

August 22: Exploring causal assumptions with string diagrams

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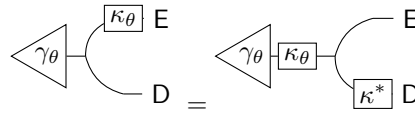
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1 Recoverability

- A natural assumption suggested by the notion of a CSDP is that of *recoverability* - that a causal theory $\mathcal{T} : E \times D \rightarrow E$ permits some decision function that reproduces the distribution of the observed data. That is, we assume that for every $(\kappa_\theta, \mu_\theta) := \theta \in \mathcal{T}$ there exists $\gamma_\theta \in \Delta(\mathcal{D})$ such that

$$\gamma_\theta \kappa_\theta = \mu_\theta \quad (1)$$

- Suppose also that we have some κ^* that, for all $\theta \in \mathcal{T}$, is a Bayesian inversion of γ_θ and κ_θ ; that is:



$$\quad (2)$$

- A sufficient condition for the existence of such a κ^* is the assumption that decisions correspond to *variable setting* - that is, there is some variable $X : E \rightarrow X$ such that for all $a \in D$, $\theta \in \mathcal{T}$ we have $\delta_a \kappa_\theta F_X = \delta_a$ (such an assumption arises in graphical models as hard interventions, and in potential outcomes as “potential-outcome identifiers”). Indeed F_X is in this case a candidate for κ^* . It is not necessary that κ^* be deterministic, however - suppose every κ ignores D . Then choose $\gamma_\theta = \gamma$ for arbitrary $\gamma \in \Delta(\mathcal{D})$ and it can be verified that $\kappa^* : b \mapsto \gamma$ satisfies 2.

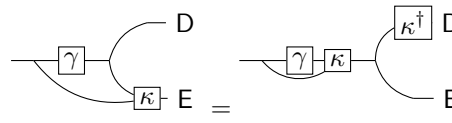
- I believe a weaker sufficient condition for the existence of a universal κ^* is that every κ_θ factorises as $\kappa_\theta = h \vee (\text{Id}_F \otimes j_\theta)$ for some fixed $h : D \rightarrow \Delta(\mathcal{F})$, but I have not yet shown this.

- We will proceed somewhat rashly: suppose that by defining $\gamma : \mathcal{T} \rightarrow \Delta(\mathcal{D})$, $\mu : \mathcal{T} \rightarrow \Delta(\mathcal{E})$ and $\kappa : \mathcal{T} \times D \rightarrow \Delta(\mathcal{E})$ by $\gamma : \theta \rightarrow \gamma_\theta$, $\mu : \theta \rightarrow \mu_\theta$ and $\kappa : (\theta, d) \rightarrow \kappa_\theta(d; \cdot)$ that all resulting objects are Markov kernels, and that \mathcal{T} is a standard measurable space.

- By previous assumptions, we have the following properties:



$$\quad (3)$$



$$\quad (4)$$



$$\quad (5)$$

18 From 4 we also have

$$\begin{array}{c}
 \begin{array}{c} \text{---} \boxed{\gamma} \text{---} \end{array} \begin{array}{c} \text{---} D \\ \text{---} \boxed{\kappa} \text{---} * \end{array} = \begin{array}{c} \text{---} \boxed{\gamma} \boxed{\kappa} \text{---} \end{array} \begin{array}{c} \text{---} \boxed{\kappa^*} \text{---} D \\ \text{---} * \end{array} \\
 \text{---} \boxed{\gamma} \text{---} D = \text{---} \boxed{\mu} \boxed{\kappa^*} \text{---} D
 \end{array} \tag{6}$$

$$\tag{7}$$

19 Where 7 follows from 1.

20 Suppose we have some $\rho : E \rightarrow \Delta(\mathcal{G})$ that is deterministic such that $\mu\rho$ and $\kappa\rho$ are also deterministic.

21 Both ρ and $\mu\rho$ therefore have left inverses ${}^\dagger(\mu\rho) : G \rightarrow \Delta(\mathcal{T})$ and ${}^\dagger\rho : G \rightarrow \Delta(\mathcal{E})$.

22 ρ is “something like” a sufficient statistic.

