Engineering Mathematics III

Discrete Mathematics

Lecture 12

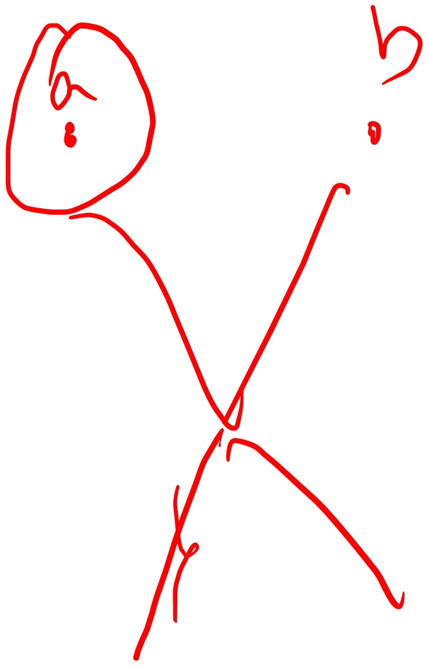
Universal lower and upper bounds & Complemented Lattice

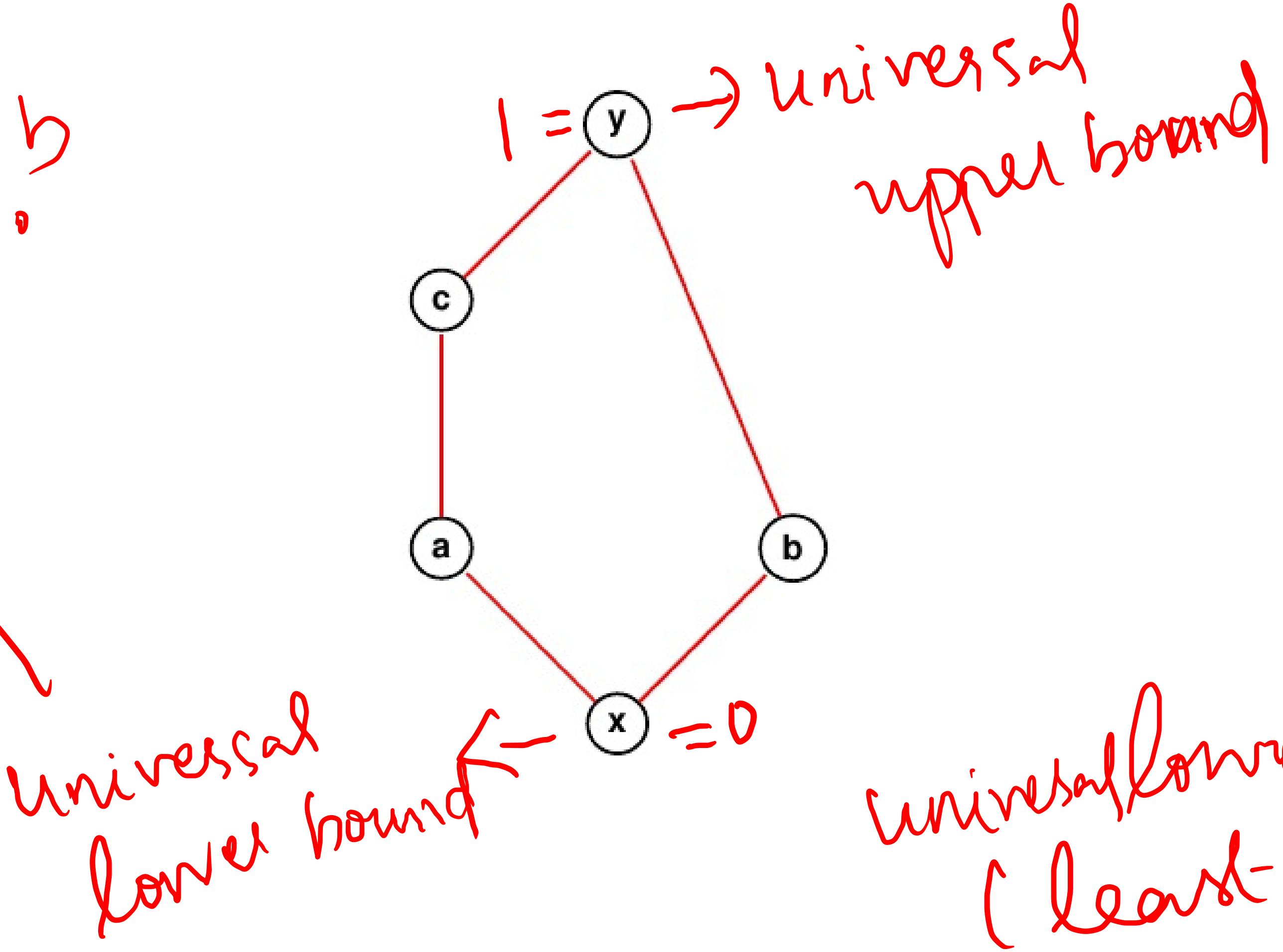
This course is taught to Computer Science Engineering students in SMIT, India during Jun-Dec, 2019.

