ELSEVIER

Signal Processing 48 (1996) 85–90

SIGNAL PROCESSING

Modified MUSIC algorithm for estimating DOA of signals

Debasis Kundu[1]

Department of Mathematics, Indian Institute of Technology, Kanpur, India

Received 12 November 1993; revised 14 April 1994 and 23 September 1994

Abstract

High-resolution signal parameter estimation is a significant problem in many signal processing applications. Such applications include direction of arrival (DOA) estimation for narrow band signals emitted by multiple sources and received by sensor arrays. It is well known that MUSIC algorithm outperforms any other method existing in the literature. In this article a modified MUSIC algorithm is proposed using the conjugate data. Strong consistency of the modified method is established. It is observed that the modified MUSIC works significantly better than the ordinary MUSIC at different SNR in terms of the mean squared error.

Zusammenfassung

Die hochauflösende Schätzung von Signalparametern ist ein wesentliches Problem in vielen Anwendungen der Signalverarbeitung. Solche Anwendungen schliessen die Schätzung des Einfallswinkels für Schmalbandsignale ein, die von mehreren Signalquellen ausgesendet und von einer Sensorgruppe empfangen werden. Es ist allgemein bekannt, daß der MUSIC Algorithmus jede andere publizierte Methode in seiner Leistungsfähigkeit übertrifft. In dieser Arbeit wird ein modifizierter MUSIC Algorithmus unter Verwendung konjugierter Daten vorgeschlagen. Für die modifizierte Methode wird die starke Konsistenz aufgezeigt. Man beobachtet, daß der modifizierte MUSIC Algorithmus bei verschiedenen Signal-Geräusch-Abständen hinsichtlich des mittleren quadratischen Fehlers wesentlich besser als der gewöhnliche MUSIC Algorithmus arbeitet.

Résumé

L'estimation de paramètres pour un signal de haute résolution est un problème significatif dans de nombreuses applications de traitement du signal. De telles applications incluent l'estimation de la direction d'arrivée (DA) pour des signaux à bande étroite émis par des sources multiples et reçus par des réseaux de senseurs. Il est bien connu que les algorithmes MUSIC surpassent toutes les autres méthodes existant dans la littérature. Dans cet article, un algorithme MUSIC modifié est proposé, qui utilise les données conjuguées. La grande consistence de la méthode modifiée est établie. On observe que le MUSIC modifié fonctionne significativement mieux que le MUSIC habituel à différents rapports signal-sur-bruit, en termes d'erreur quadratique moyenne.

Keywords: Direction of arrival of signals; MUSIC algorithm; Consistent estimates and mean squared errors

[1] The work is partially supported by a Grant No. SR/OY/M-06/93 by the Department of Science and Technology Govt. of India.

SSDI 0165-1684(95)00126-3

1. Introduction

A considerable amount of work has focused in the past two decades on eigendecomposition based methods for estimating direction of arrival (DOA) for narrow band signals emitted by multiple sources and received by sensor arrays. See, for example, the following literature [2, 4, 8, 10, 14–17, 19] and the references cited there. In this situation the $p \times 1$ observation vector $\boldsymbol{Y}(t)$ is successfully described by the following model:

$$\boldsymbol{Y}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{n}(t), \quad t = 1, \dots, N, \tag{1.1}$$

where at time t, $\boldsymbol{Y}(t)$ is a p-complex vector of observations received by p-sensors uniformly spaced. $\boldsymbol{x}(t)$ is a q-complex vector of unobservable signals emitted from q $(< p)$ -sources and $\boldsymbol{n}(t)$ is the additive noise p-vector. It is assumed that $\boldsymbol{n}(t)$ and $\boldsymbol{x}(t)$, $t = 1, \dots, N$, are independent identically distributed random variables such that $\boldsymbol{x}(t)$ and $\boldsymbol{n}(t)$ are independent and

$$\begin{aligned} &E(\boldsymbol{n}(t)) = 0, \qquad E(\boldsymbol{n}(t)\boldsymbol{n}(t)^{\mathrm{H}}) = \sigma^2 \boldsymbol{I}, \\ &E(\boldsymbol{x}(t)) = 0, \qquad E(\boldsymbol{x}(t)\boldsymbol{x}(t)^{\mathrm{H}}) = \Gamma, \\ &\mathrm{rank}(\Gamma) = q, \end{aligned} \tag{1.2}$$

where 'H' indicates the conjugate transpose of a matrix or of a vector. The $p \times q$ matrix $\boldsymbol{A}$ has the special structure

$$\begin{aligned} &\boldsymbol{A} = [\boldsymbol{a}_1, \dots, \boldsymbol{a}_q], \\ &\boldsymbol{a}_k^{\mathrm{T}} = \boldsymbol{a}^{\mathrm{T}}(\tau_k) = (1, \mathrm{e}^{\mathrm{j}\omega_0 \tau_k}, \dots, \mathrm{e}^{\mathrm{j}\omega_0 (p-1)\tau_k}), \end{aligned} \tag{1.3}$$

where $\tau_k = c^{-1} \Delta \sin\theta_k$, c is the speed of propagation, $\mathrm{j} = \sqrt{-1}$, Δ is the spacing between the sensors, and θ_k is the DOA of signals from the kth source.

Usually, ω_0 is assumed to be known and, without loss of generality [11], we can take $\omega_0 = 1$. The maximum likelihood (ML) approach, which is asymptotically optimal, is computationally expensive. In comparison, the eigendecomposition methods are relatively economical and simple to implement. So there is an interest in exploring performance relative to the ML approach. It is observed [3, 15] that the MUSIC algorithm outperforms all the methods and it has a high statistical efficiency, i.e., the variances of the estimated DOAs are very close to the Cramer–Rao lower bound.

It is well known that the estimation of $\tau = (\tau_1, \dots, \tau_q)$ is very similar to the estimation of the unknown frequencies in an undamped superimposed exponential model. It is observed [18] that in estimating the frequencies in an undamped superimposed exponential model better results (in terms of lower mean squared errors) can be obtained using both the ordinary and the conjugate data than just using the ordinary one for finite length data sequence, although they are asymptotically equivalent. Observe that the model (1.1) can also be written in the following form:

$$\bar{\boldsymbol{Y}}(t) = \bar{\boldsymbol{A}}\bar{\boldsymbol{x}}(t) + \bar{\varepsilon}(t), \quad t = 1, \dots, N. \tag{1.4}$$

Here an overbar denotes the conjugate of each element of a vector or of a matrix. In [2], Bai and Rao have mentioned that better DOA estimates can be obtained if the information of $\boldsymbol{Y}(t)$ and $\bar{\boldsymbol{Y}}(t)$ can be combined. Recently [6], it is observed that using the centro-Harmitian sample covariance matrix (i.e. basically using the information of $\boldsymbol{Y}(t)$ and $\bar{\boldsymbol{Y}}(t)$) the performance of the ESPRIT estimators can be improved drastically compared to ordinary ESPRIT, in terms of lower mean squared errors, in DOA estimation. No such attempts have been made so far, at least not known to the author, to improve MUSIC estimates using both $\boldsymbol{Y}(t)$ and $\bar{\boldsymbol{Y}}(t)$. It is the papers [2, 6] that motivate the author to search for an improved MUSIC estimator using both the information of $\boldsymbol{Y}(t)$ and $\bar{\boldsymbol{Y}}(t)$.

In this article we propose a modified MUSIC (MMUSIC) algorithm to estimate DOA of signals using both ordinary and conjugate data. It is known that MUSIC estimates are asymptotically efficient and it reaches the Cramer–Rao lower bound [12] as N tends to infinity. So it is not possible to beat MUSIC estimates asymptotically. In fact, MUSIC and MMUSIC are asymptotically equivalent. Since MMUSIC estimate uses the ordinary as well as the conjugate data it is expected that it will give better results than the MUSIC estimate for finite sample. We show that the new estimates are consistent. Some simulations are performed. It is observed that the mean squared errors (MSE) of

MMUSIC are significantly lower than that of ordinary MUSIC.

2. Modified MUSIC algorithm

We denote the $p \times p$ dispersion matrix of $\boldsymbol{Y}(t)$ by $\Sigma = E(\boldsymbol{Y}(t)\boldsymbol{Y}(t)^{\mathrm{H}}) = \boldsymbol{A}\Gamma\boldsymbol{A}^{\mathrm{H}} + \sigma^2\boldsymbol{I}$ (by the assumption (1.2)) and the $p \times N$ data matrix by $\boldsymbol{Y} = (\boldsymbol{Y}(1), \dots, \boldsymbol{Y}(N))$. Let $\boldsymbol{J}$ be the $p \times p$ exchange matrix whose entries all are zero except the one in the (i, $p - i + 1$)th position for $i = 1, \dots, p$. Therefore, the dispersion matrix of $\boldsymbol{J}\bar{\boldsymbol{Y}}(t)$ is

$$\boldsymbol{J}E(\bar{\boldsymbol{Y}}(t)\bar{\boldsymbol{Y}}(t)^{\mathrm{H}})\boldsymbol{J} = \boldsymbol{J}\bar{\Sigma}\boldsymbol{J} = \boldsymbol{J}\bar{\boldsymbol{A}}\bar{\Gamma}\bar{\boldsymbol{A}}^{\mathrm{H}}\boldsymbol{J} + \sigma^2\boldsymbol{I}. \tag{2.1}$$

Observe that (2.2) is true because $\boldsymbol{J}$ is a symmetric matrix and $\boldsymbol{J}^2 = \boldsymbol{I}$. We define the matrix $\boldsymbol{R}$ as follows:

$$\boldsymbol{R} = \boldsymbol{A}\Gamma\boldsymbol{A}^{\mathrm{H}} + \boldsymbol{J}\bar{\boldsymbol{A}}\bar{\Gamma}\bar{\boldsymbol{A}}^{\mathrm{H}}\boldsymbol{J} + 2\sigma^2\boldsymbol{I}. \tag{2.2}$$

Let us denote the eigenvalue/eigenvector decomposition of $\boldsymbol{R}$ by

$$\boldsymbol{R} = \sum_{i=1}^{p} \lambda_i \boldsymbol{S}_i \boldsymbol{S}_i^{\mathrm{H}}, \tag{2.3}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_q = \cdots = \lambda_p = 2\sigma^2$. Let

$$D(\tau) = \boldsymbol{a}(\tau) \sum_{i=q+1}^{p} \boldsymbol{S}_i\boldsymbol{S}_i^{\mathrm{H}}\boldsymbol{a}(\tau). \tag{2.4}$$

We show in the next section that $D(\tau) = 0$ for $\tau = \tau_1, \dots, \tau_q$. This means that $D(\tau)$ has q local minima at $\tau = \tau_1, \dots, \tau_q$. MMUSIC algorithm uses this property to estimate the source parameters. We suggest the following procedure to estimate the source parameters.

(1) Compute the sample estimate of $\boldsymbol{R}$ and its eigenvalue/eigenvector decomposition, i.e.

$$\hat{\boldsymbol{R}} = \frac{1}{N}\sum_{t=1}^{N} \boldsymbol{Y}(t)\boldsymbol{Y}(t)^{\mathrm{H}} + \frac{1}{N}\sum_{t=1}^{N} \boldsymbol{J}\bar{\boldsymbol{Y}}(t)\bar{\boldsymbol{Y}}(t)^{\mathrm{H}}\boldsymbol{J}$$
$$= \frac{1}{N}(\boldsymbol{Y}\boldsymbol{Y}^{\mathrm{H}} + \boldsymbol{J}\bar{\boldsymbol{Y}}\bar{\boldsymbol{Y}}^{\mathrm{H}}\boldsymbol{J}) = \sum_{i=1}^{p} \hat{\lambda}_i\hat{\boldsymbol{S}}_i\hat{\boldsymbol{S}}_i^{\mathrm{H}}. \tag{2.5}$$

(2) Compute the estimated null spectrum $\hat{D}(\tau)$, where

$$\hat{D}(\tau) = \boldsymbol{a}(\tau) \sum_{i=q+1}^{p} \hat{\boldsymbol{S}}_i\hat{\boldsymbol{S}}_i^{\mathrm{H}}\boldsymbol{a}(\tau). \tag{2.6}$$

(3) Using the value of $\hat{D}(\tau)$ at the grid points, compute the q largest, say $\tau_1, \dots, \tau_q$, local minima of $\hat{D}(\tau)$ and take $\tau_1, \dots, \tau_q$ as the estimate of the source parameters.

Observe that in the MUSIC algorithm instead of using $\hat{\boldsymbol{R}}$, $(1/N)\boldsymbol{Y}\boldsymbol{Y}^{\mathrm{H}}$ is being used. Here we are using the conjugate data through $(1/N)(\boldsymbol{J}\bar{\boldsymbol{Y}}\bar{\boldsymbol{Y}}^{\mathrm{H}}\boldsymbol{J})$ with a hope that it may give better results. Theoretical justifications for using $\hat{\boldsymbol{R}}$ are provided in the next section.

3. Consistency of MMUSIC estimates

It is well known [13] that for any $\tau_1, \dots, \tau_q$, there exists $g_0, g_1, \dots, g_q$ such that $z_1 = \mathrm{e}^{\mathrm{j}\tau_1}, \dots, z_q = \mathrm{e}^{\mathrm{j}\tau_q}$ are the roots of the following polynomial equation:

$$P(z) = g_0 + g_1 z + \cdots + g_q z^q = 0. \tag{3.1}$$

Since the roots of the polynomial $p(z)$ are of unit modulus, one must have

$$\bar{z}_k = z_k^{-1} \quad \text{for } k = 1, \dots, q. \tag{3.2}$$

Define the polynomial $Q(z)$ by

$$Q(z) = z^q\overline{P(z)} = \bar{g}_q + \bar{g}_{q-1}z + \cdots + \bar{g}_0 z^q. \tag{3.3}$$

Using (3.2), one can see that the polynomials $P(z)$ and $Q(z)$ have the same roots. Comparing the coefficients of the two polynomials, we obtain

$$\frac{g_k}{g_q} = \frac{\bar{g}_{q-k}}{\bar{g}_0} \quad \text{for } k = 0, \dots, q. \tag{3.4}$$

Using (3.4) and (3.1), we can say that for any $\tau_1, \dots, \tau_q$, there exists $b_k = g_k(\bar{g}_0/g_q)^{1/2}$, $k = 0, \dots, q$, such that

$$b_0 + b_1\mathrm{e}^{\mathrm{j}\tau_k} + \cdots + b_q e^{\mathrm{j}q\tau_k} = 0 \quad \text{for } k = 1, \dots, q, \tag{3.5}$$

where

$$b_i = \bar{b}_{q-i}, \quad i = 0, \dots, q. \tag{3.6}$$

Observe that condition (3.6) is the conjugate symmetry property and can be written compactly as

$$\boldsymbol{b} = \boldsymbol{J}\bar{\boldsymbol{b}}, \tag{3.7}$$

where $\boldsymbol{J}$ is the exchange matrix defined as (2.1) and $\boldsymbol{b} = (b_0, \dots, b_q)$; see [7]. Before proving the main result we need the following lemmas.

Lemma 1. *$\boldsymbol{u}$ is an eigenvector of $A\Gamma A^{\mathrm{H}}$ corresponding to the zero eigenvalue if and only if $\boldsymbol{u}$ is an eigenvector $\boldsymbol{J}\bar{A}\bar{\Gamma}\bar{A}^{\mathrm{H}}\boldsymbol{J}$ corresponding to the zero eigenvalue.*

Proof. Let $\boldsymbol{u}$ be an eigenvector of $A\Gamma A^{\mathrm{H}}$ corresponding to the zero eigenvalue, i.e.

$$A\Gamma A^{\mathrm{H}}\boldsymbol{u} = 0 \;\Rightarrow\; A^{\mathrm{H}}\boldsymbol{u} = 0 \quad \text{since } \operatorname{rank}(\Gamma) = q. \tag{3.8}$$

From (3.5), (3.6) and (3.7) it is clear that for any $\boldsymbol{u}$, there exists a non-zero complex constant c such that,

$$\boldsymbol{b} = c\boldsymbol{u}, \quad \text{where } \boldsymbol{b} = \boldsymbol{J}\boldsymbol{b}. \tag{3.9}$$

Therefore, from (3.8), it is clear that

$$\begin{aligned} A^{\mathrm{H}}\boldsymbol{u} = \boldsymbol{0} \;&\Rightarrow\; A^{\mathrm{H}}\boldsymbol{b} = \boldsymbol{0} \\ &\Rightarrow\; \boldsymbol{J}\bar{A}\bar{\Gamma}\bar{A}^{\mathrm{H}}\boldsymbol{J}\boldsymbol{J}\bar{\boldsymbol{b}} = 0 \quad \text{since } \boldsymbol{J}^2 = \boldsymbol{I} \\ &\Rightarrow\; \boldsymbol{J}\bar{A}\bar{\Gamma}\bar{A}^{\mathrm{H}}\boldsymbol{J}\boldsymbol{u} = 0 \quad \text{from (3.9).} \end{aligned} \tag{3.10}$$

The reverse part follows similarly. □

From Lemma 1 and using the property that both the matrix $A\Gamma A^{\mathrm{H}}$ and $\boldsymbol{J}\bar{A}\bar{\Gamma}\bar{A}^{\mathrm{H}}\boldsymbol{J}$ are positive semidefinite, we can conclude that the null space of $A\Gamma A^{\mathrm{H}}$, $\boldsymbol{J}\bar{A}\bar{\Gamma}\bar{A}^{\mathrm{H}}\boldsymbol{J}$ and $A\Gamma \mathrm{A}^{\mathrm{H}} + \boldsymbol{J}\bar{A}\bar{\Gamma}\bar{A}^{\mathrm{H}}\boldsymbol{J}$ are all the same.

Lemma 2. *Under the assumptions* (1.2) *of the model* (1.1) $D(\tau) = 0$ *for* $\tau = \tau_1, \ldots, \tau_q$.

Proof. Observe that the eigenvalues of $\boldsymbol{R}$ are of the form

$$\lambda_1 > \lambda_2 > \cdots > \lambda_{q-1} > \lambda_q = \cdots = \lambda_p = 2\sigma^2. \tag{3.11}$$

Therefore the result follows immediately. □

Finally, we can state the main result.

Theorem 1. *Under the same assumptions as Lemma 2, the MMUSIC estimators are strongly consistent estimators of the true DOA of signals.*

Proof. From [1], it follows that

$$\sum_{i=q+1}^{p} \hat{\boldsymbol{S}}_i\hat{\boldsymbol{S}}_i^{\mathrm{H}} \;\rightarrow\; \sum_{i=q+1}^{p} \boldsymbol{S}_i\boldsymbol{S}_i^{\mathrm{H}} \quad \text{almost surely.} \tag{3.12}$$

Therefore,

$$\hat{D}(\tau) \;\rightarrow\; D(\tau) \quad \text{almost surely for all } \tau. \tag{3.13}$$

Since $D(\tau)$ has q local minima at $\tau_1, \ldots, \tau_q$, therefore the q-local minima of $\hat{D}(\tau)$ with proper rearrangement will converge to $\tau_1, \ldots, \tau_q$ as N tends to infinity. □

4. Numerical experiments

We have considered an example to compare the performance of MUSIC and MMUSIC at different noise levels. We consider the following model:

$$p = 5,\; q = 2,\; \tau_1 = 1.0,\; \tau_2 = 2.0 \text{ and } N = 100. \tag{4.1}$$

The covariance matrix of the real and imaginary part of $\boldsymbol{x}(t)$ is

$$\begin{bmatrix} 1.25 & 1.0 \\ 1.0 & 1.25 \end{bmatrix}. \tag{4.2}$$

The real and imaginary part of $\boldsymbol{x}(t)$ are taken to be independent. We use the error random variables to be normally distributed with $\sigma^2 = 0.50$ (SNR $\simeq 3.33$), 1.125 (SNR $\simeq -0.19$), 2.0 (SNR $\simeq -2.68$), 3.125

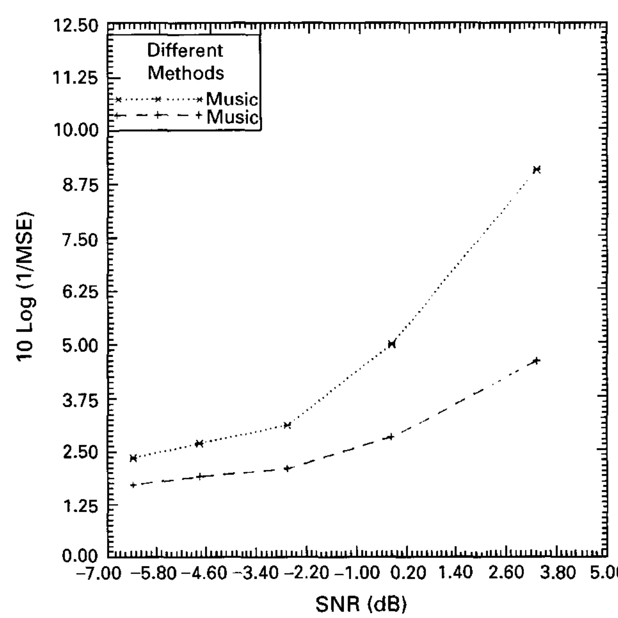

Fig. 1. Mean squared errors of τ_1.

(SNR $\simeq -4.62$), 4.50 (SNR $\simeq -6.21$). For each data set we estimated the DOA of signals by both the methods and the mean squared errors over 100 simulations are calculated. Since both are similar in nature, we reported the MSE of τ_1 only in Fig. 1.

5. Conclusions

From Fig. 1 it is obvious that MMUSIC works significantly better than MUSIC at all SNR level, in terms of MSE.

Now I would like to give some justifications to enhance the superiority of MMUSIC estimates over MUSIC estimates. We have already seen that the null spaces of $\boldsymbol{A\Gamma A}^{\mathrm{H}}$ and $(\boldsymbol{A\Gamma A}^{\mathrm{H}} + \boldsymbol{J\bar{A}\bar{\Gamma}\bar{A}}^{\mathrm{H}}\boldsymbol{J})$ are the same. The basic idea of both MUSIC and MMUSIC is to obtain vectors of the form $a(\tau)$ (as defined in (1.3)) which are orthogonal to the corresponding null spaces. So naturally the performance of the methods depend on how good approximation of the null spaces can be made from the given data. If we know exactly Σ or $\boldsymbol{R}$ then we can estimate the DOAs without any error. Since the matrix Σ or $\boldsymbol{R}$ is not known so it is estimated by $\hat{\Sigma} = (1/N)\boldsymbol{YY}^{\mathrm{H}}$ or $\hat{\boldsymbol{R}}$, respectively. Observe that as N tends to infinity then by the law of large numbers, $\hat{\Sigma}$ and $\hat{\boldsymbol{R}}$ converge to Σ and $\boldsymbol{R}$, respectively, and their orders of convergence are also same, which follows from the law of iterated logarithm. Therefore, asymptotically the estimates obtained using $\hat{\Sigma}$ or $\hat{\boldsymbol{R}}$ are equivalent and both of them attain the Cramer–Rao lower bound. But it is interesting to observe that both $\hat{\boldsymbol{R}}$ and $\boldsymbol{R}$ satisfy a conjugate symmetry constraint, namely $\boldsymbol{R} = \boldsymbol{J\bar{R}J}$ and $\hat{\boldsymbol{R}} = \boldsymbol{J\bar{\hat{R}}J}$, whereas Σ or $\hat{\Sigma}$ does not satisfy any such constraint. It is mentioned by Orfanidis [9, p. 357] (see also [5, 7]) that if constrained estimates are used then it is expected to obtain better results than the ordinary estimates for finite sample. Therefore, it is not surprising that better estimates are obtained using $\hat{\boldsymbol{R}}$ rather than $\hat{\Sigma}$. It is worth mentioning that all the results available so far for the properties of the MUSIC estimates are for large sample, and no theoretical work has been done regarding the finite sample behavior of these estimates. Unless we have a theory available for the finite sample properties of the MUSIC estimator, the two methods, namely MUSIC and MMUSIC, cannot be compared theoretically. It seems more work is needed in this direction.

Finally, since both the methods involve same amount of computations, it is recommended to use MMUSIC algorithm in estimating DOA of signals.

Acknowledgements

I would like to thank the referees for some very valuable and constructive suggestions.

References

[1] Z.D. Bai, B.Q. Miao and C.R. Rao, "Estimation of direction of arrival of signals: Asymptotic results", in: S. Haykins, ed., *Advances in Spectrum Analysis and Array Processing*, Prentice-Hall, Englewood Cliffs, NJ, Vol. II, pp. 327–347.

[2] Z.D. Bai and C.R. Rao, "Spectral analytic methods for the estimation of number of signals and direction of arrivals," in: S. Prasad and R.L. Kashyap, eds., *Spectral Analysis in One or Two Dimensions*, Oxford & IBH Publishing Co., New Delhi, 1989, pp. 493–507.

[3] A.J. Barabell et al., Performance comparison of super-resolution array processing algorithms, Proj. Rep. TST-72, Lincoln Laboratory, MIT, Lexington, MA, 1984.

[4] G. Bienvenu and L. Kopp, "Optimality of high resolution array processing using the eigensystem approach", *IEEE Trans. Acoust. Speech Signal Process.*, Vol. 31, 1983, pp. 1235–1248.

[5] M. Crowder, "On constrained maximum likelihood estimates with non IID observations", *Ann. Inst. Statist. Math.*, Vol. 36, 1984, pp. 239–249.

[6] A. Errikson, P. Stoica and T. Soderstorm, "Markov-based eigenanalysis method for frequency estimation", *IEEE Trans. Signal Processing*, Vol. 42, No. 3, 1994, pp. 586–594.

[7] N. Kannan and D. Kundu, "On modified EVLP and ML methods for estimating superimposed exponential signals", *Signal Processing*, Vol. 39, No. 3, September 1994, pp. 223–233.

[8] R. Kumaresan and D.W. Tufts, "Estimating the angles of arrival of multiple plane waves", *IEEE Trans. Aerospace Electronic Systems*, Vol. AES-19, 1983, pp. 134–139.

[9] S.J. Orfanidis, *Optimum Signal Processing: An Introduction*, McMillan, New York, 1985, 2nd Edition.

[10] A. Paulraj, R. Roy and T. Kailath, "Estimation of signal parameters via rotational invariance techniques – ESPRIT", *Proc. 19th. Asilomar Conf.*, Pacific Grove, CA, 1985.

[11] S.U. Pillai, *Array Signal Processing*, Springer, Berlin, 1989.

[12] B. Porat and B. Friedlander, "Analysis of the asymptotic relative efficiency of the MUSIC algorithm", *IEEE Trans. Acoust. Speech Signal Process.*, Vol. ASSP-36, No. 4, 1988, pp. 532–544.

[13] R. Prony, "Essai experimentable at analytique", *J. Ecole Polytechnique (Paris)*, Vol. 1, 1795, pp. 24–76.

[14] R. Roy and T. Kailath, "ESPRIT – Estimation of signal parameters via rotational invariance techniques", in: E.F. Deprettere, *SVD and Signal Processing, Algorithms, Applications and Architectures*, Elsevier, Amsterdam, 1988, pp. 233–265.

[15] R.O. Schmidt, Multiple emitter location and signal parameter estimation, Ph.D. Dissertation, Stanford, Univ., CA, 1981.

[16] R.O. Schmidt, "New mathematical tools in direction finding and spectral analysis", *Proc. SPIE 27th Annual Symp.*, San Diego, CA, 1983.

[17] R.O. Schmidt, "Multiple emitter location and signal parameter estimation", *IEEE Trans. Antennas Propagation*, Vol. AP-34, 1986, pp. 276–280.

[18] D.W. Tufts and R. Kumaresan, "Frequency estimation of multiple sinusoids: Making linear prediction perform like maximum likelihood", *Proc. IEEE*, Vol. 70, 1982, pp. 975–990.

[19] M. Wax, R.O. Schmidt and T. Kailath, "Eigenstructure method for retrieving the poles from natural responses", *Proc. 22nd CDC*, San Antonio, TX, 1983, pp. 1343–1344.

[20] M. Wax, T.J. Shan and T. Kailath, "Spatio temporal spectral analysis by eigenstructure methods", *IEEE Trans. Acoust. Speech Signal Process.*, Vol. ASSP-32, 1984, pp. 817–827.