23 A corollary of Lemma 2.5 is that left inverses have the following property:

$$\begin{array}{c} \boxed{{}^\dagger A} \end{array} \begin{array}{c} \text{---} \\ \text{---} \boxed{A} \text{---} \end{array} = \begin{array}{c} \text{---} \boxed{{}^\dagger A} \end{array} \tag{8}$$

24 We then have

$$\begin{array}{c}
\text{---} \mu \rho^\dagger(\mu \rho) \text{---} \mu \rho \text{---} \dagger \rho \kappa^* \text{---} \kappa \rho \text{---} \dagger \rho \kappa^* \text{---} \\
= \hspace{10em} (9)
\end{array}$$

$$\begin{array}{c}
\text{---} \mu \rho \text{---} \dagger(\mu \rho) \kappa^* \text{---} \kappa \rho \kappa^* \text{---} \\
\text{---} \mu \rho \text{---} \dagger(\mu \rho) \kappa^* \text{---} \kappa \rho \text{---} \\
= \hspace{10em} (10)
\end{array}$$

$$\begin{array}{c}
\text{---} \mu \rho \text{---} \kappa^* \text{---} \\
\text{---} \mu \rho \text{---} \dagger(\mu \rho) \kappa^* \text{---} \kappa \rho \text{---} \\
= \hspace{10em} (11)
\end{array}$$

$$\begin{array}{c}
E \\
\downarrow \\
\kappa \\
\downarrow \\
\kappa^* \\
\downarrow \\
\mu \\
\downarrow \\
\dagger \mu \\
\downarrow \\
\mu \\
= \hspace{10em} (12)
\end{array}$$

$$\begin{array}{c}
E \quad E \\
\downarrow \quad \downarrow \\
\kappa \quad \kappa \\
\downarrow \quad \downarrow \\
\kappa^* \quad \kappa^* \dagger \mu \\
\downarrow \quad \downarrow \\
\mu \quad \mu \\
= \hspace{10em} (13)
\end{array}$$

25 A key question is does 13 imply anything non-trivial regarding the following “identifiability” condition
 26 for arbitrary $J : E \rightarrow \Delta(\mathcal{D})$:

$$\begin{array}{c}
E \quad E \\
\downarrow \quad \downarrow \\
\kappa \quad \kappa \\
\downarrow \quad \downarrow \\
J \quad J \dagger \mu \\
\downarrow \quad \downarrow \\
\mu \quad \mu \\
= \hspace{10em} (14)
\end{array}$$

27 I call this identifiability because the left hand side is the kernel $\mathcal{T} \rightarrow \Delta(\mathcal{E})$ that computes the “result”
 28 of a given state and decision function, while the right hand side implies it is possible to find a J that
 29 minimises the expected utility κu independent of the causal state (this is because we see one of the
 30 input wires and control the other).

31 We may be able to get some insight into this by asking, given matrices A, B, C, D of appropriate
 32 shapes, if $BA = CA$ when does $BDA = CDA$?

33 2 Notes on category theoretic probability and string diagrams

34 Category theoretic treatments of probability theory often start with *probability monads* (for a good
 35 overview, see [Jacobs, 2018]). A monad on some category C is a functor $T : C \rightarrow C$ along with
 36 natural transformations called the unit $\eta : 1_C \rightarrow T$ and multiplication $\mu : T^2 \rightarrow T$. Roughly,
 37 functors are maps between categories that preserve identity and composition structure and natural
 38 transformations are "maps" between functors that also preserve composition structure. The monad
 39 unit is similar to the identity element of a monoid in that application of the identity followed by
 40 multiplication yields the identity transformation. The multiplication transformation is also (roughly
 41 speaking) associative.

42 An example of a probability monad is the discrete probability monad given by the functor $\mathcal{D} : \mathbf{Set} \rightarrow$
 43 \mathbf{Set} which maps a countable set X to the set of functions from $X \rightarrow [0, 1]$ that are probability
 44 measures on X , denoted $\mathcal{D}(X)$. \mathcal{D} maps a measurable function f to $\mathcal{D}f : X \rightarrow \mathcal{D}(X)$ given by
 45 $\mathcal{D}f : x \mapsto \delta_{f(x)}$. The unit of this monad is the map $\eta_X : X \rightarrow \mathcal{D}(X)$ given by $\eta_X : x \mapsto \delta_x$ (which
 46 is equivalent to $\mathcal{D}1_X$) and multiplication is $\mu_X : \mathcal{D}^2(X) \rightarrow \mathcal{D}(X)$ where $\mu_X : \Omega \mapsto \sum_{\phi} \Omega(\phi)\phi$.

47 For continuous distributions we have the Giry monad on the category \mathbf{Meas} of measurable spaces
 48 given by the functor \mathcal{G} which maps a measurable space X to the set of probability measures on X ,
 49 denoted $\mathcal{G}(X)$. Other elements of the monad (unit, multiplication and map between morphisms) are
 50 the "continuous" version of the above.

51 Of particular interest is the Kleisli category of the monads above. The Kleisli C_T category of a
 52 monad T on category C is the category with the same objects and the morphisms $X \rightarrow Y$ in C_T is
 53 the set of morphisms $X \rightarrow TY$ in C . Thus the morphisms $X \rightarrow Y$ in the Kleisli category $\mathbf{Set}_{\mathcal{D}}$ are
 54 morphisms $X \rightarrow \mathcal{D}(Y)$ in \mathbf{Set} , i.e. stochastic matrices, and in the Kleisli category $\mathbf{Meas}_{\mathcal{G}}$ we have
 55 Markov kernels. Composition of arrows in the Kleisli categories correspond to Matrix products and
 56 "kernel products" respectively.

57 Both \mathcal{D} and \mathcal{G} are known to be *commutative* monads, and the Kleisli category of a commutative
 58 monad is a symmetric monoidal category.

59 Diagrams for symmetric monoidal categories consist of wires with arrows, boxes and a couple of
 60 special symbols. The identity object (which we identify with the set $\{*\}$) is drawn as nothing at all
 61 $\{*\} := \square$ and identity maps are drawn as bare wires:

$$\text{Id}_X := \uparrow_X \quad (15)$$

62 We draw Kleisli arrows from the unit (i.e. probability distributions) $\mu : \{*\} \rightarrow X$ as triangles and
 63 Kleisli arrows $\kappa : X \rightarrow Y$ (i.e. Markov kernels $X \rightarrow \Delta(\mathcal{Y})$) as boxes. We draw the Kleisli arrow
 64 $1_X : X \rightarrow \{*\}$ (which is unique for each X) as below

$$\mu := \begin{array}{c} \uparrow X \\ \triangle \mu \end{array} \quad \kappa := \begin{array}{c} \uparrow Y \\ \boxed{\kappa} \end{array} \quad (16)$$

65 The product of objects in \mathbf{Meas} is given by $(X, \mathcal{X}) \cdot (Y, \mathcal{Y}) = (X \times Y, \mathcal{X} \otimes \mathcal{Y})$, which we will
 66 often write as just $X \times Y$. Horizontal juxtaposition of wires indicates this product, and horizontal
 67 juxtaposition also indicates the tensor product of Kleisli arrows. Let $\kappa_1 : X \rightarrow W$ and $\kappa_2 : Y \rightarrow Z$:

$$(X \times Y, \mathcal{X} \otimes \mathcal{Y}) := \begin{array}{c} \uparrow X \quad \uparrow Y \end{array} \quad \kappa_1 \otimes \kappa_2 := \begin{array}{c} \uparrow W \quad \uparrow Z \\ \boxed{\kappa_1} \quad \boxed{\kappa_2} \\ \downarrow X \quad \downarrow Y \end{array} \quad (17)$$

68 Composition of arrows is achieved by "wiring" boxes together. For $\kappa_1 : X \rightarrow Y$ and $\kappa_2 : Y \rightarrow Z$
 69 we have

$$\kappa_1 \kappa_2(x; A) = \int_Y \kappa_2(y; A) \kappa_1(x; dy) := \begin{array}{c} \nearrow Z \\ \boxed{\kappa_2} \\ \downarrow \\ \boxed{\kappa_1} \\ \downarrow \\ X \end{array} \quad (18)$$

70 Symmetric monoidal categories have the following coherence theorem[Selinger, 2010]:

71 **Theorem 2.1** (Coherence (symmetric monoidal)). *A well-formed equation between morphisms in*
 72 *the language of symmetric monoidal categories follows from the axioms of symmetric monoidal*
 73 *categories if and only if it holds, up to isomorphism of diagrams, in the graphical language.*

74 Isomorphism of diagrams for symmetric monoidal categories (somewhat informally) is any planar
 75 deformation of a diagram including deformations that cause wires to cross. We consider a diagram
 76 for a symmetric monoidal category to be well formed only if all wires point upwards.

77 In fact the Kleisli categories of the probability monads above have (for each object) unique *copy*:
 78 $X \rightarrow X \times X$ and *erase*: $X \rightarrow \{*\}$ maps that satisfy the *commutative comonoid axioms* that (thanks
 79 to the coherence theorem above) can be stated graphically. These differ from the copy and erase
 80 maps of *finite product* or *cartesian* categories in that they do not necessarily respect composition of
 81 morphisms.

$$\text{Erase} = \mathbb{1}_X := \begin{array}{c} * \\ \downarrow \end{array} \quad \text{Copy} = x \mapsto \delta_{x,x} := \begin{array}{c} \nearrow \searrow \\ \downarrow \end{array} \quad (19)$$

$$\begin{array}{c} \nearrow \searrow \\ \downarrow \end{array} = \begin{array}{c} \nearrow \searrow \\ \downarrow \end{array} := \begin{array}{c} \nearrow \searrow \\ \downarrow \end{array} \quad (20)$$

$$\begin{array}{c} * \\ \downarrow \end{array} = \begin{array}{c} \nearrow * \\ \downarrow \end{array} = \begin{array}{c} \uparrow \end{array} \quad (21)$$

$$\begin{array}{c} \nearrow \searrow \\ \downarrow \end{array} = \begin{array}{c} \nearrow \searrow \\ \downarrow \end{array} \quad (22)$$

82 Finally, $\{*\}$ is a terminal object in the Kleisli categories of either probability monad. This means
 83 that the map $X \rightarrow \{*\}$ is unique for all objects X , and as a consequence for all objects X, Y and all
 84 $\kappa : X \rightarrow Y$ we have

$$\begin{array}{c} * \\ \downarrow \end{array} \boxed{\kappa} \downarrow X = \begin{array}{c} * \\ \downarrow \end{array} X \quad (23)$$

85 This is equivalent to requiring for all $x \in X$ $\int_Y \kappa(x; dy) = 1$. In the case of $\mathbf{Set}_{\mathcal{D}}$, this condition is
 86 what differentiates a stochastic matrix from a general positive matrix (which live in a larger category
 87 than $\mathbf{Set}_{\mathcal{D}}$).

88 Thus when manipulating diagrams representing Markov kernels in particular (and, importantly, not
 89 more general symmetric monoidal categories) diagram isomorphism also includes applications of 20,
 90 21, 22 and 23.

91 A particular property of the copy map in $\mathbf{Meas}_{\mathcal{G}}$ (and probably $\mathbf{Set}_{\mathcal{D}}$ as well) is that it commutes with
 92 Markov kernels iff the markov kernels are deterministic [Fong, 2013].

93 2.1 Disintegration and Bayesian inversion

94 *Disintegration* is a key operation on probability distributions (equivalently arrows $\{*\} \rightarrow X$) in
 95 the categories under discussion. It corresponds to “finding the conditional probability” (though
 96 conditional probability is usually formalised in a slightly different way).

97 Given a distribution $\mu : \{*\} \rightarrow X \otimes Y$, a disintegration $c : X \rightarrow Y$ is a Markov kernel that satisfies

$$\begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ \triangle \mu \end{array} = \begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ \triangle \mu^* \end{array} \quad (24)$$

98 Disintegrations always exist in **Set_D** but not in **Meas_G**. They do exist in the latter if we restrict
 99 ourselves to standard measurable spaces. If c_1 and c_2 are disintegrations $X \rightarrow Y$ of μ , they are equal
 100 μ -A.S. In fact, this equality can be strengthened somewhat - they are equal almost surely with respect
 101 to any distribution that shares the “ X -marginal” of μ .

102 Given $\sigma : \{*\} \rightarrow X$ and a channel $c : X \rightarrow Y$, a Bayesian inversion of (σ, c) is a channel $d : Y \rightarrow X$
 103 such that

$$\begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ \triangle \sigma \end{array} \quad c = \begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ \triangle \sigma \end{array} \quad d \quad (25)$$

104 We can obtain disintegrations from Bayesian inversions and vice-versa.

105 Clerc et al. [2017] offer an alternative view of Bayesian inversion which they claim doesn’t depend
 106 on standard measurability conditions, but there is a step in their proof I didn’t follow.

107 2.2 Generalisations

108 Cho and Jacobs [2019] make use of a larger “CD” category by dropping 23. I’m not completely clear
 109 whether you end up with arrows being “Markov kernels for general measures” or something else (can
 110 we have negative arrows?). This allows for the introduction of “observables” or “effects” of the form

$$\begin{array}{c} \triangle f \\ \uparrow \end{array}$$

112 Jacobs et al. [2019] make use of an embedding of **Set_D** in **Mat(\mathbb{R}^+)** with morphisms all positive
 113 matrices (I’m not totally clear on the objects, or how they are self-dual - this doesn’t seem to be
 114 exactly the same as the category of finite dimensional vector spaces). This latter category is compact
 115 closed, which - informally speaking - supports the same diagrams as symmetric monoidal categories
 116 with the addition of “upside down” wires.

117 2.3 Key questions for Causal Theories

118 We will first define *labeled diagrams*. Rather than labelling the wires of our diagrams with *spaces* (as is
 119 typical [Selinger, 2010]), we assign a unique label to each “wire segment” (with some qualifications).
 120 That is, we assign a unique label to each bare wire in the diagram with the following additional
 121 qualifications:

- 122 • If we have a box in the diagram representing the identity map, the incoming and outgoing
 123 wires are given the same label
- 124 • If we have a wire crossing in the diagram, the diagonally opposite wires are given the same
 125 label

I’m sure one of the papers I read mentioned labeled diagrams, I just couldn’t find it when I looked for it

Since writing this, I found Kissinger [2014] as an example of a diagrammatic

126 • The input wire and the *two* output wires of the copy map are given the same label

127 Given two diagrams G_1 and G_2 that are isomorphic under transformations licenced by the axioms of
 128 symmetric monoidal categories and commutative comonoid axioms, suppose we have a labelling of
 129 G_1 . We can label G_2 using the following translation rule:

130 • For each box in G_2 , we can identify a corresponding box in G_1 via labels on each box. For
 131 each such pair of boxes, we label the incoming wires of the G_2 box with the labels of the
 132 G_1 box preserving the left-right order. We do likewise for outgoing wires.

133 These rules will lead to a unique labelling of G_2 with all wire segments are labelled. We would like
 134 for these rules to yield the following:

135 • For any sequence of diagram isomorphisms beginning with G_1 and ending with G_2 , we end
 136 up with the same set of labels

137 • If we label G_2 according to the rules above then relabel G_1 from G_2 according to the same
 138 rules we retrieve the original labels of G_1

139 We do not prove these properties here, but motivate them via the following considerations:

140 • These properties obviously hold for the wire segments into and out of boxes

141 • The only features a diagram may have apart from boxes and wires are wire crossings, copy
 142 maps and erase maps

143 • The labeling rule for wire crossings respects the symmetry of the swap map

144 • The labeling rule for copy maps respects the symmetry of the copy map and the property
 145 described in Equation 22

146 We will follow the convention whereby “internal” wire labels are omitted from diagrams.

147 Note also that each wire that terminates in a free end can be associated with a random variable.
 148 Suppose for $N \in \mathbb{N}$ we have a kernel $\kappa : A \rightarrow \Delta(\times_{i \in N} X_i)$. Define by p_j ($j \in [N]$) the projection
 149 map $p_j : \times_{i \in N} X_i \rightarrow X_j$ defined by $p_j : (x_0, \dots, x_N) \mapsto x_j$. p_j is a measurable function, hence
 150 a random variable. Define by π_j the projection kernel $\mathcal{G}(\pi_j)$ (that is, $\pi_j : \mathbf{x} \mapsto \delta_{p_j(\mathbf{x})}$). Note that
 151 $\kappa(y; p_j^{-1}(A)) = \int_{X_j} \delta_{p_j(\mathbf{x})}(A) \kappa(y; d\mathbf{x}) = \kappa \pi_j$. Diagrammatically, π_j is the identity map on the j -th
 152 wire tensored with the erase map on every other wire. Thus the j -th wire carries the distribution
 153 associated with the random variable p_j . We will therefore consider the labels of the “outgoing” wires
 154 of a diagram to denote random variables (though there are obviously many random variables not
 155 represented by such wires). We will additionally distinguish wire labels from spaces by font - wire
 156 labels are sans serif A, B, C, X, Y, Z while spaces are serif A, B, C, X, Y, Z .

Wire labels appear to have a key advantage over random variables: they allow us to “forget” the sample space as the correct typing is handled automatically by composition and erasure of wires

158 **generalised disintegrations** : Of key importance to our work is generalising the notion of disinte-
 159 gration (and possibly Bayesian inversion) to general kernels $X \rightarrow Y$ rather than restricting ourselves
 160 to probability distributions $\{*\} \rightarrow Y$. We will define generalised disintegrations as a straightforward
 161 analogy regular disintegrations, but the conditions under which such disintegrations exist are more
 162 restrictive than for regular disintegrations.

163 **Definition 2.2** (Label signatures). If a kernel $\kappa : X \rightarrow \Delta(Y)$ can be represented by a diagram
 164 G with incoming wires X_1, \dots, X_n and outgoing wires Y_1, \dots, Y_m , we can assign the kernel a “label
 165 signature” $\kappa : X_1 \otimes \dots \otimes X_n \dashrightarrow Y_1 \otimes \dots \otimes Y_m$ or, for short, $\kappa : X_{[n]} \dashrightarrow Y_{[m]}$. Note that this
 166 signature associates each label with a unique space - the space of X_1 is the space associated with the
 167 left-most wire of G and so forth. We will implicitly leverage this correspondence and write with X_1
 168 the space associated with X_1 and so forth. Note that while X_1 is by construction always different from
 169 X_2 (or any other label), the space X_1 may coincide with X_2 - the fact that labels always maintain
 170 distinctions between wires is the fundamental reason for introducing them in the first place.

There might actually be some sensible way to consider κ to be transforming the measurable functions of a type similar to $\otimes_{i \in [n]} X_i$ to functions of a type similar to $\otimes_{i \in [m]} Y_i$ (or vice versa - perhaps related to Clerc et al. [2017]), but wire labels are all we need at this point

Definition 2.3 (Generalised disintegration). Given a kernel $\kappa : X \rightarrow \Delta(Y)$ with label signature $\kappa : X_{[n]} \dashrightarrow Y_{[m]}$ and disjoint subsets $S, T \subset [m]$ such that $S \cup T = [m]$, a kernel c is a *g-disintegration from S to T* if it's type is compatible with the label signature $c : Y_S \dashrightarrow Y_T$ and we have the identity (omitting incoming wire labels):

$$\text{Diagram (26): } \kappa \text{ with inputs } Y_S, Y_T = \kappa \text{ with input } c(Y_S, Y_T)$$

I have introduced without definition additional labeling operations here: first, each label has a particular space associated with it (in order to license the notion of “type compatible with label signature”), and we have supposed labels can be “bundled”.

In contrast to regular disintegrations, generalised disintegrations “usually” do not exist. Consider $X = \{0, 1\}$, $Y = \{0, 1\}^2$ and κ has label signature $X_1 \dashrightarrow Y_{\{1,2\}}$ with

$$\kappa : \begin{cases} 1 \mapsto \delta_1 \otimes \delta_1 \\ 0 \mapsto \delta_1 \otimes \delta_0 \end{cases} \quad (27)$$

κ imposes contradictory requirements for any disintegration $c : \{0, 1\} \rightarrow \{0, 1\}$ from $\{1\}$ to $\{2\}$: equality for $X_1 = 1$ requires $c(1; \cdot) = \delta_1$ while equality for $X_1 = 0$ requires $c(1; \cdot) = \delta_0$. Subject to some regularity conditions (similar to standard Borel conditions for regular disintegrations), we can define g-disintegrations of a canonically related kernel that do generally exist; intuitively, g-disintegrations exist if they take the “input wires” of κ as input wires themselves.

Lemma 2.4. Given $\kappa : X \rightarrow \Delta(Y)$, a kernel κ^\dagger is a right inverse iff we have for all $x \in X$ $\kappa^\dagger(y; A) = \delta_x(A)$, $\kappa(x; \cdot)$ -almost surely.

Proof. Suppose κ^\dagger satisfies the almost sure equality for all $x \in X$. Then for all $x \in X$, $A \in \mathcal{Y}$ we have $\kappa \kappa^\dagger(x; A) = \int_Y \kappa^\dagger(y; A) \kappa(x; dy) = \int_Y \delta_x(A) \kappa(x; dy) = \delta_x(A)$; that is, $\kappa \kappa^\dagger = \text{Id}_X$, so κ^\dagger is a right inverse of κ .

Suppose we have a right inverse κ^\dagger . By definition, for all $x \in X$ and $A \in \mathcal{Y}$ we have $\int_Y \kappa^\dagger(y; A) \kappa(x; dy) = \delta_x(A)$. Suppose $x \notin A$ and let $B_\epsilon = \kappa_A^{\dagger-1}((\epsilon, 1])$ for some $\epsilon > 0$. We have $\int_Y \kappa^\dagger(y; A) \kappa(x; dy) \geq \epsilon \kappa(x; B)$. For any $\epsilon > 0$ we have $\kappa(x; B_\epsilon) = 0$. Consider the set $B_0 = \kappa_A^{\dagger-1}((0, 1])$. For some sequence $\{\epsilon_i\}_{i \in \mathbb{N}}$ such that $\lim_{i \rightarrow \infty} \epsilon_i = 0$ we have $B_0 = \cup_{i \in \mathbb{N}} B_{\epsilon_i}$. By countable additivity, $\kappa(x; B_0) = 0$. Suppose $x \in A$ and let $B^{1-\epsilon} = \kappa_A^{\dagger-1}([0, 1 - \epsilon))$. By an argument analogous to the above, we have $\kappa(x; B^1) = 0$. Thus the $\kappa(x; \cdot)$ measure of the set on which $\kappa^\dagger(y; A)$ disagrees with $\delta_x(A)$ is $\kappa(x; B_0) + \kappa(x; B^1) = 0$ and hence $\kappa^\dagger(y; A) = \delta_x(A)$ $\kappa(x; \cdot)$ -almost surely. \square

I haven't shown that any map inverting κ implies the existence of a Markov kernel that does so

I am using countable sets below to get my general argument in order without getting too hung up on measurability; I will try to lift it to standard measurable once it's all there

Lemma 2.5. Given $\kappa : X \rightarrow \Delta(Y)$ and a right inverse κ^\dagger , we have

$$(28)$$

200 *Proof.* Let the diagram on the left hand side be L and the diagram on the right hand side be R .

$$L(x; A \times B) = \int_Y \int_{Y \times Y} \text{Id}_Y \otimes \kappa_S^\dagger(y, y'; A \times B) \delta_{(z, z)}(dy \times dy') \kappa \pi_S(x; dz) \quad (29)$$

$$= \int \text{Id}_Y \otimes \kappa^\dagger(z, z; A \times B) \kappa \pi_S(x; dz) \quad (30)$$

$$= \int \delta_z(A) \kappa_S^\dagger(z; B) \kappa \pi_S(x; dz) \quad (31)$$

$$= \int_A \kappa_S^\dagger(z; B) \kappa \pi_S(x; dz) \quad (32)$$

$$= \delta_x(B) \kappa \pi_S(x; A) \quad (33)$$

201 Where 33 follows from Lemma 2.4.

$$R(x; A \times B) = \int \delta_{(x, x)}(dy \times dy') \kappa \pi_S \otimes \text{Id}_X(y, y'; A \times B) \quad (34)$$

$$= \kappa \pi_S(x; A) \delta_x(B) = L \quad (35)$$

202

□

203 **Theorem 2.6.** Given countable X and standard measurable Y , $n, m \in \mathbb{N}$, $S, T \subset [m]$, κ with label
204 signature $X_{[n]} \dashrightarrow Y_{[m]}$ a g -disintegration exists from S to T if $\kappa \pi_S$ is right-invertible

205 via a Markov kernel

206 *Proof.* In addition, as R is a composition of Markov kernels, and hence a Markov kernel itself, L
207 must also be a Markov kernel even if κ^\dagger is not.

208 For all $x \in X$ we have a (regular) disintegration $c_x : Y_S \rightarrow \Delta(Y_T)$ of $\kappa(x; \cdot)$ by standard mea-
209 surability of Y . Define $c : X \otimes Y_S \rightarrow \Delta(Y_T)$ by $c : (x, y_S) \mapsto c_x(y_S)$. Clearly, $c(x, y_S)$ is a
210 probability distribution on Y_T for all $(x, y_S) \in X \otimes Y_S$. It remains to show $c(\cdot)^{-1}(B)$ is measurable
211 for all $B \in \mathcal{B}([0, 1])$. But $c(\cdot)^{-1}(B) = \cap_{x \in X} c_y(\cdot)^{-1}(B)$. The right hand side is measurable by
212 measurability of $c_y(\cdot)^{-1}(B)$ countability of X , so c is a Markov kernel.

213 By the definition of c_x , we have for all $x \in X$

$$\begin{array}{c}
Y_S \quad Y_T \\
\downarrow \quad \downarrow \\
\boxed{\kappa} \\
\downarrow \\
\triangle \delta_x
\end{array}
=
\begin{array}{c}
Y_S \quad Y_T \\
\downarrow \quad \downarrow \\
\boxed{C_x} \\
\downarrow \\
\boxed{\kappa}^* \\
\downarrow \\
\triangle \delta_x
\end{array}
\quad (36)$$

$$=
\begin{array}{c}
Y_S \quad Y_T \\
\downarrow \quad \downarrow \\
\boxed{C} \\
\downarrow \\
\boxed{\kappa}^* \\
\downarrow \\
\triangle \delta_x
\end{array}
\quad (37)$$

214 Which implies

$$\begin{array}{c}
Y_S \quad Y_T \\
\downarrow \quad \downarrow \\
\boxed{\kappa}
\end{array}
=
\begin{array}{c}
Y_S \quad Y_T \\
\downarrow \quad \downarrow \\
\boxed{C} \\
\downarrow \\
\boxed{\kappa}^* \\
\downarrow \\
\triangle \delta_x
\end{array}
\quad (38)$$

215 Finally, we have

$$\begin{array}{c}
Y_S \quad Y_T \\
\downarrow \quad \downarrow \\
\boxed{C} \\
\downarrow \\
\boxed{\kappa_S^\dagger} \\
\downarrow \\
\boxed{\kappa}^* \\
\downarrow \\
\triangle \delta_x
\end{array}
=
\begin{array}{c}
Y_S \quad Y_T \\
\downarrow \quad \downarrow \\
\boxed{C} \\
\downarrow \\
\boxed{\kappa_S^\dagger} \\
\downarrow \\
\boxed{\kappa}^* \\
\downarrow \\
\triangle \delta_x
\end{array}
\quad (39)$$

$$=
\begin{array}{c}
Y_S \quad Y_T \\
\downarrow \quad \downarrow \\
\boxed{C} \\
\downarrow \\
\boxed{\kappa}^* \\
\downarrow \\
\triangle \delta_x
\end{array}
\quad (40)$$

216 Where the first line follows from 21 and the second line from 28. If κ_S^\dagger is a Markov kernel, then
217 $\forall (\text{Id}_{Y_S} \otimes \kappa_S^\dagger) c$ is a g-disintegration. \square

218 In the reverse direction, suppose κ is such that $\kappa \pi_T = \text{Id}_X$; that is, π_T is a right inverse of κ . If
219 $\kappa \pi_S$ is not right invertible then, by definition, there is no d such that $\kappa \pi_S d \pi_T = \text{Id}_X$. However, if a
220 g-disintegration of κ exists then there is a d such that $\kappa \pi_S d = \kappa$, a contradiction. Thus if $\kappa \pi_S$ is not
221 right invertible then there is *in general* no g-disintegration from S to T .

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