# Thesis Proposal Review: How Hard is a Causal Inference Problem

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## 1 Introduction: Consequences of Decisions

This thesis is concerned with understanding a particular kind of decision problem: we are given a set of feasible decisions and a set of observed data, we know the potential consequences these decisions may have and we know how desirable these consequences are. We wish to develop strategies for selecting decisions that are likely to lead to favourable consequences. For example, the decisions may be a set of possible medical treatments, consequences are states of health and data are from published medical trials; we also assume that some states of health are known to be more desirable than others.

This general kind of problem seems to me to be a reasonable description of a type of problem that people often face (allowing that it may be somewhat simplified). But I need not rely only on an appeal to intuition to argue that this is an important class of problem, as decision problems of this type have a long and extensive history of study: Von Neumann and Morgenstern (1944) considers the problem of choosing between consequences directly with some means of evaluating their desirability, Weirich (2016) discusses decision problems featuring decisions, consequences and desirability but no explicit consideration of data. Wald (1950) considers the problem of selecting a favourable decision given a set of data and a desirability function, though he eschews explicitly considering consequences, and Savage (1972) develops Wald's theory to also include consequences of decisions, yielding a class of decision problems very similar to those discussed here. Many of the solutions presented by these authors have "entered the water supply" - in particular, the expected utility theory of Von Neumann and Morgenstern (1944) underpins an enormous amount of the work on decision problems of any type, and the risk functionals of Wald (1950) are fundamental to much of statistics and machine learning. Even theories that reject the particulars proposed by these authors build on the foundations laid by them - in short, the type of problem studied here is widely accepted to be a very important class of problem.

This type of problem has particular practical relevance to the field of *causal inference*. A Google Scholar search for "causal inference" found, in the top five results:

- Holland (1986) and Frangakis and Rubin (2002) discuss causal inference as the project of relating *treatments* to *responses* via *observations*. If we postulate an implicit desirability of responses, we have a decision problem of the type outlined

- Morgan and Winship (2014) provide in their opening paragraph three examples of causal problems. Two of them have clear interpretations as decision problems where decisions involve funding of charter schools and engaging in or encouraging college study, while the third is perhaps more concerned with *responsibility* and *remedy*:

  - Do charter schools increase test scores?
  - Does obtaining a college degree increase an individual's labor market earnings?
  - Did the use of a butterfly ballot in some Florida counties in the 2000 presidential election cost Al Gore votes?

- Pearl (2009a) begins with four examples of causal questions. The first appears to be part of a decision problem, while the second to fourth are questions of responsibility and remedy:

  - What is the efficacy of a given drug in a given population?
  - Whether data can prove an employer guilty of hiring discrimination?
  - What fraction of past crimes could have been avoided by a given policy?
  - What was the cause of death of a given individual, in a specific incident?

- Robins et al. (2000) is again concerned with estimating responses to treatments via observations

From this informal survey we have six out of ten example problems that correspond directly to the type of decision problem studied here. While decision problems are a substantial class of causal inference problems, we find that questions of responsibility also figure prominently. While the approach built in this thesis may have eventual applications to questions of responsibility and other causal questions, we take the attitude that in the worst case it will only be appliccable to decision problems and this is a large and important enough class of problems that a clearer understanding of just these problems will still be very valuable.

One key difference between CSDT and existing popular approaches to causal inference is that we stipulate that *the set of decisions is a feature of the problem*, and does not depend in any way on how we choose to analyse the problem. Existing approaches provide "standard" objects (e.g. counterfactual random variables) or operations (e.g. intervening on the value of some random variable) which, if they are to be interpreted as decisions, impose some presuppositions on the nature of the decisions available. Even if these presuppositions correspond to very common regularities of decision problems, we take the view that such regularities should be included as assumptions rather than be part of the language used to express the problem.

This difference is illustrated by the question of *external validity*. Given a randomised controlled trial (RCT), under ideal conditions existing causal inference approaches agree that certain causal effects can be consistently estimated. However, as reported by Deaton and Cartwright (2018):

> Trials, as is widely noted, often take place in artificial environments which raises well recognized problems for extrapolation. For instance, with respect to economic development, Drèze (J. Drèze, personal communications, November 8, 2017) notes, based on extensive experience in India, that when a foreign agency comes in with its heavy boots and deep pockets to administer a treatment, whether through a local NGO or government or whatever, there tends to be a lot going on other than the treatment. There is also the suspicion that a treatment that works does so because of the presence of the treators, often from abroad, and may not do so with the people who will work it in practice.

Here, Drèze is describing the problem of determining the consequences of the "treatment in practice", and why these may differ from the "causal effects of treatment in the trial" - the question of external validity is, loosely, the question of how informative the latter are about the former. The usual approach of causal inference is to determine conditions under which the latter can be estimated and then, maybe, consider some additional assumptions that might allow for the latter estimate to inform the former. CSDT inverts the priority of these questions: the question of treatment in practice is primary and the question of causal effects in the trial may be a subproblem of interest under particular conditions.

Bareinboim and Pearl (2012) have claimed to have a complete solution to the problem of "[identifying] conditions under which causal information learned from experiments can be reused in a different environment where only passive observations can be collected", a claim made with more force in Pearl (2018). A complete solution to the transportability of causal information is *not* a claim of a complete solution to the problem of determining the effects of "treatment in practice" or the problem of making decisions with causal information. These latter problems ask when causal effects are informative about the consequences of decisions in the given problem, a question that doesn't even make sense without our insistence that decisions are a feature of the problem.

Key features (/aims - not all are realised yet) of CSDT are:

- Conceptual clarity:
  - CSDT separates of those aspects of a problem that are fixed by non-causal considerations (objectives, feasible decisions) and causal assumptions

- Unification and extension of existing approaches to causal inference for decision problems
  - Faithful translation from any existing approach to CSDT (including the derivation of key results)
  - Exact and approximate comparison of arbitrary causal theories
  - Quantification of the *difficulty* of a causal problem
  - Necessary conditions for key results
  - Novel approaches/assumptions for causal inference

the following seems like a reasonable point, but not sure where to put it right now

The core features of CSDT are that it is a new approach to causality that is strictly more capable of representing decision problems than existing approaches, and that it allows for novel and fundamental questions to be asked. However, a secondary feature of CSDT is that its statements can be clearly resolved to statements in the underlying theory of probability. This may also be true of some counterfactual approaches, but I don't think it is true of interventional graphical models. For example, Causal Bayesian Networks feature an elementary operation notated $P(\cdot|do(\mathsf{X}_k = a))$ where $\mathsf{X}_k$ is a random variable on some implicit sample space $E$. We can ask: what does $P(\cdot|do(\mathsf{X}_k = a))$ mean in more elementary terms? $do(\mathsf{X}_k = a)$ itself *looks* like a function, and the conventional interpretation of $\mathsf{X}_k = a$ is the preimage of $a$ under $\mathsf{X}_k$. Thus, $do()$ appears to be a function typed like a measure on $\mathcal{E}$ with the domain being the sigma algebra generated by all statements $\mathsf{X}_i = a$ for all $\mathsf{X}_i$ associated with some graph $\mathcal{G}$, which we will denote $\sigma(\underline{\bigotimes}_{i\in\mathcal{G}}\mathsf{X}_i)$. We might surmise that the "conditional probability" $P(\cdot|do(\mathsf{X}_k = \cdot))$ might then be the conditional probability on $\sigma(\underline{\bigotimes}_{i\in\mathcal{G}}\mathsf{X}_i)$. However, CBNs in general support models where $P(\cdot|do(\mathsf{X}_k = \cdot))$ is not equal to $P(\cdot|\mathsf{A})$ for any $\mathsf{A} \in \sigma(\underline{\bigotimes}_{\mathcal{G}}\mathsf{X}_i)$, so our attempt to parse this notation by "conventional reading" has failed.

