SIKKIM MANIPAL UNIVERSITY
SIKKIM MANIPAL INSTITUTE OF TECHNOLOGY

Engineering Mathematics III

Discrete Mathematics

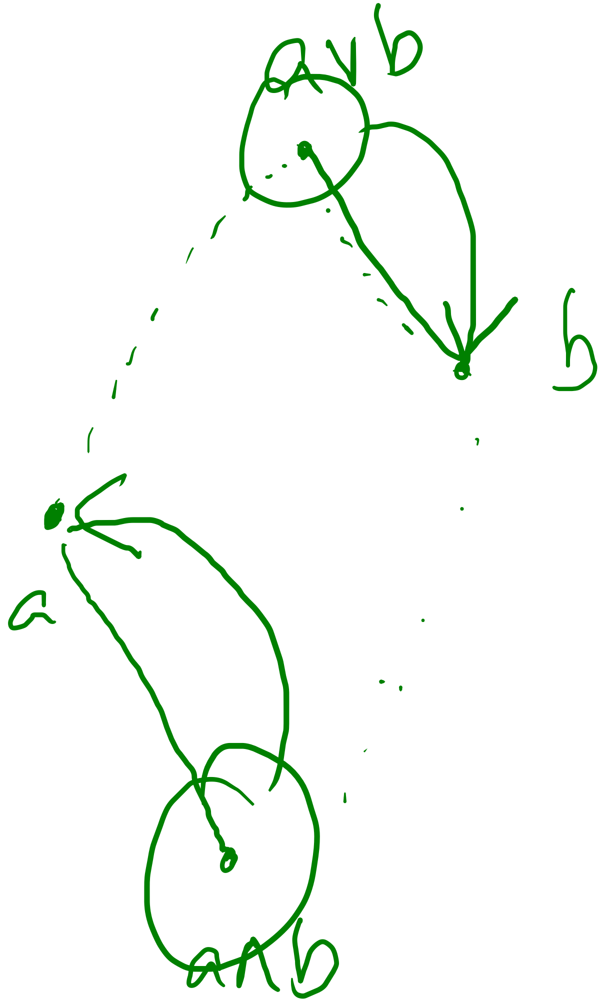
Lecture 11

Properties of Lattice & Distributive Lattice, Complemented Lattice

This course is taught to Computer Science Engineering students in SMIT, India during Jun-Dec, 2019.

3.23. In a lattice $(L, \preccurlyeq)$, show that $a \wedge b = a$ if and only if $a \vee b = b$.

Exercise



3.24. **Commutative Property:**

(a) In a lattice $(L, \preccurlyeq)$, show that $a \wedge b = b \wedge a$, for all $a, b \in L$.

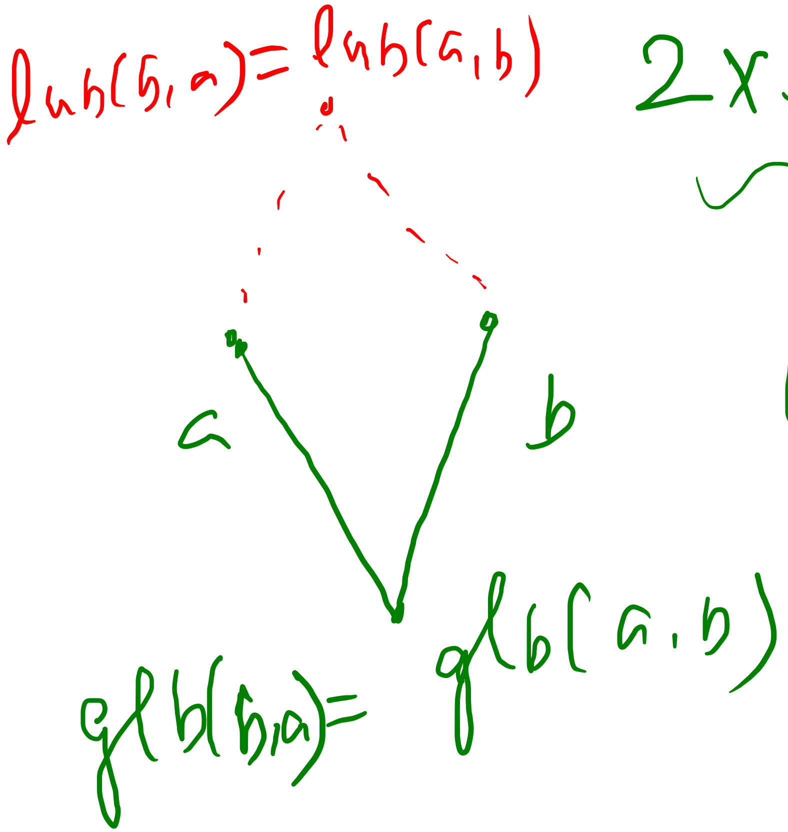
(b) In a lattice $(L, \preccurlyeq)$, show that $a \vee b = b \vee a$, for all $a, b \in L$.

Let $a, b \in L$

(a) $a \wedge b = glb(a, b)$

$= glb(b, a)$

$= b \wedge a$



$\underbrace{2 \times 3 = 3 \times 2}$

Commutative property

(b) Let $a, b \in L$,

$a \vee b = lub(a, b)$

$= lub(b, a) = b \vee a$

$(A \cup B) \cup C = A \cup (B \cup C)$

3.25. **Associative Property:**

(a) In a lattice $(L, \preccurlyeq)$, show that $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, for all $a, b, c \in L$.

Exercise (b) In a lattice $(L, \preccurlyeq)$, show that $a \vee (b \vee c) = (a \vee b) \vee c$, for all $a, b, c \in L$.

(a) $a \wedge (b \wedge c) = a \wedge (glb(b, c))$

$= glb(a, glb(b, c))$

$= glb(glb(a, b), c)$

$= glb(a \wedge b, c)$

$= (a \wedge b) \wedge c$

$(2+3)+4 = 2+(3+4)$ } Associative property

a b c

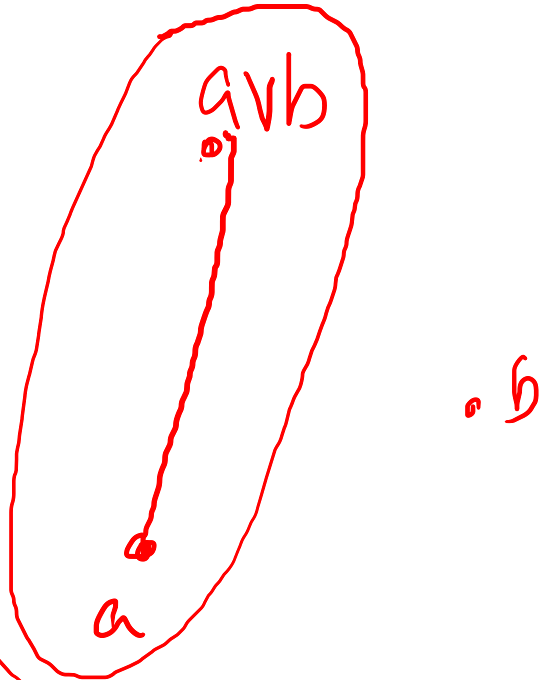
3.26. **Absorption:** In a lattice $(L, \preccurlyeq)$, show that $a \wedge (a \vee b) = a \vee (a \wedge b) = a$, for all $a, b \in L$.

$a \wedge (a \vee b) = a$

$a \wedge (a \vee b) = glb(a, a \vee b)$

$= a \quad \{\because a \leq a \vee b\}$

(a is a lower bound of a and a∨b and there is no other c such that lower bound a ≤ c.)



$a \vee (a \wedge b) = lub(a, a \wedge b)$

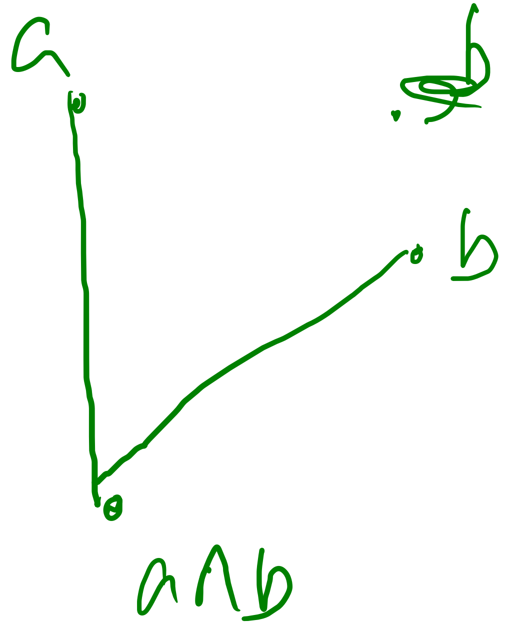
$= a \quad \{\because a \wedge b \leq a\}$

$$(a \wedge b) \vee b = b$$

$$(a \wedge b) \vee a = a$$

What is $(a \vee b) \wedge a$?

$$= a.$$



$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap$

A

Definition 3.9. A lattice $(L, \preccurlyeq)$ is said to be distributive lattice if for any $a, b, c \in L$, the follwoing identies hold:

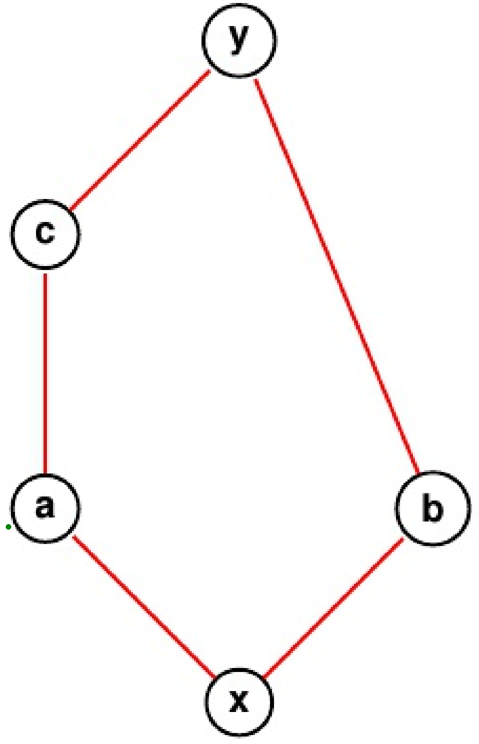
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{1}$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{2}$$

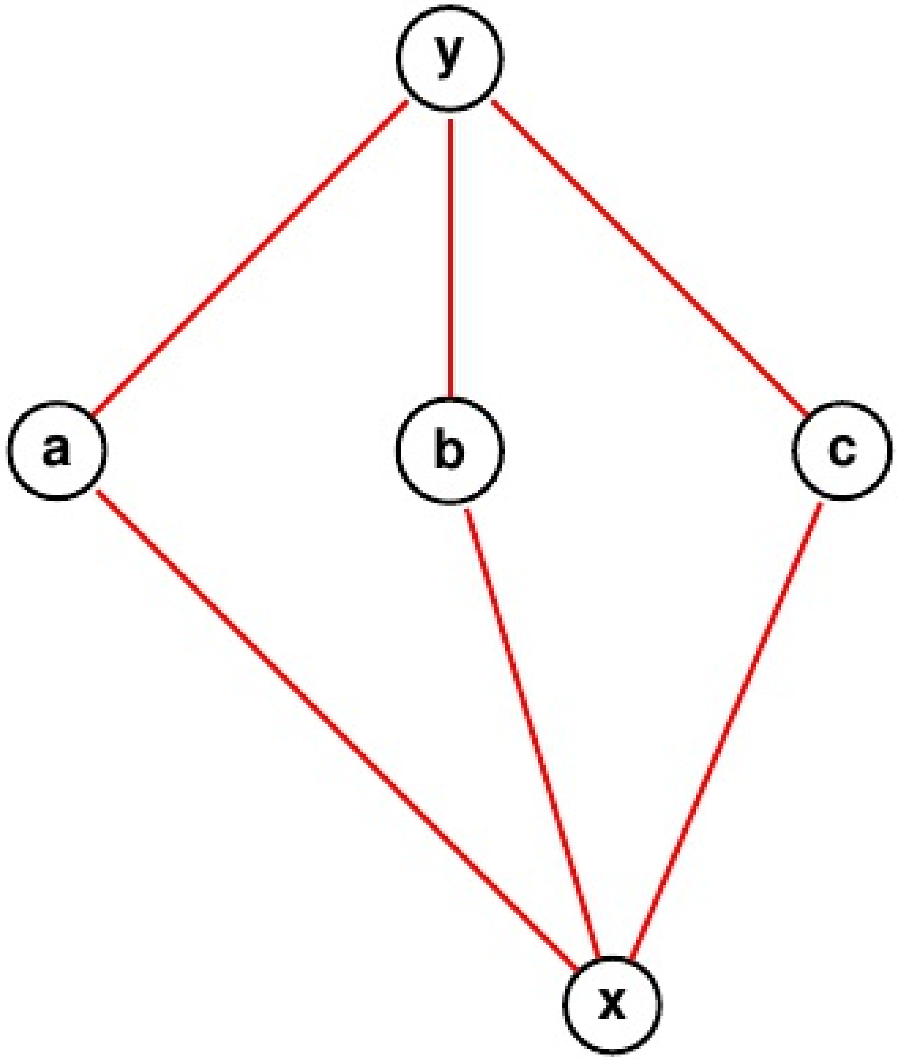
$(a \vee b) \wedge (a \vee c) = y \wedge c = c$

$a \vee (b \wedge c) = a \vee (x) = a$

$\Rightarrow$ This lattice is not distributive



Is this distributive lattice?



$a \vee (b \wedge c) = a \vee (x) = a \rightarrow ①$

$(a \vee b) \wedge (a \vee c) = y \wedge y$

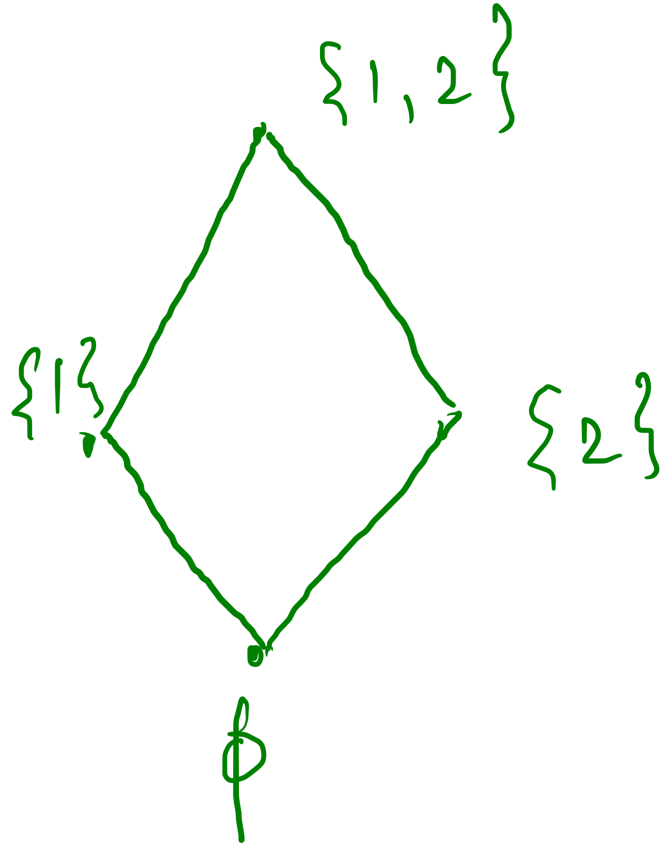
$= y \rightarrow ②$

from ① & ②,

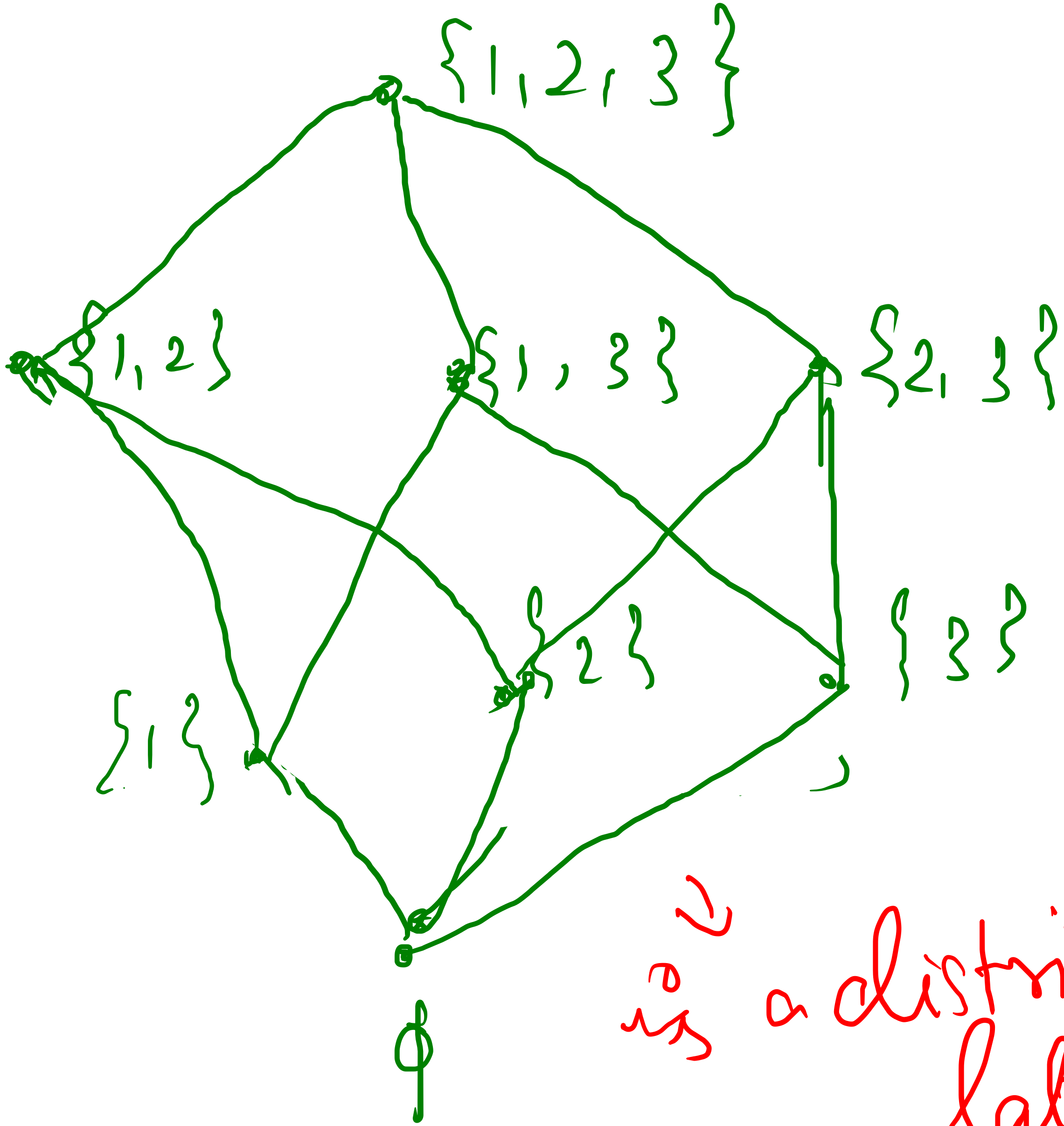
$a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$

$\Rightarrow$ this is not distributive lattice.

$(\mathcal{P}(X), \subseteq)$ $X = \{1, 2\}$



$(\mathcal{P}(X), \subseteq)$, $X = \{1, 2, 3\}$



is a distributive lattice

3.29. Show that in a lattice, if the meet operation is distributive over the join operation, then join operation is distributive over the meet operation.

verify with lec. Notes

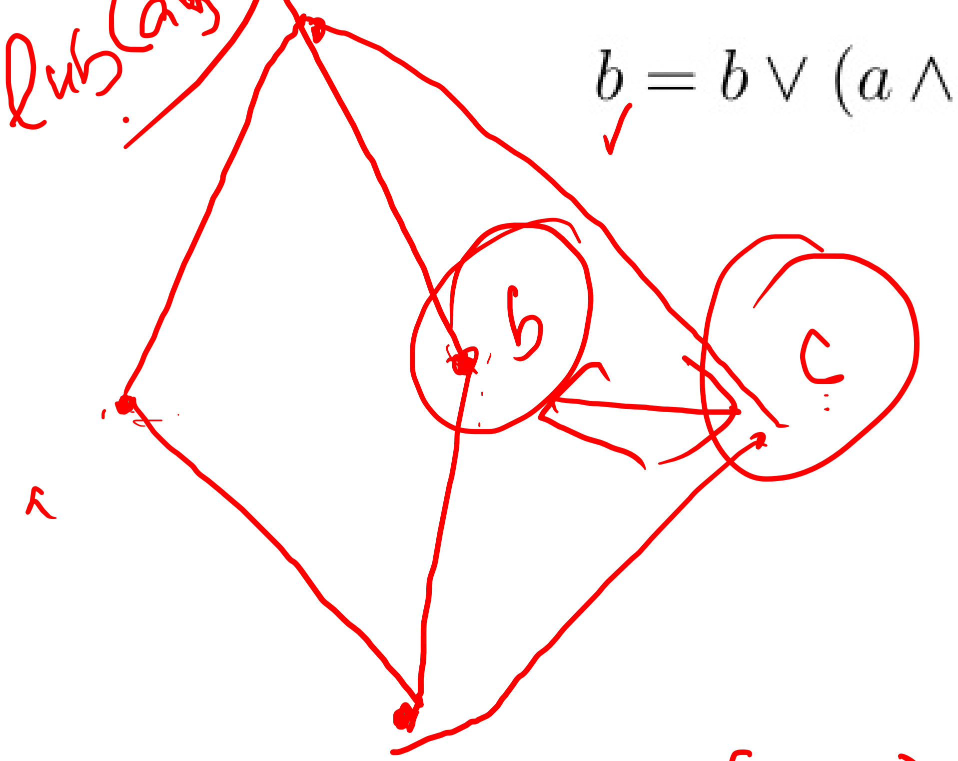
3.30. Show that in a lattice, if the join operation is distributive over the meet operation, then meet operation is distributive over the join operation.

Exercise

3.31. In a distributive lattice $(L, \preccurlyeq)$, if $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$, then show that $b = c$.

Hint:

$lub(a, c) = lub(a, b)$



$glb(a, b) = glb(a, c)$

$$
\begin{aligned}
b = b \vee (a \wedge b) = b \vee (a \wedge c) &= (b \vee a) \wedge (b \vee c) \\
&= (a \vee b) \wedge (b \vee c) \\
&= (a \vee c) \wedge (b \vee c) \\
&= (a \wedge b) \vee c \\
&= (a \wedge c) \vee c \\
&= c
\end{aligned}
$$

(by absorption)