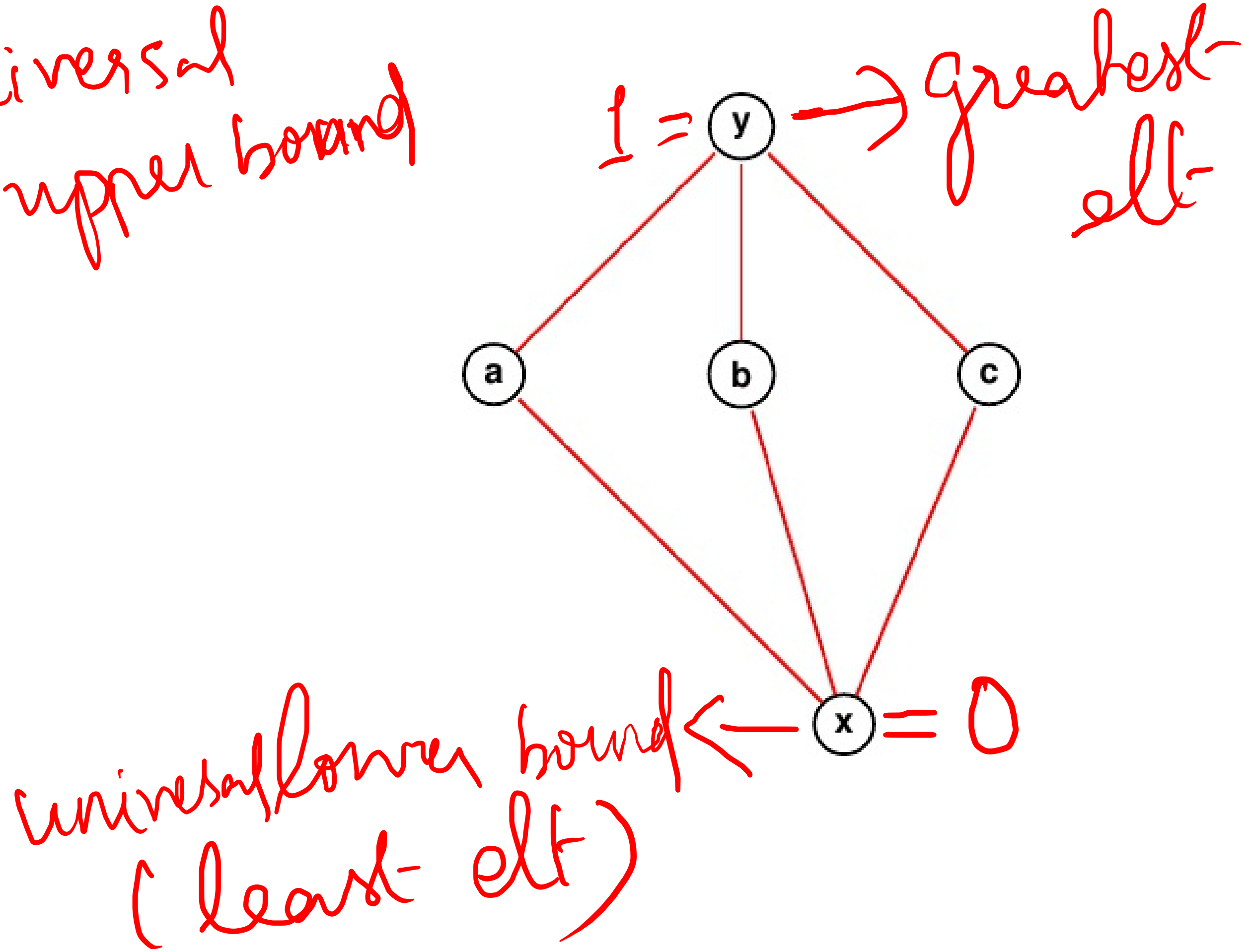
Definition 3.10. An element y is called as greatest element or universal upper bound of a lattice if for any $x \in L$,

