\left|T\Big(\frac{\delta x}{2||x||} \cdot \frac{2||x||}{\delta}\Big)\right|\right| = \frac{2||x||}{\delta}\left|\left|T\Big(\frac{\delta x}{2||x||}\Big)\right|\right| \tag{2}$$

Note then that $\left|\left|T(\frac{\delta x}{2||x||})\right|\right| < \epsilon$ as $\left|\left|\frac{\delta x}{2||x||}\right|\right| = \frac{\delta}{2}\left|\left|\frac{x}{||x||}\right|\right| = \frac{\delta}{2}$. Thus:

$$\frac{2||x||}{\delta}\left|\left|T\Big(\frac{\delta x}{2||x||}\Big)\right|\right| < \frac{2||x||}{\delta}\epsilon \tag{3}$$

Taking $c = \frac{2||x||}{\delta}$ thus suffices. For $(c) \to (a)$, we assume boundedness and take c in the definition. Consider $x_n \to x_0 \in X$ then and take c in the definition of boundedness, we want to show that:

$$\lim_{n\to\infty} ||T(x_n) - T(x_0)|| = 0$$

For this sake, note we have the following:

$$||T(x_n) - T(x_0)|| = ||T(x_n - x_0)|| \leq c||x_n - x_0||$$

But $x_n \to x_0$, so the latter quantity gets small. Applying the squeeze theorem thus has the desired result, and so all given conditions are equivalent. $\square$

Theorem (Completeness Characterization). Let $(X, ||\cdot||_X)$ be a normed space.

X is complete if and only if every absolutely convergent series in X converges in X, i.e. if $\{x_n\}_{n=1}^{\infty}$ is a sequence then:

$$\sum_{n=1}^{\infty} ||x_n||_X < \infty \longrightarrow \sum_{n=1}^{\infty} x_n \text{ converges in } X \tag{4}$$

Note here that $\sum_{n=1}^{\infty}$ converges in X means for $S_k = \sum_{n=1}^{k} x_n$, we have $\lim_{k\to\infty} S_k = x \in X$.

Proof. We start with the forward implication, assuming X is complete. Let $\{x_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} ||x_n||_X < \infty$. Let $\epsilon > 0$, then using Cauchy's criteria for series convergence, we have some $N \in \mathbb{N}$ such that:

$$\sum_{k=N}^{\infty} ||x_k||_X < \epsilon$$

Define S_n by $S_n = \sum_{k=1}^{n} x_n$. Consider $n, m \geq N$ with $n \geq m$ without loss of generality. Then we have the following:

$$\begin{aligned} ||S_n - S_m||_X = \Big|\Big| \sum_{k=1}^{n} x_n - \sum_{k=1}^{m} x_n \Big|\Big|_X \\ = \Big|\Big| \sum_{k=m}^{n} x_n \Big|\Big|_X \leq \sum_{k=m} ||x_n||_X \leq \sum_{k=N}^{\infty} ||x_n||_X < \epsilon \end{aligned} \tag{5}$$

Thus the sequence of partial sums is Cauchy, and so as X is complete it is convergent. This is of course precisely that $\sum_{n=1}^{\infty} x_n$ is convergent though, so this implication is finished.

For the reverse implication, let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X. We want to show it converges. Using the definition of a Cauchy sequence, we generate some n_js in the following fashion:

$$\begin{aligned} \exists n_1 < n_2 < ... \\ ||x_n - x_m||_X < 2^{-j}, \forall n, m \geq n_j \end{aligned} \tag{6}$$

Define a new sequence $\{y_j\}_{j=1}^{\infty}$ by taking $y_1 = x_{n_1}, y_j = \sum_{k=1}^{j} x_{n_k} - x_{n_{k-1}}$. Observe then the following:

$$\sum_{j=1}^{\infty} ||y_1||_X + \sum_{j=2}^{\infty} ||x_{n_k} - x_{n_{k-1}}||_X < ||y_1||_X + \sum_{j=1}^{\infty} 2^{-j} = ||y_1||_X + 1 < \infty \tag{7}$$

Thus, as we are operating under the assumption that all absolutely convergent series converge, $\sum_{j=1}^{\infty} y_j$ converges in X, but this just means $\lim\limits_{j \to \infty} x_{n_j}$ converges. So x_n is a Cauchy sequence with a convergent subsequence, and thus it is convergent. $\square$

Definition (Space of Linear Bounded Operators). Let X, Y be normed spaces. We define:

$$L(X, Y) = \{T : X \to Y : T \text{ is linear and bounded}\} \tag{8}$$

Definition (Operator Norm). If $T \in L(X, Y)$, then we define:

$$\begin{aligned} ||T||_{L(X,Y)} = \sup\{||T(x)||_Y, ||x||_X = 1\} \\ = \sup\left\{\frac{||T(x)||_Y}{||x||_X}, x \in X, x \neq 0\right\} \\ = \inf\{c > 0 : ||T(x)||_Y \leq c||x||_X, \forall x \in X\} \end{aligned} \tag{9}$$