1 Forests

Let Δ be a d-dimensional simplicial complex. Denote its set of i-faces by Δ_i, and define $f_i := |\Delta_i|$.

Definition 1.1. A *spanning forest of* Δ is a d-dimensional subcomplex $\Upsilon \subseteq \Delta$ with $\mathrm{Skel}_{d-1}(\Upsilon) = \mathrm{Skel}_{d-1}(\Delta)$ and satisfying the three conditions:

1. $\widetilde{H}_d(\Upsilon) = 0$;
2. $\tilde{\beta}_{d-1}(\Upsilon) = \tilde{\beta}_{d-1}(\Delta)$;
3. $f_d(\Upsilon) = f_d(\Delta) - \tilde{\beta}_d(\Delta)$.

If $0 \leq k < d$, a *k-dimensional spanning forest* of Δ is a k-dimensional spanning forest of the k-skeleton $\mathrm{Skel}_k(\Delta)$. In the case where $\tilde{\beta}_{d-1}(\Delta) = 0$, a d-dimension spanning forest called a *spanning tree*. The complex Δ is called a *forest* if it is a spanning forest of itself, i.e., if $\widetilde{H}_d(\Delta) = 0$.

Remarks. Let Υ be a subcomplex of Δ sharing the same $(d-1)$-skeleton.

1. The condition $\widetilde{H}_d(\Upsilon) = 0$ is equivalent to the elements of the set
$$A := \{\partial_{\Upsilon,d}(f) : f \in \Upsilon_d\}$$
being linearly independent (over $\mathbb{Z}$ or, equivalently, over $\mathbb{Q}$).

2. Since Υ and Δ have the same $(d-1)$-skeleton, $\partial_{\Delta,d-1} = \partial_{\Upsilon,d-1}$, and hence, $\tilde{\beta}_{d-1}(\Upsilon) = \tilde{\beta}_{d-1}(\Delta)$ is equivalent to $\operatorname{rank} \operatorname{im} \partial_{\Upsilon,d} = \operatorname{rank} \operatorname{im} \partial_{\Delta,d}$.

3. It follows from the above two remarks that Υ is a spanning forest if and only if A, defined above, is a basis for $\operatorname{im} \partial_{\Delta,d}$ over $\mathbb{Q}$, i.e, the columns of the matrix $\partial_{\Delta,d}$ corresponding to the d-faces of Υ are a $\mathbb{Q}$-basis for the column space of $\partial_{\Delta,d}$. In particular, spanning forests always exist.

Proposition 1.2. *Any two of the three conditions defining a spanning forest implies the remaining condition.*

Definition 1.3. Define the *complexity* or *forest number* of Δ to be
$$\tau := \tau_d(\Delta) := \sum_{\Upsilon \subseteq \Delta} |\mathbf{T}(\widetilde{H}_{d-1}(\Upsilon))|^2$$
where the sum is over all (d-dimensional) spanning forests of Δ.

Proposition 1.4. $\tau_d(\Delta) = 1$ *if and only if* Δ *is a forest.*

Proof. Suppose that $\tau_d(\Delta) = 1$. Then Δ possesses a unique spanning forest Υ. It follows that the set columns of ∂_d has a unique maximal linearly independent subset—those columns corresponding to the faces of Υ. Since the columns of Δ are all nonzero, it must be that the columns corresponding to Υ are the only columns, i.e., $|\Upsilon_d| = |\Delta_d|$, and hence $\Upsilon = \Delta$.

Conversely, suppose that Δ is a forest and that $\Upsilon \subseteq \Delta$ is a spanning forest. Since $\widetilde{H}_d(\Delta) = 0$, it follows that

$$|\Upsilon_d| = |\Delta_d| - \tilde{\beta}_d(\Delta) = |\Delta_d|.$$

Hence, $\Upsilon = \Delta$, and $\tau_d(\Delta) = |\mathbf{T}\widetilde{H}_{d-1}(\Delta)|^2 = 1$. $\square$

Definition 1.5. The *i-th Laplacian* of Δ is $L_i := \partial_{i+1} \circ \partial_{i+1}^t$. The *i-th critical group* of Δ is

$$\mathcal{K}_i(\Delta) := \ker \partial_i / \operatorname{im} L_i.$$

Theorem 1.6. *Suppose that* Υ *is an* i*-dimensional spanning forest of* Δ *such that* $\widetilde{H}_{i-1}(\Upsilon) = \widetilde{H}_{i-1}(\Delta)$. *Let* $\Theta := \Delta_i \setminus \Upsilon_i$. *Define the* reduced Laplacian $\tilde{L}$ *of* Δ *with respect to* Υ *to be the square submatrix of* L_i *consisting of the rows and columns indexed by* Θ. *Then there is an isomorphism*

$$\mathcal{K}_i(\Delta) \to \mathbb{Z}\Theta / \operatorname{im} \tilde{L}$$

obtained by setting the faces of Υ_i *equal to* 0.

Proof. Considering the commutative diagram

$$\begin{tikzcd}
\mathbb{Z}\Upsilon_i \arrow[r, "\partial_{\Upsilon,i}"] \arrow[d] & \mathbb{Z}\Upsilon_{i-1} \arrow[r, "\partial_{\Upsilon,i-1}"] \arrow[d, equal] & \mathbb{Z}\Upsilon_{i-2} \arrow[d, equal] \\
\mathbb{Z}\Delta_i \arrow[r, "\partial_{\Delta,i}"] & \mathbb{Z}\Delta_{i-1} \arrow[r, "\partial_{\Delta,i-1}"] & \mathbb{Z}\Delta_{i-2}
\end{tikzcd}$$

we see

$$\operatorname{im} \partial_{\Upsilon,i} \subseteq \operatorname{im} \partial_{\Delta,i} \subseteq \ker \partial_{\Delta,i-1} = \ker \partial_{\Upsilon,i-1}.$$

Thus, there is an exact

$$0 \to \operatorname{im} \partial_{\Delta,i} / \operatorname{im} \partial_{\Upsilon,i} \to \widetilde{H}_{i-1}(\Upsilon) \to \widetilde{H}_{i-1}(\Delta) \to 0.$$

By hypothesis $\widetilde{H}_{i-1}(\Upsilon) = \widetilde{H}_{i-1}(\Delta)$, and hence

$$\operatorname{im} \partial_{\Upsilon,i} = \operatorname{im} \partial_{\Delta,i}.$$

We now describe a basis for $\ker_{\Delta,i}$. For each $\theta \in \Theta$, since $\operatorname{im}\partial_{\Upsilon,i} = \operatorname{im}\partial_{\Delta,i}$,

$$\partial_{\Delta,i}(\theta) = \sum_{\tau\in\Upsilon_i} a_\theta(\tau)\partial_{\Upsilon,i}(\tau) \tag{1}$$

for some $a_\theta(\tau) \in \mathbb{Z}$. Since $\widetilde{H}_i(\Upsilon) = 0$, the boundary mapping $\partial_{\Upsilon,i}$ is injective, and thus the coefficients $a_{\tau,\theta}$ are uniquely determined. Define

$$\alpha(\theta) := \sum_{\tau\in\Upsilon_i} a_\theta(\tau)\tau$$

and extend linearly to get a mapping $\alpha : \mathbb{Z}\Theta \to \mathbb{Z}\Upsilon$. For each $\theta \in \Theta$, let

$$\hat{\theta} := \theta - \alpha(\theta).$$

We claim

$$\ker\partial_{\Delta,i} = \{\hat{\theta} : \theta \in \Theta\}.$$

The $\hat{\theta}$ are linearly independent elements of the kernel. To show they span, suppose that $\gamma = \sum_{\sigma\in\Delta_i} b_\sigma\sigma \in \ker_{\Delta,i}$. Consider

$$\gamma' := \gamma - \sum_{\theta\in\Theta} b_\sigma\hat{\sigma} = \sum_{\sigma\in\Upsilon_i} b_\sigma\sigma + \sum_{\sigma\in\Theta} b_\sigma(\sigma - \hat{\sigma}) = \sum_{\sigma\in\Upsilon_i} b_\sigma\sigma + \sum_{\sigma\in\Theta} b_\sigma\alpha(\sigma).$$

Then since γ and the $\hat{\sigma}$ are in $\ker_{\Delta,i}$, so is γ'. Further, since each $\alpha(\sigma) \in \mathbb{Z}\Upsilon_i$, so is γ'. But $\partial_{\Delta,i}$ restricted to Υ_i is equal to $\partial_{\Upsilon,i}$, which is injective. It follows that

$$\gamma = \sum_{\sigma\in\Delta_i} b_\sigma\sigma = \sum_{\sigma\in\Theta} b_\sigma\hat{\sigma}.$$

We thus have an isomorphism

$$\pi\colon \mathbb{Z}\Theta \xrightarrow{\sim} \ker\partial_{\Delta,i}$$

determined by $\sigma \mapsto \hat{\sigma}$ with inverse given by setting elements of Υ_i equal to 0:

$$\sum_{\sigma\in\Delta_i} b_\sigma\sigma \longmapsto \sum_{\sigma\in\Theta} b_\sigma\sigma.$$

Next, we claim there is a commutative diagram with exact rows

$$\begin{tikzcd}
\mathbb{Z}\Theta \arrow[r, "\tilde{L}"] \arrow[d, "\iota"] & \mathbb{Z}\Theta \arrow[r] \arrow[d, "\pi", "\wr"'] & \operatorname{cok}\tilde{L} \arrow[r] \arrow[d, dashed] & 0 \\
\mathbb{Z}\Delta_i \arrow[r, "L_i"] & \ker\partial_{\Delta,i} \arrow[r] & \mathcal{K}_i(\Delta) \arrow[r] & 0
\end{tikzcd}$$

where ι is the natural inclusion. To check commutativity of the square on the left, let $\theta \in \Theta$. Then

$$L_i\iota(\theta) = \rho + \tilde{L}\theta$$

for some $\rho \in \mathbb{Z}\Upsilon$. We then have $\pi^{-1}(\rho + \tilde{L}\theta) = \tilde{L}\theta$. Hence, there is a well-defined vertical mapping on the left. By the snake lemma, that mapping is an isomorphism if and only if

$$\mathbb{Z}\Theta \xrightarrow{\iota} \mathbb{Z}\Delta_i / \ker L_i$$

is surjective. Therefore, to finish the proof, we must show that for all $\gamma \in \Upsilon_i$, there exists $\delta \in \mathbb{Z}\Theta$ such that $\gamma + \delta \in \ker L_i$.

Now $\ker L_i = \ker \partial_{\Delta,i}\partial^t_{\Delta,i} = \ker \partial^t_{\Delta,i}$. To get a description of $\ker^t_{\Delta,i}$, consider the exact sequence

$$\mathbb{Z}\Delta_{i+1} \xrightarrow{\partial_{\Delta,i+1}} \mathbb{Z}\Delta_i \to \operatorname{cok} \partial_{\Delta,i+1} \to 0.$$

Applying the left-exact functor $\operatorname{Hom}(\,\cdot\,,\mathbb{Z})$, gives the exact sequence

$$\mathbb{Z}\Delta_{i+1} \xleftarrow{\partial^t_{\Delta,i+1}} \mathbb{Z}\Delta_i \leftarrow (\operatorname{cok} \partial_{\Delta,i+1})^* \leftarrow 0, \tag{2}$$

where we have identified each $\mathbb{Z}\Delta_j$ with its dual using dual bases. There is an exact sequence,

$$0 \to \ker \partial_{\Delta,i} / \operatorname{im} \partial_{\Delta,i+1} \to \mathbb{Z}\Delta_i / \operatorname{im} \partial_{\Delta,i+1} \to \mathbb{Z}\Delta_i / \ker \partial_{\Delta,i} \to 0,$$

i.e,

$$0 \to \widetilde{H}_i(\Delta) \to \operatorname{cok} \partial_{\Delta,i+1} \to \mathbb{Z}\Delta_i / \ker \partial_{\Delta,i} \to 0. \tag{3}$$

However,

$$\mathbb{Z}\Delta_i / \ker \partial_{\Delta,i} \xrightarrow{\sim} \operatorname{im} \partial_{\Delta,i} = \operatorname{im} \partial_{\Upsilon,i} \approx \mathbb{Z}\Upsilon$$

since $\partial_{\Upsilon,i}$ is injective. Since $\mathbb{Z}\Upsilon$ is free, sequence (3) splits:

$$\operatorname{cok} \partial_{\Delta,i+1} \approx \widetilde{H}_i(\Delta) \oplus \mathbb{Z}\Upsilon.$$

Given $\gamma \in \Upsilon$, let γ^* be the dual function. Using the above isomorphism, we identify γ^* with an element of $(\operatorname{cok} \partial_{\Delta,i+1})^*$. Using equation (1), its image in $\mathbb{Z}\Delta_i$ under the mapping in (2) is

$$\gamma + \sum_{\theta \in \Theta} a_\theta(\gamma)\theta,$$

which by exactness of (2) is an element of $\ker \partial^t_{\Delta,i+1}$. Letting $\delta := \sum_{\theta\in\Theta} a_\theta(\gamma)\theta$, we see $\gamma + \delta \in \ker \partial_{\Delta,i+1}$, as required. □

Question. Is there still a mapping between these two groups in the case $\widetilde{H}_{d-1}(\Delta) \neq \widetilde{H}_{d-1}(\Upsilon)$?

