Let Δ be a d-dimensional simplicial complex. Denote its set of i-faces by F_i, and define $f_i := |F_i|$.

Definition 0.1. A *spanning tree of* Δ is a d-dimensional subcomplex $\Upsilon \subseteq \Delta$ with $\mathrm{Skel}_{d-1}(\Upsilon) = \mathrm{Skel}_{d-1}(\Delta)$ and satisfying the three conditions:

1. $\widetilde{H}_d(\Upsilon) = 0$;
2. $\tilde{\beta}_{d-1}(\Upsilon) = 0$;
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta)$.

If $0 \le k < d$, a *k-dimensional (simplicial) spanning tree* of Δ is a spanning tree of the k-skeleton $\mathrm{Skel}_k(\Delta)$.

Proposition 0.2. *Suppose that Δ is a d-dimensional simplicial complex. Then Δ possesses a simplicial spanning tree if and only if $\tilde{\beta}_{d-1}(\Delta) = 0$. We say that such complexes are* acyclic in codimension 1.

Proof. First suppose that Υ is a spanning tree of Δ. Then $C_d(\Upsilon) \subset C_d(\Delta)$, $C_{d-1}(\Upsilon) = C_{d-1}(\Delta)$, and $\mathrm{im}(\partial_{\Upsilon,d}) \subseteq \mathrm{im}(\partial_{\Delta,d})$. It follows that there is a surjection

$$\widetilde{H}_{d-1}(\Upsilon) = C_{d-1}(\Upsilon)/\,\mathrm{im}(\partial_{\Upsilon,d}) \twoheadrightarrow C_{d-1}(\Delta)/\,\mathrm{im}(\partial_{\Delta,d}) = \widetilde{H}_{d-1}(\Delta).$$

Since $\widetilde{H}_{d-1}(\Upsilon)$ is finite, it follows that $\widetilde{H}_{d-1}(\Delta)$ is also finite, hence $\tilde{\beta}_{d-1}(\Delta) = 0$.

Now suppose that Δ is acyclic in codimension 1. To construct a spanning tree, start with $\Upsilon = \Delta$. If $\widetilde{H}_d(\Upsilon) = 0$, then Υ is a spanning tree, and we are done. If not, then there is an integer-linear combination of facets σ_i in the kernel of ∂_d:

$$a_1\sigma_1 + a_2\sigma_2 + \cdots + a_k\sigma_k,$$

where we assume that $a_1 \neq 0$. If we work over the rational numbers, then we may assume $a_1 = 1$. Still working over the rationals, we see that

$$\partial_d(\sigma_1) = -\sum_{i=2}^{k} a_i \partial_d(\sigma_i).$$

Hence, if we remove the facet σ_1 from Υ, we obtain a smaller subcomplex Υ' without changing the image of the rational boundary map: $\mathrm{im}(\partial_{\Upsilon,d}) = \mathrm{im}(\partial_{\Upsilon',d})$. It follows from rank-nullity that

$$\tilde{\beta}_d(\Upsilon') = f_d(\Upsilon') - \text{rank im}(\partial_{\Upsilon',d}) = f_d(\Upsilon) - 1 - \text{rank im}(\partial_{\Upsilon,d}) = \tilde{\beta}_d(\Upsilon) - 1,$$

and $\tilde{\beta}_{d-1}(\Upsilon') = \tilde{\beta}_{d-1}(\Upsilon) = 0$. Continuing to remove facets in this way, we eventually obtain a spanning tree of Δ. $\square$

Corollary 0.3. *If $0 \le i \le d$, then Δ has an i-dimensional spanning tree if and only if $\tilde{\beta}_{i-1}(\Delta) = 0$.*

Proof. This follows from the previous proposition since $\tilde{\beta}_{i-1}(\mathrm{Skel}(\Delta)) = \tilde{\beta}_{i-1}(\Delta)$. $\square$

Proposition 0.4. *If Δ has a unique d-dimensional spanning tree Ψ, then $\Delta = \Psi$.*

Proof. To come. $\square$

Let $0 \le i \le d$. The *i-th tree number* for Δ is

$$\tau_i := \tau_i(\Delta) := \sum_{\Psi} |H_{i-1}(\Psi)|^2,$$

where the sum is over all i-dimensional spanning trees of Δ.

Corollary 0.5. *$\tau_d(\Delta) = 1$ if and only if Δ is a tree and $\widetilde{H}_{d-1}(\Delta) = 0$, i.e., if and only if $\widetilde{H}_d(\Delta) = \widetilde{H}_{d-1}(\Delta) = 0$.*

Proof. This follows directly from the definition of $\tau_d(\Delta)$ and Proposition 0.4. $\square$

Suppose that $\tilde{\beta}_{i-1}(\Delta) = 0$. It follows that Δ has an i-dimensional spanning tree Υ. Let

$$\widetilde{F}_i := F_i(\Delta) \setminus F_i(\Upsilon),$$

the i-faces of Δ not contained in Υ, and let $\widetilde{L}_i$ be the i-Laplacian of Δ with the rows and columns corresponding to faces in $F_i(\Upsilon)$ removed.

Theorem 0.6 (Duval, Klivans, Martin)**.**

(i) *If $\widetilde{H}_{i-1}(\Upsilon) = 0$, there is an isomorphism*

$$\psi\colon\ \mathcal{K}_i(\Delta) \to \mathbb{Z}\widetilde{F}_i/\operatorname{im}(\widetilde{L}_i),$$

defined by dropping i-faces of Υ: if $c = \sum_{f\in F_i(\Delta)} c_f \cdot f \in \ker\partial_i$ represents an element of $\mathcal{K}_i(\Delta)$, then

$$\psi(c) = \sum_{f\in\widetilde{F}_i} c_f \cdot f \mod \operatorname{im}(\widetilde{L}_i).$$

(ii) *(Simplicial matrix-tree theorem)*

$$\tau_{i+1} = \frac{|\widetilde{H}_{i-1}(\Delta)|^2}{|\widetilde{H}_{i-1}(\Upsilon)|^2}\,\det(\widetilde{L}_i).$$

Definition 0.7. Let $0 \le i \le d$. The *i-th positive kernel* for Δ is the pointed cone

$$\ker^+ L_i := \{D \in \mathbb{Z}F_i : D(f) \ge 0 \text{ for all } f \in F_i\}.$$

Fixing an ordered Hilbert basis $H_i = (h_1, \dots, h_\ell)$ for $\ker^+ L_i \cap \mathbb{Z}^{f_i(\Delta)}$, define the *degree* of $D \in \mathbb{Z}F_i$ by

$$\deg(D) := (D \cdot h_1, \dots, D \cdot h_\ell)$$

where $D \cdot h_j := \sum_{f \in F_i} D(f) h_j(f)$. We say $d \in \mathbb{Z}^\ell$ is a *winning degree (in dimension i)* if each $D \in \mathbb{Z}F_i$ with $\deg(D) \ge d$ is winnable.

Proposition 0.8. *For $0 \le i \le d$,*

$$(\ker L_i)^\perp / \operatorname{im}(L_i) \approx \mathbf{T}(\mathcal{K}_i(\Delta)).$$

This says the group of i-chains of degree 0 modulo firing rules is isomorphic to the torsion part of the i-th critical group of Δ.[1]

Theorem 0.9. *Suppose that $\tau_d(\Delta) = 1$. Then 0 is a winning degree in dimension $d-1$ if and only if $\widetilde{H}^d(\Delta)$ is free. (NOTE: check on whether reduced cohomology is important here.)*

Proof. Since $\tau_d(\Delta) = 1$, by Proposition 0.5, we have $\widetilde{H}_d(\Delta) = \widetilde{H}_{d-1}(\Delta) = 0$. Thus, ∂_d is injective, and $\operatorname{im} \partial_d = \ker \partial_{d-1}$.

First suppose that $\widetilde{H}^d(\Delta)$ is free. To show 0 is a winning degree in dimension $d-1$, we must show $\mathcal{K}_{d-1}(\Delta)$ is also torsion-free. So let $\alpha \in \ker \partial_{d-1}$ and suppose there exists an integer $n > 0$ such that $n\alpha \in \operatorname{im}(L_{d-1})$. So there exists $\beta \in C_{d-1}(\Delta)$ such that

$$n\alpha = L_{d-1}\beta = \partial_d \partial_d^t \beta.$$

Since $\widetilde{H}_{d-1}(\Delta) = 0$, there exists $\gamma \in C_d(\Delta)$ such that $\partial_d \gamma = \alpha$. Since ∂_d is injective, $n\gamma = \partial_d^t \beta$. Since $\widetilde{H}^d(\Delta) = C^d(\Delta)/\operatorname{im} \partial_d^t = 0$, it follows that $\gamma = \operatorname{im} \partial_d^t$, and hence, $\alpha \in \operatorname{im} L_{d-1}$, as required.

Conversely, suppose that 0 is a winning degree in dimension $d-1$. We would like to show that $\widetilde{H}^d(\Delta)$ is free. So say $n\sigma = \partial_d^t \mu \in \operatorname{im} \partial_d^t$ for some $\sigma \in C^d(\Delta)$ and some integer $n > 0$ It follows that $n\partial_d \sigma = 0$ as an element of $\mathcal{K}_{d-1}(\Delta)$. However, $\mathcal{K}_{d-1}(\Delta)$ is torsion-free. So $\partial_d \sigma = L_{d-1}\nu = \partial_d \partial_d^t \nu$ for some $\nu \in C_{d-1}(\Delta)$. Since ∂_d is injective, it follows that $\sigma = \partial_d^t \nu$. □

Example 0.10. Figure 1 illustrates a 2-dimensional complex P which is a triangulation of the real projective plane. We have $\widetilde{H}_0(P) = \widetilde{H}_2(P) = 0$, and $\widetilde{H}_1(P) \approx \mathbb{Z}/2\mathbb{Z}$.

[1] Need to say why the perp is the group of i-chains of degree 0.

From the definition of the tree number, $\tau_2(P) = 4$, and it follows from the matrix-tree theorem that $\det(\widetilde{L}_0) = 4$ with respect to any 1-dimensional spanning tree of P. (There are 6^4 such spanning trees since the 1-skeleton of P is the completer graph K_6.) The cycle $C := \overline{01} + \overline{12} - \overline{02}$ has degree $(0, 0, 0, 0, 0)$ but is not winnable. We have $\mathcal{K}_1(P) \approx \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and hence $2C$ is winnable.

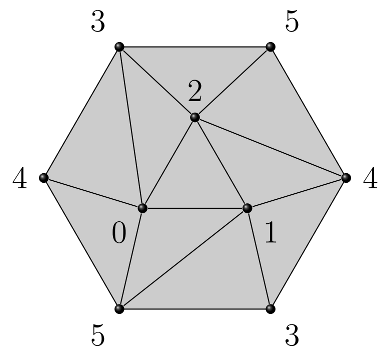

Figure 1: A triangulation of the real projective plane, $\mathbb{RP}^2$.

Question. Can we find an example where $\tau_d(\Delta) = 1$ and 0 is not a winning degree in dimension $d - 1$? The 2-dimensional chessboard complex is an example of a non-tree for which 0 is a winning degree in dimension 1.