In fact, the situation is even more dire: we may view $do(\mathsf{X}_k = a)$ as a relation between probability measures on $E$ which is not, in general, functional – an interpretation compatible with the definitions in Pearl (2009b). If $do()$ were functional, we could define $P(\cdot|(\mathsf{X}_k = a)$ to be the element of $\Delta(\mathcal{E})$ related to $P$ by $(\mathsf{X}_k = a)$. However, because $do(\mathsf{X}_k = a)$ is not functional, "conditioning" on $do(\mathsf{X}_k = \cdot)$ is ambiguous - does $P(\cdot|do(\mathsf{X}_k = a))$ refer to the set of probability measures related to $P$? A distinguished member of this set? In contrast to regular conditioning, where a similar ambiguity prevails but the ambient measure guarantees that disagreement can only happen on sets of measure zero, $P(\cdot|do(\mathsf{X}_k = a))$ can under different interpretations assign different measures to the same set. Causal Bayesian Network notational conventions suggest interpretations that do not make sense, and their meaning may be ambiguous even if we dig more deeply into the matter.

# 2 Definitions and key notation

We use three notations for working with probability theory. The “elementary” notation makes use of regular symbolic conventions (functions, products, sums, integrals, unions etc.) along with the expectation operator $\mathbb{E}$. This is the most flexible notation which comes at the cost of being verbose and difficult to read. Secondly, we use a semi-formal string diagram notation extending the formal diagram notation for symmetric monoidal categories Selinger (2010). Objects in this diagram refer to stochastic maps, and by interpreting diagrams as symbols we can, in theory, be just as flexible as the purely symbolic approach. However, we avoid complex mixtures of symbols and diagrams elements, and fall back to symbolic representations if it is called for. Finally, we use a matrix-vector product convention that isn't particularly expressive but can compactly express some common operations.

## 2.1 Standard Symbols

| Symbol | Meaning |
|---|---|
| $[n]$ | The natural numbers $\{1, ..., n\}$ |
| $f : a \mapsto b$ | Function definition, equivalent to $f(a) := b$ |
| Dots appearing in function arguments: $f(\cdot, \cdot, z)$ | The “curried” function $(x, y) \mapsto f(x, y, z)$ |
| Capital letters: $A, B, X$ | sets |
| Script letters: $\mathcal{A}, \mathcal{B}, \mathcal{X}$ | $\sigma$-algebras on the sets $A, B, X$ respectively |
| Script $\mathcal{G}$ | A directed acyclic graph made up of nodes $V$ and edg |
| Greek letters $\mu, \xi, \gamma$ | Probability measures |
| $\delta_x$ | The Dirac delta measure: $\delta_x(A) = 1$ if $x \in A$ and 0 oth |
| Capital delta: $\Delta(\mathcal{E})$ | The set of all probability measures on $\mathcal{E}$ |
| Bold capitals: $\mathbf{A}$ | Markov kernel $\mathbf{A} : X \times \mathcal{Y} \to [0, 1]$ (stochastic map |
| Subscripted bold capitals: $\mathbf{A}_x$ | The probability measure given by the curried Markov kern |
| $A \to \Delta(\mathcal{B})$ | Markov kernel signature, treated as equivalent to $A \times \mathcal{B}$ |
| $\mathbf{A} : x \mapsto \nu$ | Markov kernel definition, equivalent to $\mathbf{A}(x, B) = \nu(B)$ f |
| Sans serif capitals: $\mathsf{A}, \mathsf{X}$ | Measurable functions; we will also call them random va |
| $\mathbf{F}_{\mathsf{X}}$ | The Markov kernel associated with the function $\mathsf{X}$: $\mathbf{F}_{\mathsf{X}} \equiv$ |
| $\mathbf{N}_{\mathsf{A}\mid\mathsf{B}}$ | The conditional probability (disintegration) of $\mathsf{A}$ given $\mathsf{B}$ |
| $\nu\mathbf{F}_{\mathsf{X}}$ | The marginal distribution of $\mathsf{X}$ under $\nu$ |

## 2.2 Probability Theory

Given a set $A$, a $\sigma$-algebra $\mathcal{A}$ is a collection of subsets of $A$ where

- $A \in \mathcal{A}$ and $\emptyset \in \mathcal{A}$
- $B \in \mathcal{A} \implies B^C \in \mathcal{A}$
- $\mathcal{A}$ is closed under countable unions: For any countable collection $\{B_i | i \in Z \subset \mathbb{N}\}$ of elements of $\mathcal{A}$, $\cup_{i \in Z} B_i \in \mathcal{A}$

A measurable space $(A, \mathcal{A})$ is a set $A$ along with a $\sigma$-algebra $\mathcal{A}$. Sometimes the sigma algebra will be left implicit, in which case $A$ will just be introduced as a measurable space.

**Common $\sigma$ algebras** For any $A$, $\{\emptyset, A\}$ is a $\sigma$-algebra. In particular, it is the only sigma algebra for any one element set $\{*\}$.

For countable $A$, the power set $\mathcal{P}(A)$ is known as the discrete $\sigma$-algebra.

Given $A$ and a collection of subsets of $B \subset \mathcal{P}(A)$, $\sigma(B)$ is the smallest $\sigma$-algebra containing all the elements of $B$.

Let $T$ be all the open subsets of $\mathbb{R}$. Then $\mathcal{B}(\mathbb{R}) := \sigma(T)$ is the *Borel $\sigma$-algebra* on the reals. This definition extends to an arbitrary topological space $A$ with topology $T$.

A *standard measurable set* is a measurable set $A$ that is isomorphic either to a discrete measurable space $A$ or $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For any $A$ that is a complete separable metric space, $(A, \mathcal{B}(A))$ is standard measurable.

Given a measurable space $(E, \mathcal{E})$, a map $\mu : \mathcal{E} \to [0, 1]$ is a *probability measure* if

- $\mu(E) = 1$, $\mu(\emptyset) = 0$
- Given countable collection $\{A_i\} \subset \mathcal{E}$, $\mu(\cup_i A_i) = \sum_i \mu(A_i)$

Write by $\Delta(\mathcal{E})$ the set of all probability measures on $\mathcal{E}$.

Given a second measurable space $(F, \mathcal{F})$, a *stochastic map* or *Markov kernel* is a map $\mathbf{M} : E \times \mathcal{F} \to [0, 1]$ such that

- The map $\mathbf{M}(\cdot; A) : x \mapsto \mathbf{M}(x; A)$ is $\mathcal{E}$-measurable for all $A \in \mathcal{F}$
- The map $\mathbf{M}_x : A \mapsto \mathbf{M}(x; A)$ is a probability measure on $F$ for all $x \in E$

Extending the subscript notation above, for $\mathbf{C} : X \times Y \to \Delta(\mathcal{Z})$ and $x \in X$ we will write $\mathbf{C}_x$ for the "curried" map $y \mapsto \mathbf{C}_{x,y}$.

The map $x \mapsto \mathbf{M}_x$ is of type $E \to \Delta(\mathcal{F})$. We will abuse notation somewhat to write $\mathbf{M} : E \to \Delta(\mathcal{F})$, which captures the intuition that a Markov kernel maps from elements of $E$ to probability measures on $\mathcal{F}$. Note that we "reverse" this idea and consider Markov kernels to map from elements of $\mathcal{F}$ to measurable functions $E \to [0, 1]$, an interpretation found in Clerc et al. (2017), but (at this stage) we don't make use of this interpretation here.

