Notes on divisors on simplicial complexes.

Let Δ be a d-dimensional simplicial complex, and suppose that 0 is a winnable degree in dimension $d-1$. This means that all chains of dimension $d-1$ with all-zero degrees are winnable. This is the same as saying that

$$(\ker L_{d-1})^{\perp}/\operatorname{im}(L_{d-1}) = 0.$$

However, we know that

$$(\ker L_{d-1})^{\perp}/\operatorname{im}(L_{d-1}) \approx \mathbf{T}(K_{d-1}(\Delta)) := \mathbf{T}(\ker \partial_{d-2}/\operatorname{im}(L_{d-1}))$$

where $\mathbf{T}$ stands for "torsion part".

Next, suppose that Δ has a $(d-1)$-dimensional spanning tree Υ, which is the same as assuming $\tilde{\beta}_{d-2}(\Delta) = 0$, i.e., $\widetilde{H}_{d-2}(\Delta)$ is torsion (QUESTION: is this true?). Since Υ is a $(d-1)$-tree, we have

$$\tilde{H}_{d-1}(\Upsilon) = 0, \quad \text{and} \quad \tilde{\beta}_{d-2}(\Upsilon) = 0,$$

and from this it follows that

$$f_{d-1}(\Upsilon) = f_{d-1}(\Delta) - \tilde{\beta}_{d-1}(\Delta) + \tilde{\beta}_{d-2}(\Delta) = f_{d-1}(\Delta) - \tilde{\beta}_{d-1}(\Delta).$$

If we further assume that $\widetilde{H}_{d-2}(\Upsilon) = 0$, then there is an isomorphism

$$K_{d-1}(\Delta) \to \mathbb{Z}\tilde{F}_{d-1}/\operatorname{im}(\tilde{L}_{d-1})$$

where $\tilde{F}_{d-1}$ are the $(d-1)$-dimensional faces that are not in Υ and $\tilde{L}_{d-1}$ is the reduced Laplacian with respect to Υ. (What happens if $\widetilde{H}_{d-2}(\Upsilon) \neq 0$? Is there still a map? In our case, we know that $K_{d-1}(\Delta)$ has no torsion. This implies that $\det(\tilde{L}_{d-1}) = 1$. Is it surjective?)

The d-th tree number for Δ is

$$\tau_n(\Delta) := \sum_{\Psi} |\widetilde{H}_{d-1}(\Psi)|^2$$

where the sum is over all $(d-1)$-dimensional spanning trees of Δ. By the simplicial matrix-tree theorem,

$$\tau_n(\Delta) = \frac{|\widetilde{H}_{d-2}(\Delta)|^2}{|\widetilde{H}_{d-2}(\Upsilon)|^2}\det(\tilde{L}_{d-1}) = |\widetilde{H}_{d-2}(\Delta)|^2.$$

(since $\det(\tilde{L}_{d-1}) = 1$ and $\widetilde{H}_{d-2}(\Upsilon) = 0$).

Now, assume further that $\widetilde{H}_{d-2}(\Delta) = 0$. It then follows that $\tau_n(\Delta) = 1$. It follows that Δ has a unique d-dimensional spanning tree Ψ. Since Δ has a d-dimensional spanning tree, it follows that $\tilde{\beta}_{d-1}(\Delta) = 0$, and hence,

$$f_d(\Psi) = f_d(\Delta) - \tilde{\beta}_d(\Delta) + \tilde{\beta}_{d-1}(\Delta) = f_d(\Delta) - \tilde{\beta}_d(\Delta).$$

Since Ψ is a d-spanning tree, it has the same $(d-1)$-skeleton. It will equal Δ if it has the same facets. For sake of contradiction, suppose it does not. Then the above formula says that $\tilde{\beta}_d(\Delta) > 0$. This means there is a nonzero d-chain in the kernel of ∂_d. This chain contains at least two facets σ_1 and σ_2 in its support. Fix σ_1 and remove σ_2 from Δ. We get a new complex Δ'. Working over the rational numbers, we may assume