Reinforcement Learning Basics Any% Speedrun

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ARENA

Thursday, 8th June 2023

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  5. **Online only:** No clear distinction between training and testing. Agent gets dumped in the environment and must learn on the fly.

Y. LeCun

How Much Information is the Machine Given during Learning?

- **"Pure" Reinforcement Learning (cherry)**
  - The machine predicts a scalar reward given once in a while.
  - **A few bits for some samples**
- **Supervised Learning (icing)**
  - The machine predicts a category or a few numbers for each input
  - Predicting human-supplied data
  - **10→10,000 bits per sample**
- **Self-Supervised Learning (cake génoise)**
  - The machine predicts any part of its input for any observed part.
  - Predicts future frames in videos
  - **Millions of bits per sample**

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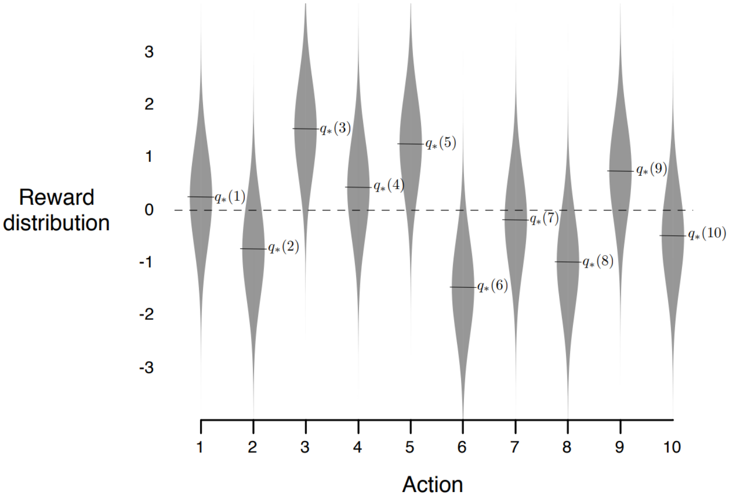

Figure 2.1: An example bandit problem from the 10-armed testbed. The true value $q_*(a)$ of each of the ten actions was selected according to a normal distribution with mean zero and unit variance, and then the actual rewards were selected according to a mean $q_*(a)$, unit-variance normal distribution, as suggested by these gray distributions.

Bandito Algorithm

- Keep track of $\hat{Q}(a)$, the estimated value of each arm after t arm-pulls

$$\hat{Q}_t(a) = \frac{\text{sum of rewards when } a \text{ taken up to } t}{\text{number of times } a \text{ taken prior to } t} = \frac{\sum_{i=1,a_i=a}^{t-1} r_t}{\sum_{i=1,a_i=a}^{t-1} 1}$$

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$$\hat{Q}_{t+1}(a) \leftarrow \frac{\hat{R}_{t+1}(a)}{N_{t+1}(a)}$$

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- **Problem:** Can get stuck in local minima.

Encouraging Exploration

1. **First approach:** Just do random stuff every now and again, hope for the best

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4. Add $\ln t$ to numerator to ensure every action is sampled infinitely often (in case you get an unlucky run). $\ln t$ is optimal because math. $c = 2$ works good in practice.

Agent-Environment Interaction Loop (MDPs)

- Environment has *states* $\mathcal{S}$, *actions* $\mathcal{A}$, *rewards* $\mathcal{R}$, *environment distribution* $p : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \times \mathcal{R} \to [0, 1]$.

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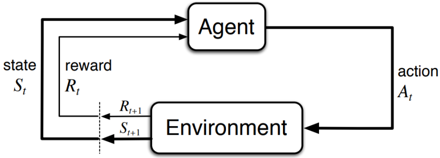

Figure: Agent-Environment interaction loop

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- **Problems:** (for continuing environments)
  - The return may diverge or be undefined (compare $2, 2, 2, 2, \dots$ with $1, 1, 1, 1, \dots$).
  - The agent might be lazy (compare $1, 1, 1 \dots$ with $0, 0, \dots, 0, 1, 1, 1, \dots$).
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  - May desire rewards now to be more valuable than rewards later: \$100 now? Or \$110 in a year?
- **Solutions:**
  - Add a discount factor $\gamma \in [0, 1)$ so rewards more imminent are worth more, and the return is always well defined.

$$G_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \dots$$

  - Want agent to choose actions to maximise the *expected* return.

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4. **Reward Hypothesis:**

   > *"That all of what we mean by goals and purposes can be well thought of as the maximization of the expected value of the cumulative sum of a received scalar signal (called reward)." -Rich Sutton*

   - *Reward alone is sufficient to communicate any possible goal or desired behaviour*

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Value Function

- Want to define the "goodness" (*value*) of a state, so the agent can take actions to move towards "good" states, and away from "bad" states.
- The value of a state depends also on how the agent chooses actions, called a *policy* $\pi : \mathcal{S} \times A \to [0, 1]$. Actions are sampled $a \sim \pi(\cdot|s)$.

Value Function

$$\begin{aligned} V_\pi(s) =& \mathbb{E}_\pi[G_t | s_t = s] \\ =& \mathbb{E}_\pi[r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \ldots | s_t = s] \end{aligned}$$

(Expectation is also with respect to the environment p.)

Bellman Equation

We note that since

$$\begin{aligned} G_t &= r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \dots \\ &= r_{t+1} + \gamma(r_{t+2} + \gamma r_{t+3} + \dots) \\ &= r_{t+1} + \gamma G_{t+1} \end{aligned}$$

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Policy Evaluation

This gives the **Bellman equation**

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- Computing V_π from π is called **policy evaluation**.

Value Function (simplified)

Assume policy $\pi : S \to A$ is deterministic, define *transition probability* $T(s' \mid s, a) := \sum_{r \in \mathcal{R}} p(s', r|s, a)$ and assume reward $r_{t+1} := R(s_t, a_t, s_{t+1})$ is deterministic function of s_t, a_t, s_{t+1}.

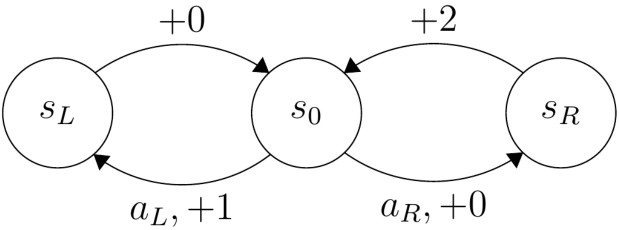
Bellman Equation

$$V_\pi(s) = \sum_{s'} T(s'|s, a)\left(R(s, a, s') + \gamma V_\pi(s')\right)$$

where $a = \pi(s)$.

Example Environment

- States $\mathcal{S} = s_0, s_L, s_R$, actions $\mathcal{A} = \{a_L, a_R\}$, rewards $\mathcal{R} = \{0, 1, 2\}$.
- Each transition indicates if an action is taken, the reward returned and which state to transition to
- What is the best action from state s_0?



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- Policy π_1 is **better** than π_2 ($\pi_1 \geq \pi_2$) if $\forall s. V_{\pi_1}(s) \geq V_{\pi_2}(s)$. A policy is **optimal** if it is better than all other policies.

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Can't just compute V_π using policy evaluation for all π, as there are $|\mathcal{A}|^{|\mathcal{S}|}$ many to choose from.

Policy Improvement

- Obviously we have that

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V_*(s) &= \max_a \sum_{s'} T(s'|s, a)\,(R(s, a, s') + \gamma V_*(s')) \\
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Theorem: Policy iteration converges to optimal policy in finitely many steps!

Problems with Policy Iteration

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For the moment, we weaken only the first assumption, and assume the environment is now a black box, from which state-reward pairs (s', r) can be sampled given state-action pairs (s, a) as input.

Temporal Difference Learning

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This then gives us an update rule to improve on our estimate $\hat{V}_\pi$ of V_π, similar to SGD, called *TD(0)*.

$$\begin{aligned} \hat{V}_\pi(s_t) &\leftarrow \hat{V}_\pi(s_t) + \alpha \delta_t \\ &\equiv \hat{V}_\pi(s_t) + \alpha \left(r_{t+1} + \gamma \hat{V}_\pi(s_{t+1}) - \hat{V}_\pi(s_t) \right) \end{aligned}$$

where $\alpha \in (0,1]$ is the **learning rate**.

Q-Value

- *Q-value* is the expected return from state s, taking action a, and thereafter following policy π.
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which has it's own Bellman equation

Q-value Bellman

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where $a' = \pi(s')$

Optimal Q-value Bellman

$$Q_*(s, a) = \sum_{s'} T(s'|s, a) \left(R(s, a, s') + \max_{a'} Q_*(s', a')\right)$$

Q-value vs. Value

Can state Q in terms of V, and vice-versa.

$$Q_\pi(s,a) = \sum_{s'} T(s'|s,a)\,(R(s,a,s') + \gamma V_\pi(s'))$$

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(exercise to the reader...)

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- **Idea:** Learn Q_* instead, recover policy π_*

SARSA: On-Policy TD Control

Apply same argument as TD(0) to the Q-Value

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SARSA Update Rule

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where $\alpha \in (0, 1]$ is the **learning rate**.

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$$\hat{Q}_*(s_t, a_t) \leftarrow \hat{Q}_*(s_t, a_t) + \alpha \left(r_{t+1} + \gamma \hat{Q}_*(s_{t+1}, a_{t+1}) - \hat{Q}_*(s_t, a_t) \right)$$

where $\alpha \in (0, 1]$ is the **learning rate**.

Actions drawn from ε-greedy strategy

$$\pi^{\varepsilon\text{-greedy}}(s) = \begin{cases} \text{random action} & \text{prob } \varepsilon \\ \arg\max_a \hat{Q}_*(s,a) & \text{prob } 1 - \varepsilon \end{cases}$$

SARSA: On-Policy TD Control

Apply same argument as TD(0) to the Q-Value

$$Q_*(s,a) = \mathop{\mathbb{E}}_{s' \sim T(\cdot|s,a)} \left[R(s,a,s') + \gamma Q_*(s', \pi_*(s'))\right]$$

On timestep t, this is (*on average*), equal to the actual reward r_{t+1}, plus the discounted Q-value of the actual next state-action pair s_{t+1}, a_{t+1}.

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Theorem: Under "niceness" conditions SARSA guaranteed to converge to Q_*.

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- Q-Learning tends to converge faster than SARSA, and chooses more aggressive/risky moves (SARSA learns from the moves that were actually taken, including any exploration).
- Can generalise further by including the action taken by SARSA in expectation (Expected SARSA).

Sarsa (on-policy TD control) for estimating $Q \approx q_*$

Algorithm parameters: step size $\alpha \in (0, 1]$, small $\varepsilon > 0$
Initialize $Q(s, a)$, for all $s \in \mathcal{S}^+, a \in \mathcal{A}(s)$, arbitrarily except that $Q(\mathit{terminal}, \cdot) = 0$

Loop for each episode:
  Initialize S
  Choose A from S using policy derived from Q (e.g., ε-greedy)
  Loop for each step of episode:
    Take action A, observe R, S'
    Choose A' from S' using policy derived from Q (e.g., ε-greedy)
    $Q(S, A) \leftarrow Q(S, A) + \alpha\big[R + \gamma Q(S', A') - Q(S, A)\big]$
    $S \leftarrow S'$; $A \leftarrow A'$;
  until S is terminal

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    - Bootstrap from current estimates (i.e. Q-Learning)

Deep Q-Networks (DQN)

- The Q-Value estimate $\hat{Q}_*(s, a; \theta)$ is now stored as a network, with parameters θ. Recall the TD-error for Q-Learning

$$\delta_t = r_{t+1} + \gamma \max_{a'} Q_*(s_{t+1}, a'; \theta) - Q_*(s_t, a_t; \theta)$$

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Then, perform gradient update step over parameters

$$\theta \leftarrow \theta - \eta \nabla_\theta L(\theta)$$

Algorithm 1 Deep Q-Learning with Replay Buffer

Input: Environment p, Number of episodes M, replay buffer size N

Initialise replay buffer $\mathcal{D}$ to capacity N
for episode $= 1$ to M **do**
  Sample initial state s from environment
  $d \leftarrow$ False
  Initalize target parameters $\theta_{\text{target}} \leftarrow \theta$
  while $d =$ False **do**

$$a \leftarrow \begin{cases} \text{random action} & \text{prob } \varepsilon \\ \arg\max_{a'} Q(s, a'; \theta) & \text{prob } 1 - \varepsilon \end{cases}$$

    Sample $(s_{\text{new}}, r, d) \sim p(\cdot | s, a)$
    Store experience $(s, a, r, s_{\text{new}}, d)$ in $\mathcal{D}$
    $s_{\text{new}} \leftarrow s$
    if Learning on this step **then**
      Sample minibatch $B \leftarrow \{(s^i, a^i, r^i, s^i_{\text{new}}, d^i)\}_{i=1}^{|B|}$ from $\mathcal{D}$
      for $j = 1$ to $|B|$ **do**

$$y^j \leftarrow \begin{cases} r^j & d^j = \text{True} \\ r^j + \gamma \max_{a'} Q(s^j_{\text{new}}, a'; \theta_{\text{target}}) & d^j = \text{False} \end{cases}$$

      end for
      Define loss $L(\theta) = \frac{1}{|B|} \sum_{i=1}^{|B|} \left(y^i - Q(s^i, a^i; \theta)\right)^2$
      Gradient descent step $\theta \leftarrow \theta - \eta \nabla_\theta L(\theta)$
    end if
    if Update target this step **then**
      $\theta_{\text{target}} \leftarrow \theta$
    end if
  end while
end for

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- Update step $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \eta \widehat{\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})}$
- Learn preferences $h(s, a, \boldsymbol{\theta})$, and (assuming $|\mathcal{A}|$ "small") define softmax policy

$$\pi_{\boldsymbol{\theta}}^{\text{softmax}}(a|s) = \frac{\exp(h(s, a, \boldsymbol{\theta})/T)}{\sum_{a'} \exp(h(s, a', \boldsymbol{\theta})/T)}$$

  where T is temperature (hyperparamter).
    - Use neural network to learn $h(s, a, \boldsymbol{\theta})$

Softmax vs. greedy

- **Advantages**
  - $\pi_{\varepsilon\text{-greedy}}$ always does uniformly random actions. $\pi_{\boldsymbol{\theta}}^{\text{softmax}}$ is still stochastic, but biased towards good moves
  - $\pi_{\boldsymbol{\theta}}^{\text{softmax}}$ is continuous w.r.t preferences $h(s, a, \boldsymbol{\theta})$. $\pi_{\varepsilon\text{-greedy}}$ might dramatically change behaviour in response to small perturbations in $\hat{Q}_* \equiv$ better convergence
  - π is a simpler function than Q. Learning π directly learns faster(?)
- **Disadvantages**
  - More computationally expensive/more complex
  - $\pi_{\boldsymbol{\theta}}^{\text{softmax}}$ will play near uniform for two states with similar values. $\pi_{\varepsilon\text{-greedy}}$ will choose the best
  - $\pi_{\boldsymbol{\theta}}^{\text{softmax}}$ will only converge to deterministic policy with a temperature schedule, hard to choose a priori/requires domain knowledge

Log-derivative trick

Note that

$$\frac{d}{dx}\log f(x) = \frac{1}{f(x)} \cdot \frac{d}{dx} f(x)$$

Hence,

$$\frac{d}{dx} f(x) = f(x) \frac{d}{dx} \log f(x)$$

Or, in the form we will use it

$$\nabla_{\boldsymbol{\theta}} P_{\boldsymbol{\theta}}(x) = P_{\boldsymbol{\theta}}(x) \nabla_{\boldsymbol{\theta}} \log P_{\boldsymbol{\theta}}(x)$$

Policy Gradient Framework

- Assume episodic environment, length t', no discount $\gamma = 1$. WLOG starting state s_0.
- Define $J(\boldsymbol{\theta}) = V_{\pi_{\boldsymbol{\theta}}}(s_0)$.
- Let $\tau = s_0, a_0, r_1, s_1, \ldots, s_{t'}$ denote a trajectory
- $G(\tau) = \sum_{t=0}^{t'} r_t$ the return.
- $\Pr(\tau|\boldsymbol{\theta}) = \prod_{k=t}^{t'} \pi_{\boldsymbol{\theta}}(a_k|s_k) T(s_{k+t}|s_k, a_k)$ is the probability of sampling τ from environment given policy params. $\boldsymbol{\theta}$.

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&= \sum_{\tau} \nabla_{\boldsymbol{\theta}} \Pr(\tau|\boldsymbol{\theta}) G(\tau) \\
&= \sum_{\tau} \Pr(\tau|\boldsymbol{\theta}) \left(\nabla_{\boldsymbol{\theta}} \log \Pr(\tau|\boldsymbol{\theta}) G(\tau)\right) \quad \text{(Log Derivative trick)} \\
&= \mathbb{E}_{\tau\sim\pi_{\boldsymbol{\theta}}}\left[\nabla_{\boldsymbol{\theta}} \log \Pr(\tau|\boldsymbol{\theta}) G(\tau)\right]
\end{aligned}
$$

Note that

$$\nabla_{\boldsymbol{\theta}} \log \Pr(\tau|\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} \log \prod_{k=t}^{t'} \pi_{\boldsymbol{\theta}}(a_k|s_k) T(s_{k+t}|s_k, a_k)$$

Note that

$$
\begin{aligned}
\nabla_{\boldsymbol{\theta}} \log \Pr(\tau | \boldsymbol{\theta}) &= \nabla_{\boldsymbol{\theta}} \log \prod_{k=t}^{t'} \pi_{\boldsymbol{\theta}}(a_k | s_k) T(s_{k+t} | s_k, a_k) \\
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Vanilla Policy Gradient (VPG)

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \mathbb{E}_{\tau \sim \pi_{\boldsymbol{\theta}}} \left[\nabla_{\boldsymbol{\theta}} \sum_{k=t}^{t'} \log \pi_{\boldsymbol{\theta}}(a_k | s_k) G(\tau) \right]$$

Clever trick 1: The future cannot affect the past

- $\pi_{\boldsymbol{\theta}}(a_i | s_i)$ gets bumped by the full return $G(\tau)$. Obviously a_t has no effect on $r_0, r_1, \ldots, r_{t-1}$
- At timestep t, swap full return $G(\tau)$ with partial return $\sum_{j=t}^{t'} r_j$

$$\begin{aligned}
\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) &= \mathbb{E}_{\tau \sim \pi_{\boldsymbol{\theta}}} \left[\nabla_{\boldsymbol{\theta}} \sum_{k=t}^{t'} \log \pi_{\boldsymbol{\theta}}(a_k | s_k) \sum_{j=k}^{t'} R(s_j, a_j, s_{j+1}) \right] \\
&= \mathbb{E}_{\tau \sim \pi_{\boldsymbol{\theta}}} \left[\nabla_{\boldsymbol{\theta}} \sum_{k=t}^{t'} \log \pi_{\boldsymbol{\theta}}(a_k | s_k) Q_{\pi_{\boldsymbol{\theta}}}(s_k, a_k) \right]
\end{aligned}$$

Algorithm 1 Vanilla Policy Gradient ($\gamma = 1$)

Input: Environment p, Number of episodes M

1: **for** episode = 1 to M **do**

2: Generate episode $s_0, a_0, r_1, s_1, \ldots, s_{T-1}, a_{T-1}, r_T$

3: Define $G_t = \sum_{i=t+1}^{T} r_i$ for $0 \leq t \leq T-1$

4: Define gain $J(\boldsymbol{\theta}) = \sum_{t=0}^{T-1} G_t \log \pi_{\boldsymbol{\theta}}(a_t | s_t)$

5: Gradient **ascent** step $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \eta \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$

6: **end for**

¯_(ツ)_/¯

Algorithm 2 Efficient Vanilla Policy Gradient ($\gamma = 1$)

Input: Environment p, Number of episodes M

for episode $= 1$ to M **do**

  Generate episode $s_0, a_0, r_1, s_1, \ldots, s_{T-1}, a_{T-1}, r_T$

  Initalise array $G = \{G_0, G_1, \ldots, G_{T-1}\}$

  $G_{T-1} \leftarrow r_T$

  for timestep in episode $t = T-2, T-1$ to 0 **do**

    $G_t \leftarrow r_{t+1} + G_{t+1}$

  end for

  Define gain $J(\boldsymbol{\theta}) = \sum_{t=0}^{T-1} G_t \log \pi_{\boldsymbol{\theta}}(a_t|s_t)$

  Gradient **ascent** step $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \eta \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$

end for

¯_(ツ)_/¯

Expected Grad-Log-Prob (EGLP) Lemma

Let $\mathrm{P}_{\boldsymbol{\theta}}$ be a parameterised probability distribution over random variable x. Then

$$\mathbb{E}_{x \sim \mathrm{P}_{\boldsymbol{\theta}}}[\nabla_{\boldsymbol{\theta}} \log \mathrm{P}_{\boldsymbol{\theta}}(x)] = 0$$

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Corollary of EGLP: For any function b that depends only on state s_t,

$$\mathbb{E}_{a_t \sim \pi_{\boldsymbol{\theta}}}[\nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(a_t | s_t) b(s_t)] = 0$$

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So, can add/subtract any such **baseline function** b into VPG without changing the result (in expectation),

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \mathbb{E}_{\tau \sim \pi_{\boldsymbol{\theta}}} \left[\nabla_{\boldsymbol{\theta}} \sum_{k=t}^{t'} \log \pi_{\boldsymbol{\theta}}(a_k|s_k) \left(Q_{\pi_{\boldsymbol{\theta}}}(s_k, a_k) - b(s_k) \right) \right]$$

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Clever trick 2: Choose $b(s_t) = V_{\pi_{\boldsymbol{\theta}}}(s_t)$, the on-policy value function

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where $A_\pi(s, a) := Q_\pi(s, a) - V_\pi(s)$ is the **advantage** function

Corollary of EGLP: For any function b that depends only on state s_t,

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- $V_{\pi_{\boldsymbol{\theta}}}$ learned by separate critic network.
- Reduces variance, only update policy when critic disagrees

Empirical Policy Gradient

Note: Police gradient uses gradient **ascent**, so we actually **maximise** loss!

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Define policy gradient "loss" (gain?)

$$L^{PG}(\boldsymbol{\theta}) = \hat{\mathbb{E}}\left[\sum_{k=t}^{t'} \log \pi_{\boldsymbol{\theta}}(a_k|s_k) A_{\pi_{\boldsymbol{\theta}}}(s_t, a_t)\right]$$

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