Discrete Mathematics
Engineering Mathematics III (MA 1308)
B.Tech Computer Science Engineering (III Sem)

Lecture Notes

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Note: Please note that, this notes is incomplete, may have printing mistakes, and its available online only for the reference.

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1 Set Theory

Definition 1.1. Set is a collection of distinct and well defined objects.

Any given set can be represented in two ways.

1. **Roaster Form**: Listing all the elements explicitly is called as Roaster form. For example, $\{1, 2, 3, 4, 5\}$ is a roaster form of set of intergers from 1 to 5.

2. **Set-Builder Form**: The above set also can be written in the following form.

$$\{x \in N : 1 \leq x \leq 5\},$$

which is known as *Set-Builder form.*

Depends of the number of elements in the set we identify the sets into different kinds. First of all, we define the cardinal number of a set.

Definition 1.2 (Cardinal Number)**.** Number of distinct elements in a given set A is called as *cardinal number of* A. It is denoted by $n(A)$ or $\#A$ or $|A|$.

In this text, we alternatively use the notations between $n(A)$ and $|A|$. As we have mentioned, we divide the sets in to two types, namely, finite and infinite sets. A set with finite number of elements is called as *finite set* and a set with infinite number of elements is called as *infinite set*. Note that, *empty set* is a set with no elements. That is, empty set is a finite set with cardinal number 0. Similarly, *singleton set* is a set with only one element, and hence, it is also a finite set. Set of all real numbers is an infinte set since it has infinitely many elements. One can also further divide the infiteset into *countably infinite sets* and *uncountable infite sets* depends on the countablity of the elements in the set. We may not discuss much on that side in this text for some reasons.

Let us define some more variety of sets which we may use in the text.

Definition 1.3 (Equal and Equivalent Set)**.** If every element of a set A is in set B and vice versa, then the sets A and B ß are called *equal sets*. Two sets having same cardinal number are called *equivalent set.*

1.1. All equal sets are equivalent but not all equivalent sets are equal. Justify.

Quite often, we look for a portion of set. For example, from all customers in our store, we would like to know the customers who have bought a particular product. That brings to define a subcollection from a given set as subset of the set.

Definition 1.4 (Subset)**.** Let A be a set. A set B is called as *subset* of the set A if all the elements in B is also an element is A. Mathematically, if

$$\text{for all } x \in B, \quad x \in A,$$

then we say that B is a subset of A and we denote it by $B \subseteq A$.

Example 1.1. Consider A=$\{1, 2, 3, 4, 5\}$ and B=$\{3, 5\}$. If all the elements of B are also present in A, then we can say that B is a subset of A. Here the element of B are 3 and 5 which are also elements of A. That is, $B \subseteq A$.

Definition 1.5 (Power set)**.** Collection of all subsets of a set A is called as power set of A. We denote it by $\mathscr{P}(A)$.

For example, consider a set A=$\{1, 2\}$. Then, $\mathscr{P}(A) = \{\{\}, \{1\}, \{2\}, \{1, 2\}\}$

Question 1.1. The number of elements in any power set of A can be obtained by finding $2^{n(A)}$.

Definition 1.6. Let A and B be two sets. The *Cartesian Product* of A and B, denoted by $A \times B$ is the set of all ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

For example,

$$\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

Definition 1.7. A *binary relation* (also known as relation) from A to B is a subset of $A \times B$.

In this lecture notes, we avoid the word 'binary' and simply use relation, since we are going to deal only binary relations.

So, relation is nothing but an intuitive formulation, so that one can say who is related with whom from the set A to the set B. So, if a in A is related to b in B, we say (a, b) is in the relation $\mathcal{R}$, and denoted it as $(a, b) \in R$. And similarly, when $(a, b) \in \mathcal{R}$, we say that a and b are related.

Besides the list of ordered pairs, we can represent a relation in tabular form, matrix form and also graphical form. Lets look at an example: Let us say, we have actor names in the set A and the movie names in the set B. Our relation is defined such that we say an actor from set A is related to a movie in set B, if he acted in that movie. So, if

$$A = \{\text{Tony Stark}, \text{Steve Rogers}, \text{Bruce Banner}, \text{Peter Parker}\}$$

and

$$B = \{\text{Spider Man}, \text{Iron Man}, \text{Captain America}, \text{Avengers Infinity War}, \text{Avengers Endgame}\}$$

So, the relation

$$\mathcal{R} = \left\{ \begin{array}{l} (\text{Tony Stark}, \text{Iron Man}), (\text{Tony Stark}, \text{Avengers Infinity War}), (\text{Tony Stark}, \text{Avengers Endgame}), \\ (\text{Steve Rogers}, \text{Captain America}), (\text{Steve Rogers}, \text{Avengers Infinity War}), \\ (\text{Steve Rogers}, \text{Avengers Endgame}), \\ (\text{Bruce Banner}, \text{Avengers Infinity War}), (\text{Bruce Banner}, \text{Avengers Endgame}), \\ (\text{Peter Parker}, \text{Spider Man}), (\text{Peter Parker}, \text{Avengers Infinity War}), \\ (\text{Peter Parker}, \text{Avengers Endgame}) \end{array} \right\}.$$

We can represent the same relation in a tabular form as given the Table 1.

	Spider Man	Ironman	Captain America	Avengers Infinity war	Avengers Endgame
Tony Stark		✓		✓	✓
Steve Rogers			✓	✓	✓
Bruce Banner				✓	✓
Peter Parker	✓			✓	✓

Table 1: Tabular form for a relation

One can also represent the same using matrix form as given in Figure 1.

$$\begin{array}{r} \\ \text{Tony Stark} \\ \text{Steve Rogers} \\ \text{Bruce Banner} \\ \text{Peter Parker} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Spiderman} & \text{Ironman} & \text{Captain America} & \text{Avengers Infinity war} & \text{Avengers Endgame} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Figure 1: Matrix Form of Relation

One can also represent the same using a graph as given in Figure 2.

1.1. Let $\mathbb{Z}$ be the set of all integers. Define $\mathcal{R}$ on $\mathbb{Z} \times \mathbb{Z}$ such that $a\mathcal{R}b \iff (a-b)$ is divisible by 5, $a, b \in \mathbb{Z}$. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$. Find the equivalence class of 3.

Solution:

(a) Reflexive:

Let $a \in \mathbb{Z}$.

We have to prove that $a\mathcal{R}a$. That is, ($a - a$ should be divisible by 5)
Now, $a - a = 0$ which is divisible by 5. Therefore $\mathcal{R}$ is reflexive.

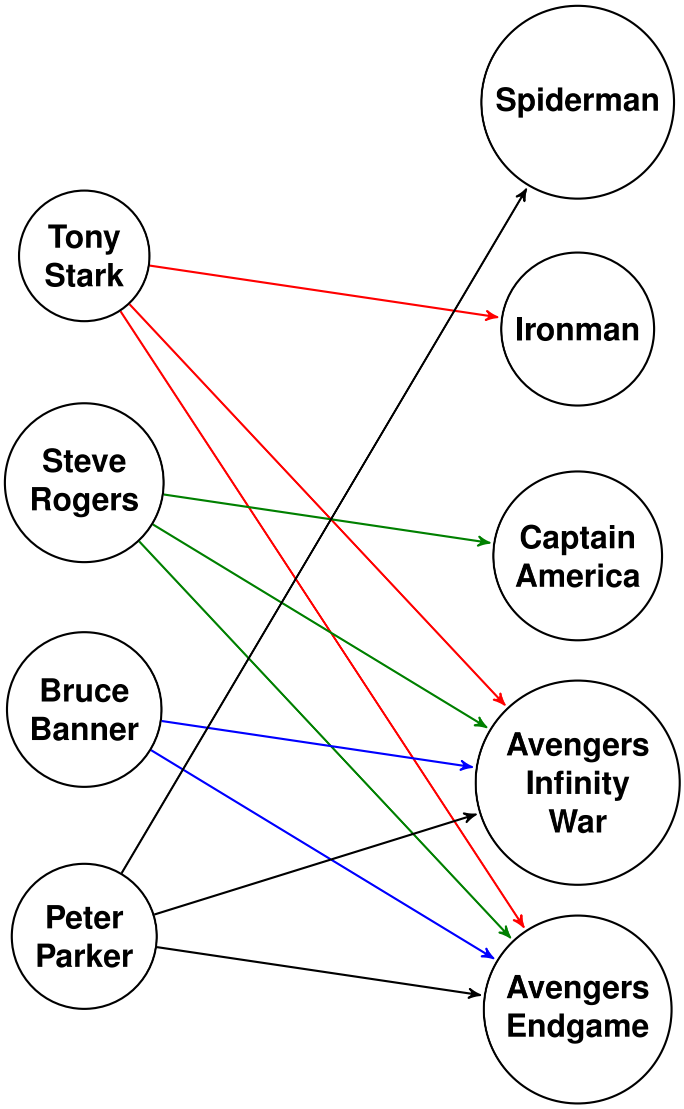


Figure 2: Graphical Representation of Relations

(b) Symmetric:

Let $a, b \in \mathbb{Z}$ and $a\mathcal{R}b$. We have to prove that $b\mathcal{R}a$. That is, $b - a$ should be divisible by 5. We know that, $a - b$ is divisible by 5, which implies that $-(a - b)$ is also divisible by 5. That is, $b - a$ is divisible by 5. Therefor, $b\mathcal{R}a$ which proves that $\mathcal{R}$ is symmetric.

(c) Transitivity:

Let $a, b, c \in \mathbb{Z}$ and $a\mathcal{R}b$ and $b\mathcal{R}c$. That is, both $a - b$ and $b - a$ is divisible by 5. We have to prove that $a\mathcal{R}c$. That is to prove that $a - c$ is divisible by 5. Note that, $(a - b) + (b - c)$ is divisible by 5. Thus, $a - c$ is divisible by 5. Therefore, $a\mathcal{R}c$ and so $\mathcal{R}$ is transitive. Hence $\mathcal{R}$ is an equivalence relation.

Now, lets find the equivalence class of 3. Note that,

$$[a] = \text{Set of all elements who are related to } a.$$

Therefore,

$$\begin{aligned}[3] = &\text{ Set of all integers who are related to } 3\\ &= \{x \in \mathbb{Z} : 3 - x \text{ is divisible by } 5\}\\ &= \{x \in \mathbb{Z} : 3 - x = 5k, \text{ for some } k \in \mathbb{Z}\}\\ &= \{3 - 5k : k \in \mathbb{Z}\}.\end{aligned}$$

1.2. Let $\mathbb{Z}$ be the set of all integers. Define $\mathcal{R}$ on $\mathbb{Z} \times \mathbb{Z}$ such that $a\mathcal{R}b \iff (a+b)$ is even, $a, b \in \mathbb{Z}$. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

1.3. Let $\mathbb{N}$ be the set of all positive integers. Define $\mathcal{R}$ on $\mathbb{N} \times \mathbb{N}$ such that $a\mathcal{R}b \iff |a+b|+2$ is a prime, $a, b \in \mathbb{N}$. Examine whether $\mathcal{R}$ is an equivalence relation on $\mathcal{N}$.

Solution: Hint: Not Transitive. Take $a = 1$, $b = 2$, and $c = 3$.

1.4. Using Warshall's algorithm, find the transitive closure of the relation

$$\mathcal{R} = \{(1,2), (2,1), (2,3), (3,4)\} \text{ on the set } A = \{1,2,3,4\}.$$

Solution:

$$W_0 = M_R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

$W_1 =$ Look at the 1st column and identify the indices where '1' is there - $\{2\}$
Look at the 1st row and identify the indices having '1' - $\{2\}$ $\therefore$ include 1 in place where the indices are given *i.e*

$$\{2\} \times \{2\} = \{2, 2\} \tag{2}$$

$$W_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

$$W_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

The indices where the 2nd column has '1' are $\{1, 2\}$
The indices where the 2nd column has '1' are $\{1, 2, 3\}$
$\therefore$ include '1' in place where the indices are given by the set

$$\{1, 2\} \times \{1, 2, 3\} = \{(1, 1), (1, 2), (1, 3), (2, 1)(2, 2), (2, 3)\} \tag{5}$$

$$W_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

The indices where the 3rd column has '1' are $\{1, 2\}$
The indices where the 3rd row has '1' are $\{4\}$
$\therefore$ include '1' in place where the indices are given by the set-

$$\{1, 2\} \times \{4\} = \{(1, 4), (2, 4)\} \tag{7}$$

$$W_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

The indices where the 4th column has '1' are $\{1, 2, 3\}$
The indices where the 4th row has '1' are $\{\}$
$\therefore$ There is nothing to be included newly for W_4
Thus the transitive closure is -

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2)(2, 3)(2, 4), (3, 4)\} \tag{9}$$

1.5. Using Warshall's algorithm, find the transitive closure of the relation

$$\mathcal{R} = \{(1,4),(2,1),(2,3),(3,1),(3,4),(4,3)\} \text{ on the set } A = \{1,2,3,4\}.$$

1.6. Using Warshall's algorithm, find the transitive closure of the relation $\mathcal{R}$ given by the directed graph in Figure 3:

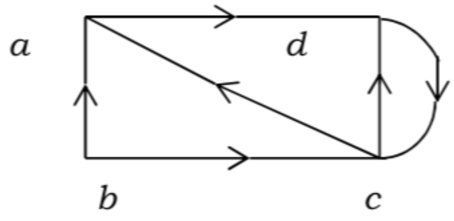


Figure 3

1.7. Let $\mathbb{N}$ be the set of all positive integers. Define $\preceq$ on $\mathbb{N} \times \mathbb{N}$ such that $a \preceq b \iff a$ divides b $a, b \in \mathbb{N}$. Examine whether $\preceq$ is a partial order relation on $\mathcal{N}$.

1.8. Let A be any set. Then consider the power set of A. Define $\mathcal{R}$ on P(A) such that $A\mathcal{R}B$ if and only if A⊆B.

Solution:

(a) For reflexive:
Let, $a \in A \implies a \in A$
$\implies A \subseteq A \implies A\mathcal{R}A$
$\therefore \mathcal{R}$ is reflexive.

(b) For symmetric:
Let $A, B \in P(A)$ and $A \subseteq B$ and $A \neq B$ T.P: B is not a subset of A $A \subseteq B$ and $A \neq B$
For $x \in B$ such that $x \in A$
$\implies$ B is not a subset of A
$\therefore \mathcal{R}$ is anti-symmetric.

(c) For transitive:
Let $A, B, C \in P(A)$ and $A \subseteq B$,$B \subseteq C$.

T.P: $A \subseteq C$
Consider an element 'a'. If $A \subseteq B$ and $B \subseteq C$ then $a \in A, B, C$
Thus $a \in A$, $a \in C \implies A \subseteq C$
$\therefore \mathcal{R}$ is transitive Hence $\mathcal{R}$ is a partial ordering relation or poset

Hence $\mathcal{R}$ is a partial ordering relation or poset

1.9. Let $\mathbb{Z}$ be the set of integers. Define $\preceq$ on $\mathbb{Z} \times \mathbb{Z}$ such that $a \preceq b \iff b = a^m$ for some $m \in \mathbb{N}$ and $a, b \in \mathbb{N}$. Show that $\preceq$ is a partial order relation on $\mathcal{Z}$.

1.10. Let $\mathbb{Z}$ be the set of integers. Define $\preceq$ on $\mathbb{Z} \times \mathbb{Z}$ such that $a \preceq b \iff a^3 - b^3$ is non-negative, $a, b \in \mathbb{N}$. Show that $\preceq$ is a partial order relation on $\mathcal{Z}$.

Definition 1.8. A relation $\mathcal{R}$ is said to be antisymmetric if for all $a, b \in \mathcal{R}$
$a\mathcal{R}b$ means b is not related to a unless $a = b$.

Definition 1.9. Set A together with a partial ordering relation $\mathcal{R}$ on A, is called a *partially ordered set* and its is denoted by $(A, \mathcal{R})$.

It is also called as *poset*. There is also an alternative notation for a poset, we represent $a\mathcal{R}b$ as $a \leq b$ under the relation $\mathcal{R}$ and so we represent $(A, \mathcal{R})$ as $(A, \leq)$.

Definition 1.10. A subset of A is called + *chain* if every two element in the subset are related.

You know what, because of anti-symmetry and transitivity every finite chain of n elements should be of the form

$$a_1 \leq a_2 \leq \ldots \leq a_n.$$

Can you think, why it is?

Quiz Questions

1.11. Let A be any arbitrary set. Then, if $A \times B = \emptyset$, what can you say about A and B?

1.12. Let R and S be any two equivalence relations on a non-empty set A. Which one of the following statements is TRUE? **GATE-CS-2010**

A. $R \cup S$, $R \cap S$ are both equivalence relations

B. $R \cup S$ is an equivalence relation

C. $R \cap S$ is an equivalence relation

D. Neither $R \cup S$ nor $R \cap S$ is equivalence relation

Verify.

1.13. For any nonempty set A, is it possible that $A \subseteq A \times A$?

A. Yes

B. No

Hint: Look at a general element in A and $A \times A$. Is it comparable?

1.14. Is it true that the transitive closure of a symmetric relation is symmetric?

A. Yes

B. No

Assignment 1

1.15. Let $A = \{1, 2\}$, construct $\mathscr{P}(A) \times A$ where $\mathscr{P}(A)$ is the power set of A.

1.16. Let A be a set of Books. Let $\mathcal{R}$ be a relation on A such that book a is related book b if and only if book a costs more and contains fewer pages than the book b. Check whether the relation $\mathcal{R}$ is reflexive, symmetric, antisymmetric or transitive? Does this relation form a poset or equivalence class?

1.17. Let $\mathcal{R}$ be a binary relation on the set of strings of 0s and 1s such that

$$\mathcal{R} = \{(a, b) : a \text{ and } b \text{ are strings that have same number of 0s}\}$$

Is R an equivalence relation? or a partial ordering relation?

1.18. In a college, there are three student clubs. Sixty students are only in the Drama club, 80 students are only in the Dance club, 30 students are only in the Maths club, 40 students are in both Drama and Dance clubs, 12 students are in both Dance and Maths clubs, 7 students are in both Drama and Maths clubs, and 2 students are in all the clubs. If 75% of the students in the college are not in any of these clubs, then what is the total number of students in the college? **[GATE-CS-2019]**

1.19. Let $S = \{v_1, v_2, \ldots, v_n\}$ be set of all variables used in a program. Define a relation $\mathcal{R}$, on S such that two variables are related if their values are same. (for example, if $a = 5$ and $b = 5$, we say a is related to b). Prove that, during the course of running the program, $\mathcal{R}$ is an equivalence relation.

1.1 Hesse Diagram

Note: This section is incomplete.

1.20. Show that if P and Q are posets defined on set X, then so is $P \cap Q$.

1.21. Draw the Hasse diagram of all natural numbers less than 10 ordered by the relation divides.

1.22. Draw the Hasse diagram of all positive divisors of 70.

1.2 Lattice

Definition 1.11. A partiallly ordered set is called *totally ordered set* is A is a chain. In this case, the binary relation $\leq$ is called as a *total ordering relation*.

Definition 1.12. Let $(A, \leq)$ be a partially ordered set. An element $a \in A$ is called as a *maximal element* if for no $b \in A$, $a \leq b$ unless $a = b$. An element $a \in A$ is called a *minimal element* if for no $b \in A$, $b \leq a$ unless $b = a$.

Definition 1.13. Let $(A, \leq)$ be a poset. An element $a \in A$ is said to *cover* an element $b \in A$ if $b \leq a$ and for no other element $c \in A$, $b \leq c \leq a$.

Definition 1.14. Let $(A, \leq)$ be a poset. An element $c \in A$ is called as *upper bound* of two element a and b if $a \leq c$ and $b \leq c$.

One may call, an element $c \in A$ is said to be a *least upper bound* of a and b if c is an upper bound of a and b and if there is no other upper bound d of a and b such that $d \leq c$. But with this definition, lub wont be unique. For example, in Figure 4a, f and g has h and i as lubs. So, we will slightly change the definition intuitively and expect the lub to be unique by defining in the following way.

Definition 1.15. Let $(A, \leq)$ be a poset. An element $c \in A$ is called as *least upper bound* of two elements a and b if

(i) c is an upper bound of a and b and

(ii) if there exists an upper bound d of a and b then $c \leq d$.

Definition 1.16. Let $(A, \leq)$ be a poset. An element $c \in A$ is called as *lower bound* of two elements a and b if $c \leq a$ and $c \leq b$.

Similar to lub, one might also define glb in the following way: an element $c \in A$ is said to be a *greatest lower bound*(glb) of a and b if c is an lower bound of a and b and if there is no other lower bound d of a and b such that $c \leq d$. But, for the same reason that we expect the uniqueness of the glb, we define the glb as follows:

Definition 1.17. Let $(A, \leq)$ be a poset. An element $c \in A$ is said to be a *greatest lower bound*(glb) of a and b if

(i) c is an lower bound of a and

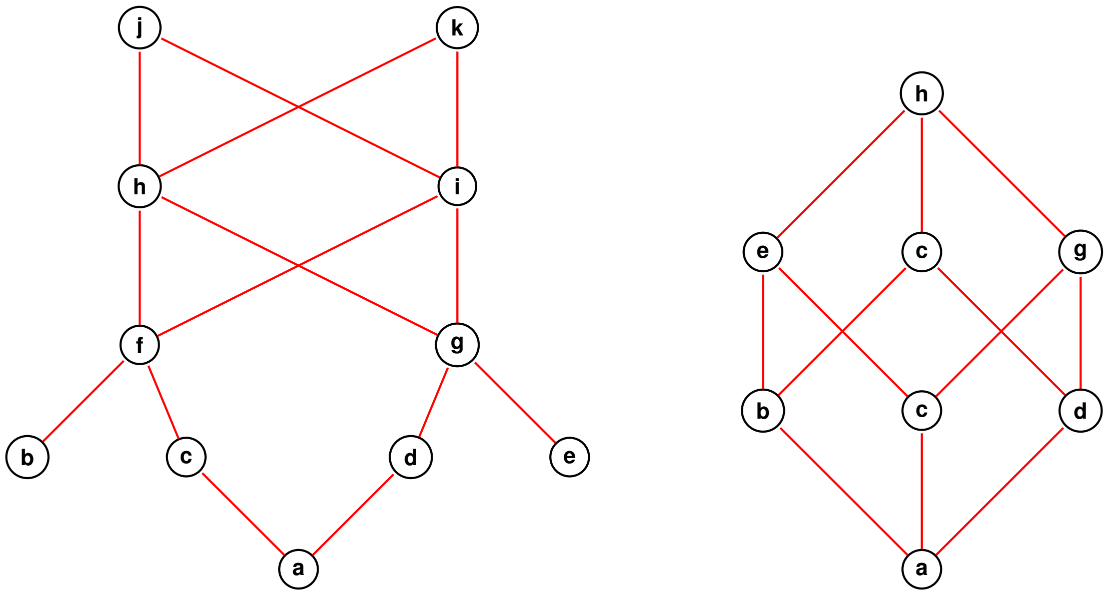
(ii) if there exists a lower bound $d \in A$ of a and b then $d \leq c$.

So, from the definitions of lub and glb, we get the following theorem.

Theorem 1. *For given any two elements a and b in a poset $(P, \preccurlyeq)$, least upper bound and greatest lower bound is unique.*

Definition 1.18. A partially ordered set is said to be a *lattice* if every two elements in the set have a unique least upper bound and a unique greatest lower bound.

1.23. Check whether the figures 4a, 4b and Figure 5 are lattices are not.



(a) (b)

Figure 4

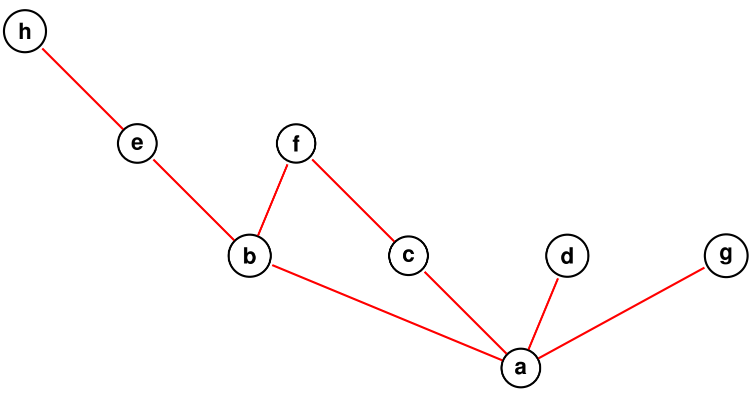


Figure 5

Solution: Figure 4a is not a lattice (because h and i has f and g as glb.) and Figure 4b is a lattice.

1.24. If X ia a nonempty set, then show that $(P(X), \preccurlyeq)$ is lattice with respect to the relation $A \preccurlyeq B \Longleftrightarrow A \subseteq B$, $A, B \in P(X)$.

Hint: Show that $A \vee B = A \cup B,\ A \wedge B = A \cap B$.

1.25. Prove that the set of natural numbers under the relation divides forms a lattice.

Hint:
Given any two natual numbers, the greatest common divisor (gcd) and the least common multiple (lcm) of those two numbers correspond to the lub and glb respectively.

1.26. The set of rationals or reals forms a lattice with the $\leq$ relation. This is a totally ordered set and lub and glb correspond to the $\max$ and $\min$ for a finite subset.

1.27. In a lattice $(L, \preccurlyeq)$, for any $a, b \in L$, show that $a \preccurlyeq a \vee b$ and $a \wedge b \preccurlyeq a$.

1.28. In a lattice $(L, \preccurlyeq)$, if $a \preccurlyeq b$, show that $a \wedge b = a, \quad a \vee b = b$.

1.29. In a lattice $(L, \preccurlyeq)$, show that $a \wedge b = a$ if and only if $a \vee b = b$.

1.30. **Idempotent Law:** In a lattice $(L, \preccurlyeq)$, show that $a \wedge a = a$ if and only if $a \vee a = a$.

1.31. **Commutative Property:**

(a) In a lattice $(L, \preccurlyeq)$, show that $a \wedge b = b \wedge a$, for all $a, b \in L$.

(b) In a lattice $(L, \preccurlyeq)$, show that $a \vee b = b \vee a$, for all $a, b \in L$.

1.32. **Associative Property:**

(a) In a lattice $(L, \preccurlyeq)$, show that $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, for all $a, b, c \in L$.

(b) In a lattice $(L, \preccurlyeq)$, show that $a \vee (b \vee c) = (a \vee b) \vee c$, for all $a, b, c \in L$.

1.33. **Absorption:** In a lattice $(L, \preccurlyeq)$, show that $a \wedge (a \vee b) = a \vee (a \wedge b) = a$, for all $a, b \in L$.

1.34. In a lattice $(L, \preccurlyeq)$, show that $(a \wedge b) \vee (a \wedge c) \preccurlyeq a \wedge (b \vee c)$, for all $a, b, c \in L$.

Hint: $a \wedge b \preccurlyeq a,\ b$ and so, $a \wedge b \preccurlyeq a,\ b \vee c$ i.e. $a \wedge b$ is a lower bound of a and $b \vee c$. Hence, $a \wedge b \preccurlyeq a \wedge (b \vee c)$.

1.35. In a lattice $(L, \preccurlyeq)$, show that $(a \wedge b) \vee (a \wedge c) \preccurlyeq a \wedge (b \vee c)$, for all $a, b, c \in L$.

Hint: $a \wedge b \preccurlyeq a,\ b$ and so, $a \wedge b \preccurlyeq a,\ b \vee c$ i.e. $a \wedge b$ is a lower bound of a and $b \vee c$. Hence, $a \wedge b \preccurlyeq a \wedge (b \vee c)$.

Similarly, show $a \wedge c \preccurlyeq a \wedge (b \vee c)$. Hence, $(a \wedge b) \vee (a \wedge c) \preccurlyeq a \wedge (b \vee c)$.

1.3 Distributive Lattice

Definition 1.19. A lattice $(L, \preccurlyeq)$ is said to be distributive lattice if for any $a, b, c \in L$, the follwoing identies hold:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{10}$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{11}$$

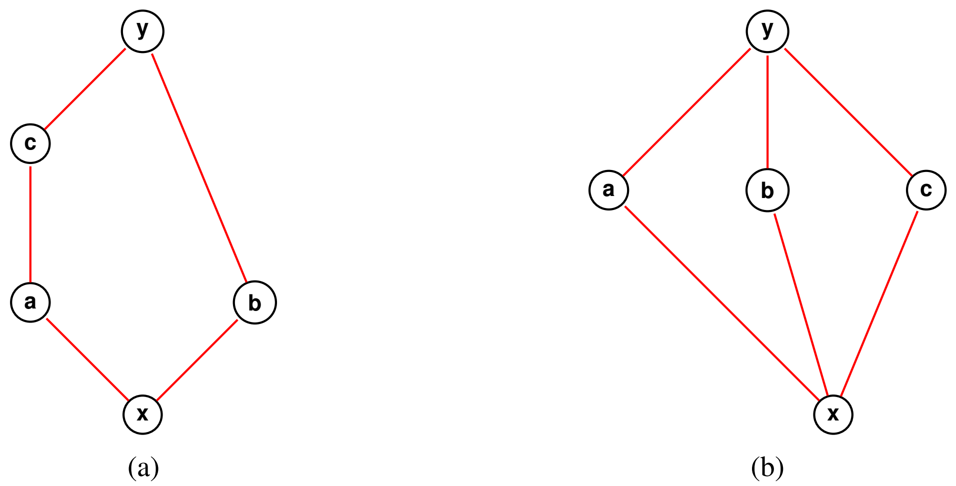


Figure 6: Example of non-Distributive Lattice

consider Figure 6a. Note that, this poset is a lattice (verify?)

$$a \vee (b \wedge c) = a \vee x = a \tag{12}$$

and

$$(a \vee b) \wedge (a \vee c) = y \wedge y = y. \tag{13}$$

Thus, $a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$. So, the lattice is not a distributive lattice.

Similarly, consider the Figure 6b. Note that, this poset is a lattice (verify). But, it is not distributive. Since,

$$a \vee (b \wedge c) = a \vee x = a \tag{14}$$

and

$$(a \vee b) \wedge (a \vee c) = y \wedge c = c, \tag{15}$$

$$a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c). \tag{16}$$

1.36. Show that in a lattice, if the join operation is distributive over the meet operation, then meet operation is distributive over the join operation.

Solution: Let us assume the the join operation is distributive over the meet operation. That is,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{17}$$

Now let us prove that the meet operation is distributive over join. That is, we need to prove that

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{18}$$

We will prove the RHS of the Equation 18 is equal to the LHS of the same.

$$\begin{aligned}
(a \wedge b) \vee (a \wedge c) &= ((a \wedge b) \vee a) \wedge ((a \wedge b) \vee c) \quad (\text{ by Equation 17}) \\
&= a \wedge ((a \wedge b) \vee c)) \quad (\text{by absorption law}) \\
&= a \wedge (c \vee (a \wedge b))) \quad (\text{ by commutative property}) \\
&= a \wedge ((c \vee a) \wedge (c \vee b)) \quad (\text{by Equation 17}) \\
&= (a \wedge (c \vee a)) \wedge (c \vee b) \quad (\text{by associative property}) \\
&= a \wedge (b \vee c) \quad (\text{by absorption law})
\end{aligned}$$

Hence, the proof.

1.37. Show that in a lattice, if the meet operation is distributive over the join operation, then join operation is distributive over the meet operation.

1.38. In a distributive lattice $(L, \preccurlyeq)$, if $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$, then show that $b = c$.

Hint:

$$\begin{aligned}
b = b \vee (a \wedge b) = b \vee (a \wedge c) &= (b \vee a) \wedge (b \vee c) \\
&= (a \vee b) \wedge (b \vee c) \\
&= (a \vee c) \wedge (b \vee c) \\
&= (a \wedge b) \vee c \\
&= (a \wedge c) \vee c \\
&= c
\end{aligned}$$

Definition 1.20. An element y is called as greatest element or universal upper bound of a lattice if for any $x \in L$,

$$x \leq y.$$

We denote this y as 1.

Similarly, one can define the least element in a lattice as follows.

Definition 1.21. An element y is called as *least element* or *universal lower bound* of a lattice if for any $x \in L$,

$$y \leq x.$$

We denote this y as 0.

Definition 1.22. A lattice $(L, \preccurlyeq)$ is said to be *bounded lattice* if it has a greatest and least element.

Definition 1.23. Let $(L, \preccurlyeq)$ be a bounded lattice. Let $x \in L$. An element $\bar{x} \in L$ is said to be complement of $x \in L$ if $x \vee \bar{x} = 1$ and $x \wedge \bar{x} = 0$.

1.39. Show that in a distributive lattice, if the complement of an element exists, then it is unique.

Definition 1.24. A bounded lattice is said to be a complemented lattice if every element has a complement.

1.40. Maximum element and minimum element of a lattice is unique.

1.41. In a distributive lattice if a complement of an element exists, then it is unique.

Solution: Let an element $a \in L$ has two complements b and c. Then,

$$a \vee b = 1 \quad \text{and} \quad a \wedge b = 0 \tag{19}$$

similarly,

$$a \vee c = 1 \quad \text{and } a \wedge c = 0. \tag{20}$$

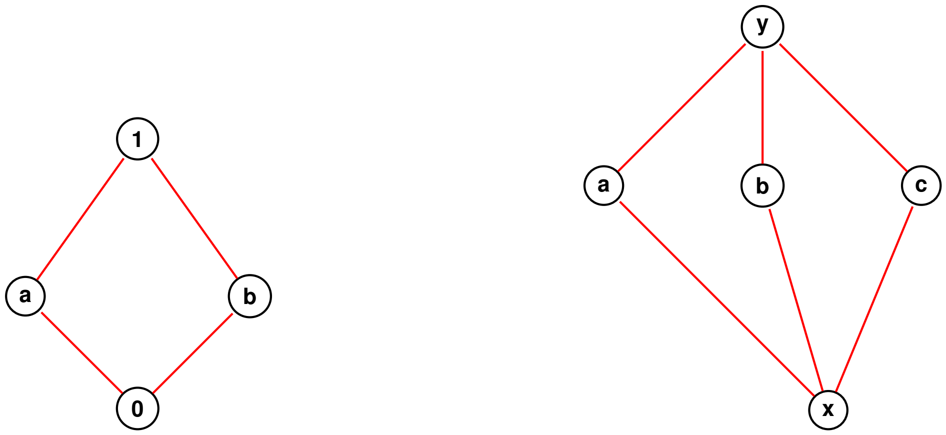
Now,

$$\begin{aligned}
b &= b \wedge 1 \\
&= b \wedge (a \vee c) \\
&= (b \wedge a) \vee (b \wedge c) \quad \text{by distributive property} \\
&= (a \wedge b) \vee (b \wedge c) \quad \text{by commutative property} \\
&= 0 \vee (b \wedge c) \\
&= (a \wedge c) \vee (b \wedge c) \\
&= (a \vee b) \wedge c \\
&= (a \vee b) \wedge c \quad \text{by distributive property} \\
&= 1 \wedge c \\
&= c.
\end{aligned}$$

Hence the proof.

1.42. Give an example of a complemented lattice.

Solution: Consider Figure 7a,



(a) Complemented Lattice

(b) Complemented but not distributive lattice

Figure 7: Example of Complemented Lattice

1.43. Give an example of a complemented lattice but not distributive.

Solution: Consider Figure 7b.

1.44. Give an example of a lattice which is distributed but not complemented.

Definition 1.25. A lattice $(L, \preccurlyeq)$ is said to be a *Boolean Lattice* if it is complemented and distributive.

Figure 4b is a boolean lattice with greatest element as h and the least element as a.

Definition 1.26. The algebraic system $(L, \vee, \wedge, ^{-})$ is known as a boolean algebra. Any boolean algebra is denoted by $\mathscr{B}$.

DeMorgan's Law:

1.45. Show that in a Boolean lattice $(L, \vee, \wedge, ^{-})$, $\overline{a \vee b} = \overline{a} \wedge \overline{b}$, for all $a, b \in L$.

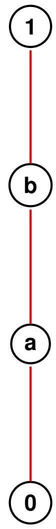


Figure 8

Solution: Let $a, b \in L$. We need to prove that the complement of $a \vee b$ is $\bar{a} \wedge \bar{b}$. That is, we need to prove that

$$(a \vee b) \vee (\bar{a} \wedge \bar{b}) = 1 \quad \text{and} \tag{21}$$

$$(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = 0 \tag{22}$$

First let us prove Equation 21.

$$\begin{aligned}
(a \vee b) \vee (\bar{a} \wedge \bar{b}) &= ((a \vee b) \vee \bar{a}) \wedge ((a \vee b) \wedge \bar{b}) && \text{(by distributive property)} \\
&= ((b \vee a) \vee \bar{a}) \wedge ((a \vee b) \wedge \bar{b}) && \text{(by commutative property)} \\
&= (b \vee (a \vee \bar{a}) \wedge (a \vee (b \wedge \bar{b})) && \text{(by associative property)} \\
&= (b \vee 1) \wedge (a \vee 0) && \text{(by definition of complement)} \\
&= 1 \wedge a && \text{(by definition of complement)} \\
&= 1 && \text{(by definition of complement)}
\end{aligned}$$

which proves the Equation 21. Similarly,

$$\begin{aligned}
(a \vee b) \wedge (\bar{a} \wedge \bar{b}) &= ((a \vee b) \wedge \bar{a}) \wedge ((a \vee b) \wedge \bar{b}) && \text{(by distributive property)} \\
&= ((b \vee a) \wedge \bar{a}) \wedge ((a \vee b) \wedge \bar{b}) && \text{(by commutative property)} \\
&= (b \vee (a \wedge \bar{a})) \wedge (a \vee (b \wedge \bar{b})) && \text{(by associative property)} \\
&= (b \vee (0)) \wedge (a \vee (0)) && \text{(by definition of complement)} \\
&= 0 \wedge 0 && \text{(again by definition of complement)} \\
&= 0 && \text{(again by definition of complement),}
\end{aligned}$$

which proves Equation 22. Hence the complement of $a \vee b$ is $\bar{a} \wedge \bar{b}$. That is,

$$\overline{a \vee b} = \bar{a} \wedge \bar{b}.$$

1.46. Show that in a Boolean lattice $(L, \vee, \wedge, ^{-})$, $\overline{a \wedge b} = \overline{a} \vee \overline{b}$, for all $a, b \in L$.

Solution: Left as excercise.

Definition 1.27 (Atom)**.** An element $x \in \mathscr{B}$ is said to be an *atom* if x covers 0. That is,

(i) $0 \preccurlyeq x$ and

(ii) there does not exist any $c \in \mathscr{B}$ such that $0 < c < x$.

For example, in Figure 4b, b, c and d are atoms.

1.47. In a distributive lattice, if $b \wedge \bar{c} = 0$ then $b \preccurlyeq c$.

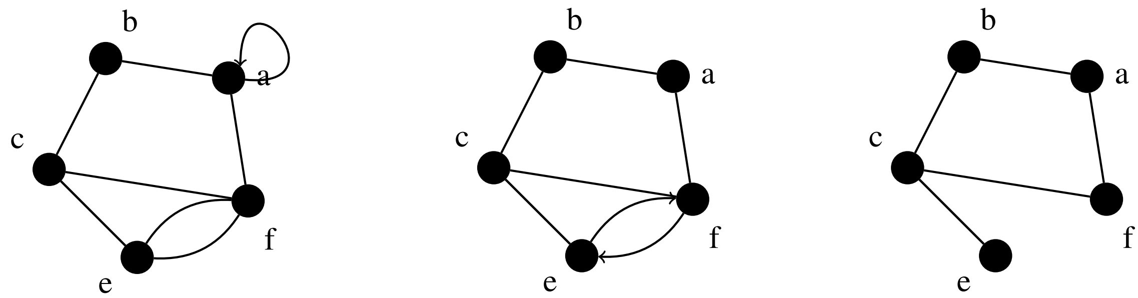
2 Graph Theory

Definition 2.1 (Graph)**.** A *graph* $G = (V, E)$ consists of nonempty set of vertices V (or nodes) and set of edges E such that each edge e_k is identified with an unordered pair of vertices (v_i, v_j).

The vertices v_i and v_j associated with the edge e_k is called as *end vertices* of the edge e_k. The definition of a graph allows that a edge can same vertex as end vertices. Such an edge is called as *self loop*. Similary, a graph can also have more than one edge between two vertices. We call those edges as *parallel edges*. In general, we can divide the graphs into three types, namely, Pseudo graph (or general graph), Multi Graph and Simple Graph, as defined in the following definition.

Definition 2.2 (Pseudo, Multi, Simple graphs)**.** A graph with no self loops and no parallel edges is called as *Simple Graph*. A graph with parallel edges is called as *Multi Graph* and the graph with both parallel edges and self loops is called as *Pseudo Graph or General Graph*.

Refer Figure 9 for different types of graphs.



(a) Pseudo Graph (General Graph) (b) Multi Graph (c) Simple Graph

Figure 9: Example of Graphs

Remark 1. Although in our definitions a graph need not have only finite number of vertices nor finite number of edges, in most of the applications (in fact almost all), vertices and edges are finite. Hence, throughout the lectures, we assume that the graph we consider is a finite graph.

For the same reason, we say a graph is *null graph* if there is no edge present. For example, refer Figure 10.

Question 2.1. Draw all simple graphs of one, two, three and four vertices.

Definition 2.3 (Degree of a vertex)**.** The degree of a vertex v in a graph G is the number of edges incident to the vertex. We denote it as $\deg(v)$. A vertex is said to be odd or even according to its degree is odd or even.

For example, the degree of the vertex a in Figure 9c is 2 and hence it is an even vertex. Similarly, the degree of the vertex c is 3 and hence it is an odd vertex. Remember that the self loops incidence a vertex twice and hence the degree contributed by a self loop is 2. For example, in Figure 9a the degree of the vertex a is 4 since the self-loop contribute 2 and the edge between a and b & a and f contributes the rest.

A sequence of degrees of all the vertices written in a non-increasing order is called as a *degree sequence of a graph.*

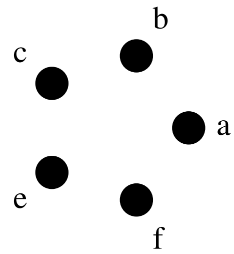


Figure 10: Example of a null graph

Question 2.2. Write the degree sequence of the graph in Figure 9c.

Solution: We know that, $\deg(a) = 2$, $\deg(b) = 2$, $\deg(c) = 3$, $\deg(c) = 1$ and $\deg(f) = 2$. Therefore the degree sequence is,

$$3, 2, 2, 2, 1$$

Note that, we haven't left the degree of any vertex.

2.1. Which of the following is not a degree sequence of any graph? [GATE 2010]

A. 7, 6,5,4,4,3,2,1

B. 6,6,6,3,3,2,2

C. 7,6,6,4,4,3,2,2

D. 8,7,7,6,4,2,1,1

Question 2.3. Write the degree sequence of the graph in Figure 9a, Figure 10.

Theorem 2 (Handshaking Lemma)**.** *The sum of the degrees of the vertices of a graph G is twice the number of edges in G. Mathematically,*

$$\sum_{v \in V(G)} \deg(v) = 2 \times |E(G)|.$$

Proof. Let G be a graph with n vertices and q edges. Note that, each edge is incident with two vertices. That is, each edge contributes 2 to the sum of the $\deg$ of vertices. Therefore, the sum of the degrees of vertices is 2 times of the number of edges. □

Question 2.4. Verify Theorem 2 for the graphs in Figure 9.

Question 2.5. An undirected graph has 10 vertices labeled 1,2,. . . 10 and 37 edges. Vertices 1,3,5,7, 9 have degree 8 and vertices 2,4,6,8 have degree 7. What is the degree of vertex 10? [CMI 2015]

Question 2.6. Prove that the number odd vertices in a graph is even.

Proof. Let $G(p, q)$ be a graph with p vertices and q edges. Note that by handshaking lemma (Theorem 2),

$$\sum_{v \in V(G)} \deg(v) = 2q.$$

We write the sum of all the degrees as sum of all even and odd degrees.

$$\begin{aligned}
\sum_{v \in V(G)} \deg(v) &= \text{sum of even degree} + \text{ sum of odd degree} \\
2 \times q &= \sum_{\substack{v \in v(G) \\ \deg(v) \text{is even}}} \deg(v) + \sum_{\substack{v \in v(G) \\ \deg(v) \text{is odd}}} deg(v) \\
\text{even} &= \text{even} + \sum_{\substack{v \in v(G) \\ \deg(v) \text{is odd}}} deg(v) \\
\text{even} &= \sum_{\substack{v \in v(G) \\ \deg(v) \text{is odd}}} deg(v)
\end{aligned}$$

So, the sum of odd number is even implies that the number of odd vertices must be even. Hence the proof. □

Question 2.7. The maximum degree of any vertex in a simple graph with n vertices is $n - 1$.

Question 2.8. What is the maximum number of edges in a $n-$node undirected graph without self loops? [GATE 2002]

2.1 Pigeonhole principle

Look at the Figure 11, how many pigeonholes are there? Clearly there are 9 pigeonholes are there. But there are 10 pigeons. So the figure clearly says if there are 9 pigeon hole and 10 pigeons, then one of the pigeonhole should have two or more pigeons. This is what pigeon hole principle says. That is,

Definition 2.4 (Pigeon Hole Principle)**.** If there are n or more pigeos are to be distributed in $n - 1$ pigeon holes, then atleast one pigeon hole contains two or more pigeons.

Figure 11: Pigeon Hole Principle

Question 2.9. In a party with atleast two people, there are alteast two people who have same number of friends.

Proof. In a group of N people, a person may have $0, 1, 2, \ldots, N-1$ friends. First lets us prove that all the people have atleast one friend. To prove that assume to the contrary that all N people have different number of friends. Then for each number in the sequence $0, 1, 2, \ldots, N-1$ there must be a person with exactly this number of friends. In particular, there is at least one with $N-1$ friends. But, if so, all others have this person as a friend, implying that there is no one with no friends at all. Therefore, the only possible numbers of friends come from the shortened sequence: $1, 2, 3, \ldots, N-1$. By the Pigeonhole Principle, there are at least two with the same number of friends, so that our assumption that this is not true proved wrong, thus it must be indeed true. $\square$

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