$$x \leq y.$$

We denote this y as 1.



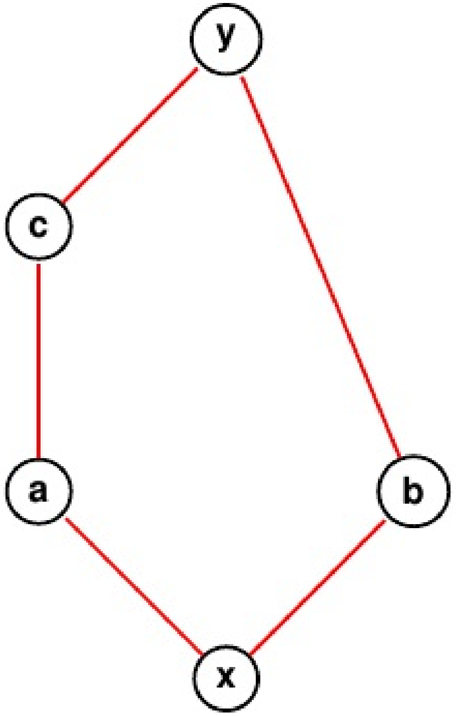


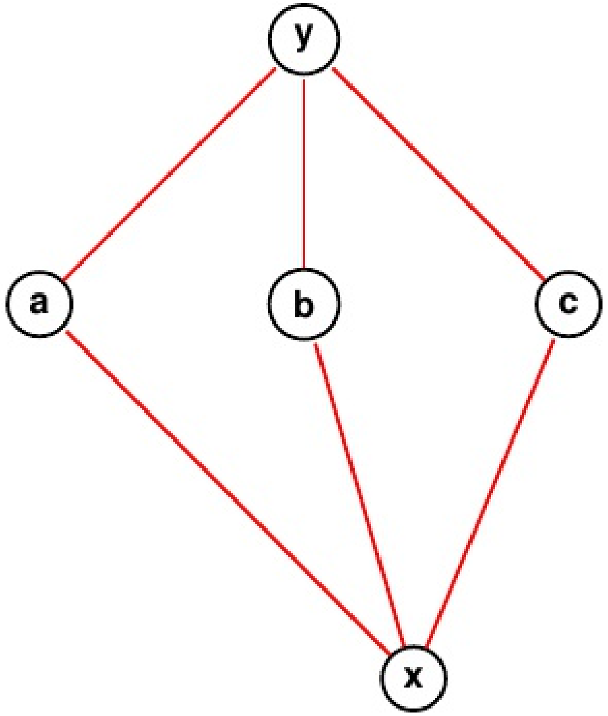


Definition 3.11. An element y is called as *least element* or *universal lower bound* of a lattice if for any $x \in L$,

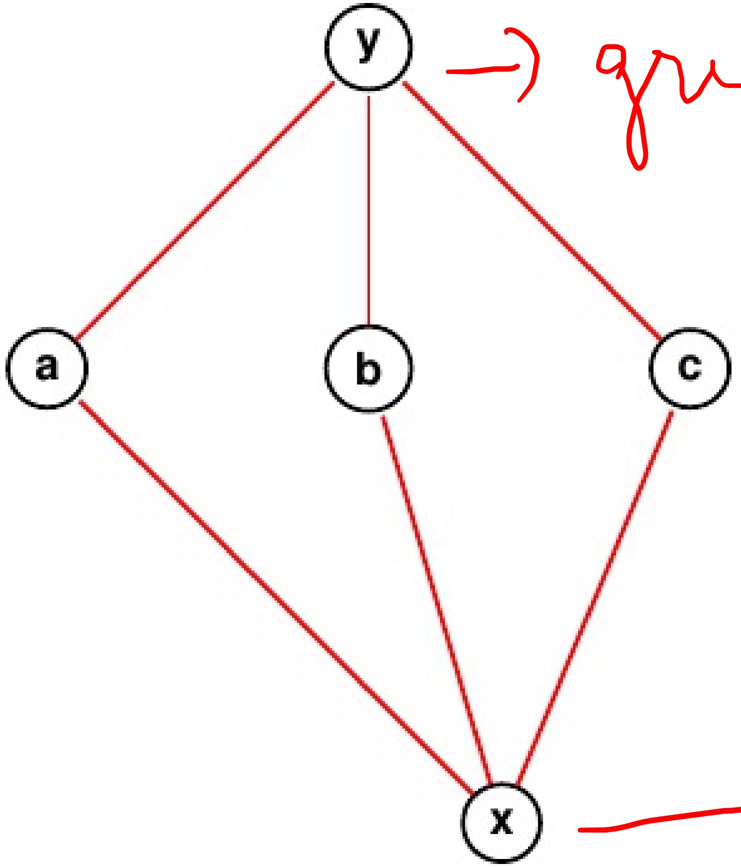
$$y \leq x.$$

We denote t





Definition 3.12. A lattice $(L, \preccurlyeq)$ is said to be *bounded lattice* if it has a greatest and least element.



→ greatest elt

→ Bounded lattice.

→ least elt

Maximum element and minimum element of a lattice is unique.

Definition 3.13. Let $(L, \preccurlyeq)$ be a bounded lattice. Let $x \in L$. An element $\bar{x} \in L$ is said to be complement of $x \in L$ if $x \vee \bar{x} = 1$ and $x \wedge \bar{x} = 0$.

$x \in L$, $x \vee \bar{x} = 1$

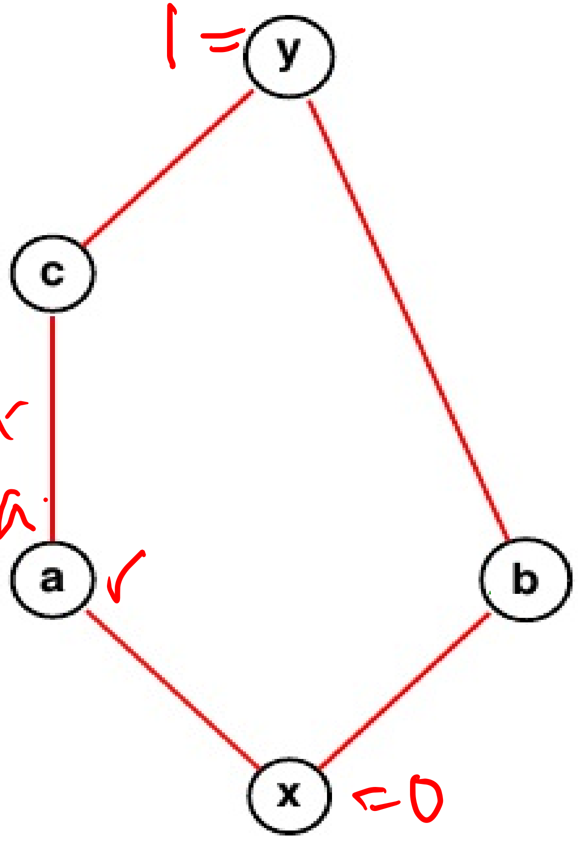
and $x \vee \bar{x} = 0$

I call $\bar{x}$ is a Complement of x.

