Sikkim Manipal Institute of Technology

III SEMESTER B.Tech(CSE) SAMPLE QUESTIONS, SET-3

DISCRETE MATHEMATICS (MA 1308)

Problem Sheet on Graph Theory

Note: A graph G with p vertices and q edges can be written as $G(p,q)$ for convenience.

1. Show that a graph contains even number of odd degree vertices.

2. Let G be a (p,q) graph whose vertices are of degree k or $k+1$. If G has t number of vertices of degree k, then show that $t = p(k+1) - 2q$.

3. In a party of six or more people, show that there are at least three mutual acquaintances or three mutual nonacquaintances.

4. Show that in group of two or more people, at least two of them having same number of friends.

5. If G is a graph with p-vertices and $\delta(G) \geq \dfrac{p-1}{2}$, then show that G is connected. Give an example to show that the converse is not true.

6. If a graph has exactly two vertices of odd degree, show that there must be path joining these two vertices.

   Hint: If G is connected then there is nothing to prove.

   If G is disconnected then these two vertices of G must be in the same connected component of G, otherwise G will have subgraphs with only one odd degree vertex. Hence the result.

7. The minimum degree among all the vertices of the graph is denoted as $\delta(G)$. If G is a graph such that $\delta(G) > 1$, then show that G contains a cycle.

8. If G is a (p,q) graph, then show that the following statements are equivalent:

   (i) G is a tree.

   (ii) G is acyclic and $q = p - 1$.

   (iii) G is connected and $q = p - 1$.

9. If G is a tree, show that every two vertices of G are joined by a unique path.

10. Show that every nontrivial tree has at least two vertices of degree 1.

    Hint: Let G be a nontrivial tree with p vertices and q edges. Then $q = p - 1$. Suppose that G has vertices $v_1, v_2, \cdots, v_m$ of degree 1 and $v_{m+1}, v_{m+2}, \cdots, v_p$ of degree different from 1. Then, $\deg_G v_i \begin{cases} = & 1, \text{ if } i = 1, 2, \cdots, m \\ \geq & 2, \text{ if } i = m+1, m+2, \cdots, p \end{cases}$

    Then $\sum\limits_{i=1}^{p} \deg_G v_i = \sum\limits_{i=1}^{m} \deg_G v_i + \sum\limits_{i=m+1}^{p} \deg_G v_i \geq m + 2(p-m) = 2p - m.$

    But, $\sum\limits_{i=1}^{p} \deg_G v_i = 2q = 2(p-1)$ and so, $2(p-1) \geq 2p - m \Rightarrow m \geq 2$. Hence the result.

11. A tree has two vertices of degree 2, one vertex of degree 3 and three vertices of degree 4. How many vertices of degree 1 does it have?

    Hint: Suppose that it has k vertices of degree 1. Then total number of vertices is $p = k + 6$. So, total number of edges is $q = p - 1 = k + 5$. Then use $\sum_{i=1}^{p} \deg_G v_i = 2q$ and find the value of k.

12. Show that the graph $G(p, q)$ with $q \geq p$ contains a cycle.

    Hint:

    Case 1: Suppose that G is connected with $q \geq p$. Then G can not be a tree as for a tree $q = p - 1$ and so, $q < p$. So, G must contain a cycle.

    Case 2: Next, assume G be a disconnected graph with $q \geq p$. Let $G_i(p_i, q_i), i = 1, 2, \cdots, n$ be the connected components of G. Then $\sum_{i=1}^{n} p_i = p, \sum_{i=1}^{n} q_i = q$.

    Then there exists at least one connected component G_k in G such that $q_k \geq p_k$. Otherwise, if $q_i < p_i, \forall i = 1, 2, \cdots, n$, then $q = \sum_{i=1}^{n} q_i < \sum_{i=1}^{n} p_i = p$, a contradiction. By the **Case 1**, G_k must contain a cycle and so, G contains a cycle.