SMIT SIKKIM MANIPAL UNIVERSITY
SIKKIM MANIPAL INSTITUTE OF TECHNOLOGY

Engineering Mathematics III

Discrete Mathematics

Lecture 31

Generating functions: Problems

Using generating functions, solve the following recurrence relations:

(i) $a_n - 9a_{n-1} + 20a_{n-2} = 0$, $a_0 = -3$, $a_1 = -10$.

$n \geq 2$

$$\sum_{n=2}^{\infty} \left(a_n x^n - 9 a_{n-1} x^n + 20 a_{n-2} x^n \right) = 0$$

$$\sum_{n=2}^{\infty} a_n x^n - 9 \sum_{n=2}^{\infty} a_{n-1} x^n + 20 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

First term:

$$\sum_{n=2}^{\infty} a_n x^n = a_2 x^2 + a_3 x^3 + \cdots$$

$$= \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) - \left(a_0 + a_1 x\right)$$

$$= A(x) - a_0 - a_1 x$$

$$= A(x) + 3 + 10x$$

Second term:

$$\sum_{n=2}^{\infty} a_{n-1} x^n = \left(a_1 x^2 + a_2 x^3 + \cdots\right)$$

$$= x\left(a_1 x + a_2 x^2 + \cdots\right)$$

$$= x\left(-a_0 + a_0 + a_1 x + a_2 x^2 + \cdots\right)$$

$$= x\left(3 + A(x)\right)$$

$$= 3x + xA(x)$$

Third term:

$$\sum_{n=2}^{\infty} a_{n-2} x^n = a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots$$

$$= x^2\left(a_0 + a_1 x + a_2 x^2 + \cdots\right)$$

$$= x^2 A(x)$$

$$A(x) + 3 + 10x - 9\left[3x + x A(x)\right] + 20x^2 A(x) = 0$$

$$A(x) + 3 + 10x - 27x - 9x A(x) + 20x^2 A(x) = 0$$

$$A(x)\left(1 - 9x + 20x^2\right) + 3 - 17x = 0$$

$$(5x-1)(4x-1)$$

$$A(x) = \frac{17x - 3}{20x^2 - 9x + 1} = \frac{17x - 3}{(5x-1)(4x-1)}$$

$$\frac{17x-3}{(5x-1)(4x-1)} = \frac{A}{5x-1} + \frac{B}{4x-1}$$

$$17x-3 = A(4x-1) + B(5x-1)$$

$x = 1/4$

$$\frac{17}{4} - 3 = B\left(\frac{1}{4}\right)$$

$$\frac{5}{4} = \frac{1}{4}B$$

$$\Rightarrow \boxed{B = 5}$$

$x = 1/5$

$\vdots$

$$\boxed{A = -2}$$

$$\frac{17x-3}{(5x-1)(4x-1)} = \frac{-2}{5x-1} + \frac{5}{4x-1}$$

$$= \frac{2}{1-5x} - \frac{5}{1-4x}$$

$$A(x) = 2(1-5x)^{-1} - 5(1-4x)^{-1}$$

$$a_n = 2(5)^n - 5(4)^n$$

$$\boxed{a_n = 2 \times 5^n - 5 . 4^n}$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots x^n +$$

$$(4x)^n \rightarrow 4^n x^n$$

Using generating functions, solve the following recurrence relations

(iii) $a_{n+2} - 5a_{n+1} + 6a_n = 2, \ a_0 = 1, \quad a_1 = 2.$

$a_{n-2} - 5a_{n-1} + 6a_n = 2, \quad a_0 = 1, \quad a_1 = 2$

$$\sum_{n=0}^{\infty} a_{n+2}x^n - 5\sum_{n=0}^{\infty} a_{n+1}x^n + 6\sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} x^n$$

$$\sum_{n=0}^{\infty} a_{n+2}x^n = a_2x^0 + a_3x^1 + a_4x^2 + \ldots = \frac{1}{x^2}\left[a_2x^2 + a_3x^3 + \ldots\right]$$

$$= \frac{1}{x^2}\left\{-a_0 - a_1x + a_0 + a_1x + a_2x^2 + \ldots\right\}$$

$$= \frac{1}{x^2}\left\{-a_0 - a_1x + A(x)\right\}.$$

Using generating functions, solve the following recurrence relations

(iv) $a_n - 7a_{n-1} + 10a_{n-2} = 0, \ a_0 = a_1 = 3.$