Definition 1.7. Define the *cut and flow spaces* and the *cut and flow lattices* of Δ by:

$$\operatorname{Cut}(\Delta) := \operatorname{im}_{\mathbb{R}} \partial_d^t, \qquad \operatorname{Flow}(\Delta) := \ker_{\mathbb{R}} \partial_d$$
$$\mathcal{C}(\Delta) := \operatorname{im}_{\mathbb{Z}} \partial_d^t, \qquad \mathcal{F}(\Delta) := \ker_{\mathbb{Z}} \partial_d.$$

(NOTE: Need to define the cocritical group.)

Theorem 1.8. *There are exact sequences*

$$0 \to \mathbb{Z}^n/(\mathcal{C}+\mathcal{F}) \to \mathcal{C}^\sharp\,/\,\mathcal{C} \approx \mathbf{T}(\mathcal{K}_{d-1}(\Delta)) \to \mathbf{T}(\widetilde{H}_{d-1}(\Delta)) \to 0$$

$$0 \to \mathbf{T}(\widetilde{H}_{d-1}(\Delta)) \to \mathbb{Z}^n/(\mathcal{C}+\mathcal{F}) \to \mathcal{F}^\sharp\,/\,\mathcal{F} \approx \mathbf{T}(\mathcal{K}^*_{d-1}(\Delta)) \to 0$$

Further, $|\mathbf{T}(\mathcal{K}_{d-1}(\Delta))| = \tau(\Delta)$.

OLD STUFF

Fix the standard inner product on $\mathbb{R}^n$. If $\mathcal{L} \subseteq \mathbb{Z}^n$, define the dual lattice of $\mathcal{L}$ by

$$\mathcal{L}^\sharp := \{v \in \mathcal{L} \otimes \mathbb{R} : \langle v, w\rangle \in \mathbb{Z}\ \forall w \in \mathcal{L}\}\,.$$

Proposition 1.9. *With notation as above, we have*

1. $\mathcal{L}^\sharp \approx \operatorname{Hom}_{\mathbb{Z}}(\mathcal{L}, \mathbb{Z})$.
2. $(\mathcal{L}^\sharp)^\sharp = \mathcal{L}$.

Proof. □

Proposition 1.10. *Suppose that Δ is a d-dimensional simplicial complex. Then Δ possesses a simplicial spanning tree if and only if $\tilde{\beta}_{d-1}(\Delta) = 0$. We say that such complexes are* acyclic in codimension 1*.*

Proof. First suppose that Υ is a spanning tree of Δ. Then $C_d(\Upsilon) \subset C_d(\Delta)$, $C_{d-1}(\Upsilon) = C_{d-1}(\Delta)$, and $\operatorname{im}(\partial_{\Upsilon,d}) \subseteq \operatorname{im}(\partial_{\Delta,d})$. It follows that there is a surjection

$$\widetilde{H}_{d-1}(\Upsilon) = C_{d-1}(\Upsilon)/\operatorname{im}(\partial_{\Upsilon,d}) \twoheadrightarrow C_{d-1}(\Delta)/\operatorname{im}(\partial_{\Delta,d}) = \widetilde{H}_{d-1}(\Delta).$$

Since $\widetilde{H}_{d-1}(\Upsilon)$ is finite, it follows that $\widetilde{H}_{d-1}(\Delta)$ is also finite, hence $\tilde{\beta}_{d-1}(\Delta) = 0$.

Now suppose that Δ is acyclic in codimension 1. To construct a spanning tree, start with $\Upsilon = \Delta$. If $\widetilde{H}_d(\Upsilon) = 0$, then Υ is a spanning tree, and we are done. If not, then there is an integer-linear combination of facets σ_i in the kernel of ∂_d:

$$a_1\sigma_1 + a_2\sigma_2 + \cdots + a_k\sigma_k,$$

where we assume that $a_1 \neq 0$. If we work over the rational numbers, then we may assume $a_1 = 1$. Still working over the rationals, we see that

$$\partial_d(\sigma_1) = -\sum_{i=2}^{k} a_i \partial_d(\sigma_i).$$

Hence, if we remove the facet σ_1 from Υ, we obtain a smaller subcomplex Υ' without changing the image of the rational boundary map: $\operatorname{im}(\partial_{\Upsilon,d}) = \operatorname{im}(\partial_{\Upsilon',d})$. It follows from rank-nullity that

$$\tilde{\beta}_d(\Upsilon') = f_d(\Upsilon') - \operatorname{rank}\operatorname{im}(\partial_{\Upsilon',d}) = f_d(\Upsilon) - 1 - \operatorname{rank}\operatorname{im}(\partial_{\Upsilon,d}) = \tilde{\beta}_d(\Upsilon) - 1,$$

and $\tilde{\beta}_{d-1}(\Upsilon') = \tilde{\beta}_{d-1}(\Upsilon) = 0$. Continuing to remove facets in this way, we eventually obtain a spanning tree of Δ. $\square$

Corollary 1.11. *If $0 \leq i \leq d$, then Δ has an i-dimensional spanning tree if and only if $\tilde{\beta}_{i-1}(\Delta) = 0$.*

Proof. This follows from the previous proposition since $\tilde{\beta}_{i-1}(\operatorname{Skel}(\Delta)) = \tilde{\beta}_{i-1}(\Delta)$. $\square$

Proposition 1.12. *If Δ has a unique d-dimensional spanning tree Ψ, then $\Delta = \Psi$.*

Proof. To come. $\square$

Let $0 \leq i \leq d$. The *i-th tree number* for Δ is

$$\tau_i := \tau_i(\Delta) := \sum_{\Psi} |H_{i-1}(\Psi)|^2,$$

where the sum is over all i-dimensional spanning trees of Δ.

Corollary 1.13. $\tau_d(\Delta) = 1$ *if and only if Δ is a tree and $\widetilde{H}_{d-1}(\Delta) = 0$, i.e., if and only if $\widetilde{H}_d(\Delta) = \widetilde{H}_{d-1}(\Delta) = 0$.*

Proof. This follows directly from the definition of $\tau_d(\Delta)$ and Proposition 1.12. $\square$

Suppose that $\tilde{\beta}_{i-1}(\Delta) = 0$. It follows that Δ has an i-dimensional spanning tree Υ. Let

$$\widetilde{D}_i := \Delta_i(\Delta) \setminus \Delta_i(\Upsilon),$$

the i-faces of Δ not contained in Υ, and let $\widetilde{L}_i$ be the i-Laplacian of Δ with the rows and columns corresponding to faces in $\Delta_i(\Upsilon)$ removed.

Theorem 1.14 (Duval, Klivans, Martin)**.**

(i) *If* $\widetilde{H}_{i-1}(\Upsilon) = 0$*, there is an isomorphism*

$$\psi\colon\ \mathcal{K}_i(\Delta) \to \mathbb{Z}\widetilde{D}_i/\operatorname{im}(\widetilde{L}_i),$$

defined by dropping i*-faces of* Υ*: if* $c = \sum_{f\in\Delta_i(\Delta)} c_f \cdot f \in \ker\partial_i$ *represents an element of* $\mathcal{K}_i(\Delta)$*, then*

$$\psi(c) = \sum_{f\in\widetilde{D}_i} c_f \cdot f \mod \operatorname{im}(\widetilde{L}_i).$$

(ii) *(Simplicial matrix-tree theorem)*

$$\tau_{i+1} = \frac{|\widetilde{H}_{i-1}(\Delta)|^2}{|\widetilde{H}_{i-1}(\Upsilon)|^2}\det(\widetilde{L}_i).$$

2 Degrees

Definition 2.1. Let $0 \le i \le d$. The *i-th positive kernel* for Δ is the pointed cone

$$\ker^+ L_i := \{D \in \mathbb{Z}\Delta_i : D(\sigma) \ge 0 \text{ for all } \sigma \in \Delta_i\}.$$

Fixing an ordered Hilbert basis $H_i = (h_1, \dots, h_\ell)$ for $\ker^+ L_i$, define the *degree* of a chain $D \in \mathbb{Z}\Delta_i$ by

$$\deg(D) := (D\cdot h_1, \dots, D\cdot h_\ell)$$

where $D\cdot h_j := \sum_{\sigma\in\Delta_i} D(\sigma)h_j(\sigma)$. We say $d \in \mathbb{Z}^\ell$ is a *winning degree (in dimension i)* if each $D \in \mathbb{Z}\Delta_i$ with $\deg(D) \ge d$ is winnable.

Proposition 2.2. *The degree of a chain depends only on its linear equivalence class.*

Proof. It suffices to show that every element of $\operatorname{im} L_i$ has degree zero. If $\tau \in \ker L_i$ and $\sigma \in \mathbb{Z}\Delta_i$,

$$\langle\tau, L_i\sigma\rangle = \langle L_i^t\tau, \sigma\rangle = \langle L_i\tau, \sigma\rangle = 0,$$

since L_i is symmetric. □

Corollary 2.3. *If a chain D is winnable, then* $\deg(D) \ge 0$.

Proof. If D is winnable, then $D \sim E$ for some $E \ge 0$. Then $\deg(D) = \deg(E)$, and since each element of the Hilbert basis has nonnegative coefficients, $\deg(E) \ge 0$. □

Proof. See Jesse Kim's thesis for a "minimal criminal" argument using the *stars*, $\partial_i^t(\sigma)$, of elements $\sigma \in \Delta_{i-1}$. □

Lemma 2.4. *For each i, there is a strictly positive element $\alpha \in \ker L_i$, i.e., such that $\alpha_j > 0$ for all j.*

Corollary 2.5. *For each i, the $\mathbb{Z}$-span of $\ker^+ L_i$ is $\ker L_i$. Hence,*

$$(\ker^+ L_i)^\perp = (\ker L_i)^\perp = (\ker \partial_{i+1}^t)^\perp.$$

Proof. Take a strictly positive element $\alpha \in \ker L_i$ that is primitive, i.e., such that the components of α are relatively prime. We can then complete $\{\alpha\}$ to a basis $\{\alpha, \beta_1, \ldots, \beta_k\}$. (To see this, consider the exact sequence

$$0 \to \mathbb{Z}\alpha \to \mathbb{Z}^n \to \mathbb{Z}^n/\mathbb{Z}\alpha \to 0.$$

Since $\mathbb{Z}^n/\mathbb{Z}\alpha$ is torsion free, so the sequence splits.) Then, for each nonzero $N \in \mathbb{Z}$, the set $\{\alpha, \beta_1 + N\alpha, \ldots, \beta_k + N\alpha\}$ is still a basis for $\ker L_i$. By taking $N \gg 0$, this basis will consist solely of elements of $\ker^+ L_i$. □

Proposition 2.6. *For $0 \le i \le d$,*

$$(\ker L_i)^\perp / \operatorname{im}(L_i) \approx \mathbf{T}(\mathcal{K}_i(\Delta)).$$

Hence, the group of i-chains of degree 0 modulo firing rules is isomorphic to the torsion part of the i-th critical group of Δ.

Proof. To see that $\operatorname{im} L_i \subseteq (\ker L_i)^\perp$, let $\sigma \in \mathbb{Z}\Delta_i$ and $\tau \in \ker L_i = \ker \partial_{i+1}^t$. Then

$$\langle \tau, L_i\sigma \rangle = \langle \tau, \partial_{i+1}\partial_{i+1}^t \sigma \rangle = \langle \partial_{i+1}^t \tau, \partial_{i+1}^t \sigma \rangle = \langle 0, \partial_{i+1}^t \sigma \rangle = 0.$$

We also have $(\operatorname{im} \partial_i^t) \subseteq \ker \partial_i$. To see this, take $\sigma \in (\operatorname{im} \partial_i^t)^\perp$ and $\tau \in \mathbb{Z}\Delta_i$. Then

$$0 = \langle \sigma, \partial_i^t \tau \rangle = \langle \partial_\sigma, \tau \rangle.$$

Since τ is arbitrary, $\partial_i \sigma = 0$.

Next,

$$\operatorname{im} \partial_i^t \subseteq \ker \partial_{i+1}^t \Rightarrow (\ker L_i)^\perp = (\ker \partial_{i+1}^t)^\perp \subseteq (\operatorname{im} \partial_i^t)^\perp \subseteq \ker \partial_i.$$

Hence,

$$(\ker L_i)^\perp / \operatorname{im} L_i \subseteq \ker \partial_i / \operatorname{im} L_i =: \mathcal{K}_i(\Delta).$$

Since $\dim_{\mathbb{Q}} (\ker L_i)^\perp = \dim_{\mathbb{Q}} \operatorname{im} L_i$, the group $(\ker L_i)^\perp / \operatorname{im} L_i$ is finite, and hence torsion. So it is a subset of $\mathbf{T}(\mathcal{K}_i(\Delta))$. To show the opposite inclusion, let $\sigma \in \ker \partial_i$, and suppose there exists a positive integer n such that $n\sigma \in \operatorname{im} L_i$. Say $n\sigma = L_i \tau$, and let $\nu \in \ker L_i = \ker \partial_{i+1}^t$. Then

$$n\langle \nu, \sigma \rangle = \langle \nu, n\sigma \rangle = \langle \nu, L_i\tau \rangle = \langle \partial_{i+1}^t \nu, \partial_{i+1}^t \tau \rangle = 0.$$

Therefore, $\langle \nu, \sigma \rangle = 0$. So each torsion element of $\mathcal{K}_i(\Delta)$ is an element of $(\ker L_i)^\perp$. □

Proposition 2.7. *All divisors (cycles of codimension 1) of degree* 0 *on* Δ *are winnable if and only if* $\tau(\Delta) = 1$, *i.e., if and only if* Δ *is a forest.*

Proof. The divisors of degree 0 are $(\ker L_{d-1})^{\perp}$, and hence, by Proposition 2.6, these divisors are winnable if and only if $\mathbf{T}(\mathcal{K}_{d-1})(\Delta) = 0$. The result then follows from Theorem 1.8. $\square$

Definition 2.8. The *positive lattice* is

$$\Lambda(\Delta) = \mathrm{Span}_{\mathbb{N}} \left\{ \partial_{d-1}(e) : e \in \Delta_{d-1} \right\}.$$

Lemma 2.9. *The natural surjection* $\mathcal{K}_i(\Delta) \to \widetilde{H}_i(\Delta)$ *is an isomorphism when restricted to the free parts of* $\mathcal{K}_i(\Delta)$ *and* $\widetilde{H}_i(\Delta)$ *and a surjection when retricted to the torsion parts.*

Proof. Consider the exact sequence

$$0 \to \mathrm{im}\, \partial_{i+1} / \, \mathrm{im}\, L_i \to \mathcal{K}_i(\Delta) \to \widetilde{H}_i(\Delta) \to 0.$$

We have

$$\mathrm{im}\, L_i \subseteq \mathrm{im}\, \partial_{i+1} \subseteq (\ker L_i)^{\perp}.$$

So from Proposition 2.6, it follows that $\mathrm{im}\, \partial_{i+1} / \, \mathrm{im}\, L_i$ is finite. Tensoring the sequence by $\mathbb{Q}$ then gives the result about the free parts, and since the torsion functor $\mathbf{T}(\,\cdot\,)$ is left-exact, there is a surjection for the torsion parts. $\square$

Proposition 2.10. *Let*

$$X := \left\{ D \in \mathbb{Z}\Delta_{d-1} : \partial_{d-1}(D) \in \Lambda(\Delta) \right\}.$$

Then every $D \in X$ *is winnable if and only if* Δ *is a forest, acyclic in codimension* 1, *i.e., if and only if* $\widetilde{H}_d(\Delta) = \widetilde{H}_{d-1}(\Delta) = 0$.

Proof. Note that $D \in \Lambda(\Delta)$ if and only if $D = E + A$ for some $E \geq 0$ and $A \in \ker \partial_{d-1}$. Thus, the condition that every $D \in \Lambda(\Delta)$ is winnable is equivalent to the condition that $\ker \partial_{d-1} = \mathrm{im}\, L_{d-1}$, i.e, $\mathcal{K}_{d-1}(\Delta) = 0$.

($\Rightarrow$) Suppose every $D \in \Lambda(\Delta)$ is winnable so that $\mathcal{K}_{d-1}(\Delta) = 0$. By Theorem 1.8, we have $|\mathbf{T}(\mathcal{K}_{d-1})| = \tau_d(\Delta)$. Thus, $\tau_d(\Delta) = 1$, so that Δ is a forest, or equivalently, $\widetilde{H}_d(\Delta) = 0$. Further, there is a surjection $\mathcal{K}_{d-1}(\Delta) \twoheadrightarrow \widetilde{H}_{d-1}(\Delta)$. So $\widetilde{H}_{d-1}(\Delta) = 0$.

($\Leftarrow$) Conversely, suppose that $\widetilde{H}_d(\Delta) = \widetilde{H}_{d-1}(\Delta) = 0$. The fact that $\widetilde{H}_d(\Delta) = 0$ says Δ is a forest. So $\tau_d(\Delta) = 1$, which implies that $\mathcal{K}_{d-1}(\Delta)$ is free by Theorem 1.8. However, the free part of $\mathcal{K}_{d-1}(\Delta)$ is the same as the free part of $\widetilde{H}_{d-1}(\Delta)$. Therefore, $\mathcal{K}_{d-1}(\Delta) = 0$. $\square$

Example 2.11. Consider the simplicial complex Δ pictured in Figure 1. Since $\widetilde{H}_2(\Delta) = 0$, the complex is a forest, but since $\widetilde{H}_1(\Delta) = \mathbb{Z} \neq 0$, it is not a tree. Thus, by Proposition 2.7, a divisor is winnable if and only if its degree is 0. Order the edges lexicographically. Consider a generator for the first homology such as

$$D = (0, 1, -1, 0, 0, 1) = \overline{02} + \overline{23} - \overline{03}.$$

The Hilbert basis for the positive kernel is given by the rows of the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Therefore, $\deg(D) = (-1, 1, 1, 1)$, so D is not winnable. However, $\partial_1(D) = 0 \in \Lambda(\Delta)$.

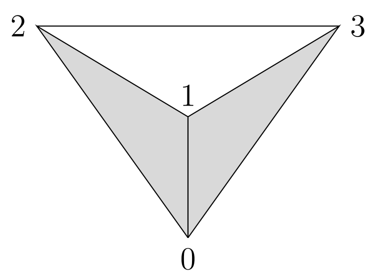

Figure 1: A hollow tetrahedron with two missing facets.

OLD STUFF

Theorem 2.12. *If $\tau_d(\Delta) = 1$, then 0 is a winning degree in dimension $d - 1$.*

Proof. Since $\tau_d(\Delta) = 1$, by Proposition 1.13, we have $\widetilde{H}_d(\Delta) = \widetilde{H}_{d-1}(\Delta) = 0$. Thus, ∂_d is injective, and $\operatorname{im} \partial_d = \ker \partial_{d-1}$. The universal coefficient theorem gives an exact sequence

$$0 \to \operatorname{Ext}^1(H_{d-1}(\Delta), \mathbb{Z}) \to H^d(\Delta) \to \operatorname{Hom}(H_d(\Delta), \mathbb{Z}) \to 0,$$

which implies that $H^d(\Delta) = 0$. We have $\widetilde{H}^d(\Delta) = \widetilde{H}^d(\Delta)$ for $d > 0$. In the case where $d = 0$, we have that $\tau_d(\Delta) = 1$ implies that Δ is a single point, ∂_0 is an isomorphism, and thus, every $(d-1)$-cycle is linearly equivalent to 0. So we will assume $d > 0$ from now on.

To show 0 is a winning degree in dimension $d-1$, we must show $\mathcal{K}_{d-1}(\Delta)$ is also torsion-free. So let $\alpha \in \ker \partial_{d-1}$ and suppose there exists an integer $n > 0$ such that $n\alpha \in \operatorname{im}(L_{d-1})$. So there exists $\beta \in C_{d-1}(\Delta)$ such that

$$n\alpha = L_{d-1}\beta = \partial_d \partial_d^t \beta.$$

Since $\widetilde{H}_{d-1}(\Delta) = 0$, there exists $\gamma \in C_d(\Delta)$ such that $\partial_d \gamma = \alpha$. Since ∂_d is injective, $n\gamma = \partial_d^t \beta$. Since $\widetilde{H}^d(\Delta) = C^d(\Delta)/\operatorname{im}\partial_d^t = 0$, it follows that $\gamma = \operatorname{im}\partial_d^t$, and hence, $\alpha \in \operatorname{im} L_{d-1}$, as required. $\square$

Example 2.13. Figure 2 illustrates a 2-dimensional complex P which is a triangulation of the real projective plane. We have $\widetilde{H}_0(P) = \widetilde{H}_2(P) = 0$, and $\widetilde{H}_1(P) \approx \mathbb{Z}/2\mathbb{Z}$. From the definition of the tree number, $\tau_2(P) = 4$, and it follows from the matrix-tree theorem that $\det(\widetilde{L}_0) = 4$ with respect to any 1-dimensional spanning tree of P. (There are 6^4 such spanning trees since the 1-skeleton of P is the completer graph K_6.) The cycle $C := \overline{01} + \overline{12} - \overline{02}$ has degree $(0,0,0,0,0)$ but is not winnable. We have $\mathcal{K}_1(P) \approx \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and hence $2C$ is winnable.

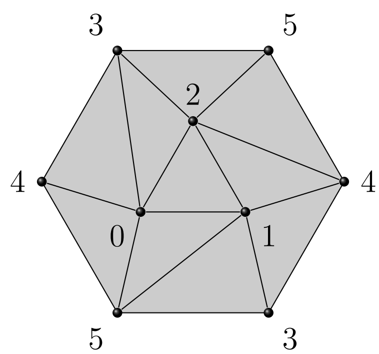

Figure 2: A triangulation of the real projective plane, $\mathbb{RP}^2$.

Question. Can we find an example where $\tau_d(\Delta) = 1$ and 0 is not a winning degree in dimension $d-1$? The 2-dimensional chessboard complex is an example of a non-tree for which 0 is a winning degree in dimension 1.

Question.

NOTE: The dual of an intersection of cones is the minkowski sum of the duals of the cones in the intersection. For example, over the rationals,

$$(\ker L_i \cap \mathcal{O}^+)^* = (\ker L_i)^* + (\mathcal{O}^+)^* = (\ker L_i)^\perp + \mathcal{O}^+.$$

The expression on the left is the dual of the positive kernel (as a cone), which is the set of divisors of nonnegative degree. The expression on the right is the set of divisors having the same degree as an effective divisor.