$a + (-a) = 0$

$a \times \left(\frac{1}{a}\right) = 1$

$1 =$



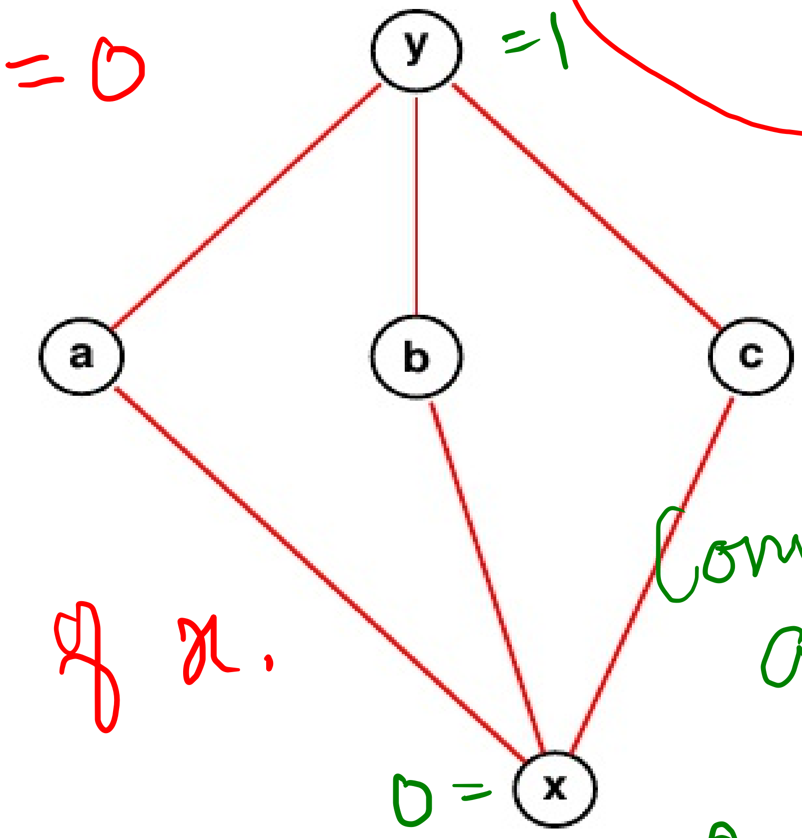
$= 0$

b is a complement of a.

$a \vee b = 1$

$a \wedge b = 0$

$= 1$



$0 =$

Complement of $a = \{b, c\}$

Complement of $0 = 1$

In a distributive lattice if a complement of an element exists, then it is unique.

Proof:

Let $a \in L$, where L is distributive lattice.

Assume that b and c are two complements of a.

$\Rightarrow a \vee b = 1 \qquad a \wedge b = 0$

$a \vee c = 1 \qquad a \wedge c = 0$

TP: $b = c$

Now,

$b = glb(b, 1)$

$= b \wedge 1$

$= b \wedge (a \vee c) \quad \{\because a \vee c = 1\}$

$= (b \wedge a) \vee (b \wedge c) \quad \{\because$ distributive property$\}$

$= (a \wedge b) \vee (b \wedge c)$

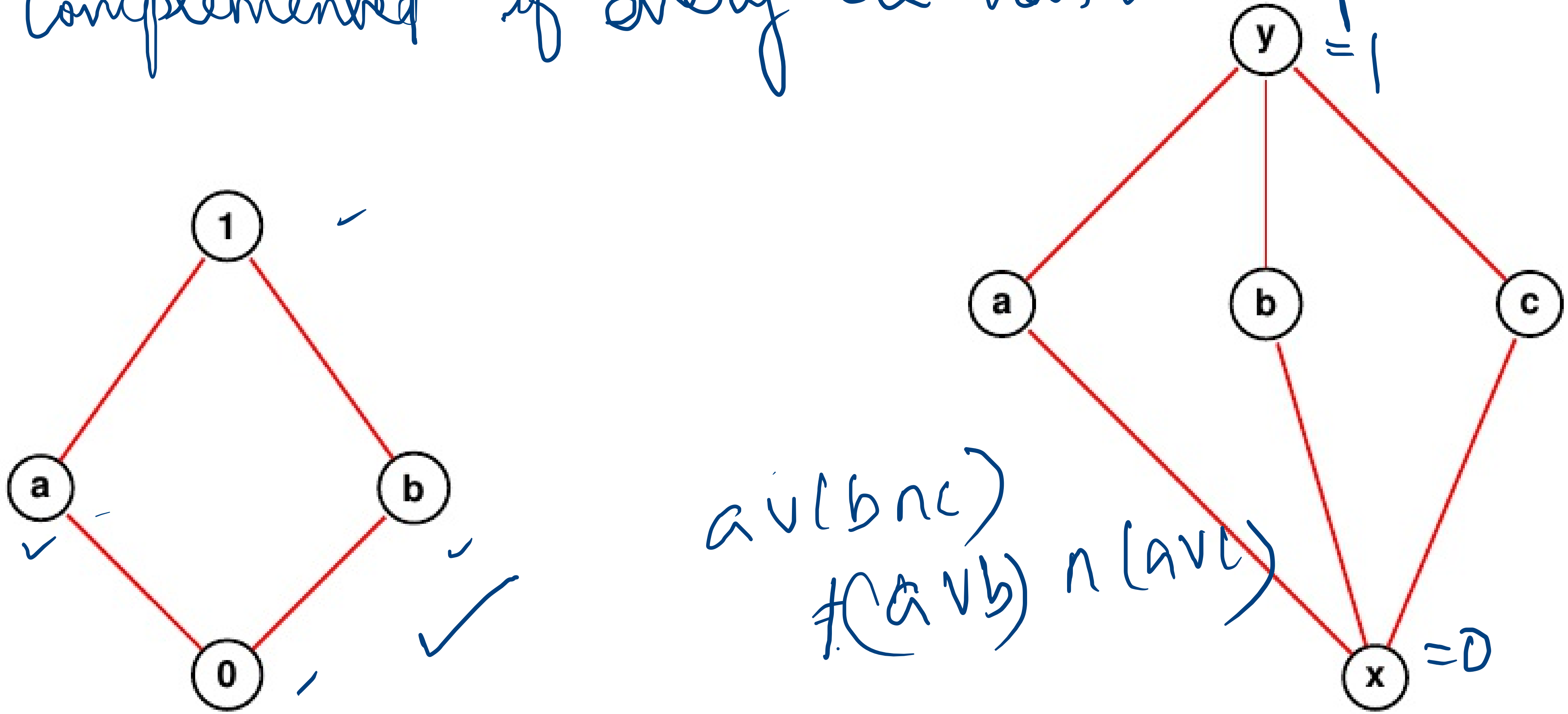
$= 0 \vee (b \wedge c)$

$= (a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge c$

$= 1 \wedge c$

$= c$ //.

Def: (Complemented) A bounded lattice is called as complemented if every elt has a complement.

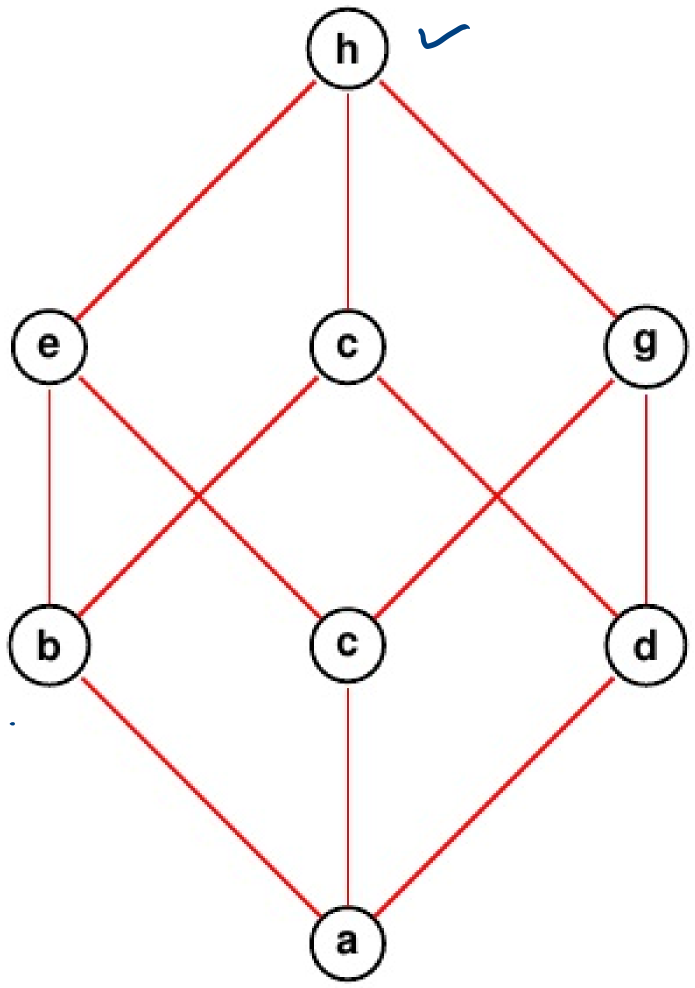


(a) Complemented Lattice

(b) Complemented but not distributive lattice

Figure 7: Example of Complemented Lattice

Definition 3.15. A lattice $(L, \preccurlyeq)$ is said to be a *Boolean Lattice* if it is complemented and distributive.



→ ~~Bot~~ Boolean lattice.

Show that in a Boolean lattice $(L, \vee, \wedge, ^{-})$, $\overline{a \vee b} = \overline{a} \wedge \overline{b}$, for all $a, b \in L$.

$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

$\overline{a \vee b} = \overline{a} \wedge \overline{b}$

Exercise

TP: $(a \vee b) \wedge (\overline{a} \wedge \overline{b}) = 0$

$(a \vee b) \vee (\overline{a} \wedge \overline{b}) = 1$

Show that in a Boolean lattice $(L, \vee, \wedge, \bar{\ })$, $\overline{a \wedge b} = \overline{a} \vee \overline{b}$, for all $a, b \in L$.

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

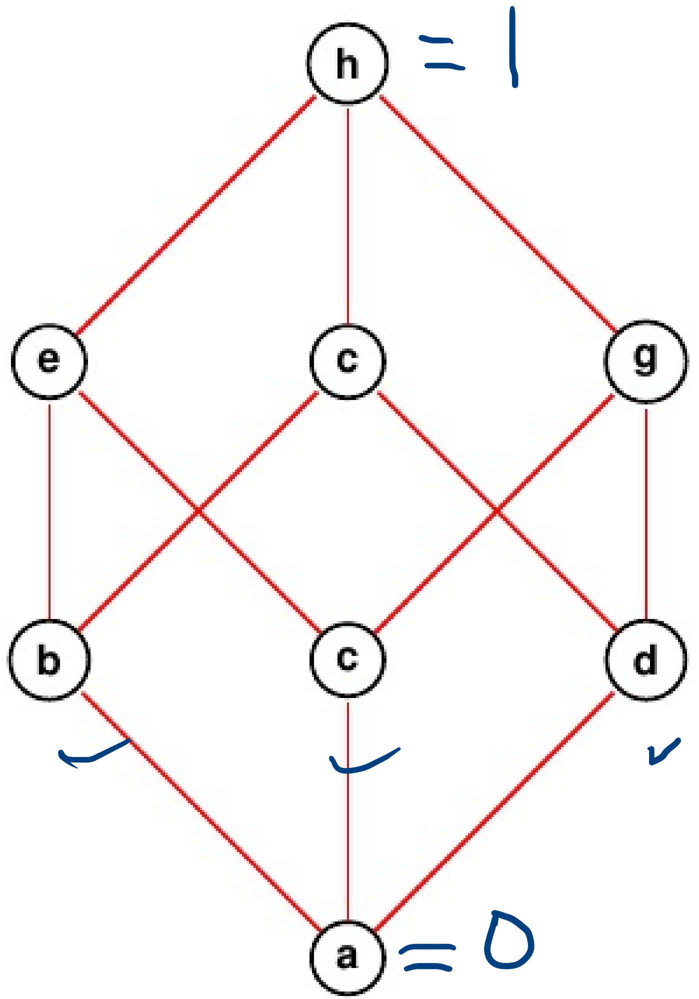
Exercise.

Definition 3.17 (Atom). An element $x \in \mathscr{B}$ is said to be an *atom* if x covers 0. That is,

(i) $0 \preccurlyeq x$ and

(ii) there does not exist any $c \in \mathscr{B}$ such that $0 < c < x$.

atoms = { b , c , d }



. In a distributive lattice, if $b \wedge \bar{c} = 0$ then $b \preccurlyeq c$.

Excercise: