Causal Statistical Decision Theory|What are interventions?

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1 Ignorability Does Not Identify the Average Causal Effect

There is an error in derivations of the identification of the Average Causal Effect from the assumption of ignorability in Potential Outcomes. I will present a counterexample to the claim that ignorability + positivity leads to the identification of Average Causal Effect. The problem arises because authors assume that the average of an arbitrary sequence of random variables - absent any assumptions of IID or exchangeability - converges to something meaningful. I think the actual condition they want is "treatment exchangeability".

For the setup, I refer to Angrist and Pischke (2014)

(although I should probably actually refer to Rubin (2005)).

> We use the letter Y as shorthand for health, the outcome variable of interest. To make it clear when were talking about specific people, we use subscripts as a stand-in for names: Y_i is the health of individual i. The outcome Y_i is recorded in our data. But, facing the choice of whether to pay for health insurance, person i has two potential outcomes, only one of which is observed. To distinguish one potential outcome from another, we add a second subscript: The road taken without health insurance leads to Y_{0i} (read this as y-zero-i) for person i, while the road with health insurance leads to Y_{1i} (read this as y-onei) for person i. Potential outcomes lie at the end of each road one might take. The causal effect of insurance on health is the difference between them, written $\mathsf{Y}_{1i} - \mathsf{Y}_{0i}$.
>
> [...]

Actual health outcomes: Y_i, treatment: D_i

[...]

κ is both the individual and average causal effect on health outcomes.

[...]

$$\mathrm{Avg}_n[\mathsf{Y}_{1i} - \mathsf{Y}_{0i}] = \frac{1}{n}\sum_{i=1}^{n}[\mathsf{Y}_{1i} - \mathsf{Y}_{0i}] \tag{1}$$

[...]

$$\begin{aligned}\mathrm{Avg}_n[\mathsf{Y}_{1i}|\mathsf{D}_i = 1] - \mathrm{Avg}_n[\mathsf{Y}_{0i}|\mathsf{D}_i = 0] = \\ \kappa + \mathrm{Avg}_n[\mathsf{Y}_{0i}|\mathsf{D}_i = 1] - \mathrm{Avg}_n[\mathsf{Y}_{0i}|\mathsf{D}_i = 0]\end{aligned} \tag{2}$$

[...]

Random assignment eliminates selection bias When D_i is randomly assigned, $\mathbb{E}[\mathsf{Y}_{0i}|\mathsf{D}_i = 1] = \mathbb{E}[\mathsf{Y}_{0i}|\mathsf{D}_i = 0]$, and the difference in expectations by treatment status captures the causal effect of treatment:

$$\begin{aligned}\mathbb{E}[\mathsf{Y}_i|\mathsf{D}_i = 1] - \mathbb{E}[\mathsf{Y}_i|\mathsf{D}_i = 0] &= \mathbb{E}[\mathsf{Y}_{1i}|\mathsf{D}_i = 1] - \mathbb{E}[\mathsf{Y}_{0i}|\mathsf{D}_i = 0] && (3)\\ &= \mathbb{E}[\mathsf{Y}_{0i} + \kappa|\mathsf{D}_i = 1] - \mathbb{E}[\mathsf{Y}_{0i}|\mathsf{D}_i = 0] && (4)\\ &= \kappa + \mathbb{E}[\mathsf{Y}_{0i}|\mathsf{D}_i = 1] - \mathbb{E}[\mathsf{Y}_{0i}|\mathsf{D}_i = 0] && (5)\\ &= \kappa && (6)\end{aligned}$$

Provided the sample size is large enough for the law of large numbers to work its magic (so we can replace the conditional averages in equation 2 with conditional expectations), selection bias disappears in a randomized experiment

The problem is that the expectations in Equation 6 *cannot* be replaced with conditional averages as defined in 1, even in the infinite limit. From the strong law of large numbers we can deduce that, given IID variables $(\mathsf{Y}^j_{i0}, \mathsf{D}^j_i) \sim \mathbb{P}(\mathsf{Y}_{i0}, \mathsf{D}_i)$ for $j \in \mathbb{N}$,

$$\lim_{n\to\infty}\sum_{j}^{n}\frac{\mathsf{Y}^j_{i0}[\![\mathsf{D}^j_i = 1]\!]}{\sum_j^n[\![\mathsf{D}^j_i = 1]\!]} \overset{\mathbb{P}-a.s.}{=} \mathbb{E}[\mathsf{Y}_{1i}|\mathsf{D}_i = 1] \tag{7}$$

Note that Angrist and Pischke do *not* assume that $(\mathsf{Y}_{i0}^j, \mathsf{D}_i^j)$ are given - in their conventions, this would refer to repeated samples of the "same individual".

The quantity given by their "conditional average" is an average of random variables that share similar names, but are otherwise unrelated:

$$\sum_i^n \frac{\mathsf{Y}_{i0}[\![\mathsf{D}_i = 1]\!]}{\sum_i^n [\![\mathsf{D}_i = 1]\!]} \tag{8}$$

This counterexample satisfies the stronger assumptions presented in Rubin 2005, hence it doesn't quite line up with the assumptions from Angrist and Pischke; need to incorporate Rubin.

Suppose we have random variables $(\boldsymbol{D}, \boldsymbol{Y}_0, \boldsymbol{Y}_1) := ([\mathsf{D}_0, \mathsf{D}_1, ...], [\mathsf{Y}_{00}, \mathsf{Y}_{10}, ...], [\mathsf{Y}_{01}, \mathsf{Y}_{11}, ...]) \in [0,1]^{3\mathbb{N}}$ and

$$\mathbb{P}(\boldsymbol{D} = \mathbf{d}, \boldsymbol{Y}_0 = \mathbf{y}_0, \boldsymbol{Y}_1 = \mathbf{y}_1) = \prod_{i \in \mathbb{N}} \left((1-\epsilon)\delta_{(i \mod 2)}(d_i) + \epsilon\right) \delta_{(i \mod 2)}(y_{i0})\delta_{(1-i \mod 2)}(y_{i1}) \tag{9}$$

By construction $\boldsymbol{Y}_1, \boldsymbol{Y}_0 \perp\!\!\!\perp_{\mathbb{P}} \boldsymbol{D}$, and $\mathbb{P}(\mathsf{D}_i = d_i) > 0$ for all d_i, which implies for all i $\mathbb{E}[\mathsf{Y}_{0i}|\mathsf{D}_i = 1] = \mathbb{E}[\mathsf{Y}_{0i}|\mathsf{D}_i = 0]$. However

$$\begin{align}
\lim_{n\to\infty} \sum_i^n \frac{\mathsf{Y}_{i1}[\![\mathsf{D}_i = 1]\!]}{\sum_i^n [\![\mathsf{D}_i = 1]\!]} - \lim_{n\to\infty} \sum_i^n \frac{\mathsf{Y}_{i0}[\![\mathsf{D}_i = 0]\!]}{\sum_i^n [\![\mathsf{D}_i = 0]\!]} &= 1 - \frac{\epsilon}{2} - \frac{\epsilon}{2} \tag{10}\\
&= 1 - \epsilon \tag{11}\\
\lim_{n\to\infty} \sum_i^n \frac{\mathsf{Y}_{i0}}{n} - \lim_{n\to\infty} \sum_i^n \frac{\mathsf{Y}_{i1}}{n} &= \frac{1}{2} - \frac{1}{2} \tag{12}\\
&= 0 \tag{13}\\
&= \text{"the average causal effect"} \tag{14}\\
&\neq \lim_{n\to\infty} \sum_i^n \frac{\mathsf{Y}_{i1}[\![\mathsf{D}_i = 1]\!]}{\sum_i^n [\![\mathsf{D}_i = 1]\!]} - \lim_{n\to\infty} \sum_i^n \frac{\mathsf{Y}_{i0}[\![\mathsf{D}_i = 0]\!]}{\sum_i^n [\![\mathsf{D}_i = 0]\!]} \tag{15}
\end{align}$$

Contradicting the claim made by Eq. 6.

I don't know the *necessary* conditions for Equation 6 to hold. However, I think the following is sufficient: for any finite permutation $\pi : [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$:

$$\lim_{n\to\infty} \sum_i^n \frac{\mathsf{Y}_{i1}[\![\mathsf{D}_i = 1]\!]}{\sum_i^n [\![\mathsf{D}_i = 1]\!]} = \lim_{n\to\infty} \sum_i^n \frac{\mathsf{Y}_{i1}[\![\mathsf{D}_{\pi(i)} = 1]\!]}{\sum_i^n [\![\mathsf{D}_{\pi(i)} = 1]\!]} \tag{16}$$

A *sufficient* condition is is the RVs $\mathsf{Y}_{i0}, \mathsf{Y}_{i1}$ are IID according to some $\mathbb{P}(\mathsf{Y}_0, \mathsf{Y}_1)$. In this case the potential outcomes model along with SUTVA assumption collapses to a Causal Bayesian Network. show this

Alternatively, I think the desired result might also hold given *treatment-exchangeability*

References

Joshua D. Angrist and Jörn-Steffen Pischke. *Mastering 'Metrics: The Path from Cause to Effect*. Princeton University Press, Princeton ; Oxford, with french flaps edition edition, December 2014. ISBN 978-0-691-15284-4.

Donald B. Rubin. Causal Inference Using Potential Outcomes. *Journal of the American Statistical Association*, 100(469):322–331, March 2005. ISSN 0162-1459. doi: 10.1198/016214504000001880. URL `https://doi.org/10.1198/016214504000001880`.

Appendix: