Acoustic Wave

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The discussion of different modes and their interactions provides important information on sound sources. Once the sound sources are determined, the next task is to obtain the sound fields by solving the acoustic wave equations. In this chapter we study how to solve the sound field in an open space. (Sound in ducts is to be discussed in a separate chapter.) The sound field from a pulsating sphere in a three-dimensional (3-D) space is first investigated. The concepts of monopole, dipole and quadrupole are introduced. Based on the monopole sound solution, one of the most important techniques to solve acoustic equations is introduced: the 3-D Green's function and the formal integral solutions. Using the integral solutions, the sound field from moving sources, especially rotating sources, is studied. The two-dimensional (2-D) Green's function is also briefly discussed.

Acoustic Wave Equations

For the acoustic mode in a uniform ideal flow:

$$-\frac{1}{\rho_0 a_0^2}\frac{D_0 p}{Dt} = \nabla \cdot \vec{u}\,, \tag{1}$$

$$\frac{D_0 \vec{u}}{Dt} = -\frac{1}{\rho_0}\nabla p\,, \tag{2}$$

$$S = 0,\ \rho = \frac{1}{a_0^2}p, \tag{3}$$

$$\nabla \times \vec{u} = 0\,. \tag{4}$$

These are the irrotational, isentropic, linear Euler equations. Variables with subscript '0' are mean flow quantities, and $D_0 / Dt = \partial / \partial t + \vec{u}_0 \cdot \nabla$. All other variables are perturbation variables.

Velocity Potential and Acoustic Wave Equations

The most important consequence of Eq.(4) ($\nabla \times \vec{u} = 0$) is the introduction of the velocity potential. Shown in Fig. 1 is a control volume *V* bounded by inner surface S_i and outer surface S_o. *C* is a closed contour in *V*. *C* is reducible if it can shrink to a point without having to cross any boundaries. Its shrinking path forms a surface S_c. If in *V* any contour is reducible, the region is *singly-connected*. Otherwise, it is multiply-connected. If the volume in Fig.1 is three-dimensional and the dimensions of the inner body are finite, then *V* if singly-connected. On the other hand, if the volume is two-dimensional, the inner body extends to infinity in the third direction. Any contour enclosing the body is not reducible; therefore V is multiply-connected. This argument shows the difference of the mathematical treatment when dealing with two-dimensional and three-dimensional problems.

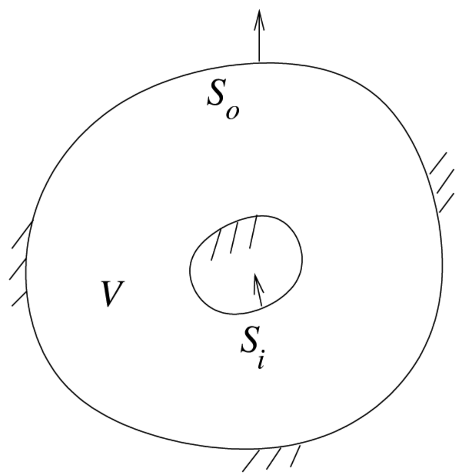


Fig. 1, Control volume V bounded by inner surface S_i and outer surface S_o.

Suppose in a singly-connected region there are two points O and P, and two paths C_1 and C_2 joining the two points. C_1 and C_2 forms a closed contour $C = C_1 + (-C_2)$ and there exists a surface S_c with contour C as the boundary. By applying the Stokes' theorem, we know that circulation Γ around the contour is zero since $\nabla \times \vec{u} = 0$. Therefore,

$$\int_{C_1} \vec{u} \cdot d\vec{l} = \int_{C_2} \vec{u} \cdot d\vec{l} \; . \tag{5}$$

That means the line integration is only a function of the positions of O and P. It does not depend on the paths. Therefore, we can define a function of position $\phi(\vec{x})$:

$$\phi(\vec{x}_P) = \phi(\vec{x}_O) + \int_{C_1} \vec{u} \cdot d\vec{l} \; . \tag{6}$$

When P approaches O, $d\vec{l} \approx \vec{x}_p - \vec{x}_O$, we have:

$$\left[1 + d\vec{l} \cdot \nabla + \frac{1}{2!}(d\vec{l} \cdot \nabla)^2 + \cdots\right]\phi(\vec{x}_O) \approx \phi(\vec{x}_O) + \vec{u} \cdot d\vec{l} \; .$$

As $d\vec{l} \to 0$, $d\vec{l} \cdot \nabla\phi(\vec{x}_O) = \vec{u} \cdot d\vec{l}$, then,

$$\vec{u} = \nabla \phi \; . \tag{7}$$

$\phi(\vec{x})$ is the velocity potential function. Acoustic velocity is the gradient of the velocity potential.

In a multiply-connected region, contour $C = C_1 + (-C_2)$ may not be reducible. Under this circumstance, the Stokes' Theorem doesn't apply and circulation Γ around C may not be zero even for $\nabla \times \vec{u} = 0$. Then,

$$\int_{C_1} \vec{u} \cdot d\vec{l} = \int_{C_2} \vec{u} \cdot d\vec{l} + n\Gamma.$$

The line integration depends on the path going from O to P. Velocity potential ϕ defined by Eq.(6) has multiple values at a point depending on how many rounds the path goes. ϕ is no longer a function of position since it may have multiple values at the same position. In this case, an artificial barrier, such as S_b in Fig. 2, must be inserted to make the region singly-connected.

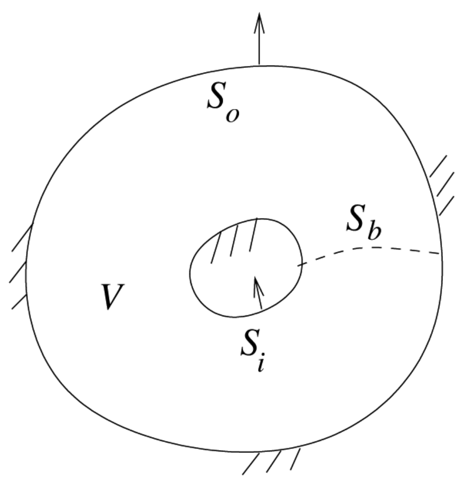


Fig.2, Artificial barrier in a multiply-connected control volume.

The benefit of introducing the velocity potential is twofold: the number of variables is reduced, and the irrotational requirement is automatically satisfied since $\nabla \times \nabla\phi = 0$. One disadvantage is that mathematically the velocity potential requires higher order smoothness than the velocity itself.

It is the gradient of velocity potential, $\nabla\phi$, not the potential itself, that has physical meaning. Since ϕ is defined by the spatial integral [Eq.(6)], any function of time $\phi^*(t)$ added to ϕ does not affect the equation and analysis.

With the newly introduced velocity potential, Eqs.(1) and (2) can be written as:

$$-\frac{1}{\rho_0 a_0^2}\frac{D_0 p}{Dt} = \nabla^2 \phi, \tag{8}$$

$$\frac{D_0}{Dt}(\nabla\phi) = -\frac{1}{\rho_0}\nabla p, \tag{9}$$

Assume the acoustic medium is uniform. Then the derivatives on the left side of Momentum equation Eq.(9) can be interchanged. Eq.(9) can be reduced to:

$$\frac{D_0 \phi}{Dt} = -\frac{1}{\rho_0} p + f(t),$$

i.e.,

$$\frac{D_0}{Dt}\left(\phi - \int^t f(\tau) d\tau\right) = -\frac{1}{\rho_0} p, \tag{10}$$

where $f(t)$ is any function of time. According to Eq.(7), any function of time added to ϕ doesn't affect the acoustic velocity. Therefore, Eq.(10) can be written simply as:

$$\frac{D_0 \phi}{Dt} = -\frac{1}{\rho_0} p. \tag{11}$$

Substituting it into continuity equation (9), we have:

$$\frac{1}{a_0^2}\frac{D_0}{Dt}\left(\frac{D_0 \phi}{Dt}\right) - \nabla^2 \phi = 0, \tag{12}$$

$$\frac{1}{a_0^2}\frac{D_0}{Dt}\left(\frac{D_0 p}{Dt}\right) - \nabla^2 p = 0. \tag{13}$$

where

$$\frac{D_0}{Dt}\left(\frac{D_0}{Dt}\right) = \frac{\partial^2}{\partial t^2} + 2u_{0i}\frac{\partial^2}{\partial t \partial x_i} + u_{0i}u_{0j}\frac{\partial^2}{\partial x_i \partial x_j}.$$

These are the acoustic equations in a uniform mean flow. It is also satisfied by any other acoustic variables such as ρ. The nontrivial solution of (12) or (13) is the acoustic wave. The trivial solution, $p = 0$ or $\phi = 0$, corresponds to vorticity waves or entropy waves.

There are no sources in linear Euler equations (1) ~ (3). The corresponding acoustic equation (13) or (12) is a homogeneous partial differential equation. It describes the propagation of sound waves in a uniform flow. At any point in the medium, $\frac{1}{a_0^2}\frac{D_0}{Dt}\left(\frac{D_0 p}{Dt}\right) - \nabla^2 p$ must be zero. Otherwise, there are sound sources at this point. These sources are from external action such as flow injection, or from scattering of other modes due to nonuniformity in the flow, or from nonlinear interactions between different modes.

Solution Uniqueness and Boundary Conditions

In a singly connected region, Eq.(13) has a unique solution if the normal velocity on all the boundary surfaces (S_i and S_o in Fig.1) enclosing the area of interest is prescribed (cht7.doc). The uniqueness of the solution can be proved by the energy integral method (Pierce1989 p.171, Bachelor Chapter 2.7&2.8). If the outside surface S_o is at infinity, the sound field vanishes according to the causality requirement. Causality means there is no sound before it reaches the observer.

If the region is multiply connected, we first make it singly connected by inserting barrier(s) as in Fig. 2. Then the solution is unique when the normal velocity distribution is prescribed on all surfaces: S_i, S_o and barrier S_b. Prescribing normal velocity on S_b is equivalent to set the flux across the artificial barrier, or circulation around the inner body. What is the correct circulation for a particular problem? The model we began with [Eqs.(1)~(4)] assumes inviscid medium. If the inner body surface is smooth, zero circulation is often assumed (Morse&Ingard1968, p.400). If there is a sharp edge on the inner surface, velocity at the sharp edge goes to infinity if the medium is inviscid. In this case viscosity can not be neglected near the sharp edge. Vortexes evolve and shed at the sharp edge due to the effect of viscosity; the circulation is induced around the inner boundary. Therefore enforcing the Kutta condition at the sharp edge is the way to set the exact circulation due to the viscous effect. But this condition can only be applied once at one sharp edge, such as the trailing edge of an airfoil. Discontinuity/infinity is allowed at other sharp edges such as the leading edge of the airfoil.

Prescribing normal velocity is only a sufficient, not necessary, boundary condition. A unique solution can also be rendered if velocity potential, or pressure, is set on the surfaces. Actually, a linear combination of the normal velocity and pressure can be given at the boundary surfaces. This is the impedance boundary condition, which will be discussed in another chapter.

In an open space, the outer surface S_o is at infinity. In numerical analyses, it is impossible to set the computation domain to infinity. Usually the boundary is placed far away from the source region and approximate equations are used to ensure causality, such as the Sommerfeld Radiation Boundary equations:

$$\lim_{r\to\infty}\left[r^{1/2}\left(\frac{\partial p}{\partial r}+\frac{1}{a_0}\frac{\partial p}{\partial t}\right)\right]=0 \text{ for a 2-D stationary medium,} \tag{14}$$

$$\lim_{r\to\infty}\left[r\left(\frac{\partial p}{\partial r}+\frac{1}{a_0}\frac{\partial p}{\partial t}\right)\right]=0 \text{ for a 3-D stationary medium,} \tag{15}$$

or the 2D radiation boundary condition considering a mean flow by Tam:

$$\frac{1}{V(\theta)}\frac{\partial p}{\partial t}+\frac{\partial p}{\partial r}+\frac{p}{2r}=0\,,\ V(\theta)=a_0\left[M\cos\theta+\left(1-M^2\sin^2\theta\right)^{1/2}\right]. \tag{16}$$

It is noted that Eq.(16) is different from (14) when M=0. These boundary conditions formally make an open region finite.

Sound Generated by Vibrating Spheres

Sound Generated by Radially Vibrating Sphere in Stationary Medium (Mass Fluctuation)

Here we will discuss the sound field in a stationary medium generated by an external source: a vibrating sphere. Consider a sphere with radius R. The sphere surface vibrates radially with uniform amplitude and phase. When the vibration amplitude is small, the boundary condition can be represented by a uniform velocity at the nominal surface:

$$u_r(r,t)=v_s(t)\ \text{ at }\ r=R\,. \tag{17}$$

According to the uniqueness of solution discussed in the previous section, the sound field is uniquely determined with the boundary condition in (17).

In the spherical coordinate system, the gradient and the Laplacian are respectively:

$$\nabla=\vec{i}_r\frac{\partial}{\partial r}+\vec{i}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}+\vec{i}_\theta\frac{1}{r}\frac{\partial}{\partial\theta},$$
$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{\sin^2\theta}\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$

The boundary velocity in (17) and the sound field are spherically symmetric. Therefore wave equation (12) and (11) can be simplified:

$$\frac{1}{a_0^2}\frac{\partial^2(r\phi)}{\partial t^2}-\frac{\partial^2(r\phi)}{\partial r^2}=0\,, \tag{18}$$

$$\frac{\partial\phi}{\partial t}=-\frac{1}{\rho_0}p. \tag{19}$$

The general solution (the d'Alembert's solution) of (18) is:

$$\phi(r,t)=r^{-1}\left[F(t-r/a_0)+E(t+r/a_0)\right]. \tag{20}$$

F and E are two arbitrary functions representing respectively the out-going and in-coming spherical waves. The requirement of causality excludes function E. Therefore the solution is:

$$\phi(r,t) = r^{-1}F(t - r/a_0), \tag{21}$$

$$u_r(r,t) = \partial\phi/\partial r = -r^{-1}\dot{F}(t - r/a_0)/a_0 - r^{-2}F(t - r/a_0),\ u_\varphi = 0,\ u_\theta = 0, \tag{22}$$

$$p(r,t) = -\rho_0 r^{-1}\dot{F}(t - r/a_0), \tag{23}$$

where $\dot{F}(t - r/a_0) \equiv \left.\dfrac{dF(\tau)}{d\tau}\right|_{\tau = t - r/a_0}$.

The sound source is the mechanical vibration of the sphere. The volume of the sphere changes as its surface vibrates. It is equivalent to the injection of fluid into the medium. The rate of the injected volume is

$$Q(t) = 4\pi R^2 v_s(t). \tag{24}$$

Substituting (22) into boundary condition (17), we have

$$\dot{F}(\tau) + a_0 R^{-1} F(\tau) = -\frac{a_0}{4\pi R} Q(\tau + R/a_0). \tag{25}$$

The first order linear ordinary differential equation (ODE) (25) can be solved using the *method of separation of variables* and the *method of variation of parameter* (c.f. Appendix). Suppose there is no vibration at $t = -\infty$, then the solution is:

$$F(\tau) = -\frac{a_0}{4\pi R}\int_{-\infty}^{\tau} Q(\tau' + R/a_0) e^{-(\tau - \tau')a_0/R} d\tau'. \tag{26}$$

Therefore the transient velocity potential is:

$$\phi(r,t) = -\frac{a_0}{4\pi R r}\int_{-\infty}^{t-(r-R)/a_0} Q(\tau) e^{-[t-(r-R)/a_0 - \tau]a_0/R} d\tau,\ r \ge R; \tag{27}$$

and the sound pressure:

$$p(r,t) = \frac{\rho_0 a_0}{4\pi R r}\left\{-\frac{a_0}{R}\int_{-\infty}^{t-(r-R)/a_0} Q(\tau) e^{-[t-(r-R)/a_0 - \tau]a_0/R} d\tau + Q(t - (r - R)/a_0)\right\},\ r \ge R. \tag{28}$$

$t - (r - R)/a_0$ is the retarded time at the surface point closest to the observer. (27) and (28) give the sound field generated by a vibrating sphere. They apply anywhere as long as $r \ge R$. There is no physical and mathematical meaning for $r < R$. The net force of the sphere acting on the fluid is zero since the pressure is spherically symmetric.

Monopole

The source is compact when its dimension is small compared with sound wavelength λ, *i.e.*, $R/\lambda << 1$. It is useful to investigate the sound field for a compact source. We assume the amplitude of volume injection rate $Q(t)$ is constant when the sphere shrinks into a point $R/\lambda \to 0$. Performing integration by parts we have,

$$\begin{aligned}
&\frac{a_0}{R}\int_{-\infty}^{t-(r-R)/a_0} Q(\tau)e^{-[t-(r-R)/a_0-\tau]a_0/R}d\tau \\
&= Q(t-(r-R)/a_0) - \frac{R}{a_0}\dot{Q}(t-(r-R)/a_0) \\
&+\left(\frac{R}{a_0}\right)^2\left[\ddot{Q}(t-(r-R)/a_0) - \int_{-\infty}^{t-(r-R)/a_0}\dddot{Q}(\tau)e^{-[t-(r-R)/a_0-\tau]a_0/R}d\tau\right]
\end{aligned}.$$

Note

$$|\dot{Q}(t-(r-R)/a_0)| \sim 2\pi a_0 |Q(t-(r-R)/a_0)|/\lambda,$$

then, as $R/\lambda \to 0$,

$$\frac{a_0}{R}\int_{-\infty}^{t-(r-R)/a_0} Q(\tau)e^{-[t-(r-R)/a_0-\tau]a_0/R}d\tau \approx Q(t-(r-R)/a_0). \tag{29}$$

This approximation is very important and will be used repeatedly hereafter.

Apply (29) in (27) and (28), then $R/\lambda \to 0$,

$$\phi(r,t) = -\frac{1}{4\pi r}Q(t-r/a_0), \tag{30}$$

$$u_r(r,t) = \frac{1}{4\pi r^2}Q(t-r/a_0) + \frac{1}{4\pi a_0 r}\dot{Q}(t-r/a_0),\ u_\theta(r,t) = u_\varphi(r,t) = 0, \tag{31}$$

$$p(r,t) = \frac{\rho_0}{4\pi r}\dot{Q}(t-r/a_0), \tag{32}$$

where $r > 0$. Eqs.(30)~(32) describe the sound field from a compact source, *i.e.,* source pulsating at low frequencies. Actually they apply for any frequency, since the source region is reduced to a conceptual point. This source is called monopole. The directivity of the sound field from a monopole is omnidirectional.

According to (31), the acoustic velocity has two different characteristics in two regions. In the far field ($r >> 1$) the dominant effect is from the second term on the right hand side of Eq.(31): the volume injection rate $\dot{Q}(t)$, or the acceleration of the vibrating surface. The acceleration is balanced by acoustic pressure [Eq.(32)]. The far field is the sound field. In the near field ($r << 1$), the dominant effect is from the volume injection $Q(t)$ in the first term. Velocity in the near field is much larger than the acoustic pressure. Acoustic pressure is a higher order quantity. The reason for the stronger velocity around the source is that we kept $Q(t)$ constant when shrinking the source to meet the boundary condition Eq.(24). We call this kind of source the velocity driver. Velocity cannot be balanced by sound pressure in the source region, where Eqs(1)&(3) reduce to $\nabla\cdot\vec{u}\approx 0$ and $\rho\approx 0$. The flow is nearly incompressible in this region. Note a vorticity wave in a uniform flow support no fluctuating pressure. Is it possible this term represents a vorticity wave? It can be shown that the curl of velocity is also zero. Therefore the 1st term on the right hand side of Eq.(31) represents an solenoidal, irrotational velocity field. This region is often referred as the potential field of the body. The potential field decays fast away from the body. It has effect on another body only when they are in proximity.

The Nonhomogeneous Acoustic Equation

Acoustic equations such as (13) and solutions (30)~(32) are valid everywhere except at the source point. Now we are to write the wave equation that also formally applies at this source point. The wave equation should have this form:

$$\frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2}-\nabla^2 p=S\delta(\vec{r}-\vec{r}_0), \tag{33}$$

with solution:

$$p(\vec{r},t)=\frac{\rho_0}{4\pi r}\dot{Q}(t-|\vec{r}-\vec{r}_0|/a_0)\,,\ \vec{r}\neq\vec{r}_0\,. \tag{34}$$

$\vec{r}_0$ is the source position. The source function $S\delta(\vec{r}-\vec{r}_0)$ is not a regular function. It can only be defined in the integral sense. The integration of $\frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2}-\nabla^2 p$ in any sphere that doesn't include the source point is zero. If the sphere with radius ε is centered at the source point, from the Gauss' Divergence Theorem we have:

$$\begin{aligned}
S &= \iiint\left(\frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2}-\nabla^2 p\right)dV \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{\varepsilon}\frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2}r^2\sin\theta\, dr\, d\theta\, d\varphi-\int_0^{2\pi}\int_0^{\pi}\nabla p\cdot\vec{i}_r\varepsilon^2\sin\theta\, d\theta\, d\varphi \\
&= \frac{\rho_0}{a_0^2}\int_0^{\varepsilon}\left.\frac{d^3Q(\tau)}{d\tau^3}\right|_{\tau=t-r/a_0} r\,dr+\rho_0\left(\left.\frac{dQ(\tau)}{d\tau}\right|_{\tau=t-\varepsilon/a_0}+\frac{R}{a_0}\left.\frac{d^2Q(\tau)}{d\tau^2}\right|_{\tau=t-\varepsilon/a_0}\right)
\end{aligned}\quad. \tag{35}$$

Although the integrand in the volume integration is singular at the source point, it is integrable.

As $R\to 0$,

$$S \to \rho_0 \dot{Q}(t)\,. \tag{36}$$

The inhomogeneous wave equation is:

$$\frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \rho_0 \dot{Q}(t)\delta(\vec{r}-\vec{r}_0)\,, \tag{37}$$

The sound source is the time derivative of *mass injection rate* $\rho_0 \dot{Q}(t)$ at $\vec{r}_0$. It is noted that in the inhomogeneous equation, the spatial coordinate is $\vec{r}$ (or $\vec{x}$), the field or observer coordinate. $\vec{r}_0$ is the source coordinate. It is a parameter, not the spatial variable. ∇ operates with respect to $\vec{r}$, not $\vec{r}_0$.

Superposition applies to linear equation (37). Assume the distribution of volume injection $Q(\vec{r},t)$, then the inhomogeneous equation and its solution are:

$$\frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \rho_0 \frac{\partial Q(\vec{r},t)}{\partial t}\,, \tag{38}$$

$$p(\vec{r},t) = \frac{\rho_0}{4\pi}\iiint \frac{1}{r_0}\left[\frac{\partial Q(\vec{r}_0,\tau)}{\partial \tau}\right]_{\tau = t-|\vec{r}-\vec{r}_0|/a_0} d\vec{r}_0\,. \tag{39}$$

Note it is the partial time derivations, not the whole time derivative, of the source in (38) and (39). Both the equation and the solution apply in the whole field.

Sound Generated by Transversely Oscillating Sphere in Stationary Medium (Momentum Fluctuation) and Dipole

Consider a *rigid* sphere transversely oscillating along the *z*-axis. The oscillation speed of the sphere center is $v_c(t)$ in *z* direction. The volume of the sphere doesn't change. The total volume of the medium are constant at any time. So it is not a monopole. The sound is generated by the translation of the sphere. It is the thickness noise in turbomachinery. The problem can be treated as a moving source problem (the details in the following section of this chapter). However, if oscillation speed $v_c(t)$ is small, to first order, it is equivalent to a vibrating sphere with fixed center and the speed at the surface:

$$u(r,\theta,t) = v_c(t)\cos\theta\,, \text{ at } r = R. \tag{40}$$

Note the $\cos\theta$ dependence of the surface speed as compared to the uniform surface speed in the monopole (17).

Boundary condition (40) and the sound field are no longer spherically symmetric. Instead they are axisymmetric about the *z*-axis. The standard method to solve the sound field from arbitrary vibration of a sphere is the separation of variables. (Morse&Ingard1968, p.332) For a linear problem like this, we can simply use the method of superposition.

Suppose we have two radially vibrating spheres, one at $z=h/2$ with rate of volume injection $Q(t)$, the other at $z=-h/2$ with rate of volume injection $-Q(t)$. The total rate of the volume injection is zero. Their difference is $2Q(t)$. According to (28), the sound pressure from the two sources is:

$$\begin{aligned} p = & \frac{\rho_0 a_0}{4\pi R r_1}\left\{-\frac{a_0}{R}\int_{-\infty}^{t-(r_1-R)/a_0} Q(\tau)e^{-[t-(r_1-R)/a_0-\tau]a_0/R}d\tau + Q(t-(r_1-R)/a_0)\right\} \\ & + \frac{\rho_0 a_0}{4\pi R r_2}\left\{\frac{a_0}{R}\int_{-\infty}^{t-(r_2-R)/a_0} Q(\tau)e^{-[t-(r_2-R)/a_0-\tau]a_0/R}d\tau - Q(t-(r_2-R)/a_0)\right\} \end{aligned} . \tag{41}$$

r_1 and r_2 are respectively the distances from the observation point to the upper and lower sphere centers:

$$r_1 = \left(r^2 - hr\cos\theta + h^2/4\right)^{1/2},\ r_2 = \left(r^2 + hr\cos\theta + h^2/4\right)^{1/2}. \tag{42}$$

As $h \to 0$ we keep amplitude of $Q(t)h$ constant, then from Eqs.(41) and (29),

$$\begin{aligned} p \to & \frac{\rho_0 a_0 \cos\theta}{4\pi R r} \\ & \cdot\left\{\left(\frac{1}{R}-\frac{1}{r}\right)\left[\frac{a_0}{R}\int_{-\infty}^{t-(r-R)/a_0} hQ(\tau)e^{-[t-(r-R)/a_0-\tau]a_0/R}d\tau - hQ(t-(r-R)/a_0)\right] + \frac{1}{a_0}h\dot{Q}(t-(r-R)/a_0)\right\} \\ \to & \frac{\rho_0 a_0 \cos\theta}{4\pi r} \\ & \cdot\left\{\left(\frac{1}{R}-\frac{1}{r}\right)\frac{R}{a_0^2}\left[h\ddot{Q}(t-(r-R)/a_0) - \int_{-\infty}^{t-(r-R)/a_0} h\dddot{Q}(\tau)e^{-[t-(r-R)/a_0-\tau]a_0/R}d\tau\right] + \frac{1}{a_0 r}h\dot{Q}(t-(r-R)/a_0)\right\} \\ & ,\ r \ge R. \end{aligned} \tag{43}$$

The radial velocity can be obtained by:

$$u_r = -\frac{1}{\rho_0}\int_{-\infty}^{t}\frac{\partial p}{\partial r}dt'. \tag{44}$$

From (43) and (44) we can see the $\cos\theta$ dependence of the sound field. The sound field has two lobes with strongest sound in the *z* direction. The two lobes have the same magnitude but 180^0 out of phase. The force of the sphere acting on the medium is in *z* direction:

$$F_z(t) = \int_0^{2\pi}\int_0^{\pi} pR^2 \cos\theta \sin\theta d\theta d\varphi = \frac{1}{3}\rho_0 h\dot{Q}(t). \quad (45)$$

As $R \to 0$, Eq.(43) becomes:

$$p \to \frac{\rho_0 h \cos\theta}{4\pi}\left\{\frac{1}{ra_0}\ddot{Q}(t - r/a_0) + \frac{1}{r^2}\dot{Q}(t - r/a_0)\right\},\ r > 0. \quad (46)$$

Compared with the monopole solution (32), (46) shows the superposed sound field of two monopoles with opposite strengths. This is called a dipole. Its strength is $Q(t)h$. To examine the physical meaning of the dipole sound field (46), we expand the volume source at $z = h/2$ for small h:

$$\begin{aligned}\dot{Q}(t-(r_1-R)/a_0) &= \dot{Q}(t-(r-R)/a_0) + h/2\cos\theta\ddot{Q}(t-(r-R)/a_0)/a_0 + O(h^2) \\ &= \dot{Q}(t-(r-R)/a_0) + (r_1-r)d\dot{Q}(t-(r-R)/a_0)/dr + O(h^2)\end{aligned}. \quad (47)$$

(Note: $d\dot{Q}(t(r))/dr = -\ddot{Q}(t)/a_0$.) The source at $z = h/2$ can be assumed to be at $z = 0$ plus a correction. The correction is due to the change of distance $r_1 - r = -h\cos\theta/2$ and the change of the retarded time $h/2\cos\theta/a_0$ when the source is repositioned. Applies the similar expansion to the source at $z = -h/2$. If we ignore the difference of $1/r$ in amplitude ($1/r_1 \approx 1/r_2 \approx 1/r$), the sum of the two sources gives the first term in (46). This term represents the sound source due to the difference of the retarded time of the two sources.

On the other hand, if we ignore the difference of the retarded time and concentrate on the difference in $1/r$, then we obtain the second term of solution (46). This term is small in the farfield. In the farfield the sound is mainly generated by the retarded time effect of the two monopoles.

We can rewrite solution (46) in the form with the spatial derivative:

$$p = -h\cos\theta \frac{d}{dr}\left[\frac{\rho_0}{4\pi r}\dot{Q}(t - r/a_0)\right]. \quad (48)$$

The spatial derivative is on the field coordinate, not the source coordinate. Because of the retardation of the sound propagation from the source to the observer, the sound field at observation time t has a spatial distribution, $\dot{Q}(t - r/a_0)/r$. The gradient of this spatial distribution at the observation point in the direction of the dipole determines the sound pressure at this point. The gradient includes two effects: the retarded time difference and the propagation distance in amplitude. In the far field, the retarded time effect is dominant; therefore one may take $1/r$ out of the spatial derivative in (48).

In terms of F_z, the wave solution for the dipole is:

$$p = -\frac{3\cos\theta}{4\pi}\frac{d}{dr}\left[\frac{1}{r}F_z(t - r/a_0)\right],\ r > 0. \tag{49}$$

Solution (46) can also be directly derived by performing the Taylor expansion on monopole solution (32). To generalize the solution, suppose the dipole is situated at $\vec{r}_s$ with the separation vector $\vec{h}$, then,

$$p(\vec{r},t) = \frac{\rho_0}{4\pi}\left[\frac{1}{r_1}\dot{Q}(t - r_1/a_0) - \frac{1}{r_2}\dot{Q}(t - r_2/a_0)\right], \tag{50}$$
$$\vec{r}_1 = \vec{r} - \vec{r}_s - \frac{1}{2}\vec{h},\ \vec{r}_2 = \vec{r} - \vec{r}_s + \frac{1}{2}\vec{h},\ r_1 = |\vec{r}_1|,\ r_2 = |\vec{r}_2|.$$

To first order Taylor expansion, we have:

$$p(\vec{r},t) \approx -\vec{h}\cdot\nabla_{\vec{r}}\left[\frac{\rho_0}{4\pi|\vec{r}-\vec{r}_s|}\dot{Q}(t - |\vec{r}-\vec{r}_s|/a_0)\right]. \tag{51}$$

In Cartesian coordinates, the solution has this form:

$$p(\vec{r},t) = -\left(h_j\frac{\partial}{\partial x_j}\right)\left[\frac{\rho_0}{4\pi|\vec{r}-\vec{r}_s|}\dot{Q}(t - |\vec{r}-\vec{r}_s|/a_0)\right]. \tag{52}$$

(51) and (52) are just generalized (48). The sound at the observation point is determined by the gradient of the monopole sound field in the direction of the dipole.

To obtain the inhomogeneous wave equation for the dipole, one may attempt to integrate $\frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p$ over a sphere centered at the source as in the monopole case. But the integration is zero since there is no net flow from the two monopoles. The correct way is to begin with the inhomogeneous wave equation for monopole (37):

$$\begin{aligned}\frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p &= \rho_0\dot{Q}(t)\delta(\vec{r}-\vec{r}_s - \frac{1}{2}\vec{h}) - \rho_0\dot{Q}(t)\delta(\vec{r}-\vec{r}_s + \frac{1}{2}\vec{h}) \\ &\approx -\rho_0\dot{Q}(t)\vec{h}\cdot\nabla_{\vec{r}}\delta(\vec{r}-\vec{r}_s)\end{aligned}. \tag{53}$$

***Quadrupole* (*Momentum Flux Fluctuation*)**

Consider two dipoles with the same strength $\dot{Q}(t)\vec{h}$ but in opposite directions. The distance between the two dipoles is $\vec{q}$ centered at $\vec{r}_s$ as in Fig.3. According to (51), the superposed sound pressure is:

$$p(\vec{r},t) = -\frac{\rho_0}{4\pi}\left(\vec{h}\cdot\nabla\right)\left[\frac{1}{r_1}\dot{Q}(t-r_1/a_0) - \frac{1}{r_2}\dot{Q}(t-r_2/a_0)\right], \tag{54}$$
$$r_1 = \left|\vec{r}-\vec{r}_s-\frac{1}{2}\vec{q}\right|,\; r_2 = \left|\vec{r}-\vec{r}_s+\frac{1}{2}\vec{q}\right|.$$

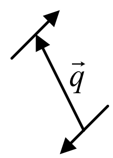


Fig.3, A quadrupole.

To first order, the Taylor expansion of (54) is:

$$p(\vec{r},t) = \left(\vec{h}\cdot\nabla\right)\left(\vec{q}\cdot\nabla\right)\left[\frac{\rho_0}{4\pi R}\dot{Q}(t-R/a_0)\right],\; R = \left|\vec{r}-\vec{r}_s\right|. \tag{55}$$

In Cartesian coordinates, we can rewrite the solution as:

$$p(\vec{r},t) = \left(h_i q_j \frac{\partial^2}{\partial x_i \partial x_j}\right)\left[\frac{\rho_0}{4\pi R}\dot{Q}(t-R/a_0)\right]. \tag{56}$$

$\vec{h}$ and $\vec{q}$ can be in any direction. There are two special cases. If $\vec{h}$ and $\vec{q}$ are parallel, the quadrupole is longitudinal. A longitudinal quadrupole has two lobes in sound field. Compared with a dipole, the two lobes from a longitudinal quadrupole are in phase instead of out of phase. If $\vec{h}$ and $\vec{q}$ are perpendicular, the quadrupole is lateral. There are four lobes in the sound field of a quadrupole.

(55) can be rewrite to:

$$p(\vec{r},t) = \left(\frac{\vec{h}+\vec{q}}{2}\cdot\nabla\right)^2\left[\frac{\rho_0}{4\pi R}\dot{Q}(t-R/a_0)\right] - \frac{\rho_0}{4\pi}\left(\frac{\vec{h}-\vec{q}}{2}\cdot\nabla\right)^2\left[\frac{\rho_0}{4\pi R}\dot{Q}(t-R/a_0)\right]. \tag{57}$$

Any quadrupole can be decomposed into two longitudinal quadrupoles. Longitudinal quadrupole is the basic type of all quadrupoles.

The inhomogeneous equation for a quadrupole is:

$$\frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2}-\nabla^2 p=-\rho_0\dot{Q}(t)\left(\vec{h}\cdot\nabla\right)\left[\delta(\vec{r}-\vec{r}_s-\frac{1}{2}\vec{q})-\delta(\vec{r}-\vec{r}_s+\frac{1}{2}\vec{q})\right]. \qquad (58)$$
$$\approx\rho_0\dot{Q}(t)(\vec{h}\cdot\nabla)\left(\vec{q}\cdot\nabla\right)\delta(\vec{r}-\vec{r}_s)$$

In summary, any sound sources can be represented by superposition of monopoles. When monopoles are close to each other, models such as dipoles or quadrupoles are more appropriate. A quadrupole can always be separated into two longitudinal quadrupoles. Longitudinal quadrupole is the basic type of quadrupoles.

Multipole Expansions

Dipole is composed of two monopoles with equal strength but opposite signs, while a quadrupole is composed of four monopoles with equal strength but opposite signs. Monopole, dipole, quadrupole, etc., are the basic types of sound sources. If a source region is compact compared to the sound wavelengths, any source can be represented by the sum of these basic sources.

Let's begin with the sound field from two monopoles: one with strength $\rho_0\dot{Q}_1(t)$ at $z=h/2$ and the other with $\rho_0\dot{Q}_2(t)$ at $z=-h/2$. According to (34), the total sound pressure is:

$$\begin{aligned}p(\vec{r},t)&=\frac{\rho_0}{4\pi}\left[\frac{1}{r_1}\dot{Q}_1(t-r_1/a_0)+\frac{1}{r_2}\dot{Q}_2(t-r_2/a_0)\right]\\&=\frac{\rho_0}{4\pi r}\left[\dot{Q}_1(t-r/a_0)+\dot{Q}_2(t-r/a_0)\right]\\&\quad-h\cos\theta\frac{d}{dr}\left[\frac{\rho_0}{4\pi r}\frac{1}{2}(\dot{Q}_1(t-r/a_0)-\dot{Q}_2(t-r/a_0))\right]+O(h^2)\end{aligned} \qquad . \qquad (59)$$

To order of $O(h^2)$, the acoustic field is generated by a monopole with strength $\rho_0[\dot{Q}_1(t)+\dot{Q}_2(t)]$ at their geometric center, and a dipole with strength $h\rho_0[\dot{Q}_1(t)-\dot{Q}_2(t)]/2$. According to (37), the inhomogeneous equation is:

$$\begin{aligned}\frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2}-\nabla^2 p&=\rho_0\dot{Q}_1(t)\delta(\vec{r}-\vec{h}/2)+\rho_0\dot{Q}_2(t)\delta(\vec{r}+\vec{h}/2)\\&=\rho_0[\dot{Q}_1(t)+\dot{Q}_2(t)]\delta(\vec{r})-\rho_0[\dot{Q}_1(t)-\dot{Q}_2(t)]/2\vec{h}\cdot\nabla\delta(\vec{r})+O(h^2)\end{aligned} \qquad . \qquad (60)$$

Suppose there are N monopoles with strength $\rho_0 \dot{Q}_j(t)$ at $\vec{r}_j$, $j = 1,2,\cdots,N$. The total sound pressure is:

$$p(\vec{r},t) = \frac{\rho_0}{4\pi}\sum_{j=1}^{N}\frac{1}{R_j}\dot{Q}_j(t - R_j / a_0)\,,\; R_j = \left|\vec{r} - \vec{r}_j\right|. \tag{61}$$

Assume the source region is small in all dimensions compared with the sound wave length, and $\vec{r}_0$ is the geometric center of the source region. The sound pressure from the *j*-th monopole can be expanded about $\vec{r}_0$:

$$\begin{aligned} p_j(\vec{r},t) &= \frac{\rho_0}{4\pi\left|\vec{R}_0 - \vec{r}_j + \vec{r}_0\right|}\dot{Q}_j\left(t - \frac{\left|\vec{R}_0 - \vec{r}_j + \vec{r}_0\right|}{a_0}\right) \\ &= \left\{1 - \left(\vec{r}_j - \vec{r}_0\right)\cdot\nabla + \frac{1}{2!}\left[\left(\vec{r}_j - \vec{r}_0\right)\cdot\nabla\right]^2 + \cdots\right\}\left[\frac{\rho_0}{4\pi R_0}\dot{Q}_j\left(t - \frac{R_0}{a_0}\right)\right] \\ R_0 &= \left|\vec{r} - \vec{r}_0\right|. \end{aligned} \tag{62}$$

Therefore the total sound pressure from all the monopoles is:

$$\begin{aligned} p(\vec{r},t) = {} & \frac{\rho_0}{4\pi R_0}\sum_{j=1}^{N}\dot{Q}_j\left(t - R_0 / a_0\right) \\ & - \sum_{j=1}^{N}\left(\vec{r}_j - \vec{r}_0\right)\cdot\nabla\left[\frac{\rho_0}{4\pi R_0}\dot{Q}_j\left(t - R_0 / a_0\right)\right] \\ & + \sum_{j=1}^{N}\frac{1}{2!}\left[\left(\vec{r}_j - \vec{r}_0\right)\cdot\nabla\right]^2\left[\frac{\rho_0}{4\pi R_0}\dot{Q}_j\left(t - R_0 / a_0\right)\right] \\ & + \cdots \end{aligned} \tag{63}$$

To the zero-th order, the N monopoles can be represented by one monopole with the total strength of all the monopoles. The dipoles seem not having an equivalent dipole as in (60), where $\vec{r}_0$ can be chosen in the middle of the two monopoles. If all the monopoles oscillate at the same frequency ω, $\dot{Q}_j(t) = \mathrm{Real}(\hat{Q}_j e^{i\omega t})$, then,

$$
\begin{aligned}
p(\vec{r},t) = & \left(\sum_{j=1}^{N}\hat{Q}_j\right)\frac{\rho_0}{4\pi R_0}e^{i\omega(t-R_0/a_0)} \\
& -\left[\sum_{j=1}^{N}\hat{Q}_j\left(\vec{r}_j-\vec{r}_0\right)\right]\cdot\nabla\left[\frac{\rho_0}{4\pi R_0}e^{i\omega(t-R_0/a_0)}\right] \\
& +\left[\sum_{j=1}^{N}\frac{1}{2!}\left(\hat{Q}_j\left(\vec{r}_j-\vec{r}_0\right)\cdot\nabla\right)^2\right]\left[\frac{\rho_0}{4\pi R_0}e^{i\omega(t-R_0/a_0)}\right] \\
& +\cdots
\end{aligned}
\quad . \tag{64}
$$

Then the sound sources can be represented by one monopole, one dipole, and one quadrupole, etc. The equivalent monopole has the strength of the sum of all the monopoles. The dipole is in a direction of the vector sum of all the monopoles.

Green's Function in Three Dimensional Open Space with No Flow

We have discussed the monopole and its high order derivatives. There are three types of sound sources in a flow: volume fluctuation, force, and viscous stress oscillation. They can be respectively modeled as monopoles, dipoles, and quadrupoles. Since high order poles can be constructed by monopoles, it is essential to solve the acoustic field of a monopole.

The solution to the acoustic wave equation of a monopole with unit strength is called the Green's function. Setting source strength per unit volume $\rho_0\dot{Q}(t)=\delta(t-\tau)$, a pulse with unit strength emitting sound at τ, in the inhomogeneous acoustics equation (37) and its solution (34), we have:

$$
\frac{1}{a_0^2}\frac{\partial^2 G}{\partial t^2}-\nabla_{\vec{x}}^2 G=\delta(t-\tau)\delta(\vec{x}-\vec{y}), \tag{65}
$$

$$
G(\vec{x},t\mid\vec{y},\tau)=\frac{1}{4\pi|\vec{x}-\vec{y}|}\delta\left(t-|\vec{x}-\vec{y}|/a_0-\tau\right). \tag{66}
$$

$G(\vec{x},t\mid\vec{y},\tau)$ is the Green's function in a stationary medium in the open space. It represents the sound at the observation point $\vec{x}$ and time t generated by a pulse at source point $\vec{y}$ released at time τ. The Green's function in Eq.(66) is in Cartesian coordinates. Green's functions in cylindrical coordinates and in spherical coordinates are also useful in applications and will be discussed later. Operator $\nabla_{\vec{x}}^2$ has a subscript $\vec{x}$ to explicitly indicate that the left side of the equation is operated on the observation coordinate $\vec{x}$. The Green's function is often written as $G(\vec{x},t\mid\vec{y},\tau)$ to explicitly indicate the field (observer) coordinates and time and the source coordinates and emitting time. This is very important in the following context.

As $a_0 \to \infty$, one obtains the three-dimensional Laplacian equation and its Green's function:

$$\nabla^2 G = -\delta(\vec{x} - \vec{y}), \tag{67}$$

$$G(\vec{x} \mid \vec{y}) = \frac{1}{4\pi|\vec{x} - \vec{y}|}. \tag{68}$$

If the medium moves at constant velocity $\vec{u}_0$, G satisfies the more general inhomogeneous wave equation:

$$\frac{1}{a_0^2}\frac{D_0}{Dt}\left(\frac{D_0 G}{Dt}\right) - \nabla_{\vec{x}}^2 G = \delta(\vec{x} - \vec{y})\delta(t - \tau). \qquad (69)(70)$$

Obviously the Green's function satisfying (70) is different from (66).

The Green's function has three important properties (Rienstra2006 for proof):

(1)Causality: there is no sound before the source energy is released:

$$G(\vec{x}, t \mid \vec{y}, \tau) = 0, \ \frac{D_0 G(\vec{x}, t \mid \vec{y}, \tau)}{Dt} = 0, \text{ when } t < \tau. \tag{71}$$

(2)Reciprocity:

$$G(\vec{x}, t \mid \vec{y}, \tau) = G(\vec{y}, -\tau \mid \vec{x}, -t). \tag{72}$$

In this particular case (uniform medium), the Green function and its adjoint are the same function. It is self-adjoint, or symmetric. In general, the adjoint Green function is different from its original:

$$G_a(\vec{y}, -\tau \mid \vec{x}, -t) \neq G(\vec{x}, t \mid \vec{y}, \tau), \tag{73}$$

such as in the shear flow in Tam& Auriault1988.

(3) $G(\vec{x}, t \mid \vec{y}, \tau)$ also satisfies:

$$\frac{1}{a_0^2}\frac{D_0}{D\tau}\left(\frac{D_0 G}{D\tau}\right) - \nabla_{\vec{y}}^2 G = \delta(\vec{x} - \vec{y})\delta(t - \tau). \tag{74}$$

(4) $G(\vec{x}, t \mid \vec{y}, \tau)$ is singular with order $O(r^{-1})$ at the source point.

<u>Integral Representation of Acoustic Equation: Kirchhoff's Formula</u>

In the previous sections we have discussed the simple sources and the Green's functions. The sound field from continuously distributed monopoles was briefly mentioned in Eq.(38) and (39). Here we will study distributed sources in more details.

Suppose we have a continuously distributed source field $q(\vec{x},t)$ in the moving medium. The inhomogeneous acoustic wave equation is [ref.(13)&(38)]:

$$\frac{1}{a_0^2}\frac{D_0}{Dt}\left(\frac{D_0 p}{Dt}\right)-\nabla_{\vec{x}}^2 p = q(\vec{x},t)\text{, or,} \tag{75}$$

$$\frac{1}{a_0^2}\frac{D_0}{D\tau}\left(\frac{D_0 p}{D\tau}\right)-\nabla_{\vec{y}}^2 p = q(\vec{y},\tau)\,. \tag{76}$$

With the Green's function, the partial differential acoustic wave equation can be transformed into an integral acoustic wave equation. The integration will be operated on source coordinate $\vec{y}$ and time τ. The Green's function satisfying (74) instead of Eq.(70), will be used.

As in Fig.1, control volume $V(t)$ is bounded by surfaces $S(t)$, including inner surface $S_i(t)$ and outer surface $S_o(t)$. The surfaces can be arbitrary, where no boundary conditions are enforced. Subtract Eq.(74) multiplied by p from Eq.(76) multiplied by G, and integrate this equation over volume V from time t_0 to t+ (t+ means inclusive of t):

$$\begin{aligned}&\int_{t_0}^{t+}\iiint_{V(\tau)}\left\{\frac{1}{a_0^2}\left[p\frac{D_0}{D\tau}\left(\frac{D_0 G}{D\tau}\right)-G\frac{D_0}{D\tau}\left(\frac{D_0 p}{D\tau}\right)\right]-\left(p\nabla_{\vec{y}}^2 G-G\nabla_{\vec{y}}^2 p\right)\right\}d\vec{y}d\tau\\ &=\int_{t_0}^{t+}\iiint_{V(\tau)}\{p(\vec{y},\tau)\delta(\vec{x}-\vec{y})\delta(t-\tau)-G(\vec{x},t\,|\,\vec{y},\tau)q(\vec{y},\tau)\}d\vec{y}d\tau\end{aligned}\ . \tag{77}$$

According to Green's Second Identity:

$$\int_{t_0}^{t+}\iiint_{V(\tau)}\left(p\nabla_{\vec{y}}^2 G-G\nabla_{\vec{y}}^2 p\right)d\vec{y}d\tau=\int_{t_0}^{t+}\iint_{S(\tau)}\left(p\nabla_{\vec{y}}G-G\nabla_{\vec{y}}p\right)\cdot d\vec{S}d\tau\ . \tag{78}$$

Sound pressure p was chosen as the acoustic variable in Eqs.(75)&(76). p is a single-valued function of position no matter if the region is singly-connected or multiply-connected. If velocity potential ϕ is chosen as the acoustic variable, one concern is that ϕ is single-valued only in a singly-connected region. If the region is multiply-connected, artificial barrier(s) S_b should be inserted to make the region singly-connected as in Fig.(2), and the surface integration in (78) should include S_b. However, the normal direction on the two sides of the surface have opposite directions; the integration on the barrier is zero. Therefore equation (78) applies for ϕ as well.

To integrate $\int_{t_0}^{t+}\iiint_{V(\tau)}\frac{1}{a_0^2}\left[p\frac{D_0}{D\tau}\left(\frac{D_0 G}{D\tau}\right)-G\frac{D_0}{D\tau}\left(\frac{D_0 p}{D\tau}\right)\right]d\vec{y}d\tau$ in (77), we begin with the simplest case.

Stationary Control Volume in Stationary Medium

Consider a stationary control volume V (relative to the observer) without mean flow, *i.e.*, $\frac{D_0}{D\tau}\left(\frac{D_0}{D\tau}\right)=\frac{\partial^2}{\partial\tau^2}$. Since V doesn't vary with time, the integration order can be exchanged:

$$\begin{aligned}\int_{t_0}^{t+}\iiint_V\left(p\frac{\partial^2 G}{\partial\tau^2}-G\frac{\partial^2 p}{\partial\tau^2}\right)d\vec{y}d\tau &= \iiint_V\int_{t_0}^{t+}\left(p\frac{\partial^2 G}{\partial\tau^2}-G\frac{\partial^2 p}{\partial\tau^2}\right)d\tau d\vec{y} \\ &= \iiint_V\left(G\frac{\partial p}{\partial\tau}-p\frac{\partial G}{\partial\tau}\right)d\vec{y}\Bigg|_{\tau=t_0}\end{aligned}.$$

The integration at $t+$ is zero, required by the causality of the Green's function (71). Then we obtain the famous Kirchhoff's formula in stationary volume V without mean flow:

$$\begin{aligned}&\int_{t_0}^{t}\iiint_V G(\vec{x},t\,|\,\vec{y},\tau)q(\vec{y},\tau)d\vec{y}d\tau \\ &+\int_{t_0}^{t}\iint_S\left(G\nabla_{\vec{y}}p-p\nabla_{\vec{y}}G\right)\cdot d\vec{S}d\tau \\ &-\frac{1}{a_0^2}\iiint_V\left(p\frac{\partial G}{\partial\tau}-G\frac{\partial p}{\partial\tau}\right)d\vec{y}\Bigg|_{\tau=t_0} \\ &=\begin{cases}p(\vec{x},t), & \text{if } \vec{x} \text{ is in volume } V \text{ and not at the surfaces;}\\ 0, & \text{if } \vec{x} \text{ is outside of volume } V.\end{cases}\end{aligned} \qquad (79)$$

The *Kirchhoff's formula,* also called the *Kirchhoff-Helmholtz Integral Theorem,* was first published in 1882. Although the formula is presented on sound pressure, it holds for any other acoustic variables. The first term on the left side is the sound field from the distributed sources in the volume as if there were no surfaces S_i and S_o. In this case Green's function (66) applies, and the Kirchhoff's formula reduces to (39) that was based on the principal of superposition. The second term on the left side of (79) represents the effect of the (physical boundary) surfaces on the sound field (refraction, reflection, etc.), or the contribution of the sources from outside of the surfaces. The last term on the left side is the effect of the initial condition, which is zero if no sound exists initially. If $\vec{x}$ is on the surface, the volume integral about the Delta function in (77) is undefined. A limit analysis on (79) will be performed to derive a formula applicable for $\vec{x}$ on the surface.

The partial differential equation (76) describes the relationship of a source and the sound near it. On the other hand, Kirchhoff's formula (79) connects the source region to its far field sound field. It incorporates in a single equation the effects of the sources, the physical boundary or arbitrary permeable surfaces, and the initial conditions. Generally the Kirchhoff's formula is not a solution. There are unknowns (p and $\partial p / \partial \tau$) in the volume integral and in the surface integral in (77). According to the solution uniqueness theorem, p and $\partial p / \partial \tau$ cannot be prescribed independently on the surfaces. One must be solved when the other is prescribed.

Stationary Control Volume with Uniform Mean Flow

There are two ways to include a uniform mean flow $\vec{u}_0$ into the Kirchhoff's formula. The *first* is to follow the same procedure as for (79) (Goldstein1976). Note

$$\begin{aligned} & p\frac{D_0}{D\tau}\left(\frac{D_0G}{D\tau}\right)-G\frac{D_0}{D\tau}\left(\frac{D_0p}{D\tau}\right)=\frac{D_0}{D\tau}\left(p\frac{D_0G}{D\tau}-G\frac{D_0p}{D\tau}\right) \\ & =\frac{\partial}{\partial\tau}\left(p\frac{D_0G}{D\tau}-G\frac{D_0p}{D\tau}\right)+\nabla\cdot\left[\vec{u}_0\left(p\frac{D_0G}{D\tau}-G\frac{D_0p}{D\tau}\right)\right] \end{aligned}.$$

We have the Kirchhoff's formula for a stationary volume in a uniform mean flow:

$$\begin{aligned} & \int_{t_0}^{t}\iiint_V G(\vec{x},t\mid\vec{y},\tau)q(\vec{y},\tau)d\vec{y}d\tau \\ & +\int_{t_0}^{t}\iint_{S_i+S_o}\left[G\left(\nabla_{\vec{y}}p-\vec{u}_0\frac{1}{a_0^2}\frac{D_0p}{D\tau}\right)-p\left(\nabla_{\vec{y}}G-\vec{u}_0\frac{1}{a_0^2}\frac{D_0G}{D\tau}\right)\right]\cdot d\vec{S}d\tau \\ & -\frac{1}{a_0^2}\iiint_V\left(p\frac{D_0G}{D\tau}-G\frac{D_0p}{D\tau}\right)d\vec{y}\bigg|_{\tau=t_0} \\ & =\begin{cases} p(\vec{x},t), & \text{if } \vec{x} \text{ is in the volume } V \text{ and not at the surfaces;} \\ 0, & \text{if } \vec{x} \text{ is not in the volume } V. \end{cases} \end{aligned} \quad (80)$$

Note the extra terms in the surface integral compared to the no-flow Kirchhoff formula Eq.(79).

The *second* method is to make use of the *linear transformations* of the coordinates. Without loss of generality, we establish a coordinate with the x_1-axis parallel to the mean flow, then wave equation Eq.(75) is:

$$\frac{1}{a_0^2}\left(\frac{\partial^2}{\partial t^2}+2u_{01}\frac{\partial^2}{\partial t\partial x_1}+u_{01}^2\frac{\partial^2}{\partial x_1^2}\right)p-\nabla^2p=q(\vec{x},t). \quad (81)$$

The simplest linear transformation is the *Galilean transformation*:

$$\tilde{t}=t,\ \tilde{x}_1=x_1-u_0t,\ \tilde{x}_2=x_2,\ \tilde{x}_3=x_3. \tag{82}$$

With (82), (81) can be transformed to an equation in a stationary flow. All the analyses for the stationary medium apply. However, the property of the source/observer changes after the transformation. A stationary source in the coordinate system moves in the other coordinate system. The *Lorentz-type transformation* can be used to avoid it:

$$\tilde{x}_1=\gamma x_1,\ \tilde{x}_2=x_2,\ \tilde{x}_3=x_3, \text{ and } \tilde{t}=t/\gamma+\gamma u_0 x_1/a_0^2, \tag{83}$$

where $\gamma=1/\sqrt{1-M^2}$, $M=u_0/a_0$. In this system, the dimension in $\tilde{x}_1$ direction is dilated in a subsonic flow, while the time is compressed and shifted. The amount of the time shift varies with x_1. With this transformation, acoustic equation (75) and the Green's function equation become:

$$\frac{1}{a_0^2}\frac{\partial^2 p}{\partial\tilde{t}^2}-\tilde{\nabla}^2 p=q(\tilde{x}_1/\gamma,\tilde{x}_2,\tilde{x}_3,\gamma(\tilde{t}-\gamma u_0x_1/a)), \tag{84}$$

$$\frac{1}{a_0^2}\frac{\partial^2 G}{\partial\tilde{t}^2}-\tilde{\nabla}^2 G=\delta(\tilde{\vec{x}}-\tilde{\vec{y}})\delta(\tilde{t}-\tilde{\tau}). \tag{85}$$

[Note: $\delta(\tilde{x}/\gamma)=\gamma\delta(\tilde{x})$.] Acoustic equation (84) has the same form as in a stationary medium, and the property of the source/observer doesn't change. Eq.(84) is now widely used in applications. (Morino)

Transformation (83) is also called the *Prandtl-Glauert Transformation*, since the acoustic equation is reduced to the *Prandtl-Glauert* equation for compressible steady flows:

$$\left(1-M^2\right)\frac{\partial^2 p}{\partial x_1^2}+\frac{\partial^2 p}{\partial x_2^2}+\frac{\partial^2 p}{\partial x_3^2}=0.$$

Sometimes it is referred to as the *Karman-Tsian Transformation*.

Moving Control Volume with Uniform Mean Flow

Suppose control surfaces $S(t)$ moves with time. According to the *Leibniz's Rule*,

$$\frac{d}{d\tau}\left[\iiint\limits_{V(t)}\left(p\frac{D_0G}{D\tau}-G\frac{D_0p}{D\tau}\right)d\vec{y}\right]=\iiint\limits_{V(t)}\frac{\partial}{\partial\tau}\left(p\frac{D_0G}{D\tau}-G\frac{D_0p}{D\tau}\right)d\vec{y}+\iint\limits_{S(t)}\left(p\frac{D_0G}{D\tau}-G\frac{D_0p}{D\tau}\right)\vec{V}_s\cdot d\vec{S}$$

where $\vec{V}_s$ is the moving velocity of the surface. Then we have

$$\int_{t_0}^{t+}\iiint_{V(t)}\left[p\frac{D_0}{D\tau}\left(\frac{D_0 G}{D\tau}\right)-G\frac{D_0}{D\tau}\left(\frac{D_0 p}{D\tau}\right)\right]d\vec{y}d\tau$$

$$=\int_{t_0}^{t+}\iiint_{V(t)}\frac{\partial}{\partial\tau}\left(p\frac{D_0 G}{D\tau}-G\frac{D_0 p}{D\tau}\right)d\vec{y}d\tau+\int_{t_0}^{t+}\iiint_{V(t)}\nabla\cdot\left[\vec{u}_0\left(p\frac{D_0 G}{D\tau}-G\frac{D_0 p}{D\tau}\right)\right]d\vec{y}d\tau$$

$$=\int_{t_0}^{t+}\frac{d}{d\tau}\left[\iiint_{V(t)}\left(p\frac{D_0 G}{D\tau}-G\frac{D_0 p}{D\tau}\right)d\vec{y}\right]d\tau-\int_{t_0}^{t+}\iint_{S(t)}\left(p\frac{D_0 G}{D\tau}-G\frac{D_0 p}{D\tau}\right)\vec{V}_s\cdot d\vec{S}d\tau$$

$$+\int_{t_0}^{t+}\iint_{S(t)}\left(p\frac{D_0 G}{D\tau}-G\frac{D_0 p}{D\tau}\right)\vec{u}_0\cdot d\vec{S}d\tau$$

$$=-\iiint_{V(t)}\left(p\frac{D_0 G}{D\tau}-G\frac{D_0 p}{D\tau}\right)d\vec{y}\Bigg|_{\tau=t_0}+\int_{t_0}^{t+}\iint_{S(t)}\left(p\frac{D_0 G}{D\tau}-G\frac{D_0 p}{D\tau}\right)\left(\vec{u}_0-\vec{V}_s\right)\cdot d\vec{S}d\tau$$

The Kirchhoff's formula for moving surfaces in a mean flow is

$$\begin{aligned}&\int_{t_0}^{t}\iiint_{V(t)}G(\vec{x},t\mid\vec{y},\tau)q(\vec{y},\tau)d\vec{y}d\tau\\&+\int_{t_0}^{t}\iint_{S(t)}\left[G\left(\nabla_{\vec{y}}p-\left(\vec{u}_0-\vec{V}_s\right)\frac{1}{a_0^2}\frac{D_0 p}{D\tau}\right)-p\left(\nabla_{\vec{y}}G-\left(\vec{u}_0-\vec{V}_s\right)\frac{1}{a_0^2}\frac{D_0 G}{D\tau}\right)\right]\cdot d\vec{S}d\tau\\&-\frac{1}{a_0^2}\iiint_{V(t)}\left(p\frac{D_0 G}{D\tau}-G\frac{D_0 p}{D\tau}\right)d\vec{y}\Bigg|_{\tau=t_0}\\&=\begin{cases}p(\vec{x},t), & \text{if } \vec{x} \text{ is in the volume } V \text{ and not at the surfaces;}\\ 0, & \text{if } \vec{x} \text{ is not in the volume } V.\end{cases}\end{aligned}\tag{86}$$

This formula reduces to (79) or (80) without mean flow/moving surfaces. It is for general sources. To apply it for sources from flow in aeroacoustics, modifications/derivations are needed, such as Goldstein's version, and the FW-H equation.

Applications of Kirchhoff's Formulas

As we have mentioned, the Kirchhoff's formulas are the integral representations of the differential acoustic equations. Generally they are not the solutions. There is unknown variable in the surface integral of (79), (80) and (86): the surface integration is coupled with the sound field. The volume integration may also include the unknown variable if there is strong interaction between the source and the sound.

If there is no object in a stationary medium and the sound has no back effect on its source $q(\vec{y},\tau)$, (79) reduces to:

$$p(\vec{x},t)=\int_{t_0}^{t}\iiint_{V}G(\vec{x},t\mid\vec{y},\tau)q(\vec{y},\tau)d\vec{y}d\tau\,.\tag{87}$$

In this case the Kirchhoff's formula is the solution. The sound field can be calculated by this formula using the open-space Green's function (66) in the Cartesian coordinates in the open space. An example is the jet noise prediction. Noise from turbulence is modeled by the Lighthill Acoustic Analogy. The Lighthill stress is assumed known and the sound it generates assumes no back effect.

In some situations, the coupling of the source and its generated sound on the surfaces can be eliminated. If the surface is rigid, the non-penetration boundary condition ($\partial p / \partial n = 0$) must be satisfied. Then

$$\int_{t_0}^{t} \iint_{S} G \nabla_{\vec{y}} p \cdot d\vec{S} d\tau = 0 .$$

If the Green's function also satisfies the non-penetration boundary condition at the surface ($\partial G / \partial n = 0$), then

$$\int_{t_0}^{t} \iint_{S_i + S_o} p \nabla_{\vec{y}} G \cdot d\vec{S} d\tau = 0 .$$

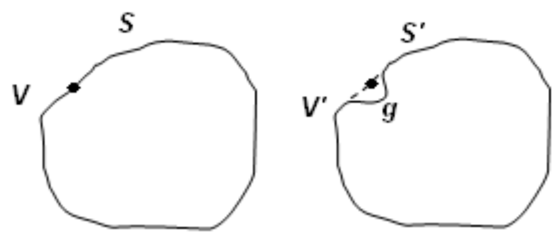
Eq. (79) reduces to Eq.(87): the surface coupling is eliminated and Eq.(87) is the solution. But the Green's function in Eq.(87) must satisfy the boundary condition at the surface. Check out Morse&Ingard1968, p. 500 for how to develop the Green's function in a duct using the normal modes. Check out Howe1998: p.61, p.65 for the compact Green's function which gives the leading order terms for the sound produced by sources near a solid body.

In general, it is difficult to find a Green's function satisfying the boundary condition at the surfaces. If there is no volume source in a stationary medium, Eq.(79) reduces to:

$$p(\vec{x}, t) = \int_{t_0}^{t} \iint_{S} \left(G \nabla_{\vec{y}} p - p \nabla_{\vec{y}} G \right) \cdot d\vec{S} d\tau . \tag{88}$$

The surface has two possible effects on the sound field. One is the diffraction: the sound propagates towards the surface and is diffracted. The other is the object surface acts as a sound source. No matter which type of effects, given p, $\partial p / \partial n$ (and $\partial p / \partial t$ when there is mean flow) on the body surface, the sound field can be calculated from Eq.(88). However, according to the solution uniqueness theorem, to render a unique solution, only one of p and $\partial p / \partial n$, or a linear combination of the two, can be prescribed on any part of the surface. They cannot be prescribed simultaneously. When one is prescribed, the other has to be solved. Therefore either p or $\partial p / \partial n$ on the surface is an unknown and needs to be solved.

To solve the integral equation (88), let's put the observation point $\vec{x}$ on the surface as in Fig.4a. When the source point approaches the observation point $\vec{x}$, the Green's function is singular. We will show the integral remains finite and is the sum of the principal value of the integration and $p(\vec{x},t)/2$.



a b

Fig.4, Observer on the surface.

First we deform the surface near the observer as in Fig.4b. The surface near the observer is pushed into the object to form two new surfaces: *S'* and *g*.(Ang2007) *S'* is the original surface *S* excluding a small round surface with radius ε. *g* is a half sphere surface with radius ε. The observer is in the new volume *V'* so Eq.(88) applies:

$$p(\vec{x},t) = \int_{t_0}^{t} \iint_{S'} \left(G\nabla_{\vec{y}} p - p\nabla_{\vec{y}} G\right)\cdot d\vec{S}d\tau + \int_{t_0}^{t} \iint_{g} \left(G\nabla_{\vec{y}} p - p\nabla_{\vec{y}} G\right)\cdot d\vec{S}d\tau . \qquad (89)$$

Applying the free-space Green's function (66), the second integral in (89) is:

$$\int_{t_0}^{t} \iint_{g} \left(G\nabla_{\vec{y}} p - p\nabla_{\vec{y}} G\right)\cdot d\vec{S}d\tau = \frac{1}{4\pi}\int_{t_0}^{t} \iint_{g} \frac{1}{\varepsilon}\delta\left(t-\varepsilon/a_0-\tau\right)\nabla_{\vec{y}} p\cdot d\vec{S}d\tau$$

$$-\frac{1}{4\pi}\int_{t_0}^{t} \iint_{g} p\nabla_{\vec{y}}\left(\frac{1}{\varepsilon}\delta\left(t-\varepsilon/a_0-\tau\right)\right)\cdot d\vec{S}d\tau$$

$$= -\frac{1}{4\pi}\iint_{g}\frac{1}{\varepsilon}\left[\frac{\partial p}{\partial n}\right]_{t-\varepsilon/a_0} dS - \frac{1}{4\pi}\int_{t_0}^{t}\iint_{g} p\delta\left(t-\varepsilon/a_0-\tau\right)\frac{\vec{x}-\vec{y}}{\varepsilon^3}\cdot d\vec{S}d\tau$$

$$-\frac{1}{4\pi}\int_{t_0}^{t}\iint_{g} p\frac{1}{\varepsilon}\nabla_{\vec{y}}\delta\left(t-\varepsilon/a_0-\tau\right)\cdot d\vec{S}d\tau$$

$$= -\frac{1}{4\pi}\varepsilon\int_0^{\pi}\int_0^{\pi}\left[\frac{\partial p}{\partial n}\right]_{t-\varepsilon/a_0}\sin\theta d\theta d\phi + \frac{1}{4\pi}\int_0^{\pi}\int_0^{\pi}[p]_{t-\varepsilon/a_0}\sin\theta d\theta d\phi$$

$$-\frac{1}{4\pi a_0}\varepsilon\iint_{g}\left[\frac{\partial p}{\partial t}\right]_{t-\varepsilon/a_0}\sin\theta d\theta d\phi$$

Suppose the observer lies on a smooth part of *S* where $\partial p/\partial n$ is smooth. As $\varepsilon\to 0$,

$$\int_{t_0}^{t}\iint_{g}\left(G\nabla_{\vec{y}}p - p\nabla_{\vec{y}}G\right)\cdot d\vec{S}d\tau \to \frac{1}{2}p\,.$$

Plugging it into (89), we have,

$$p(\vec{x},t) \to \int_{t_0}^{t}\iint_{S'}\left(G\nabla_{\vec{y}}p - p\nabla_{\vec{y}}G\right)\cdot d\vec{S}d\tau + \frac{1}{2}p(\vec{x},t) \text{ as } \varepsilon \to 0\,. \tag{90}$$

The integral is over the original surface S excluding the singular point (the observer). It is called the principal integral. This shows the integral over S is the sum of its principal value and $p(\vec{x},t)/2$. Eq.(90) is then simply:

$$\frac{1}{2}p(\vec{x},t) = \int_{t_0}^{t}\iint_{S'}\left(G\nabla_{\vec{y}}p - p\nabla_{\vec{y}}G\right)\cdot d\vec{S}d\tau\,,\ \vec{x} \text{ on } S. \tag{91}$$

This is the boundary surface integral equation on sound pressure. (Crighton, et.al.1992, pp287-291, Pierce1989, p.182) After it is solved, the sound field can be computed from (88). The surface can be discretized so Eq.(91) is solved numerically. This is the boundary element method (BEM). (Long-BEM.pdf, BEM2005.pdf)

If the inner surface is smooth, a model without circulation applies. In this model, the lift on the body is zero. If the surface has sharp edges, such as in an airfoil, the model without circulation will give singularity at sharp edges. In this case, the Kutta condition must be enforced to remove the singularity, which is equivalent to add circulation and lift to the medium and the body.

Other methods include CFD or CAA methods. One example is the sound from a jet. An artificial surface can be put around the jet, far enough so that the linear acoustic equation holds outside the surface. The flow field within the surface is known from other methods, such as experiment or numerical solutions.

Sound Field from Moving Sources

In this section we will use Kirchhoff's formula (87) to investigate the effect of moving sources in an open space. The Green's function for a stationary source in a stationary medium in the Cartesian system, (66), is used in the analysis. A stationary source oscillates but its time averaged position $\vec{y}$ doesn't move.

Sound Field from Moving Sources with Constant Velocity

The simplest case is all the sources move with constant velocity $\vec{U}_0$. Establish a frame $\vec{\eta}$ moving with the sources. The moving frame is chosen to coincide with the fixed frame $\vec{y}$ at $\tau = 0$: $\vec{y} = \vec{\eta}$. At time τ the coordinate of the source position in the fixed frame is

$$\vec{y} = \vec{\eta} + \vec{U}_0 \tau . \tag{92}$$

Suppose the strength of the source observed in the moving frame is $A(\vec{\eta}, \tau)$. To facilitate the explanation, we assume all the sources oscillate at the same angular frequency ω_s (constant-frequency assumption), while their strengths vary with position, *i.e.*,

$$A(\vec{\eta}, \tau) = \mathrm{Real}\{\hat{A}(\vec{\eta}) e^{i\omega_s \tau}\} . \tag{93}$$

To apply Green's function (66), the source must be defined in the fixed frame:

$$q(\vec{y}, \tau) = A(\vec{y} - \vec{U}_0 \tau, \tau) = \hat{A}(\vec{y} - \vec{U}_0 \tau) e^{i\omega_s \tau} . \tag{94}$$

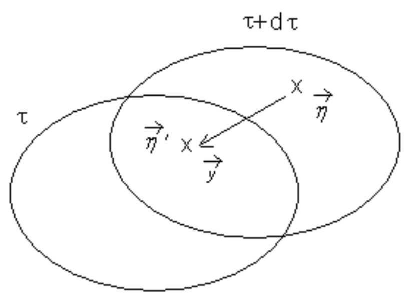


Fig.5, Effect of a moving source.

Examining the source in Eq.(94) at $\vec{y}$ in the fixed frame, one may find it not only oscillates at ω_s as in $e^{i\omega_s \tau}$, but its amplitude also varies as in $\hat{A}(\vec{y} - \vec{U}_0 \tau)$. Expression (94) formally separates the two effects of the source: the oscillation and the convection. The physical meaning can be explained by Fig.5. At time τ the source is in the elliptic region shown in the figure. Let's observe the point marked by $\times$. Its coordinates in the fixed and moving frames are $\vec{y}$ and $\vec{\eta}$ respectively. Short time $d\tau$ later, the source region and the marked point move by distance $\vec{U}_0 d\tau$ to a new position. But the fixed observation point at $\vec{y}$ in the moving frame moves by distance $-\vec{U}_0 d\tau$. At $\vec{y}$ the amplitude of the source is $\hat{A}(\vec{y} - \vec{U}_0 \tau)$. It varies only when $\hat{A}(\vec{\eta})$ is not uniform. The sound source in the fixed frame is time dependent even it is steady ($\omega_s = 0$) in the moving frame when it is nonuniform.

To generate the sound, the source strength in the fixed frame must vary with time. It is useful to investigate the time variation rate of the source at a fixed point. The source strength observed at $\vec{y}$ not only varies with τ as the source oscillates, but also varies due to the source motion as shown by the first argument ($\vec{y}-\vec{U}_0\tau$) of A in Eq.(94), therefore,

$$\left.\frac{\partial q}{\partial \tau}\right|_{\vec{y}} = \left.\frac{\partial A}{\partial \tau}\right|_{\vec{\eta}} + \left.\frac{\partial A}{\partial \eta_j}\right|_{\tau} \left.\frac{\partial \eta_j}{\partial \tau}\right|_{\vec{y}} = \left.\frac{\partial A}{\partial \tau}\right|_{\vec{\eta}} - U_{0j} \left.\frac{\partial A}{\partial \eta_j}\right|_{\tau} . \tag{95}$$

The time changing rates of a quantity in the two frames are different: $\left.\frac{\partial}{\partial t}\right|_{\vec{y}} \neq \left.\frac{\partial}{\partial t}\right|_{\vec{\eta}}$. Conventionally we denote the time rate $\left.\frac{\partial}{\partial \tau}\right|_{\vec{y}}$ observed in the fixed frame as $\frac{\partial}{\partial t}$, and the time rate $\left.\frac{\partial}{\partial \tau}\right|_{\vec{\eta}}$ observed in the moving frame as $\frac{\partial}{\partial \tau}$. Then,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - U_{0j} \frac{\partial}{\partial \eta_j} . \tag{96}$$

$\frac{\partial}{\partial \tau}$ represents the source oscillation. $-U_{0j}\frac{\partial}{\partial \eta_j}$ is the source motion effect. The difference between $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \tau}$ is the source motion effect. It can be neglected only when all of the three conditions are satisfied: (1)source oscillation ($\frac{\partial}{\partial \tau}$) is not small; (2)the source moves slowly ($\left|\vec{U}_0\right| << 1$); and (3)the source strength is fairly uniform ($\left|\partial A/\partial \eta_j\right| << 1$). If a source is steady in the moving frame, such as the steady force on a rotating blade, the first condition is not met. Since the source is on the blade surfaces, its spatial distribution is highly nonuniform, therefore neither the 3rd condition is satisfied. A steady force generates no sound if it doesn't move. The moving effect is the only sound generation mechanism in this case. It shouldn't be ignored under any circumstances in turbomachinery noise analyses.

The effect of the moving source is now examined. Substituting source strength (94) and Green function (66) into the Kirchhoff's formula (87), we have:

$$p(\vec{x},t) = \frac{1}{4\pi}\int_{t_0}^{t}\iiint_V \frac{1}{|\vec{x}-\vec{y}|} A(\vec{y}-\vec{U}_0\tau,\tau)\delta\left(t-|\vec{x}-\vec{y}|/a_0-\tau\right)d\vec{y}d\tau . \tag{97}$$

Volume V includes all the sources in the fixed frame. It occupies the whole space and is not a function of time. The first argument in the source strength ($\vec{y}-\vec{U}_0\tau$) reflects the effect of source motion. To express this effect explicitly, we may convert the spatial integral from the fixed frame $\vec{y}$ to the moving frame $\vec{\eta}$:

$$\begin{aligned} p(\vec{x},t) &= \frac{1}{4\pi}\int_{t_0}^{t}\iiint_V \frac{1}{\left|\vec{x}-\vec{\eta}-\vec{U}_0\tau\right|} A(\vec{\eta},\tau)\delta\left(t-\left|\vec{x}-\vec{\eta}-\vec{U}_0\tau\right|/a_0-\tau\right)d\vec{\eta}d\tau \\ &= \frac{1}{4\pi}\iiint_V\int_{t_0}^{t} \frac{1}{\left|\vec{x}-\vec{\eta}-\vec{U}_0\tau\right|} A(\vec{\eta},\tau)\delta\left(t-\left|\vec{x}-\vec{\eta}-\vec{U}_0\tau\right|/a_0-\tau\right)d\tau d\vec{\eta} \end{aligned}, \qquad (98)$$

The integration order can be changed in the last step since the whole space V is not a function of time.

For the two functions $f(\tau)=\dfrac{1}{\left|\vec{x}-\vec{\eta}-\vec{U}_0\tau\right|}A(\vec{\eta},\tau)$ and $g(\tau)=t-\left|\vec{x}-\vec{\eta}-\vec{U}_0\tau\right|/a_0-\tau$, we have

$$\int_{-\infty}^{\infty} f(\tau)\delta(g(\tau))d\tau = \int_{-\infty}^{\infty}\frac{f(\tau)}{|g'(\tau)|}\delta(g(\tau))dg(\tau)=\sum_n \frac{f(\tau_n)}{|g'(\tau_n)|}. \qquad (99)$$

τ_n is the *n*th root of $g(\tau)=0$, *i.e.*,

$$(t-\tau)a_0 = R(\vec{x},\vec{y}(\tau)) = \left|\vec{x}-\vec{\eta}-\vec{U}_0\tau\right|. \qquad (100)$$

τ_n is the source emitting time for the sound reaching the observer at time *t*. If the source moving speed is subsonic, there is only one root (*n*=1). If the speed is supersonic, there are two roots (*n*=2). It is not straightforward to solve τ_n from (100) even for the simple case of uniformly moving source. When the source doesn't move, τ_n is the same retarded time as before.

Note

$$\begin{aligned} g'(\tau_n) &= -1-\frac{-(\vec{x}-\vec{\eta})\cdot\vec{U}_0+|\vec{U}_0|^2\ \tau_n}{a_0\left|\vec{x}-\vec{\eta}-\vec{U}_0\tau_n\right|} \\ &= -1+\frac{\vec{U}_0}{a_0}\cdot\frac{\left(\vec{x}-\vec{\eta}-\vec{U}_0\tau_n\right)}{\left|\vec{x}-\vec{\eta}-\vec{U}_0\tau_n\right|} \\ &= -1+M_r \end{aligned}, \qquad (101)$$

$$M_r = M_0\cos\theta_n .$$

θ_n is the angle between the source moving direction and the source-observer vector *at the emission time*. $M_0 = |\vec{U}_0| / a_0$ is the *acoustic Mach number*, or *source Mach number*. It is not the actual Mach number since the source moving velocity $\vec{U}_0$ is not the flow velocity. M_r is the Mach number towards the observer.

Substituting Eq.(99) and Eq.(101) into Eq.(98), the final expression for sound pressure from moving sources in time domain is:

$$\boxed{p(\vec{x},t) = \sum_n \frac{1}{4\pi} \iiint_{V'} \frac{A(\vec{\eta},\tau_n)}{R(\tau_n)} \frac{1}{|1 - M_r(\tau_n)|} d\vec{\eta}}\ . \qquad (*5)(102)$$

For a harmonic source,

$$p(\vec{x},t) = \sum_n \frac{1}{4\pi} \iiint_{V'} \frac{\hat{A}(\vec{\eta}) e^{i\omega_s \tau_n}}{R(\tau_n)} \frac{1}{|1 - M_r(\tau_n)|} d\vec{\eta}\ . \qquad (*)$$

$R(\tau_n) = |\vec{x} - \vec{\eta} - \vec{U}_0 \tau_n|$ is the distance between the observer and the source *at emission time* τ_n. ***Formally, the sound field from a moving source is the sound field as if the source is stationary divided by*** $|1 - M_r|$**.** The motion has two major effects. The first is the amplitude factor $1/|1 - M_r|$. It is solely due to the source motion, and generated when the spatial integration over $\vec{y}$ in the fixed frame is converted to that over $\vec{\eta}$ in the moving frame. The complexity of the spatial integration over $\vec{y}$ in Eq.(97) due to $A(\vec{y} - \vec{U}_0 \tau, \tau)$ is relieved in (102) since the first argument in $A(\vec{\eta},\tau_n)$ is no longer a function of time. However, the complexity doesn't disappear. It is just moved to the amplitude factor $1/|1 - M_r|$.

The other major effect of motion is on the emission time τ_n, which affects amplitude factors $1/R$ and $1/|1 - M_r|$, and mostly importantly affects the phase and the frequency.

For a steady source, $A(\vec{\eta},t) = A(\vec{\eta})$, $R(\vec{x},t\,|\,\vec{\eta},\tau) = R(\vec{x} - \vec{\eta})$, $M_r = 0$. There is no sound since $p(\vec{x},t) = p(\vec{x})$ does not vary with time. On the other hand, if the source moves, then R and M_r are time dependent and the sound is generated.

For the harmonic source in the moving frame as in (93), if the source doesn't move, the sound amplitude at the observation point is $|\hat{A}(\vec{\eta})|\, d\vec{\eta} / (4\pi R)$. The frequency in the sound field is the same as in the source: $d\phi / dt = \omega_s$ ($\phi = \omega_s \tau_n$). The effects of the source motion on the sound amplitude and the phase are seemly separated in (102). The sound amplitude is changed through $|1 - M_0 \cos\theta_n|$ and $R(\tau_n)$. The sound received by an observer in front of the source ($\cos\theta_n > 0$) is stronger than that received by an observer behind the source ($\cos\theta_n < 0$). On the other hand, the source motion effect on frequency is through $A(\vec{\eta}, \tau_n)$. To show it, let's assume the source motion speed is subsonic and in the far field:

$$\left|\vec{x} - \vec{\eta} - \vec{U}_0\tau\right| = |\vec{x}| - \frac{\vec{x}}{|\vec{x}|} \cdot (\vec{\eta} + \vec{U}_0\tau) + O(|\vec{x}|^{-1}) . \tag{103}$$

The retarded time is:

$$\tau \approx \frac{t - (|\vec{x}| - \vec{x} \cdot \vec{\eta} / |\vec{x}|) / a_0}{1 - M_0 \cos\theta} . \tag{*5-1)(104}$$

According to Eq.(102) the sound pressure has time factor $e^{i\omega_s t/(1-M_0 \cos\theta)}$. The sound frequency measured in the fixed frame is $\omega_s / (1 - M_0 \cos\theta)$. Time in the fixed frame, *t*, is compressed therefore its frequency increases when the source moves towards the observer. This is the *Doppler effect* through phase, and $1/(1 - M_r)$ is the Doppler factor. (When $M_r = 1$ the Doppler factor and the sound field become singular and special treatment is needed.)

The observation time is compressed in (104) which can be explained further from (100). Assume $R(\vec{x}, \vec{y}(\tau)) = \left|\vec{x} - \vec{\eta} - \vec{U}_0\tau\right|$, then,

$$\left.\frac{\partial t}{\partial \tau}\right|_{\vec{\eta}} - 1 = \frac{1}{a_0} \left.\frac{\partial R}{\partial \vec{y}}\right|_{\vec{x}} \cdot \left.\frac{\partial \vec{y}}{\partial \tau}\right|_{\vec{\eta}} = -\frac{1}{a_0} \frac{\vec{R}}{R} \cdot \vec{U}_0 = -M_r(\tau) .$$

Then,

$$\left.\frac{\partial t}{\partial \tau}\right|_{\vec{\eta}} = 1 - M_r(\tau) , \quad \left.\frac{\partial}{\partial t}\right|_{\vec{\eta}} = \frac{1}{1 - M_r(\tau)} \left.\frac{\partial}{\partial \tau}\right|_{\vec{\eta}} . \tag{105}$$

Because of the source motion, $\left.\frac{\partial}{\partial t}\right|_{\vec{\eta}}$ and $\left.\frac{\partial}{\partial \tau}\right|_{\vec{\eta}}$ are no longer equal, which has also been shown by (96). Changing rates of any quantity in reception time and emission time are different.

Exactly speaking, frequency is the recurrence of an oscillation returns to the same state. That requires the oscillation has constant amplitude. Mathematically the Fourier transform of a signal gives Fourier components with constant amplitudes. Therefore the variation of amplitude due to $R(\tau_n)$ and $\left|1 - M_0 \cos\theta_n\right|$ in (102) also changes the frequency and gives the Doppler effect. We may consider this as the *Doppler* effect through amplitude. The variation of amplitude corresponds to Fourier components at different frequencies. Therefore the effects of the source motion on sound amplitude and phase are not exactly separated in (102).

The sound field in (102) is the sum over the possible emission times τ_n from (100). From now on we will omit this summation for convenience.

Sound Field from Arbitrarily Moving Sources

All analyses for the sound field from uniformly moving sources are valid for arbitrarily moving sources except that

$$\vec{y} = \vec{\eta} + \int_0^{\tau} \vec{U}_0(\vec{\eta}, t')dt' . \tag{106}$$

Approximations

Solution (102) is concise and has clear physical meaning. It is the basis for developing numerical methods in time domain. Ref. to cht22.doc for details. It can hardly be directly used in frequency domain without any approximations.

Two types of approximations are usually made. The first is on amplitude factor $1/|1-M_r|$. $1/|1-M_r| \sim 1$ when $\vec{U}_0$ is small, such as in Eq.(12-43) in Blake. However, this approximation can only be made when the other two of the three conditions as discussed previously are met: (1) the source oscillation is not small, and (2) the source strength is fairly uniform. Obviously this is not the case in turbomachinery since sources on blade surfaces are strongly discontinuous in space. $1/|1-M_r|$ appears when the integration in the fixed frame is converted to the integration in the moving frame. In time-domain methods integrations are implemented in the physical domain. It is convenient to use the moving frame. Therefore $1/|1-M_r|$ appears a lot in these methods. In frequency domain methods, solutions are expressed in modes. Sources are coupled with mode shapes, which can easily convert between the fixed and the moving frames. Therefore a fixed frame is usually adopted in frequency domain methods and handling of $1/|1-M_r|$ is avoided.

The second type of approximation is on the emission time and the distance. Most of the complexity of moving source noise prediction comes from finding the emission time τ in (100) from the reception time *t*. This is not easy even for a source moving at constant velocity. One way to get around it is to compute the sound field in *source time* τ instead of *reception time t*. *t* can be directly computed from τ by (100). This strategy is applicable in time-domain numerical simulations. (Ref. cht22.doc for time domain method) In analyses the reception time is preferred. It is not avoidable to find the emission time.

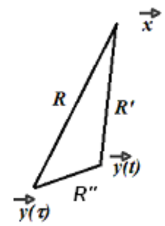


Fig.6, Approximation of $R(\vec{x},\vec{y}(\tau))$ by $R'(\vec{x},\vec{y}(t))$ for finding the retarded time.

Depending on the conditions, there are two approximations for R and τ . In the *far field*, R can be approximated by Eq.(103) and the emission time by (104). It is often used for high-speed rotors. For *a slowly moving* source. the retarded time in $R(\tau_n)$ in the phase is ignore and $R(\vec{x},\vec{y}(\tau))$ in Eq.(100) is replaced by $R'(\vec{x},\vec{y}(t))$ shown in Fig.6:

$$(t-\tau)a_0 \approx R'(\vec{x},\vec{y}(t)),$$

from which the retarded time can be obtained readily:

$$\tau \approx t - \frac{R'(\vec{x},\vec{y}(t))}{a_0}. \tag{107}$$

Note

$$R'^2 = R^2 + R''^2 - 2RR''\cos\theta,\ R'' = U_0(t-\tau) = M_0 R,$$
$$\left(\frac{R'}{R}\right)^2 = 1 + M_0{}^2 - 2M_0\cos\theta.$$

If $|M_0| << 1$, the error of the approximation is:

$$\frac{1}{R}(R-R') = M_0\cos\theta.$$

It is justified for slowly moving sources. This approximation is widely used in low speed turbomachinery noise analyses.

Another approximation is often made on amplitude factor $1/R(\tau_n)$. In the far field the moving effect on $R(\tau_n)$ is often ignored. A constant R between the observer and the geometric center of the source region can be used.

Sound Field from Rotating Sources

In this section we will apply (102) in the investigation of sound field generated by rotating sources. It is convenient to use the cylindrical coordinates shown by Fig.7 in the frame moving with the sources:

$$\vec{\eta} = (r_s, \phi_s, z_s)\,, \tag{108}$$

and in the fixed frame:

$$\vec{y} = (r_s, \theta_s, z_s)\,. \tag{109}$$

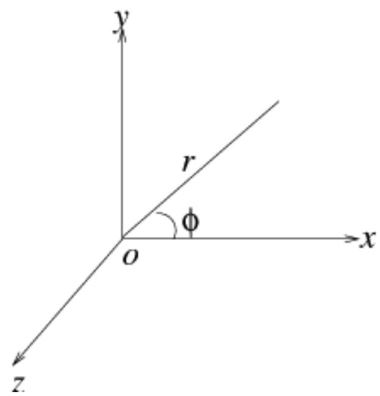


Fig.7, Cylindrical coordinates.

The source rotates at angular speed Ω on the $z = 0$ plane. The cylindrical coordinate relation between the two frames is:

$$\theta_s = \phi_s + \Omega t\,. \tag{109}$$

At $t = 0$, ϕ_s and θ_s are the same.

Ignoring the moving effect in amplitude

Since the moving effect is ignored in the amplitude, this analysis only applies for source oscillating at high frequency in the moving frame.

The sound pressure at observation point $\vec{x}$ with cylindrical coordinates (r_o, θ_o, z_o) can be evaluated by Eq.(102). The first approximation we make is in the far field to approximate $|\vec{R}|$ in the amplitude by $|\vec{x}|$. When the rotating speed is small ($M_r << 1$), it seems natural to neglect the moving effect in the amplitude, *i.e.*, $|1 - M_r(\tau_n)| \sim 1$, as in Blake sec.12.2.2. Then in the far field,

$$p(\vec{x},t) \approx \frac{1}{4\pi|\vec{x}|} \iiint_{V'} A(\vec{\eta},\tau) d\vec{\eta} \,. \qquad (110)$$

With $M_r << 1$ assumed, $\left|\vec{R}(\vec{x},\vec{y}(\tau))\right|$ in the source phase may be approximated by $\left|\vec{R}'(\vec{x},\vec{y}(t))\right|$ as shown in Fig.6 so that the retarded time can be obtained from Eq.(107) in which

$$\begin{aligned} \left|\vec{R}'\right| &= \sqrt{[r_o \cos\theta_o - r_s \cos(\Omega t + \phi_s)]^2 + [r_o \sin\theta_o - r_s \sin(\Omega t + \phi_s)]^2 + z_o^2} \\ &= \sqrt{|\vec{x}|^2 + r_s^2 - 2 r_o r_s \cos(\Omega t + \phi_s - \theta_o)} \\ &\approx |\vec{x}| - r_s \sin\beta \cos(\Omega t + \phi_s - \theta_o) \end{aligned} \,. \qquad (111)$$

β is the angle between $\vec{x}$ and the z-axis, $\sin\beta = r_o / |\vec{x}|$.

For a harmonic source in the moving frame: $\hat{A}(\vec{\eta},\omega_s) e^{i\omega_s \tau}$ (change to (93)?), (110) becomes:

$$p(\vec{x},t) \approx \frac{1}{4\pi|\vec{x}|} e^{-ik_s|\vec{x}|} \iiint_{V'} \hat{A}(\vec{\eta},\omega_s) e^{i\omega_s t + ik_s r_s \sin\beta \cos(\Omega t + \phi_s - \theta_o)} d\vec{\eta} \,,$$

where $k_s = \omega_s / a_0$.

The *Jacobi–Anger expansion:*

$$e^{iz\cos\phi} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\phi} \,, \quad e^{iz\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\phi} \,,$$

is often used to expand a plane wave as a sum of cylindrical waves. (cf. Morse&Ingard1968, Eq.(1.2.9) at p.13). With this expansion, variables about r_s and ϕ_s in the sound pressure expression can be separated and the sound pressure in cylindrical coordinates is:

$$p(\vec{x},t) \approx \frac{1}{4\pi|\vec{x}|} e^{-ik_s|\vec{x}|} \iiint_{V'} \hat{A}(\vec{\eta},\omega_s) \sum_{n=-\infty}^{\infty} i^n J_n(k_s r_s \sin\beta) e^{i(\omega_s + n\Omega)t} e^{in(\phi_s - \theta_o)} d\vec{\eta} \,. \qquad (112)$$

For the single frequency ω_s observed in the moving frame, there exist multiple frequencies in the fixed frame due to the Doppler effect through phase: $\omega_s + n\Omega$, integer

$n = -\infty, \infty$. At each frequency $\omega_s + n\Omega$, there is a spinning mode pattern in the circumferential direction with spinning mode number *n*.

Once the frequency and the amplitude of the source in the moving frame are known, sound pressure can be computed from (112). If the source is steady in the moving frame ($\omega_s = 0$), such as a steady force, no sound is computed from this formula. This is due to the assumption of $|1 - M_r(\tau_n)| \sim 1$. Solution (112) only applies for sources with high frequency oscillations. For steady sources in the moving frame, effect of $|1 - M_r(\tau_n)|$ must be retained.

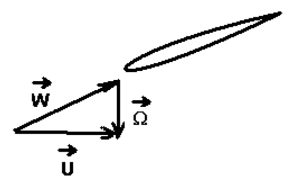


Fig.8, Airfoil subjected to incident flow.

Suppose a rotor with B blades rotates in the wake of a stator with V vanes. To investigate the source characteristics, let's concentrate on one rotating blade as in Fig.8. The aerodynamics of the flow over the blade depends primarily on the incident velocity $\vec{W}$:

$$\vec{W} = \vec{U} - \vec{\Omega} . \tag{113}$$

$\vec{U}$ is the upstream velocity in the fixed frame, $\vec{\Omega}$ is the blade rotating velocity. The source strength depends on $|\vec{W}|$ and its direction (angle of attack), therefore is a function of $\vec{U}$, the wake flow of the upstream stator. In the fixed frame, the wake flow is periodic in θ with period $2\pi/V$. It can be represented by this Fourier series:

Theta=inertial frame, phi: rotating frame

$$\vec{U}(r,\theta,z) = \sum_{m=0}^{\infty} \vec{U}_m(r,z) e^{imV\theta} . \tag{114}$$

In the rotating frame the wake becomes:

$$\vec{W}(r,\phi_s,z) = \sum_{m=0}^{\infty} \vec{U}_m(r,z) e^{imV\phi_s} e^{imV\Omega\tau} - \vec{\Omega}. \tag{115}$$

Observed at position (r,ϕ_s,z) in the moving frame, the incident velocity oscillates with amplitude $\vec{U}_m(r,z)e^{imV\phi_s}$ at frequency

$$\omega_s = mV\Omega \tag{115-1}$$

when the blade moves through the wake. The rotation of the rotor blade transforms the flow spatial nonuniformity in the stator wake into time nonuniformity. Frequency ω_s of the source observed in the moving frame is the *stator* blade passing frequency (BPF) and its harmonics, depending on the rotation speed of the stator relative to the rotor and the *number of stator vanes*, instead of the *number of the rotor blades*.

The blade at ϕ_0 responds to the incident wake with the source strength in a form similar to (115):

$$\sum_{m=0}^{\infty} \widetilde{A}_m(r,z) e^{imV\phi_s} e^{imV\Omega\tau} \delta(\phi_s - \phi_0) . \tag{116}$$

The Dirac delta function is to indicate that the sound source only exists when the blade occupies the space. The B rotor blades are separated by $2\pi/B$. The sound source at the jth blade is:

$$\sum_{m=0}^{\infty} \widetilde{A}_m(r_s,z) e^{imV\phi_s} e^{imV\Omega\tau} \delta[\phi_s - \phi_0 - (j-1)2\pi/B] . \tag{117}$$

If we concentrate on the mth Fourier component of the wake in (115), the source is:

$$\hat{A}_m(\vec{\eta},\omega_s) e^{i\omega_s\tau} = \widetilde{A}_m(r_s,z) e^{imV\phi_s} e^{imV\Omega\tau} \delta[\phi_s - \phi_0 - (j-1)2\pi/B] , \quad j = 1,\ldots, B . \tag{118}$$

The corresponding sound pressure is

$$p_m(\vec{x},t) = \frac{1}{4\pi|\vec{x}|} e^{-imV\Omega|\vec{x}|/a_0} \cdot \iint \sum_{j=1}^{B} \widetilde{A}_m(r_s,z) \sum_{n=-\infty}^{\infty} i^n J_n(mV\Omega r_s \sin\beta / a_0) e^{i(mV+n)\Omega t} e^{-in\theta_o} e^{i(n+mV)[\phi_0+(j-1)2\pi/B]} dr_s dz .$$

Since

$$\sum_{j=1}^{B} e^{i(n+mV)(j-1)2\pi/B} = B \sum_{k=-\infty}^{\infty} \delta(n + mV - kB) ,$$

(cf. Formulae.doc) then,

$$p_m(r_o,\theta_o,z_o,t) = \frac{B}{4\pi|\vec{x}|} e^{-imV\Omega|\vec{x}|/a_0} \cdot \iint \widetilde{A}_m(r_s,z) \sum_{k=-\infty}^{\infty} i^{kB-mV} J_{kB-mV}(mV\Omega r_s \sin\beta / a_0) e^{ikB\Omega t} e^{-i(kB-mV)\theta_o} e^{ikB\phi_0} dr_s dz . \tag{119}$$

Visualize this! Stator not moving wrt to fixed observer!

This result is similar to Eq.(12-14) of Blake. The sound frequencies observed in the *fixed* frame are $kB\Omega$, the *rotor blade-passing frequency* (BPF) and its harmonics. They depend on the *rotation speed of the rotor* relative to the stator, and the *number of rotor blades*. They are totally different from the source frequency $\omega_s = mV\Omega$, which is the stator blade-passing frequency and its harmonics. Corresponding to each frequency $kB\Omega$, there is a spinning mode in the circumferential direction: $kB - mV$. (Spinning mode is defined positive if it propagates in positive θ_o direction.) It can be showed that the frequency at observer $kB\Omega$ is the difference between the source frequency $\omega_s = mV\Omega$ and the spinning mode $mV - kB$ rotating frequency $(mV - kB)\Omega$. This is consistent with the fact that the time rate in the fixed frame is the difference of the time rate in the moving frame and the changing rate due to the source gradient: $\partial/\partial t = \partial/\partial\tau - U_{0j}\partial/\partial\xi_j$. It is called the frequency scattering. The radiation coefficient, $J_{kB-mV}(mV\Omega r_s \sin\beta / a_0)$, is determined by the spinning mode and the source frequency $\omega_s = mV\Omega$.

For a steady source in the rotating frame, $m = 0$, $p_m(\vec{x},t)$ is constant according to (119), *i.e.*, no sound. The Doppler effect through amplitude, or $1/|1 - M_r|$, was ignored when deriving (119). As we have mentioned, the source distribution on the rotating blade surfaces is highly nonuniform in space. Effect of $1/|1 - M_r|$ cannot be neglected even M_r is small.

Keeping the moving effect in amplitude

To keep the Doppler effect through amplitude, it is better to model the source directly in the fixed frame (Morse&Ingard1968, p.738). Then the source is stationary in the Kirchhoff integration.

In the fixed coordinate the wake is Eq.(114).Suppose at time τ, a rotator blade occupies the space at fixed point (r_s, θ_s, z_s). The source at the blade is proportional to $\vec{U}(r_s, \theta_s, z_s)$. Since the rotor is rotating at speed Ω, the occurrence frequency of a blade at this location is $B\Omega$ (rotor BPF). The time factor is then $e^{inB\Omega\tau}$. n is the harmonic of the BPF. The same argument applies at another location except that there is time lag θ_s/Ω. Therefore the correct time factor is $e^{inB\Omega(\tau - \theta_s/\Omega)}$, and the model of the source strength is:

$$q(r_s, \theta_s, z_s, \tau) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \widetilde{A}_{mn}(r_s, z_s) e^{imV\theta_s} e^{inB\Omega(\tau - \theta_s/\Omega)} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \widetilde{A}_{mn}(r_s, z_s) e^{i(mV - nB)\theta_s} e^{inB\Omega\tau} . \quad (120)$$

n begins with 1 since *n*=0 corresponds to a steady source which generates no sound. This expression reflects the source motion effect we discussed previously. (It is produced by the source motion and the source strength gradient?). The difference of the source models in the fixed frame (120) and in the moving frame (117) is that in the fixed frame, the source is continuously distributed with θ_s, while in the moving frame it is discrete with

ϕ as it exists only when the blades occupy that position. In the fixed frame the source exists at any point in the source region as the rotor rotates.

Substituting source model (120) into the Kirchhoff formula (102), we obtain the sound pressure from the mth component of the wake:

$$p_m(\vec{x},t)=\frac{1}{4\pi}\iint\int_0^{2\pi}\left[\frac{1}{|\vec{x}-\vec{y}|}\sum_{n=1}^{\infty}\widetilde{A}_{mn}(r_s,z_s)e^{i(mV-nB)\theta_s}e^{inB\Omega\tau}\right]_\tau d\theta_s dr_s dz_s \ .$$

With the approximation (111) for slowly moving sources and the Jacobi-Anger expansion, the sound pressure is

$$p_m(\vec{x},t)\approx\frac{1}{4\pi|\vec{x}|}\iint\int_0^{2\pi}\sum_{n=1}^{\infty}\widetilde{A}_{mn}(r_s,z_s)e^{i(mV-nB)\theta_s}e^{inB\Omega t}e^{-inB\Omega|\vec{x}|/a_0}e^{inB\Omega r_s\sin\beta\cos(\theta_s-\theta_o)/a_0}r_s d\theta_s dr_s dz_s$$

$$=\frac{1}{4\pi|\vec{x}|}\iint\int_0^{2\pi}\sum_{k=-\infty}^{\infty}\sum_{n=1}^{\infty}\widetilde{A}_{mn}(r_s,z_s)e^{-ik\theta_o}e^{inB\Omega t}e^{-inB\Omega|\vec{x}|/a_0}i^k J_k(nB\Omega r_s\sin\beta/a_0)e^{i(mV-nB+k)\theta_s}r_s d\theta_s dr_s dz_s$$

i.e.,

$$\begin{aligned}p_m(r_o,\theta_o,z_o,t)&=\frac{1}{2|\vec{x}|}\\ &\cdot\iint\sum_{n=1}^{\infty}\widetilde{A}_{mn}(r_s,z_s)e^{-i(nB-mV)\theta_o}e^{inB\Omega t}e^{-inB\Omega|\vec{x}|/a_0}i^{nB-mV}J_{nB-mV}(nB\Omega r_s\sin\beta/a_0)r_s dr_s dz_s\end{aligned} \ . \quad (121)$$

This is a rather different result from (119). It is the rotor blade passing frequencies in $J_{nB-mV}(nB\Omega r_s\sin\beta/a_0)$ instead of the stator BPFs as in (119).

The sound frequencies are the rotor blade passing frequency and its harmonics: $nB\Omega$. The spinning (circumferential) mode of the sound field is $mV-nB$, the difference of the stator (wake) harmonic and the rotor harmonic, also called the Sofrin-Tyler mode. For each frequency, *i.e.,* fixed n, there are multiple spinning modes for different m. For each spinning mode, the sound pressure is:

$$\begin{aligned}p_{mn}(\vec{x},t)&=\frac{1}{2|\vec{x}|}e^{-i(nB-mV)\theta_o}e^{inB\Omega t}e^{-inB\Omega|\vec{x}|/a_0}i^{nB-mV}\\ &\cdot\iint\widetilde{A}_{mn}(r_s,z_s)J_{nB-mV}(nB\Omega r_s\sin\beta/a_0)r_s dr_s dz_s\end{aligned} \ . \quad (122)$$

The circumferential period of the spinning mode is $2\pi/(mV-nB)$. The rotation speed (phase speed) of the spinning mode is $nB\Omega/(mV-nB)$.

Although the source (120) was modeled in the fixed frame to develop the solution, sometimes it is more convenient to determine the source in the rotating frame either numerically or theoretically. Substituting (109) into (120), the source in the rotating frame is:

$$q(r_s,\phi_s,z_s,\tau)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\widetilde{A}_{mn}(r_s,z_s)e^{i(mV-nB)\phi_s}e^{imV\Omega\tau}\ . \tag{123}$$

(123 vs.117?)
If we solve the gust/blade interaction problem in frequency domain in the rotating frame, we obtain the source strength $\hat{q}_m(r_s,\phi_s,z_s)$ with time factor $e^{imV\Omega\tau}$. It can be decomposed into $\widetilde{A}_{mn}(r_s,z_s)$ according to this relationship:

$$\hat{q}_m(r_s,\phi_s,z_s)=\sum_{n=1}^{\infty}\widetilde{A}_{mn}(r_s,z_s)e^{i(mV-nB)\phi_s}\ . \tag{124}$$

Once $\widetilde{A}_{mn}(r_s,z_s)$ is determined, the sound pressure of this spinning mode can be determined by the integration in the source region of this source strength multiplied by weighting function $\frac{1}{2|\vec{x}|}e^{-i(nB-mV)\theta_o}e^{-inB\Omega|\vec{x}|/a_0}i^{nB-mV}J_{nB-mV}(nB\Omega r_s\sin\beta/a_0)r_s$. r_s and $J_{nB-mV}(nB\Omega r_s\sin\beta/a_0)$ in the weighting function reflect the importance of the source location to the sound field. The source at larger radius generates stronger sound. There is no sound along the axis ($\beta=0$) except when $nB=mV$. This actually is the result of source motion effect. When $nB\neq mV$ the sound arriving at the axis cancels each other when the source rotates. On the other hand, when $nB=mV$, sound waves enforce each other, which should be avoided to control the rotor noise. At some radial locations, $J_{nB-mV}(nB\Omega r_s\sin\beta/a_0)$=0. This is also the effect of source motion. When the source at one of these locations rotates, the sound is cancelled as long as the observation point has angle β.

Sound generated by two rotors

The similar analysis also applies for a wake generated by a front rotor instead of a stator. In the frame moving with the front rotor, the wake is:

$$\vec{U}_w(r,\theta_w,z)=\sum_{m=0}^{\infty}\vec{U}_m(r,z)e^{imV\theta_w}\ . \tag{125}$$

θ_w is the circumferential angle in the frame rotating with the front rotor. Suppose the front rotor rotates at speed Ω_w, then in the fixed frame the wake is:

$$\vec{U}(r,\theta,z)=\sum_{m=0}^{\infty}\vec{U}_m(r,z)e^{imV(\theta-\Omega_w\tau)}\ . \tag{126}$$

In the fixed frame, suppose at time τ an aft rotor blade occupies the space at $\vec{y}=(r_s,\theta_s,z_s)$. Since a blade passes this point at the aft rotor Blade Passing Frequency $B\Omega$, the source has time factor:

$$e^{inB\Omega(\tau-\theta_s/\Omega)}.$$

θ_s/Ω is the time lag at a different circumferential location. n is the harmonic of the aft rotor BPF. The source strength is modeled as:

$$\begin{aligned} q(r_s,\theta_s,z_s,\tau) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\widetilde{A}_{mn}(r_s,z_s)e^{imV(\theta_s-\Omega_w\tau)}e^{inB\Omega(\tau-\theta_s/\Omega)} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\widetilde{A}_{mn}(r_s,z_s)e^{i(mV-nB)\theta_s}e^{i\omega_{mn}\tau} \end{aligned}, \qquad (127)$$

Frequency in the fixed frame is $\omega_{mn}=nB\Omega-mV\Omega_w$. Different from the stator/rotor model (123), here n begins with 0.

Substituting source model (127) into the Kirchhoff formula (102), making use of the approximation (111) and the Jacobi-Anger expansion, the sound pressure in the far-field is:

$$\begin{aligned} & p_m(r_o,\theta_o,z_o,t) \\ &= \frac{1}{4\pi|\vec{x}|}\iint\int_0^{2\pi}\sum_{n=0}^{\infty}\widetilde{A}_{mn}(r_s,z_s)e^{i(mV-nB)\theta_s}e^{i\omega_{mn}t}e^{-i(nB\Omega-mV\Omega_w)|\vec{x}|/a_0}e^{i\omega_{mn}r_s\sin\beta\cos(\theta_s-\theta_o)/a_0}r_s d\theta_s dr_s dz_s \\ &= \frac{1}{4\pi|\vec{x}|}\iint\int_0^{2\pi}\sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty}\widetilde{A}_{mn}(r_s,z_s)e^{-ik\theta_o}e^{i\omega_{mn}t}e^{-i(nB\Omega-mV\Omega_w)|\vec{x}|/a_0}i^k J_k(\omega_{mn}r_s\sin\beta/a_0)e^{i(mV-nB+k)\theta_s}r_s d\theta_s dr_s dz_s \end{aligned}$$

,
i.e.,

$$\begin{aligned} & p_m(r_o,\theta_o,z_o,t) = \frac{1}{2|\vec{x}|}\sum_{n=0}^{\infty} \\ & \cdot\iint\widetilde{A}_{mn}(r_s,z_s)e^{-i(nB-mV)\theta_o}e^{i\omega_{mn}t}e^{-i(nB\Omega-mV\Omega_w)|\vec{x}|/a_0}i^{nB-mV}J_{nB-mV}(\omega_{mn}r_s\sin\beta/a_0)r_s dr_s dz_s \end{aligned}. \qquad (128)$$

This solution reduces to (121) as $\Omega_w=0$. It reduces to the solution of a front rotor interacting with a aft stator as $\Omega=0$.

In the stator/rotor case (121), there may be multiple spinning modes at one frequency. Here for the rotor/rotor case, one spinning mode (*m,n*) corresponds to only one frequency, *vice versa*. The sound field have modes other than (*m,n*) at one specific frequency only when it is scattered by nonuniformity such as splices in a duct during propagating process. The frequency at observer $\omega_{mn}=nB\Omega-mV\Omega_w$ is the difference between the source frequency $mV(\Omega-\Omega_w)$ and the spinning mode $mV-nB$ rotating frequency $(mV-nB)\Omega$. This is consistent with the fact that the time rate in the fixed frame is the difference of the time rate in the moving frame and the changing rate due to the source gradient: $\partial/\partial t=\partial/\partial\tau-U_{0j}\partial/\partial\xi_j$. It is called the frequency scattering.

Based on solution (128), sound intensity of mode (*m,n*) depends on two factors: (1) source strength $\widetilde{A}_{mn}(r_s,z_s)$, and (2) source location as in radiation coefficient $J_{nB-mV}(\omega_{mn}r_s\sin\beta/a_0)$. Usually the lower the spinning mode number $|nB-mV|$, the higher the radiation coefficient $|J_{nB-mV}(\omega_{mn}r_s\sin\beta/a_0)|$. If $|nB-mV|$ is large, $J_{nB-mV}\sim 0$ even $\omega_{mn}r_s\sin\beta/a_0$ is large. This mode is called 'cut-off'. J_{nB-mV} becomes important only when the frequency increases to some level. Then the mode is 'cut-on'. (Blake1986, Vol.2, p.882) This is how the 'cut-on' and 'cut-off' phenomenon is explained when the Green's function in the Cartesian coordinates (66) is used. The Green's function expressed in cylindrical coordinates will be given later. The 'cut-on' phenomena will be explained in a slightly different way.

The response of the aft rotor is roughly periodic in the circumferential direction, *i.e*., $\widetilde{A}_{mn}(r_s,z_s)$ is stronger when $|nB-mV|$ is close to *B* and its harmonics. Therefore, based on *B* and *V*, we can estimate that, the mode whose $|nB-mV|$ is small and close to *B* and its harmonics has stronger sound intensity. To control noise, *B* and *V* must be selected carefully.

As an example, let’s assume $V=12$ and $B=10$. For $m=1$, the spinning mode numbers are 12,2,-8,-18,…, when *n*=0,1,2,3,… The spinning mode number (absolute value) closest to *B* is –8, therefore the strong sound can be expected at mode (*m,n*)=(1,2). For $m=2$, the spinning mode numbers are 24,14,4,-6,,… as *n*=0,1,2,3,… The spinning mode numbers (absolute value) closer to *B*=10 are 4 and –6, therefore the strong sound can be expected at mode (*m,n*)=(2,2) and (2,3). In this example, none of any combination of *m* and *n* produces a spinning number equal to 10. *V* and *B* are designed to avoid this happening.

Examining (121), we find no cutoff mechanism of the sound propagation in the free space. This is different from a rotor generating sound in a duct. In a duct only a limited number of modes can propagate. Other modes are evanescent. For the rotor-alone steady pressure generated sound, it propagates only when the tip speed reaches or exceeds supersonic. (Hubbard1995, p.167)

The source in the frame rotating with the aft rotor is:

$$q(r_s,\phi_s,z_s,\tau)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\widetilde{A}_{mn}(r_s,z_s)e^{i(mV-nB)\phi_s}e^{imV(\Omega-\Omega_w)\tau}\,. \tag{129}$$

If we solve the gust/blade interaction problem in frequency domain in the rotating frame, we obtain the source strength $\hat{q}_m(r_s, \phi_s, z_s)$ with time factor $e^{imV(\Omega-\Omega_w)\tau}$. It can be decomposed into $\widetilde{A}_{mn}(r_s, z_s)$ according to this relationship:

$$\hat{q}_m(r_s, \phi_s, z_s) = \sum_{n=0}^{\infty} \widetilde{A}_{mn}(r_s, z_s) e^{i(mV-nB)\phi_s} \ . \tag{130}$$

Once $\widetilde{A}_{mn}(r_s, z_s)$ is determined, the sound pressure of this spinning mode can be determined by Eq.(128).

Green's Functions in Cylindrical Coordinates in Open Space

When we say the Green's function in cylindrical coordinates, we mean $G = f(r)g(\phi)$.

Green's function in cylindrical coordinates: Morse&Ingard, Eq.(7.3.15). It is derived from the Cartesian Green's function using the Jacobi-Anger expansion.

General solution in cylindrical coordinates: Morse&Ingard, p.360

In application, the Green's function in Cartesian coordinates is often used, and the Jacobi–Anger expansion is used to transfer it to the cylindrical coordinates when needed. It should give the same result as using the Green's function in cylindrical coordinates directly.

J, Y: good for standing waves, therefore in ducts
H: good for propagating waves in open space

Spherical coordinate system: appropriate when the source region is concentrated in a small region of space (in all three dimensions)
Cylindrical coordinate system: appropriate when the source region is extended in one direction and concentrated in the other two.
Cartesian coordinate system: appropriate when the region extended in 2 dimensions.
Appropriate coordinate systems are also chosen based on the source integration when the source region is not compact. In this case the source integration is needed. Then appropriate coordinate system is chosen for the convenience of numerical integration

Green's function in spherical coordinates: Morse&Ingard, Eq.(7.2.31)

Green's Functions in Two Dimensions

The Green's functions in a three-dimensional stationary medium in open space are given by Eq.(66) in Cartesian coordinates and by ** in cylindrical coordinates. A point source in two dimensions can be modeled as a line source in three dimensions. Therefore the 2D

Green's function can be obtained by integrating along the line source. The 2D Green's function can also be obtained by directly solving the 2D acoustic equation. The second approach is used in this section.

The non-homogeneous two-dimensional equation for the Green's function is:

$$\frac{1}{a_0^2}\frac{D^2 G}{Dt^2} - \nabla^2 G = \delta(\vec{x}-\vec{x}_s)e^{-i\omega t}, \tag{131}$$

The point source at $\vec{x}_s$ oscillates at circular frequency ω. The solution has the same frequency so we can define:

$$G(\vec{x}\,|\,\vec{x}_s, t) = \hat{G}(\vec{x}\,|\,\vec{x}_s, \omega)e^{-i\omega t}. \tag{132}$$

2D Green's Function in Stationary Medium

We first discuss the case $u_0 = 0$. The solution is axially symmetric. Polar coordinate (r, θ) is adopted:

$$r = |\,\vec{x}-\vec{x}_s\,|,\ \theta = tg^{-1}[(x_2 - x_{2s})/(x_1 - x_{1s})].$$

The axisymmetric equation in the polar coordinate system is:

$$\frac{d^2\hat{G}}{dr^2} + \frac{1}{r}\frac{d\hat{G}}{dr} + \left(\frac{\omega}{a_0}\right)^2 \hat{G} = -\delta(\vec{r}). \tag{133}$$

The corresponding homogeneous equation of (133) (replacing the right hand side by 0) is the Bessel equation. Its general solution can be constructed by the two linearly independent Bessel functions:

$$\hat{G}(\vec{x}\,|\,\vec{x}, \omega) = AJ_0(\omega r/a_0) + BY_0(\omega r/a_0). \tag{134}$$

$J_0(\omega r/a_0)$ and $Y_0(\omega r/a_0)$ are the zeroth order first and second kind of Bessel function respectively. A and B are determined by boundary conditions at two radial locations. Solution (134) represents standing waves such as in an annular duct (cht16.doc). For open space we are discussing, (134) is not an appropriate choice. Instead, we use the third kind of Bessel functions:

$$\hat{G}(\vec{x}\,|\,\vec{x}, \omega) = AH_0^{(1)}(\omega r/a_0) + BH_0^{(2)}(\omega r/a_0). \tag{135}$$

$H_0^{(1)}(\omega r/a_0)$ and $H_0^{(2)}(\omega r/a_0)$ are Hankel functions. In the far field ($r \to \infty$),

$$H_0^{(1)}(\omega r/a_0) \to \sqrt{\frac{2a_0}{\pi\omega r}}e^{i(\omega r/a_0-\pi/4)},$$
$$H_0^{(2)}(\omega r/a_0) \to \sqrt{\frac{2a_0}{\pi\omega r}}e^{-i(\omega r/a_0-\pi/4)}.$$

Considering the time factor in (132), one can see that $H_0^{(1)}(\omega r/a_0)$ represents a wave propagating towards $r\to\infty$, and $H_0^{(2)}(\omega r/a_0)$ represents a wave propagating towards $r\to 0$.

B is determined by the boundary at $r\to\infty$. In the current case there is only outward propagating wave, therefore, B=0. A is determined by the boundary at $r\to 0$. The integration of both sides of Eq.(133) on any circular area that doesn't include the source point is zero. If the circular area with radius ε is centered at the source point, then,

$$\iint_S \left[\nabla\cdot\nabla\hat{G} + \left(\frac{\omega}{a_0}\right)^2 \hat{G}\right] dS = -1.$$

From the Gauss' Divergence Theorem we have:

$$A\int_0^{2\pi}\frac{dH_0^{(1)}(\omega r/a_0)}{dr}rd\theta + A\left(\frac{\omega}{a_0}\right)^2\int_0^{2\pi}\int_0^{\varepsilon}H_0^{(1)}(\omega r/a_0)rdrd\theta = -1$$

As $r\to 0$,

$$H_0^{(1)}(\omega r/a_0) \to \frac{2}{\pi}i[\ln(\omega r/a_0) - 0.1159],$$
$$\frac{d}{dr}H_0^{(1)}(\omega r/a_0) = -\frac{\omega}{a_0}H_1^{(1)}(\omega r/a_0) \to \frac{2i}{\pi r}.$$

Therefore,

$$A = \frac{i}{4},$$
$$\boxed{\hat{G}(\vec{x}\,|\,\vec{x}_s,\omega) = \frac{i}{4}H_0^{(1)}(\omega\,|\,\vec{x}-\vec{x}_s\,|\,/a_0)}. \tag{136}$$

This is the two-dimensional Green's function in a stationary medium in Cartesian coordinates in frequency domain. It has the logarithmic singularity at the origin. The 2D Green's function in cylindrical coordinates is referred to Eq.(7.3.18) in Morse&Ingard1968.

In the limit of incompressible medium, $a_0 \to \infty$. The non-homogeneous equation is:

$$\nabla^2 \hat{G} = -\delta(\vec{x} - \vec{x}_s) \ .$$

One may find its Green's function by performing the limiting analysis of (136) as $a_0 \to \infty$. But the correct way is to use the similar procedure from (133) to (136), which leads to this Green's function in two-dimensional stationary incompressible medium:

$$\hat{G}(\vec{x} \,|\, \vec{x}_s, \omega) = -\frac{1}{2\pi} \ln |\, \vec{x} - \vec{x}_s \,| \ . \qquad ()$$

2D Green's Function in Subsonic Mean Flow

(Cf.Howe1998, p.38)

When $u_0 \neq 0$ and $u_0 < a_0$, we can use the *Prandtl-Glauert transformation* (83) to transform Eq.(131) into:

$$\begin{aligned} \frac{1}{a_0^2} \frac{\partial^2 G}{\partial \tilde{t}^2} - \tilde{\nabla}^2 G &= \delta[(\vec{\tilde{x}} - \vec{\tilde{x}}_s)/\gamma] e^{-i\omega\gamma(\tilde{t} - u_0 \tilde{x}_1 / a_0^2)} \\ &= \gamma \delta(\vec{\tilde{x}} - \vec{\tilde{x}}_s) e^{-i\omega\gamma(\tilde{t} - u_0 \tilde{x}_{1s} / a_0^2)} \end{aligned} \ . \qquad (137)$$

According to (136), the solution to (137) in two dimensions is:

$$G(\vec{\tilde{x}} \,|\, \vec{\tilde{x}}_s, \omega\gamma) = \frac{i}{4} H_0^{(1)}(\gamma \tilde{r} \omega / a_0) \gamma e^{i\omega\gamma u_0 \tilde{x}_{1s} / a_0^2} e^{-i\omega\gamma\tilde{t}} \ .$$

In the original coordinates it is:

$$\begin{gathered} \boxed{\hat{G}(\vec{x} \,|\, \vec{x}_s, \omega) = \frac{i}{4} \gamma e^{-i\gamma^2 M_0 (x_1 - x_{1s}) \omega / a_0} H_0^{(1)}(\gamma \tilde{r} \omega / a_0)}, \\ \tilde{r} = \sqrt{\gamma^2 (x_1 - x_{1s})^2 + (x_2 - x_{2s})^2} \end{gathered} \qquad (138)$$

Eq.(138) is the Green's function in a subsonic mean flow for Eq.(131).

<u>Green's Function in Three Dimensional Open Space with Mean Flow</u>

Eq.(69) is the non-homogeneous equation for the Green's function in a three-dimensional flow:

$$\frac{1}{a_0^2} \frac{D_0}{Dt} \left(\frac{D_0 G}{Dt} \right) - \nabla_{\vec{x}}^2 G = \delta(\vec{x} - \vec{y}) \delta(t - \tau)$$

Applying the *Prandtl-Glauert transformation* (83) in Eq.(69), one obtains

$$\left(\frac{1}{a_0^2}\frac{\partial^2}{\partial \tilde{t}^2} - \nabla_{\tilde{\vec{x}}}^2\right)G = \delta(\tilde{\vec{x}} - \tilde{\vec{y}})\delta(\tilde{t} - \tilde{\tau})\,. \tag{139}$$

The Green's function in a stationary medium is provided by Eq.(66). Therefore, the solution to Eq.(139) in three dimensions is

$$G(\tilde{\vec{x}}, \tilde{t} \mid \tilde{\vec{y}}, \tilde{\tau}) = \frac{1}{4\pi\left|\tilde{\vec{x}} - \tilde{\vec{y}}\right|}\delta\left(\tilde{t} - \left|\tilde{\vec{x}} - \tilde{\vec{y}}\right|/a_0 - \tilde{\tau}\right).$$

Reverting it to the original coordinates, one obtains the Green's function for uniform mean flows:

$$\boxed{G(\vec{x}, t \mid \vec{y}, \tau) = \frac{1}{4\pi R_1}\delta\left(t - \tau - R_0/a_0\right)}. \tag{140}$$

$$\begin{aligned} R_1 &= \sqrt{(x_1 - y_1)^2 + (1 - M^2)[(x_2 - y_2)^2 + (x_3 - y_3)^2]} \\ &= \sqrt{|\vec{x} - \vec{y}|^2 - M^2[(x_2 - y_2)^2 + (x_3 - y_3)^2]} \\ &= \sqrt{(1 - M^2)|\vec{x} - \vec{y}|^2 + M^2(x_1 - y_1)^2} \end{aligned}$$

$$R_0 = \frac{R_1 - M(x_1 - y_1)}{1 - M^2}$$

Note $R_1 \le |\vec{x} - \vec{y}|$: amplification of mean flow.

C.f. Michalke&Michel1979, Eq.(3.2), Ikeda et al 2016, Eq.(8), Najafi-Yazdi et al 2011, Eq. (2.24), Howe1998, Eq.(1.7.17) for frequency domain.