Given an indiscrete measurable space $(\{*\}, \{\{*\}, \emptyset\})$, we identify Markov kernels $\mathbf{N} : \{*\} \to \Delta(\mathcal{E})$ with the probability measure $\mathbf{N}_*$. In addition, there is a unique Markov kernel $\ast : E \to \Delta(\{\{*\}, \emptyset\})$ given by $x \mapsto \delta_*$ for all $x \in E$ which we will call the "discard" map.

## 2.3 Product Notation

We can use a notation similar to the standard notation for matrix-vector products to represent operations with Markov kernels. Probability measures $\mu \in$

$\Delta(\mathcal{X})$ can be read as row vectors, Markov kernels as matrices and measurable functions $\mathsf{T} : Y \to T$ as column vectors. Defining $\mathbf{M} : X \to \Delta(\mathcal{Y})$ and $\mathbf{N} : Y \to \Delta(\mathcal{Z})$, the measure-kernel product $\mu\mathbf{A}(G) := \int \mathbf{A}_x(G) d\mu(x)$ yields a probability measure $\mu\mathbf{A}$ on $\mathcal{Z}$, the kernel-kernel product $\mathbf{MN}(x; H) = \int_Y \mathbf{B}(y; H) d\mathbf{A}_x$ yields a kernel $\mathbf{MN} : X \to \Delta(\mathcal{Z})$ and the kernel-function product $\mathbf{A}\mathsf{T}(x) := \int_Y \mathsf{T}(y) d\mathbf{A}_x$ yields a measurable function $X \to T$. Kernel products are associative (Çinlar, 2011).

The tensor product $(\mathbf{M} \otimes \mathbf{N})(x, y; G, H) := \mathbf{M}(x; G)\mathbf{N}(y; H)$ yields a kernel $(\mathbf{M} \otimes \mathbf{N}) : X \times Y \to \Delta(\mathcal{Y} \otimes \mathcal{Z})$.

## 2.4 String Diagrams

Some constructions are unwieldly in product notation; for example, given $\mu \in \Delta(\mathcal{E})$ and $\mathbf{M} : E \to (\mathcal{F})$, it is not straightforward to construct a measure $\nu \in \Delta(\mathcal{E} \otimes \mathcal{F})$ that captures the "joint distribution" given by $A \times B \mapsto \int_A \mathbf{M}(x; B) d\mu$.

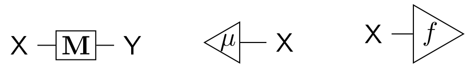
Such constructions can, however, be straightforwardly captured with string diagrams, a notation developed for category theoretic probability. Cho and Jacobs (2019) also provides an extensive introduction to the notation discussed here.

Some key ideas of string diagrams:

- Basic string diagrams can always be interpreted as a mixture of kernel-kernel products and tensor products of Markov kernels
  - Extended string diagrams can be interepreted as a mixture of kernel-kernel products, kernel-function products, tensor products of kernels and functions and scalar products
- String diagrams are the subject of a coherence theorem: taking a string diagram and applying a planar deformation yields a string diagram that represents the same kernel (Selinger, 2010). This also holds for a number of additional transformations detailed below

A kernel $\mathbf{M} : X \to \Delta(\mathcal{Y})$ is written as a box with input and output wires, probability measures $\mu \in \Delta(\mathcal{X})$ are written as triangles "closed on the left" and measurable functions (which are only elements of the "extended" notation) $\mathsf{T} : Y \to T$ as triangles "closed on the right". We additionally name every wire represented in a string diagram, which we call (following Cho and Jacobs (2019)) *meta-variables* and use the same sans-serif font as for random variables ($\mathsf{X}, \mathsf{Y}$ etc.) to emphasise their connection. See Paragraph 2.4.1 for a more detailed explanation of meta variables.

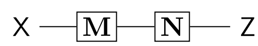
For $\mathbf{M} : X \to \Delta(\mathcal{Y})$, $\mu \in \Delta(\mathcal{X})$ and $f : X \to W$:



(1)

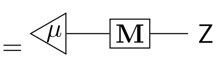
**Basic and extended notation** We canonically regard a probability measure $\mu \in \Delta(\mathcal{E})$ to be a Markov kernel $\mu : \{*\} \to \Delta(\mathcal{E})$. This allows for the definition of "basic" string diagrams for which Markov kernels are the only building blocks. Such a definition isn't possible for measurable functions. Suppose by analogy with the example probability measures and try to identify a measurable function $f : E \to \mathbb{R}$ with a Markov kernel $f' : E \times \{*\} \to \mathbb{R}$. For $x \in E$ we cannot generally have both $f'(x, *) = 1$ and $f'(x, *) = f(x)$, and so this attempt fails. This lack of normalisation is the reason we require an "extended" string diagram notation if we wish to incorporate functions and expectations which allows for the representation of scalars.

**Elementary operations** We can compose Markov kernels with appropriate spaces - the equivalent operation of the "matrix products" of product notation. Given $\mathbf{M} : X \to \Delta(\mathcal{Y})$ and $\mathbf{N} : Y \to \Delta(\mathcal{Z})$, we have

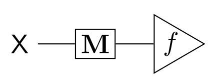
$\mathbf{MN} :=$  (2)



Probability measures are distinguished in that that they only admit "right composition" while functions only admit "left composition". For $\mu \in \Delta(\mathcal{E})$, $h : F \to X$:

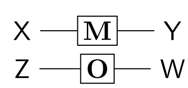
$\mu\mathbf{M} :=$  (3)



$\mathbf{M}f :=$  (4)

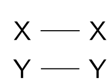


We can also combine Markov kernels using tensor products, which we represent with vertical juxtaposition. For $\mathbf{O} : Z \to \Delta(\mathcal{W})$:

$\mathbf{M} \otimes \mathbf{N} :=$  (5)



Product spaces can be represented either by two parallel wires or a single wire:

$X \times Y \cong \mathrm{Id}_X \otimes \mathrm{Id}_Y :=$  (6)

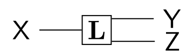


$=$  (7)



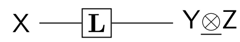
The notation $\mathsf{X}\underline{\otimes}\mathsf{Y}$ will be explained in paragraph 2.4.2 - $\mathsf{X}\underline{\otimes}\mathsf{Y}$ is a meta variable taking values in in the product space $X \times Y$.

Because a product space can be represented by parallel wires, a kernel $\mathbf{L} : X \to \Delta(\mathcal{Y} \otimes \mathcal{Z})$ can be written using either two parallel output wires or a single output wire:




$$ \tag{8} $$

$$\equiv \tag{9}$$

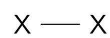



$$ \tag{10} $$

**Markov kernels with special notation** A number of Markov kernels are given special notation distinct from the generic "box" representation above. These special representations facilitate intuitive graphical interpretations.

The identity kernel $\mathbf{Id} : X \to \Delta(X)$ maps a point $x$ to the measure $\delta_x$ that places all mass on the same point:

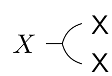
$$\mathbf{Id}_x : x \mapsto \delta_x \equiv \tag{11}$$




The identity map preserves the name of a wire.

The copy map $\curlyvee : X \to \Delta(\mathcal{X} \times \mathcal{X})$ maps a point $x$ to two identical copies of x:

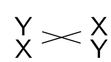
$$\curlyvee : x \mapsto \delta_{(x,x)} \equiv \tag{12}$$




Copy maps *copy* the name of a wire.

The swap map $\sigma : X \times Y \to \Delta(\mathcal{Y} \otimes \mathcal{X})$ swaps its inputs:

$$\sigma := (x, y) \to \delta_{(y,x)} \equiv \tag{13}$$

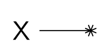



The swap map preserves the names of visually connected wires.

Apart from identity, copy and swap maps, we assign different names to the input and output wires of Markov kernels.

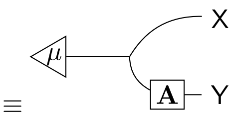
The discard map $* : X \to \Delta(\{*\})$ maps every input to $\delta_*$ which is effectively mapping every input to 1

$$* : x \mapsto \delta_* \equiv \tag{14}$$




Before introducing key rules of manipulation permitted by string diagrams, we will illustrate the correspondence between the three notations with a few simple examples. Given $\mu \in \Delta(X), \mathbf{A} : X \to \Delta(Y)$ and $A \in \mathcal{X},\ B \in \mathcal{Y}$, the following correspondences hold, where we express the same object in elementary notation, product notation and string notation respectively:

$$\nu := A \times B \mapsto \int_A A(x;B)d\mu(x) \equiv \mu\curlyvee(\mathbf{Id}_X \otimes \mathbf{A}) \equiv \tag{15}$$



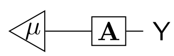

Where the resulting object is a probability measure $\nu \in \Delta(\mathcal{X} \otimes \mathcal{Y})$. Note that the elementary notation requires a function definition here, while the product and string notations can represent the measure without explicitly addressing its action on various inputs and outputs. Cho and Jacobs (2019) calls this construction "integrating $\mathbf{A}$ with respect to $\mu$".

Define the marginal $\nu_Y \in \Delta(\mathcal{Y}) : B \mapsto \nu(X \times B)$ for $B \in \mathcal{Y}$ and similarly for $\nu_X$. We can then express the result of marginalising 15 over $X$ in our three separate notations as follows:

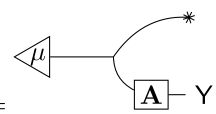
$$\nu_Y(B) = \nu(X \times B) = \int_X A(x;B)d\mu(x) \tag{16}$$

$$\nu_Y = \mu\mathbf{A} = \mu\curlyvee(\mathbf{Id}_X \otimes \mathbf{A})(* \otimes \mathbf{Id}_Y) \tag{17}$$

$$\nu_Y =$$




$$= \tag{18}$$

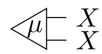



The elementary notation 16 makes the relationship between $\nu_Y$ and $\nu$ explicit and, again, requires the action on each event to be defined. The product notation 17 is, in my view, the least transparent but also the most compact in the form $\mu\mathbf{A}$, and does not demand the explicit definition of how $\nu_Y$ treats every event. The graphical notation is the least compact in terms of space taken up on the page, but unlike the product notation it shows a clear relationship to the graphical construction in15, and displays a clear graphical logic whereby marginalisation corresponds to "cutting off branches". Like product notation, it also allows for the definition of derived measures such as $\nu_Y$ without explicit definition of the handling of all events. It also features a much smaller collection of symbols than does elementary notation.

String diagrams often achieve a good balance between interpretational transparency, expressive power and symbol economy. Downsides of string diagrams are that they can be time consuming to typeset, and formal reasoning with them takes some practice.

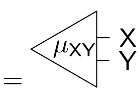
### 2.4.1 Wire Names, Random Variables and Free Random Variables

The first purpose of wire names is to unambiguously refer to particular wires in a given diagram. For example, suppose we have some $\mu \in \Delta(\mathcal{X} \times \mathcal{X})$, and we label wires with *spaces* rather than names:



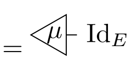

(19)

Given just only the diagram 21, we cannot easily refer to "the top wire" or "the bottom wire". Trying to say something like "the probability of the bottom wire conditional on the top wire" is very confusing, and we really do need to be able to talk about such conditional probabilities (see 2.4.2). Giving wires unique names solves this, but (as suggested by this example), there is another desirable property of wire names: they should function as de-facto random variables, so that "the probability of the bottom wire conditional on the top wire" actually refers to a conditional probability defined in terms of random variables. Suppose we have a probability space $\langle E, \mathcal{E}, \mu \rangle$, and for arbitrary random variables $\mathsf{X} : E \to X$ and $\mathsf{Y} : E \to Y$ write the joint distribution $\mu_{\mathsf{XY}}$. We want wire labels to correspond to random variables in the sense that

$\mu_{\mathsf{XY}} :=$ 

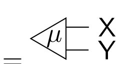

(20)

That is, the correspondence between the product space $X \times Y$ and the values taken by the random variables $\mathsf{X}$ and $\mathsf{Y}$ implicit in the definition of $\mu_{\mathsf{XY}}$ is reflected by the wire names on the right hand side of 20. Given this definition, if we have some finite set of random variables $S = \{\mathsf{X}, \mathsf{Y}, ..., \mathsf{Z}\}$ such that $\mu_{\mathsf{XY...Z}} = \mu$, then we must be able to represent $\mu$ in a diagram with $|S|$ output wires (as the joint distribution is by definition on an appropriate product space), and we should label the wires of this diagram with $\mathsf{X}, \mathsf{Y}, ...\mathsf{Z}$. Note that $\{\mathrm{Id}_E\}$ always satisfies this criterion, thus we can always draw $\mu$ with a single output wire labeled $\mathrm{Id}_E$.

$\mu =$ 

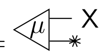

(21)

In general, if we have $E = X \times Y$, we can define $\mathsf{X} : X \times Y \to X$ and $\mathsf{Y} : X \times Y \to Y$ by the projection maps $\mathsf{X} : (x, y) \mapsto x$, $\mathsf{Y} : (x, y) \mapsto y$ and $\mu_{\mathsf{XY}} = \mu$ and we can write (see Lemma 2.9):

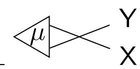
$\mu =$ 


(22)

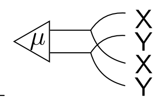
If we take 21 to *define* $\mathsf{X}$ and $\mathsf{Y}$, then Equation 20 compels the following labels for products involving $\mu$:

$\mu_{\mathsf{X}} =$ 

(23)

$\mu_{\mathsf{YX}} =$ 

(24)

$\mu_{\mathsf{XYXY}} =$ 

(25)

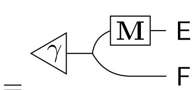
This illustrates the logic of the representations of the identity 44, the swap 13 and the copy maps 12: their representations visually preserve the identities of wires that should be identified according to Equation 20. Note that the presence of a copy map as in 25 is the *only* time when a diagram will feature identical labels on wires.

**Free Random Variables** We are interested in working with general Markov kernels, not just probability measures, and we will thus not always have an ambient probability space as in 20 to ground our wire labels.

**Definition 2.1** (Free Random Variables)**.** Given an ambient Markov kernel $\mathbf{M} : E \to \Delta(\mathcal{F})$, a free random variable $\mathsf{X}$ is a measurable function on $\mathcal{E} \otimes \mathcal{F}$.

The equivalent of marginal and joint distributions for free random variables are marginal and joint *maps*.

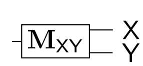
**Definition 2.2** (Marginal, joint maps)**.** Given a Markov kernel $\mathbf{M} : E \to \Delta(\mathcal{F})$, note that given $\gamma \in \Delta(\mathcal{E})$, by our definitions, $\gamma \prec \mathbf{M} := \gamma\curlyvee(\mathrm{Id}_E \otimes \mathbf{M})$ is a probability measure on $\Delta(\mathcal{E} \otimes \mathcal{F})$ and

$\gamma \prec \mathbf{M} =$ 

(26)

Given $\mathbf{M} : E \to \Delta(\mathcal{F})$ and a free random variable $\mathsf{X}$, the marginal map $\mathbf{M}_{\mathsf{X}} : E \to \Delta(\mathcal{X})$ is the unique Markov kernel such that for all $\gamma \in \Delta(\mathcal{E})$, $\gamma(\mathbf{M}_{\mathsf{X}}) = (\gamma \prec \mathbf{M})_{\mathsf{X}}$ where the right hand side is an ordinary marginal distribution. Similarly, given free random variables $\mathsf{X}$, $\mathsf{Y}$, the joint map $\mathbf{M}_{\mathsf{XY}} : E \to \Delta(\mathcal{X} \otimes \mathcal{Y})$ is the unique Markov kernel such that $\gamma \in \Delta(\mathcal{E})$, $\gamma(\mathbf{M}_{\mathsf{XY}}) = (\gamma \prec \mathbf{M})_{\mathsf{XY}}$.

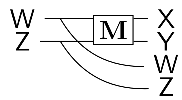
We are now placed to impose a criterion on wire labels equivalent to 20: given the ambient Markov kernel $\mathbf{M} : E \to \Delta(\mathcal{F})$, wire labels in diagrams representing joint distributions must correspond in the sense of 20:

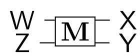
$\mathbf{M}_{\mathsf{XY}} :=$ 

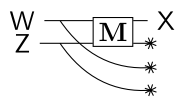
(27)

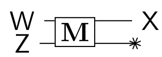
In addition, we impose the requirement of identity 44, swap 13 and copy maps 12 - “output” wires that are connected to “input” wires with no boxes in between share names.

Wire labels for general kernels behave similarly to those for probability measures. Suppose $E = X \times Y$ and $F = W \times Z$ and define $\mathsf{X}, \mathsf{Y}, \mathsf{W}, \mathsf{Z}$ as projection maps from $X \times Y \times W \times Z$ to their respective spaces with $\mathbf{M}$ as before. Then the following labels are compelled:

$\mathrm{Id}_E \prec \mathbf{M} =$  (28)


$\mathbf{M} =$  (29)


$\mathbf{M}_{\mathsf{X}} =$  (30)


$=$  (31)


As well as properties analogous to Equations 25 and 24.

**Derivations of label properties**

**Definition 2.3** (Function induced kernel)**.** Given a measurable function $\mathsf{X} : F \to X$, define the function induced kernel $\mathbf{F}_{\mathsf{X}} : F \to \Delta(\mathcal{X})$ to be the the Markov kernel $a \mapsto \delta_{\mathsf{X}(a)}$ for all $a \in X$.

**Lemma 2.4** (Pushforward measures and functional kernels)**.** *Given a kernel space $\langle F, \mathcal{F}, \mathbf{M}, G \rangle$ and a random variable $\mathsf{X} : F \to X$, for any prior $\mu \in \Delta(\mathcal{G})$ the pushforward $\mathsf{X}_{\#}\mu\mathbf{A} = \mu\mathbf{A}\mathbf{F}_{\mathsf{X}}$.*

*Proof.* For all $B \in \mathcal{F}$:

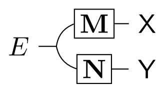
$$(\mathsf{X})_{\#}\mu\mathbf{A}(B) = \mu\mathbf{A}(\mathsf{X}^{-1}(B)) \tag{32}$$

$$= \int_F \delta_{\mathsf{X}(a)}(B) d\mu\mathbf{A}(a) \tag{33}$$

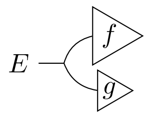
$$= \mu\mathbf{A}\mathbf{F}_{\mathsf{X}}(B) \tag{34}$$

□

**Definition 2.5** (Coupled tensor product $\underline{\otimes}$)**.** Given two Markov kernels $\mathbf{M}$ and $\mathbf{N}$ or functions $f$ and $g$ with shared domain $E$, let $\mathbf{M}\underline{\otimes}\mathbf{N} := \curlyvee(\mathbf{M} \otimes \mathbf{N})$ and $f\underline{\otimes}g := \curlyvee(f \otimes g)$ where these expressions are interpreted using standard product notation. Graphically:

$\mathbf{M}\underline{\otimes}\mathbf{N} :=$ 

(35)

$f\underline{\otimes}g :=$ 

(36)

The operation denoted by $\underline{\otimes}$ is associative (Lemma 2.6), so we can without ambiguity write $f\underline{\otimes}g\underline{\otimes}...\underline{\otimes}h$ for finite groups of functions or Markov kernels sharing a domain.

**Lemma 2.6** ($\underline{\otimes}$ is associative)**.** *For Markov kernels* $\mathbf{L}$*,* $\mathbf{M}$ *and* $\mathbf{N}$ *sharing a domain* $E$*,* $(\mathbf{L}\underline{\otimes}\mathbf{M})\underline{\otimes}\mathbf{N} = \mathbf{L}\underline{\otimes}(\mathbf{M}\underline{\otimes}\mathbf{N})$*.*

**Definition 2.7** (Joint distribution)**.** Given $\mu \in \Delta(\mathcal{E})$ and $\mathsf{X} : E \to X$ and $\mathsf{Y} : E \to Y$, the *joint distribution* of $\mathsf{X}$ and $\mathsf{Y}$ is the pushforward measure of $\mu$ by the random variable $\mathsf{X}\underline{\otimes}\mathsf{Y}$ on $\mathcal{X} \otimes \mathcal{Y}$.

This is identical to the definition in, for example, Çinlar (2011) if we note that the random variable $(\mathsf{X}, \mathsf{Y}) : \omega \mapsto (\mathsf{X}(\omega), \mathsf{Y}(\omega))$ (Çinlar's definition) is equivalent to $\mathsf{X}\underline{\otimes}\mathsf{Y}$.

**Lemma 2.8** (Joint distributions from coupled tensor products)**.** *Given a probability space* $\langle E, \mathcal{E}, \mu\rangle$ *and a finite set of random variables* $G = \{\mathsf{X}_i | i \in [n]\}$*, the joint distribution of* $G$ *is given by* $\mu(\underline{\bigotimes}_{i\in[n]}\mathbf{F}_{\mathsf{X}_i})$*.*

*Proof.* This follows directly from Definition 2.18 and Lemma 2.4. □

**Lemma 2.9** (Coupled projection maps are equal to the identity)**.** *Suppose* $E$ *is a finite Cartesian product:* $E = \prod_{i\in[n]} A_i$*. Let* $\pi_i : E \to A_i$ *be the projection map* $(a_1, .., a_i, .., a_n) \mapsto a_i$*. Then* $\underline{\bigotimes}_{i\in[n]}\pi_i = \mathrm{Id}_E$ *where* $\mathrm{Id}_E$ *is the identity function on* $E$*.*

*Proof.* Define $\pi_{[m]} : E \to \prod_{i\in[m]} A_i$ by $(a_1, ..., a_m, ..., a_n) \mapsto (a_1, ..., a_m)$. Suppose $\underline{\bigotimes}_{i\in[n-1]}\pi_i = \pi_{[n-1]}$. Then by associativity of $\underline{\otimes}$, $\underline{\bigotimes}_{i\in[n]}\pi_i = \pi_{[n-1]}\underline{\otimes}\pi_n$ and for all $(a_1, ..., a_n) \in E$, $\pi_{[n-1]}\underline{\otimes}\pi_n(a_1, ..., a_n) = (\pi_{[n-1]}(a_1, ..., a_n), \pi_n(a_1, ..., a_n)) = (a_1, ..., a_{n-1}, a_n) = \pi_{[n]}(a_1, ..., a_n)$.

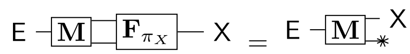
Also, $\underline{\bigotimes}_{i\in[1]}\pi_i = \pi_1$, thus $\underline{\bigotimes}_{i\in[n]}\pi_i = \pi_{[n]}$. But $\pi_{[n]} = \mathrm{Id}_E$. □

**Corollary 2.10.** *If we have a probability space* $\langle E, \mathcal{E}, \mu\rangle$ *where* $E = \prod_{i\in[n]} A_i$ *and* $\mathsf{X}_i := \pi_i$*, then* $\mu_{\underline{\bigotimes}_{i\in[n]}\mathsf{X}_i} = \mu$*.*

**Lemma 2.11** (A projection is the identity tensored with the erase map)**.** *Let* $\pi_X : X \times Y \to X$ *be the projection* $\pi_X : (x, y) \mapsto x$*. Then* $\mathbf{F}_{\pi_x} = \mathbf{F}_{\mathrm{Id}_X} \otimes *$

*Proof.* $\mathbf{F}_{\pi_X}$ is, by definition, the Markov kernel $(x, y) \mapsto \delta_x$, which is equivalent to $\mathbf{F}_{\mathrm{Id}_X} \otimes *$. □

**Corollary 2.12.** *For any* $\mathbf{M} : E \to \Delta(\mathcal{X} \otimes \mathcal{Y})$*,*



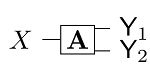

(37)

We take a random variable to be a measurable function on a *kernel space* $\langle F, G, \mathcal{F} \otimes \mathcal{G}, \mathbf{M} \rangle$ where $\mathbf{M} : G \to \Delta(\mathcal{F})$ is a Markov kernel. A random variable $\mathsf{X} : F \to X$ has a probability distribution only relative to some argument measure $\nu \in \Delta(\mathcal{G})$. Because of this, we cannot in general unambiguously talk about "the" distribution of a given random variable; in general we have only conditional probabilities (which we define in Paragraph 2.4.2). This approach mirrors to some extent the approach suggested by Hájek (2003), which also takes conditional probability to be fundamental.

This choice is largely pragmatic - it is helpful to make statements about the properties of random variables on a kernel space without quantifying over prior distributions. However, there is a connection between this choice and the philosophical field of decision theory. In particular, we use random variables to model both stochastic quantities and quantities that depend on the decision maker's choices. We treat the uncertainty associated with choosing and that associated with stochasticity as different – we do *not* suppose that uncertainty over which choice will be made should itself be modeled using probability. *Evidential decision theory*, as defended by Jeffrey (1981), proposes that is is proper to consider choices to be random variables, though it doing so rigorously may necessitate a theory that allows for the assignment of probabilities to the outcomes of mathematical deliberations such as the theory of *logical induction* introduced in Garrabrant et al. (2017). Understanding the relationship between choices and stochastic processes is a deep, interesting and difficut question, and one we sidestep by presuming that we can address nearly all common decision problems while disregarding modelling whatever process gives rise to choices. The resulting decision theory is structurally similar to *causal decision theory* (Lewis, 1981).

This definition of random variables permits the convention of identifying every output wire of a string diagram with a random variable.

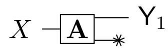
**Example 2.13** (Wire names to random variables)**.** Suppose we have a Markov kernel $\mathbf{A} : X \to \Delta(\mathcal{Y} \otimes \mathcal{Y})$:




(38)

Define $\mathsf{Y}_1' : Y \times Y \to Y$ by the projection map $\mathsf{Y}_1' : (y_1, y_2) \mapsto y_1$ and $\mathsf{Y}_2' : Y \times Y \to Y$ by the projection $\mathsf{Y}_2' : (y_1, y_2) \mapsto y_2$. Given any prior $\mu \in \Delta(\mathcal{X})$, let $(\mathsf{Y}_1')_{\#}\mu\mathbf{A}$ be the pushforward of $\mathsf{Y}_1'$ by $\mu\mathbf{A}$. Then $\mathbf{F}_{\mathsf{Y}_1'} : Y \times Y \to Y$ will be given by $a \mapsto \delta_{\mathsf{Y}_1'(a)}$.

Define $\Pi_{\mathsf{Y}_1} : Y \times Y \to \Delta(\mathcal{Y})$ by $\Pi_{\mathsf{Y}_1} = \mathrm{Id}_Y \otimes *$. $\Pi_{\mathsf{Y}_1}$ is the Markov kernel that marginalises over the second argument; i.e. it marginalises over the wire named $\mathsf{Y}_2$. Graphically:

$$\mathbf{A}\Pi_{\mathsf{Y}_1} =$$  (39)

Note that for all $(y_1, y_2) \in Y \times Y$, $(\Pi_1)_{y_1,y_2} = \delta_{y_1} = \delta_{\mathsf{Y}'_1(y_1,y_2)}$. That is, $\Pi_{\mathsf{Y}_1} = F_{\mathsf{Y}'_1}$ and so $(\mathsf{Y}'_1)_{\#}\mu\mathbf{A} = \mu\mathbf{A}\Pi_{\mathsf{Y}_1}$.

Furthermore, define the joint distribution of $\mathsf{Y}'_1$ and $\mathsf{Y}'_2$ by $(\mathsf{Y}'_1\underline{\otimes}\mathsf{Y}'_2)_{\#}\mu\mathbf{A}(B\times C) = \mu\mathbf{A}(\mathsf{Y}'^{-1}_1(B) \cap \mathsf{Y}'^{-1}_2(C))$ for all $B, C \in \mathcal{Y}$. Then, defining $\Pi_{\mathsf{Y}_1\otimes\mathsf{Y}_2} = \mathrm{Id}_Y \otimes \mathrm{Id}_Y = \mathrm{Id}_{Y\times Y}$:

$$\begin{aligned}
(\mathsf{Y}'_1\underline{\otimes}\mathsf{Y}'_2)_{\#}\mu\mathbf{A}(B\times C) &= \mu\mathbf{A}(\mathsf{Y}'^{-1}_1(B) \cap \mathsf{Y}'^{-1}_2(C)) && (40)\\
&= \int_{Y\times Y} \delta_{\mathsf{Y}'_1(y_1,y_2)}(B)\delta_{\mathsf{Y}'_2(y_1,y_2)}(C) d\mu\mathbf{A}(y_1,y_2) && (41)\\
&= \int_{B\times C} d\mu\mathbf{A}(y_1,y_2) && (42)\\
&= \mu\mathbf{A}(B\times C) && (43)\\
&= \mu\mathbf{A}\Pi_{\mathsf{Y}_1\otimes\mathsf{Y}_2}(B\times C) && (44)
\end{aligned}$$

That is, for any prior $\mu$, the joint distribution of $\mathsf{Y}'_1$ and $\mathsf{Y}'_2$ under $\mu\mathbf{A}$ is "carried" by the wires labeled $\mathsf{Y}_1$ and $\mathsf{Y}_2$, and the marginal distribution of $\mathsf{Y}_1$ is "carried" by the wire named $\mathsf{Y}_1$ alone. It's in this sense that we identify the random variable $\mathsf{Y}'_1$ with $\mathsf{Y}_1$. We will henceforth drop the distinction between a wire name and its associated random variable, letting $\mathsf{Y}_1$ denote both the wire and the random variable previously named $\mathsf{Y}'_1$.

In general, given a Markov kernel with output space $\prod_{i\in[n]} X_i$, we can identify the $j$-th output wire with the random variable given by the projection map $\pi_j : (x_1, ...x_j, ...x_n) \mapsto x_j$.

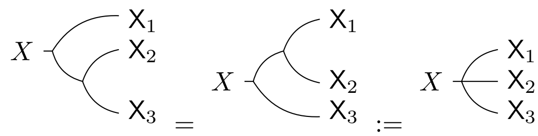
### 2.4.2 Working With String Diagrams

todo:

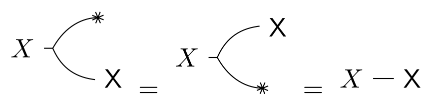
- Functional generalisation
- Conditioning
- Infinite copy map
- De Finetti's representation theorem

There are a relatively small number of manipulation rules that are useful for string diagrams. In addition, we will define graphically analogues of the standard notions of *conditional probability*, *conditioning*, and infinite sequences of exchangeable random variables.

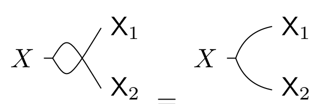
**Axioms of Symmetric Monoidal Categories** Recalling the unique Markov kernels defined above, the following equivalences, known as the *commutative comonoid axioms*, hold among string diagrams:




(45)




(46)

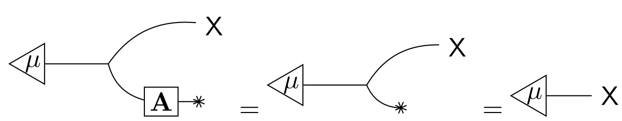



(47)

The discard map ⁎ can "fall through" any Markov kernel:


(48)

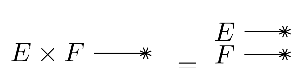
Combining 46 and 48 we can derive the following: integrating $\mathbf{A} : X \to \Delta(\mathcal{Y})$ with respect to $\mu \in \Delta(\mathcal{X})$ and then discarding the output of $\mathbf{A}$ leaves us with $\mu$:



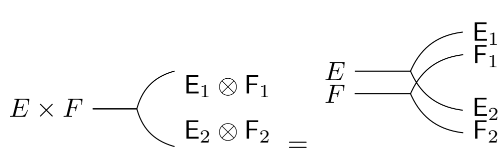

(49)

In elementary notation, this is equivalent to the fact that, for all $B \in \mathcal{X}$, $\int_B \mathbf{A}(x; B) d\mu(x) = \mu(B)$.

The following additional properties hold for ⁎ and $\curlyvee$:

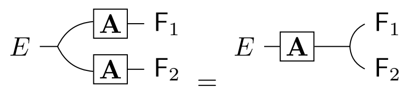



(50)




(51)

A key fact that *does not* hold in general is




(52)

In fact, it holds only when $\mathbf{A}$ is a *deterministic* kernel.

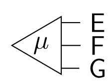
**Definition 2.14** (Deterministic Markov kernel)**.** A *deterministic* Markov kernel $\mathbf{A} : E \to \Delta(\mathcal{F})$ is a kernel such that $\mathbf{A}_x(B) \in \{0, 1\}$ for all $x \in E$, $B \in \mathcal{F}$.

**Theorem 2.15** (Copy map commutes for deterministic kernels (Fong, 2013))**.** *Equation 52 holds iff* $\mathbf{A}$ *is deterministic.*

**Disintegraion and Bayesian Inversion** We use *disintegration* to define a notion of conditional probability. It is not identical to the standard definition of conditional probability one can find in, for example, Çinlar (2011), but each can be recovered from the other.

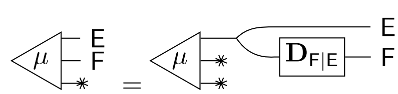
We'll proceed from an example to a general definition.

**Example 2.16** (Disintegration with respect to "convenient" random variables)**.** Given a probability measure $\mu \in \Delta(\mathcal{E} \otimes \mathcal{F} \otimes \mathcal{G})$:




(53)

A Markov kernel $\mathbf{D}_{\mathsf{F}|\mathsf{E}}$ is a $\mathsf{F}|\mathsf{E}$ ("$\mathsf{F}$ on $\mathsf{E}$")-disintegration of $\mu$ if




(54)

Equation 54 echoes the familiar property of conditional probability $P(A \cap B) = P(A|B)P(B)$; in elementary notation it states that for and disintegration $\mathbf{D}_{\mathsf{F}|\mathsf{E}}$ and all $A \in \mathcal{E}$, $B \in \mathcal{F}$, $\mu(A \times B) = \int_A \mathbf{D}_{\mathsf{F}|\mathsf{E}}(x; B) d\mu_{\mathsf{E}}(x)$ where $\mu_{\mathsf{E}} := \mu \mathbf{F}_{\mathsf{E}}$ is the marginal distribution of $\mathsf{E}$ under $\mu$.

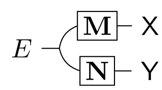
Example 2.16 defines disintegration given a probability measure $\mu$ and a pair of random variables $\mathsf{E}$ and $\mathsf{F}$ that are adapted to the product structure of the output space of $\mu$, a product structure that allows us to draw a diagram for $\mu$ featuring two wires as outputs. There are three extensions to this definition that are desirable:

1. We would like to replace individual wires $\mathsf{E}$ and $\mathsf{F}$ with arbitrary sets of wires

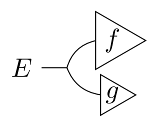
2. We would like to be able to disintegrate a probability mesure with respect to arbitrary random variables, not just sets that are adapted to the product structure of the output space
3. We would like to define disintegration for arbitrary Markov kernels rather than probability measures only

As we show, we can associate any set of wires with a random variable, so the first item is solved by a solution to the second.

**Definition 2.17** (Coupled tensor product $\underline{\otimes}$)**.** Given two Markov kernels $\mathbf{M}$ and $\mathbf{N}$ or functions $f$ and $g$ with shared domain $E$, let $\mathbf{M}\underline{\otimes}\mathbf{N} := \curlyvee(\mathbf{M} \otimes \mathbf{N})$ and $f\underline{\otimes}g := \curlyvee(f \otimes g)$ where these expressions are interpreted using standard product notation. Graphically:

$\mathbf{M}\underline{\otimes}\mathbf{N} :=$  (55)

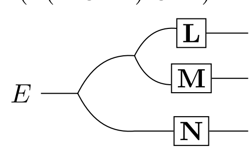


$f\underline{\otimes}g :=$  (56)

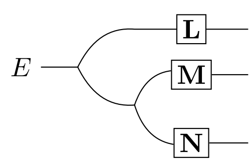


*Proof.* This follows from the commutativity of the swap map (Equation 45):

$$(\mathbf{L}\underline{\otimes}\mathbf{M})\underline{\otimes}\mathbf{N} = \curlyvee(\curlyvee(\mathbf{L} \otimes \mathbf{M}) \otimes \mathbf{N}) \quad (57)$$

$=$  (58)



$=$  (59)



$$= \mathbf{L}\underline{\otimes}(\mathbf{M}\underline{\otimes}\mathbf{N}) \quad (60)$$

□

**Definition 2.18** (Joint distribution)**.** Given $\mu \in \Delta(\mathcal{E})$ and $\mathsf{X} : E \to X$ and $\mathsf{Y} : E \to Y$, the *joint distribution* of $\mathsf{X}$ and $\mathsf{Y}$ is the pushforward measure of $\mu$ by the random variable $\mathsf{X}\underline{\otimes}\mathsf{Y}$ on $\mathcal{X} \otimes \mathcal{Y}$.

This is identical to the definition in, for example, Çinlar (2011) if we note that the random variable $(\mathsf{X}, \mathsf{Y}) : \omega \mapsto (\mathsf{X}(\omega), \mathsf{Y}(\omega))$ (Çinlar's definition) is equivalent to $\mathsf{X}\underline{\otimes}\mathsf{Y}$.

**Lemma 2.19** (Joint distributions from coupled tensor products)**.** *Given a probability space $\langle E, \mathcal{E}, \mu \rangle$ and a finite set of random variables $G = \{\mathsf{X}_i | i \in [n]\}$, the joint distribution of $G$ is given by $\mu(\underline{\otimes}_{i\in[n]} \mathbf{F}_{\mathsf{X}_i})$.*

*Proof.* This follows directly from Definition 2.18 and Lemma 2.4. □

**Lemma 2.20.** *For measurable functions $\mathsf{X} : E \to X$ and $\mathsf{Y} : E \to Y$, $\mathbf{F}_{\mathsf{X}\underline{\otimes}\mathsf{Y}} = \mathbf{F}_\mathsf{X} \underline{\otimes} \mathbf{F}_\mathsf{Y}$.*

*Proof.* $\mathcal{X} \otimes \mathcal{Y}$ is by definition generated by the rectangles $A \times B$ for $A \in \mathcal{X}$, $B \in \mathcal{Y}$. To show equivalence of kernels $E \to \Delta(\mathcal{X} \otimes \mathcal{Y})$ it is sufficient to show agreement on all $s \in E$ and rectangles $A \times B$, $A \in \mathcal{X}$, $B \in \mathcal{Y}$.

