A Simplified Model for Delays in Overtaking on a Two-lane Road

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[Received December, 1957]

SUMMARY

A theoretical formula is found for the average speed achieved over a long journey by a vehicle travelling along a road according to certain rules governing overtaking, when the other traffic on the road is postulated to behave in a particularly simple way.

1. Introduction

This paper concerns the delays experienced by a vehicle travelling along a road according to specified rules governing overtaking and passing when all other traffic on the road behaves in a specified very simple manner. It is not claimed that the theory gives a good representation of normal traffic conditions, although there may be circumstances in which it gives a fair approximation to certain aspects of real situations; this is discussed in the paper. The merits claimed for the theory are first that the model incorporates fairly realistic overtaking and passing rules; it is mainly the composition of the traffic streams that is over-simplified. Second, the theory is exact on the basis of the assumptions made. Third, the theory gives an explicit formula for the average speed maintained over a long journey.

2. The Mathematical Model

Vehicles are travelling in both directions along a long straight two-lane road. In one direction the flow is q vehicles per unit time, and each vehicle travels at a constant speed, v. In the other direction the flow is Q and the speed V. In each stream, vehicles are spaced at random, they never enter or leave the road, and they have zero length. These vehicles cannot overtake each other, and are never delayed.

The problem is concerned with a single additional vehicle travelling in the first direction. When uninterrupted, it travels at a constant speed, u, greater than v. When it wishes to overtake a v-vehicle (to use the obvious notation) it acts according to the following rules:

(i) If there is a distance of d or more from the v-vehicle to the next approaching V-vehicle, it overtakes without slowing down.

(ii) If the distance to the next V-vehicle is less than d, it slows down instantaneously to speed v (i.e., it sits on the tail of the vehicle it wishes to overtake), then waits until there is a distance of at least $D = d + (v + V)t$ to the next approaching V-vehicle, waits a further time t, and then accelerates instantaneously to speed u and overtakes.

(iii) It overtakes v-vehicles one at a time, and can enter the gap between any two of them.

The second of these rules needs a little explanation. In the first place, although the rule refers to instantaneous acceleration and braking, this is not an essential feature of the model. There is no reason why the vehicle should not be thought of as slowing down gradually as soon as its driver realizes that he will have to slow down on reaching the next v-vehicle. As far as acceleration is concerned, $d/(u+V)$ is the time required to overtake at constant speed u, whereas if the u-vehicle has slowed to v, it requires an extra time t to overtake. The essential feature is that it requires the use of the offside lane for an extra time t; it does not matter precisely how its speed changes during the course of overtaking.

An important consequence of the rules is that at the moment of overtaking, whether or not following a wait, a distance of exactly d in the opposing stream is known to be clear of vehicles; the decision to overtake is quite independent of the positions of vehicles at greater distances.

3. The Problem and Method of Solution

The analysis given in this paper has the single aim of finding the average speed $\bar{u}$ achieved by the u-vehicle in the course of a long journey along the road; any other results are incidental.

Before discussing the method of solution, it is necessary to explain the meaning attached in this paper to the term "wait". This refers to the length of time for which the u-vehicle is following a v-vehicle at reduced speed v; it should not be called delay, since it is not all lost time. It is easy to see that a wait of duration T results in a delay of time $T(u-v)/u$.

The progression of the u-vehicle along the road consists of free runs at speed u separated by overtakings before each of which there may or may not be a wait. The average distance between successive v-vehicles is v/q; the u-vehicle, when running unimpeded, is travelling at speed $u-v$ relative to the stream of v-vehicles. Therefore the average duration of the free runs is $v/q(u-v)$. If the average wait at all overtakings, including those with zero wait, is denoted by $\bar{w}$, then the u-vehicle travels alternately in free runs at speed u of average duration $v/q(u-v)$ and waits at speed v of average duration $\bar{w}$. Its average speed $\bar{u}$ is therefore

$$\bar{u}=\frac{uv+q(u-v)\,v\bar{w}}{v+q(u-v)\,\bar{w}}. \tag{1}$$

The problem has thus been resolved into the determination of $\bar{w}$, the average wait at overtakings. The wait w at any particular overtaking depends solely on the positions of the approaching V-vehicles. If the previous overtaking took place a sufficient time earlier, then the part of the stream of V-vehicles relevant to the present overtaking has not been concerned in previous overtakings, and nothing is known about the positions of vehicles in it. If, on the other hand, the previous overtaking took place only a short time previously, it will be known that some portion of the road ahead is free of V-vehicles, otherwise the previous overtaking could not have occurred. The determination of $\bar{w}$ can therefore be divided into two parts, first the determination of $\bar{w}(x)$, the average wait given that the previous overtaking took place at a distance x further forward in the opposing stream than the present demand to overtake, and secondly the averaging of $\bar{w}(x)$ over the distribution of x.

4. Solution of Main Problem

Determination of $\bar{w}(x)$

If x is greater than d, nothing is known about the positions of vehicles in the opposing stream, and $\bar{w}(x)$ is independent of x. We denote its value by $\bar{w}(d)$. If x is less than d, it is known that the opposing stream is free of vehicles for at least a distance $d-x$, but it is not known what happens beyond this distance. We shall consider the latter case, where x is less than d.

The probability of having to wait is

$$P(w > 0 \mid x) = 1 - e^{-Qx/V}. \tag{2}$$

If a wait is necessary, it lasts until the overtaking vehicle is a distance $D-d$ behind the first point at which there is a gap of length at least D in the opposing stream. The distance that the overtaking vehicle has to travel, relative to the opposing stream, is therefore made up of four components, as follows:

(a) the distance $d-x$ known to be free of V-vehicles,

(b) the distance adjacent to that in (a) to the first opposing vehicle; this is known to be of length between 0 and x, and has average length

$$\frac{\int_0^x y\, Q/V\, e^{-yQ/V}\, dy}{\int_0^x Q/V\, e^{-yQ/V}\, dy} = \frac{V(e^{xQ/V} - xQ/V - 1)}{Q(e^{xQ/V} - 1)},$$

(c) the distance from the first opposing vehicle to the point at which a gap of at least D is available; the average, which follows from a result given by Tanner (1951), is $Q(e^{QD/V} - QD/V - 1)/V$,

(d) the final distance $D-d$.

The average wait $\bar{w}(x)$ is obtained by adding these four components, multiplying by the probability (2) and dividing by $v+V$ to express the result as a time. This finally gives, for $x < d$,

$$\bar{w}(x) = \frac{V}{Q(v+V)}\,[e^{QD/V}(1 - e^{-xQ/V}) - xQ/V]. \tag{3}$$

The quantity $\bar{w}(d)$ is obtained from (3) by putting $x = d$ in the right-hand side.

Expression for $\bar{u}$

When running unimpeded, the u-vehicle takes a time $x/(u+V)$ to cover a distance x relative to the stream of V-vehicles. In this time it will cover a distance $x(u-v)/(u+V)$ relative to the stream of v-vehicles. Now the probability that the distance between two successive v-vehicles is less than y is $1 - e^{-qy/v}$, and so the probability that x is less than X is $1 - e^{-qX(u-v)/v(u+V)}$. The frequency distribution of x is therefore

$$\frac{q(u-v)}{v(u+V)} \exp\left\{-\frac{q(u-v)\,x}{v(u+V)}\right\} dx, \quad 0 < x < \infty. \tag{4}$$

If we average $\bar{w}(x)$ from (3) over the distribution (4) and substitute into (1), we get

$$\frac{z}{y} = -z + e^{-az} + \frac{az}{az + A}\, e^{B}(1 - e^{-az-A}), \tag{5}$$

where

$$y = (\bar{u} - v)/(v + V),\ z = (u - v)/(u + V),$$
$$a = qd/v,\ A = Qd/V,\ B = QD/V.$$

5. Numerical Values

No extensive tabulations of numerical values have been made, but Table 1 gives values of y calculated from (5) for selected values of a, A, B and z. All the tabulated values are for $D = 3d/2$, i.e. $B = 3A/2$. These numerical values show the expected tendencies: y decreases with increasing a, A and B, and increases with z.

TABLE 1

Numerical Values of y Calculated from (5)

$D = 3d/2$ (*i.e.* $B = 3A/2$)

		$a = 0$	1	2	3	∞
	$z = 0$	0	0	0	0	0
	$\frac{1}{4}$	0·333	0·333	0·333	0·333	0·333
$A = 0$	$\frac{1}{2}$	1·000	1·000	1·000	1·000	1·000
$B = 0$	$\frac{3}{4}$	3·000	3·000	3·000	3·000	3·000
	1	∞	∞	∞	∞	∞
	0	0	0	0	0	0
	$\frac{1}{4}$	0·333	0·214	0·165	0·138	0·059
1	$\frac{1}{2}$	1·000	0·395	0·277	0·228	0·126
$1\frac{1}{2}$	$\frac{3}{4}$	3·000	0·573	0·386	0·322	0·201
	1	∞	0·766	0·507	0·425	0·287
	0	0	0	0	0	0
	$\frac{1}{4}$	0·333	0·099	0·062	0·047	0·013
2	$\frac{1}{2}$	1·000	0·132	0·080	0·062	0·026
3	$\frac{3}{4}$	3·000	0·154	0·096	0·076	0·039
	1	∞	0·175	0·111	0·091	0·052
	0	0	0	0	0	0
	$\frac{1}{4}$	0·333	0·035	0·019	0·014	0·003
3	$\frac{1}{2}$	1·000	0·040	0·023	0·017	0·006
$4\frac{1}{2}$	$\frac{3}{4}$	3·000	0·043	0·026	0·020	0·008
	1	∞	0·048	0·029	0·023	0·011
	0	0	0	0	0	0
	$\frac{1}{4}$	—	0	0	0	0
∞	$\frac{1}{2}$	—	0	0	0	0
∞	$\frac{3}{4}$	—	0	0	0	0
	1	—	0	0	0	0

6. Discussion

Applicability of Results

If on a long straight two-lane road vehicles were instructed to behave as the v-vehicles and V-vehicles of the model, then it is thought that, with appropriate choice of constants, the theory would give a fair approximation to the average speed of a faster vehicle travelling along the road. In practice traffic in one direction does not all choose to travel at the same speed—desired speeds are spread over a very wide range. This results in bunching of vehicles which are temporarily unable to proceed as fast as they wish. The model used does not allow for any bunching of this sort. On a road with fairly light traffic, however, on which bunching is not very marked, the model should give a fair idea of the relation between desired speed and actual average speed, at least for those vehicles whose drivers are prepared to overtake one vehicle at a time, pushing into the space, however small, between two slower vehicles. The constant speeds v and V would for this purpose have to be replaced by the actual average traffic speeds. Before any such relations can be estimated, however, further discussion of the constants d and D is required.

The Constants d and D

For any specified speeds u, v and V, and for specified vehicle characteristics and a specified safety margin of the u-vehicle, there exist appropriate values of d and D, although these may be difficult to determine empirically. If u, v and V or the behaviour of the u-vehicle vary, however, d and D will also vary. For instance, if u is not much greater than v, the distance d will be large, since overtaking then takes a considerable time; if the u-vehicle has poor acceleration, then $D - d$ will become large, and so on. The expected variation of d and D with u, v and V will now be discussed.

When the u-vehicle wishes to overtake a v-vehicle at an uninterrupted speed u, the time for which the opposing lane must be clear is proportional to $1/(u - v)$. Since the relative speed of the u-vehicle and opposing vehicles is $u + V$,

$$d = s\,\frac{(u+V)}{u-v}, \tag{6}$$

where s is the sum of the "effective lengths" of the u-vehicle and the v-vehicle. A safety margin of a certain time, rather than a certain distance, may be demanded, in which case (6) would have an extra term added proportional to $u + V$; this term, however, would probably be small compared with the first, and it will be omitted. Further, the distance s will increase with v, perhaps roughly in proportion to it.

In the case where the u-vehicle has been forced to slow down to speed v, suppose that it has acceleration f (assumed independent of speed) and that it must gain a distance s on the v-vehicle before being able to pull back to its nearside. Suppose also that f is sufficiently large for the vehicle to accelerate up to speed u before completing its overtaking manoeuvre. It is required to find the minimum gap in the opposing traffic that will enable this manoeuvre to be carried out.

The acceleration from v to u occupies a time $(u - v)/f$ and a road distance $(u^2 - v^2)/(2f)$. In this time, the v-vehicle has moved a distance $v(u - v)/f$, so that the u-vehicle has gained a distance $(u^2 - v^2)/(2f) - v(u - v)/f = (u - v)^2/(2f)$ on it. To gain the remaining distance $s - (u - v)^2/(2f)$ requires a time $[s - (u - v)^2/(2f)]/(u - v)$ and so a road distance $u[s - (u - v)^2/(2f)]/(u - v)$. Thus the whole manoeuvre occupies a time

$$(u - v)/f + [s - (u - v)^2/(2f)]/(u - v) = (u - v)/(2f) + s/(u - v)$$

and a road distance $us/(u - v) + v(u - v)/(2f)$. In this total time, the opposing vehicles have been approaching at speed V, so that the total length of gap required in the opposing stream is

$$D = V[(u-v)/(2f) + s/(u-v)] + us/(u-v) + v(u-v)/(2f)$$
$$= \frac{(u+V)\,s}{u-v} + \frac{(V+v)(u-v)}{2f}. \tag{7}$$

If f is very large, (7) reduces to (6), as it should. This analysis confirms the formulation of overtaking rule (ii) in the original mathematical model. During the overtaking manoeuvre, the u-vehicle covers a road distance $us/(u-v) + v(u-v)/(2f)$ in a time $s/(u-v) + (u-v)/(2f)$, which is equivalent to travelling at speed v for a time $(u-v)/(2f)$ and then at u for a time $s/(u-v)$; in other words the extra wait t to which the time lost in accelerating is equivalent is the same as the coefficient of $(v+V)$ in the expression for D, as assumed by the model.

The expression (7) is based on the assumption that the u-vehicle completed its acceleration before finishing its overtaking manoeuvre, the condition for which is $(u-v)^2 < 2fs$. If this condition is not satisfied, then the original model needs modification; this case requires further thought.

The effect of the above analysis is to replace the two constants d and D in the mathematical model by two more fundamental constants s and f, corresponding roughly to the sum of the effective lengths of the u-vehicle and the v-vehicle and to the acceleration of the u-vehicle. These constants will still vary from vehicle to vehicle and from driver to driver, but provided the derivation of (6) and (7) is considered to be sufficiently reliable it is probably more satisfactory to use them as basic constants of the model rather than d and D. Even if formulae (6) and (7) should be found to be unrealistic, the d, D formulation of the model would remain valid.

It would be of interest from the point of view of drivers to replace d and D in (5), the expression for $\bar{u}$, by (6) and (7) and to see in quantitative terms how much increase in average speed could be obtained by increasing vehicle performance (cruising speed u and acceleration f) or by relaxing the safety margin (decreasing s), in various traffic conditions. One would expect to find smaller benefits under congested conditions and also, except as regards u, under conditions of very light traffic. From a more general point of view, it would be useful to compare the gaps d and D given by (6) and (7) with observations of normal driver behaviour, and to compare the relation between speed and flow, and between actual average speed and desired speed, with related observational data. One point worth studying in this connection is the relative impeding effects of equal numbers of vehicles in the two directions. None of these matters is studied in the present paper.

Possible Extensions of Theory

The third item to be discussed is how the model might be improved to give a more realistic representation of actual traffic. First let us consider an improvement that could probably be made without introducing any fundamental change into the nature of the model.

The v-vehicles and V-vehicles have been assumed to be spaced at random and not to be bunched; thus a u-vehicle could enter the space between any two of them. There seems to be no fundamental difficulty in generalizing this to some extent. Suppose that each stream of vehicles was fed into the road as the output from a fixed-service-time queueing process with random input; in other words, that vehicles had a certain minimum separation on the road, their positions being obtained from a random distribution by moving vehicles backwards along the stream till the minimum separations were achieved. Bunching would then occur, and it could be stipulated that the u-vehicle had to overtake a bunch in a single manoeuvre, not vehicle by vehicle. Tanner (1953) has given a formula for the average delay in waiting for a gap of given size in a stream of this character, and the same paper gives a formula for the distribution of bunch-size. If the overtaking rules are extended to specify what size gap in the opposing stream is required in order to overtake a bunch of given length, then the average delay in overtaking a random bunch can be written down without difficulty. The only difficulty foreseen is that the integration over x, the distance from the previous overtaking, might not be possible in terms of elementary functions, although this is thought unlikely.

The bunching introduced into the model just described is not of the same type as that which occurs in real traffic, although by adjustment of the minimum separation of vehicles it should be possible to obtain any desired average size of bunch. What this average size should be, in terms of the various flows and speeds in the model, is at present unknown. To extend the theoretical model to include a realistic type of bunching requires completely different and much more complicated mathematical methods; in the author's view, the theoretical analysis of a model incorporating a realistic type of bunching, arising from the variation among vehicles' desired speeds, is beyond the scope of techniques at present available.

ACKNOWLEDGMENT

This work was carried out as part of the programme of the Road Research Board of the Department of Scientific and Industrial Research and is published by permission of the Director of Road Research.

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