For all $q \in X$, $r \in Y$, $A \in \mathcal{X}$ and $B \in \mathcal{Y}$, we have

$$\delta_{(q,r)}(A \times B) = \delta_q(A)\delta_r(B) \tag{61}$$

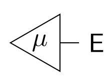
This can be verified by checking the four combinations of $q$ in or not in A and $r$ in or not in $B$.

For all $s \in E$, $A \in \mathcal{X}, B \in \mathcal{Y}$,

$$\begin{aligned}
\mathbf{F}_{\mathsf{X}\underline{\otimes}\mathsf{Y}}(s; A \times B) &= \delta_{\mathsf{X}(s),\mathsf{Y}(s)}(A \times B) & (62)\\
&= \delta_{\mathsf{X}(s)}(A)\delta_{\mathsf{Y}(s)}(B) & (63)\\
&= (\mathbf{F}_\mathsf{X} \underline{\otimes} \mathbf{F}_\mathsf{Y})(s; A \times B) & (64)
\end{aligned}$$

□

**Definition 2.21** (Disintegration and Conditional Probability)**.** Given a probability measure $\mu \in \Delta(\mathcal{E})$:

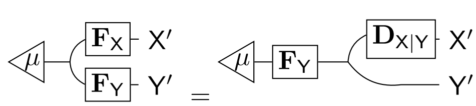



(65)

and two groups of random variables $G_X = \{\mathsf{X}_i | i \in [n]\}$ and $G_Y = \{\mathsf{Y}_i | i \in [m]\}$, define $\mathsf{X} = \underline{\otimes}_{i\in[n]} \mathsf{X}_i$, $\mathsf{Y} := \underline{\otimes}_{i\in[m]} \mathsf{Y}_i$ and $\mathsf{W} = \mathsf{X}\underline{\otimes}\mathsf{Y}$. Define the $\mathsf{X}|\mathsf{Y}$ disintegration of $\mu$ to be the $\mathsf{X}'|\mathsf{Y}'$ disintegration of $\mu\mathbf{F}_\mathsf{W}$, where $\mathsf{X}' : \prod_{i\in[n]} X_i \times \prod_{j\in[m]} Y_j \to \prod_{i\in[n]} X_i$ and $\mathsf{Y}' : \prod_{i\in[n]} X_i \times \prod_{j\in[m]} Y_j \to \prod_{i\in[n]} Y_i$ are the respective projection maps $\mathsf{X}' : (x_0, ..., x_[n], y_0, ..., y_[m]) \mapsto (x_0, ..., x_m)$ and $\mathsf{Y}' : (x_0, ..., x_[n], y_0, ..., y_[m]) \mapsto (y_0, ..., y_[m])$.

By construction, $\mathsf{X}'$ and $\mathsf{Y}'$ are such that $\mathsf{X}' \circ \mathsf{W} = \mathsf{X}$ and $\mathsf{Y}' \circ \mathsf{W} = \mathsf{Y}$.

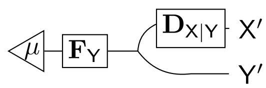
By Lemma 2.20, $\mathbf{D}_{\mathsf{X}|\mathsf{Y}}$ is a $\mathsf{X}|\mathsf{Y}$ disintegration of $\mu$ if and only if




(66)

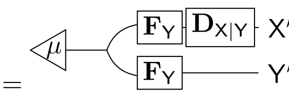
Equation 66 generalises 54 beyond disintegrations by "convenience" wire names to disintegrations by arbitrary sets of random variables. Note that where we *can* construct a diagram for $\mu$ with convenient labels for X and Y, due to the identification between random variables and wires, 66 reduces to 54, and we will use the simpler definition where possible as it yields less cluttered diagrams.

From the fact that $\mathbf{F}_{\mathsf{Y}}$ is a deterministic kernel, we also have:




(67)

$=$




(68)

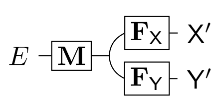
We note, without proof, that: $\mathbf{F}_{\mathsf{Y}}\mathbf{D}_{\mathsf{X}|\mathsf{Y}}$ is an ordinary conditional probability $\mu(\mathsf{X}| \otimes_{i\in[m]} \mathcal{Y}_i)$, and where such an ordinary conditional probability exists we can find a disintegration $\mathbf{D}_{\mathsf{X}|\mathsf{Y}}$ (Çinlar, 2011).

In addition, it is well known that disintegrations of $\mu$ are non-unique, and different disintegrations are equal only up to sets of $\mu$-measure 0. This ambiguity turns out to be more problematic in our causal work as sets of $\mu$-measure 0 may meaningfully impact the consequences of decisions in fairly ordinary circumstances.

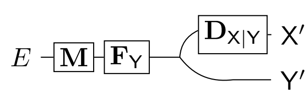
We are also interested in an analogue of disintegration that applies to an arbitrary Markov kernel $\mathbf{M} : E \to \Delta(\mathcal{F})$ rather than only to probability measures. A generalised notion of disintegraion will allow for a formal definition of Dawid (2010)'s definition of "extended conditional independence".

There are a number of choices we could make here, and we have made a particular choice that leads to generalised disintegrations usually failing to exist. This choices allows for a much cleaner treatment of conditional independence, as we will explain below.

**Definition 2.22** (Generalised Disintegration)**.** Given a Markov kernel $\mathbf{M} : E \to \Delta(\mathcal{F})$ and random variables $\mathsf{X} : F \to X$, $\mathsf{Y} : F \to Y$, a $\mathsf{X}|\mathsf{Y}$ disintegration of $\mathbf{M}$ is any kernel $\mathbf{D}_{\mathsf{X}|\mathsf{Y}}$ such that



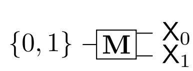

$=$




(69)

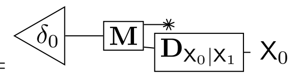
Similarly to the standard definition, this reduces in the case that $\mathbf{M}$ can be drawn with $\mathsf{X}$ and $\mathsf{Y}$ labelling output wires.

**Example 2.23** (Generalised disintegrations usually do not exist)**.** Let $\mathbf{M} : \{0,1\} \to \Delta(\{0,1\}^2)$ be defined by $\mathbf{M} : q \mapsto \delta_q \otimes \delta_1$ and let $\mathsf{X}_0, \mathsf{X}_1 : \{0,1\}^2 \to \{0,1\}$ be defined by $\mathsf{X}_0 : (r,s) \mapsto r$ and $\mathsf{X}_1 : (r,s) \mapsto s$. Note that we can write:

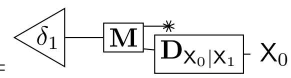



(70)

Suppose we have some $\mathbf{D}_{\mathsf{X}_0|\mathsf{X}_1}$. Then we must have

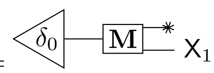
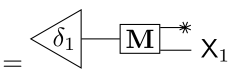
$\delta_0 =$  (71)



$\delta_1 =$  (72)



but by assumption

$\delta_1 =$  $=$  (73)



Thus no such $\mathbf{D}_{\mathsf{X}_0|\mathsf{X}_1}$ can exist.

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## Appendix: