LECTURE NOTES ON

LAPLACE TRANSFORMS FOR ENGINEERING APPLICATION

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NOTICE: this material must not be used as a substitute for attending the lectures.

Contents

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1 Introduction

In many problems, a function $f(t),\ \ t \in [a,\ b]$ is transformed to another function $F(s)$ through a relation of the type:

$$F(s) = \int_a^b K(t,s)f(t)dt$$

where $K(t,s)$ is a known function. Here, $F(s)$ is called an integral transform of $f(t)$. Thus, an integral transform sends a given function $f(t)$ in t-domain into another function $F(s)$ in s-domain. This transformation of $f(t)$ into $F(s)$ provides a method to tackle a problem more readily. In some cases, it affords solutions to otherwise difficult problems. Laplace transform is a widely used integral transform (where $f(t)$ is defined on $[0,\infty)$ and $K(s,t) = e^{-st}$) with many applications in physics and engineering. The Laplace transform is named after mathematician and astronomer Pierre-Simon Laplace.

As we will see in later sections we can use Laplace transforms to reduce a differential equation with constant coefficients to an algebraic problem. The algebra can be messy on occasion, but it will be simpler than actually solving the differential equation directly in many cases. Thus, it provides a **powerful tool** to solve differential equations.

It is important to note here that there is some sort of analogy with what we had learnt during the study of logarithms in school. That is, to multiply two numbers, we first calculate their logarithms, add them and then use the table of antilogarithm to get back the original product. In a similar way, we first transform the problem that was posed as a function of $f(t)$ to a problem in $F(s)$, make some calculations and then use the table of inverse Laplace transform to get the solution of the actual problem.

In this chapter, we shall see same properties of Laplace transform and its applications in solving differential equations.

2 Definitions and Examples

Definition 2.1. *(Piece-wise Continuous Function) A function $f(t)$ is said to be a piece-wise continuous function on a closed interval $[a,b] \subset \mathbb{R}$, if there exists finite number of points $a = t_0 < t_1 < t_2 < \cdots < t_N = b$ such that $f(t)$ is continuous in each of the intervals $(t_{i-1},\ t_i)$ for $1 \le i \le N$ and has finite limits as t approaches the end points, see the Figure 1. A function $f(t)$ is said to be a piece-wise continuous function for $t \ge 0$, if $f(t)$ is a piece-wise continuous function on every closed interval $[a,b] \subset [0,\infty)$. For example, see Figure 1.*

Definition 2.2. *(Laplace Transform) Let $f : [0,\infty) \longrightarrow \mathbb{R}$ and $s \in \mathbb{R}$. Then $F(s)$, for $s \in \mathbb{R}$ is called the LAPLACE TRANSFORM of $f(t)$, and is defined by*

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty f(t)e^{-st}dt$$

whenever the integral exists.

Recall that $\int\limits_0^\infty g(t)dt$ exists if $\lim\limits_{m\longrightarrow\infty} \int\limits_0^m g(t)d(t)$ exists and we define $\int\limits_0^\infty g(t)dt = \lim\limits_{m\longrightarrow\infty} \int\limits_0^m g(t)d(t)$. We shall use $\mathcal{L}\{f(t)\}$ instead of $\mathcal{L}\{f(t)\}(s)$ when there is no confusion about the transformed variable s. The following question arises here, naturally.

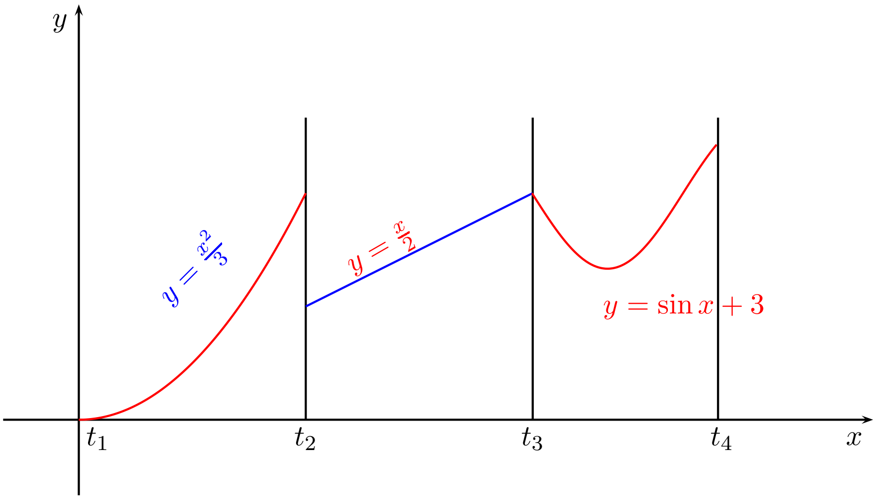

Figure 1: Piecewise Continuous Function

"Can any function $F(s)$ of s be the Laplace Transform of another function $f(t)$ of t? "

The answer is NO. The following remarks clarifies this answer.

Remark 2.1. *Suppose $F(s)$ exists for some function f. Then by definition,* $\lim_{m\longrightarrow\infty}\int_0^{\mathbf{m}} f(t)e^{-st}dt$ *exists. Now, one can use the theory of improper integrals to conclude that*

$$\lim_{s\longrightarrow\infty} F(s)=0.$$

Hence, a function $F(s)$ satisfying $\lim_{s\longrightarrow\infty} F(s)$ *does not exist or* $\lim_{s\longrightarrow\infty} F(s)\neq 0$, *cannot be a Laplace transform of a function f .*

Example 2.1. *Find $F(s)=\mathcal{L}\{f(t)\}$, where $f(t)=1,\ \ t\geq 0$.*

Solution: By definition,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}dt \\ &= \lim_{m\longrightarrow\infty}\int_0^m e^{-st}dt \\ &= \lim_{m\longrightarrow\infty}\frac{e^{-sm}-1}{-s}. \end{aligned}$$

Note that if $s>0$, then

$$\lim_{m\longrightarrow\infty}\frac{e^{-sm}}{-s}=0.$$

Thus, $F(s)=\dfrac{1}{s}, \quad \text{for} \quad s>0.$ ⋈

Example 2.2. *Find $F(s)=\mathcal{L}\{f(t)\}$, where $f(t)=\sin at,\ \ t\geq 0$.*

Solution: By definition,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} \sin at dt \\ &= \lim_{m \longrightarrow \infty} \int_0^m e^{-st} \sin at dt \\ &= \lim_{m \longrightarrow \infty} \left[\frac{e^{-st}(-s \sin at - a \cos at)}{s^2 + a^2} \right]_0^m \\ &= \lim_{m \longrightarrow \infty} \left[\frac{a}{s^2 + a^2} - \frac{e^{-sm}(s \sin am + a \cos am)}{s^2 + a^2} \right]. \end{aligned}$$

Note that if $s > 0$, then

$$\lim_{m \longrightarrow \infty} \frac{e^{-sm}(-s \sin am - a \cos am)}{s^2 + a^2} = 0. \text{ (Why?)}$$

Thus, $F(s) = \dfrac{a}{s^2 + a^2}$, for $s > 0$. ⋈

The following table lists the Laplace Transforms of various elementary functions.

Sl. No.	$f(t)$	$\mathcal{L}\{f(t)\}(s) = F(s)$
1.	1	$\frac{1}{s}$, $s > 0$
2.	t	$\frac{1}{s^2}$, $s > 0$
3.	t^n, $n = 0, 1, 2 \ldots$	$\frac{n!}{s^{n+1}}$, $s > 0$
4.	e^{at}	$\frac{1}{s - a}$, $s > a$
5.	$\sin at$	$\frac{a}{s^2 + a^2}$, $s > 0$
6.	$\cos at$	$\frac{s}{s^2 + a^2}$, $s > 0$
7.	$\sinh at$	$\frac{a}{s^2 - a^2}$, $s > \|a\|$
8.	$\cosh at$	$\frac{s}{s^2 - a^2}$, $s > \|a\|$

Problem 2.1. *Derive the Laplace Transforms of each of the elementary functions listed in the above table.*

Before we proceed to the next definition, notice that the integral $\int\limits_0^\infty x^{n-1} e^{-x} \, dx$ exist for $n > 0$ and changes its value as the value of n changes.

Definition 2.3. *For $n > 0$, the integral $\int\limits_0^\infty x^{n-1} e^{-x} \, dx$ is called the Gamma Function of n and we write $\Gamma(n) = \int\limits_0^\infty x^{n-1} e^{-x} \, dx$.*

Properties 2.1. *(a).* $\Gamma(n+1) = n\Gamma(n)$.

(b). $\Gamma(n+1) = n!$, *if n is a positive integer.*

(c). $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Example 2.3. *Show that* $\mathcal{L}\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$. *Hence deduce that* $\mathcal{L}\{t^{\frac{3}{2}}\} = \dfrac{3\sqrt{\pi}}{4s^{\frac{5}{2}}}$.

Solution: Recall that, for $n > 0$ the Gamma function is defined by

$$\Gamma(n) = \int_0^\infty u^{n-1}e^{-u}\,du.$$

$$\begin{aligned}
\therefore\ \mathcal{L}\{t^n\} &= \int_0^\infty e^{-st}t^n\,dt \\
&= \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^n \frac{du}{s}, \text{ where } u = st \\
&= \frac{1}{s^{n+1}}\int_0^\infty e^{-u}u^n\,du, \\
&= \frac{1}{s^{n+1}}\Gamma(n+1).
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{t^{\frac{3}{2}}\} &= \frac{1}{s^{\frac{3}{2}+1}}\Gamma\left(\frac{3}{2}+1\right) \\
&= \frac{1}{s^{\frac{5}{2}}}\frac{3}{2}\Gamma\left(\frac{3}{2}\right), && \because\ \Gamma(n+1) = n\Gamma(n) \\
&= \frac{1}{s^{\frac{5}{2}}}\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{3\sqrt{\pi}}{4s^{\frac{5}{2}}}, && \because\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.
\end{aligned}$$

In the remaining part of this chapter, whenever the improper integral is calculated, we will not explicitly write the limiting process. However, the students are advised to provide the details.

3 Properties of Laplace Transforms

In this section we discuss some important properties of Laplace transforms.

Theorem 3.1. *(Linear Property) If* $\mathcal{L}(f_1(t)) = F_1(s)$ *and* $\mathcal{L}(f_2(t)) = F_2(s)$, *then for any two constants* $a,\ b$,

$$\mathcal{L}(af_1 + bf_2)(t)) = aF_1(s) + bF_2(s) = a\mathcal{L}(f_1(t)) + b\mathcal{L}(f_1(t)).$$

Proof. By definition,

$$\begin{aligned}
\mathcal{L}(af_1 + bf_2)(t)) &= \int_0^\infty e^{-st}(af_1 + bf_2)(t)dt \\
&= \int_0^\infty e^{-st}(af_1(t) + bf_2(t))dt \\
&= a\int_0^\infty e^{-st}f_1(t)dt + b\int_0^\infty e^{-st}f_2(t))dt \\
&= a\mathcal{L}(f_1(t)) + b\mathcal{L}(f_1(t)).
\end{aligned}$$

□

Example 3.1. *Find the Laplace Transform of the following functions*

1. $\sin 2t \sin 3t$.
2. $\sin^3 2t$.
3. $e^{2t} + 4t^3 - 2\sin 2t + 3\cos 3t$.
4. $1 + 2\sqrt{t} + \dfrac{3}{\sqrt{t}}$.
5. $(\sin t - cost)^2$.
6. $\left(\sqrt{t} - \frac{3}{\sqrt{t}}\right)^3$
7. $\cos\sqrt{t}$, *Hint: expand and then use linear property.*

Theorem 3.2. *(First Translation or Shifting Property) If $\mathcal{L}\{f(t)\} = F(s)$, then for any constant a,*

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

Proof. By definition,

$$\begin{aligned}
\mathcal{L}\{e^{at}f(t)\} &= \int_0^\infty e^{-st}e^{at}f(t)dt \\
&= \int_0^\infty e^{-(s-a)t}f(t)dt \\
&= \int_0^\infty e^{-rt}f(t)dt, \text{ where } r = s-a \\
&= \mathcal{L}\{f(t)\}(r) \\
&= F(r) \\
&= F(s-a).
\end{aligned}$$

□

Example 3.2. *Find $F(s) = \mathcal{L}\{f(t)\}$, where $f(t) = e^{bt}\sin at, \ \ t \geq 0$.*

Solution: Let $g(t) = \sin at$. We know that, $\mathcal{L}\{g(t)\} = \dfrac{a}{s^2+a^2}, \ s > 0$. Therefore, by First Shifting Property,

$$\mathcal{L}\{e^{bt}g(t)\} = \frac{a}{(s-b)^2+a^2}, \ \ s > b,$$

that is,

$$\mathcal{L}\{e^{bt}\sin at\} = \frac{a}{(s-b)^2+a^2}, \ \ s > b.$$

⋈

Problem 3.1. *Find $\mathcal{L}(e^{at}\sin(bt))$.*

Solution: We know $\mathcal{L}(\sin(bt)) = \dfrac{b}{s^2+b^2}$. Hence $\mathcal{L}(e^{at}\sin(bt)) = \dfrac{b}{(s-a)^2+b^2}$.

Problem 3.2. *Find* $\mathcal{L}^{-1}\left(\frac{s-5}{(s-5)^2+36}\right)$.

Solution: By s -Shifting, if $\mathcal{L}(f(t)) = F(s)$ then $\mathcal{L}(e^{at}f(t)) = F(s-a)$. Here, $a = 5$ and

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+36}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+6^2}\right) = \cos(6t).$$

Hence, $f(t) = e^{5t}\cos(6t)$. ⋈

In fact, using the First Shifting Property, we can easily derive the results listed in the following table.

Sl. No.	$f(t)$	$\mathcal{L}\{f(t)\}(s) = F(s)$
1.	e^{bt}	$\frac{1}{s-b}$, $s > 0$
2.	te^{bt}	$\frac{1}{(s-b)^2}$, $s > 0$
3.	$t^n e^{bt}$, $n = 0, 1, 2 \ldots$	$\frac{n!}{(s-b)^{n+1}}$, $s > b$
4.	$e^{bt}\sin at$	$\frac{a}{(s-b)^2+a^2}$, $s > b$
5.	$e^{bt}\cos at$	$\frac{s}{(s-b)^2+a^2}$, $s > b$
6.	$e^{bt}\sinh at$	$\frac{a}{(s-b)^2-a^2}$, $s > b + \lvert a\rvert$
7.	$e^{bt}\cosh at$	$\frac{s}{(s-b)^2-a^2}$, $s > b + \lvert a\rvert$

Problem 3.3. *Find the Laplace Transform of the following functions*

1. $e^{-2t}(\sin 5t - 2\cos 5t)$.
2. $e^{-3t}(2\cos 5t - 3\sin 5t)$.
3. $e^{3t}\sin^2 t$.
4. $e^{4t}\sin 2t\cos t$.

Problem 3.4. *If* $F(s) = \mathcal{L}\{f(t)\}$, *then show that*

1. $\mathcal{L}\{\sinh at f(t)\} = \frac{1}{2}\left[F(s-a) - F(s+a)\right]$.
2. $\mathcal{L}\{\cosh at f(t)\} = \frac{1}{2}\left[F(s-a) + F(s+a)\right]$.

Hence or otherwise find $\mathcal{L}\{\sinh 2t \sin 3t\}$ *and* $\mathcal{L}\{\cosh 3t\cos 2t\}$.

Theorem 3.3. *(Second Translation or Shifting Property) If* $\mathcal{L}\{f(t)\} = F(s)$ *and*

$$g(t) = \begin{cases} f(t-a), & \text{if } t > a \\ 0, & \text{if } t < a \end{cases},$$

then

$$\mathcal{L}\{g(t)\} = e^{-as}F(s).$$

Proof.

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \int_0^\infty e^{-st} g(t)\, dt \\
&= \int_a^\infty e^{-st} f(t-a)\, dt \\
&= \int_a^\infty e^{-st} f(t-a)\, dt \quad \text{put } z = t-a \text{ so that } dt = dz \\
&= \int_0^\infty e^{-s(z+a)} f(z)\, dz \\
&= e^{-as} \int_0^\infty e^{-sz} f(z)\, dz \\
&= e^{-as} F(s)
\end{aligned}$$

□

Theorem 3.4. *(Change of Scale Property) If $\mathcal{L}\{f(t)\} = F(s)$ then*

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F(\frac{s}{a}).$$

Proof. Exercise. □

3.1 Laplace Transform of Derivatives

Theorem 3.5. *Let $\mathcal{L}\{f(t)\} = F(s)$. If $f'(t), \ldots, f^{(n-1)}(t), f^{(n)}(t)$ exist and $f^{(n)}(t)$ is continuous for $t \geq 0$, then*

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots s f^{(n-2)}(0) - f^{(n-1)}(0).$$

In particular, for $n = 1$, we have

$$\mathcal{L}\big(f'(t)\big) = sF(s) - f(0).$$

and for $n = 2$, we have

$$\mathcal{L}\big(f''(t)\big) = s^2 F(s) - s f(0) - f'(0).$$

Corollary 3.1. *Let $f'(t)$ be a piecewise continuous function for $t \geq 0$. Also, let $f(0) = 0$. Then*

$$\mathcal{L}(f'(t)) = sF(s).$$

Problem 3.5. *Find the Laplace transform of $f(t) = \cos^2(t)$.*

Solution: Note that $f(0) = 1$ and $f'(t) = -2\cos t\, \sin t = -\sin(2t)$. Also,

$$\mathcal{L}(f'(t)) = \mathcal{L}(-\sin(2t)) = \frac{-2}{s^2+4}.$$

Now, $\mathcal{L}(f(t)) = \dfrac{1}{s}\left(-\dfrac{2}{s^2+4} + 1\right) = \dfrac{s^2+2}{s(s^2+4)}$. ⋈

3.2 Multiplication by t^n

Theorem 3.6. *Let $f(t)$ be a piecewise continuous function with $\mathcal{L}(f(t)) = F(s)$. If the function $F(s)$ is differentiable, then*

$$\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} F(s).$$

Problem 3.6. *Find $\mathcal{L}(t\sin(at))$.*

Solution: We know that $\mathcal{L}(\sin(at)) = \dfrac{a}{s^2+a^2}$. Hence

$$\mathcal{L}(t\sin(at)) = -\frac{d}{ds}\frac{a}{s^2+a^2} = \frac{2as}{(s^2+a^2)^2}.$$

⋈

Problem 3.7. *Find Laplace Transform of the following functions:*

(a). $f(t) = t^3 e^{-3t}$.

(b). $f(t) = t\cos(at)$.

(c). $f(t) = t^2 \sin(at)$.

(d). $f(t) = te^{-t}\sin(3t)$.

(e). $f(t) = t^2 e^{-t}\cos t$.

3.3 Division by t

Theorem 3.7. *Let $\mathcal{L}(f(t)) = F(s)$. Then*

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(u)du.$$

Problem 3.8. *Find Laplace transform of $f(t) = \dfrac{e^{at} - \cos bt}{t}$.*

Solution: First we note that,

$$\begin{aligned}\mathcal{L}(e^{at} - \cos bt) &= \mathcal{L}(e^{at}) - \mathcal{L}(\cos bt)\\ &= \frac{1}{s-a} - \frac{s}{s^2+b^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}\left(\frac{e^{at} - \cos bt}{t}\right) &= \int_s^\infty \left[\frac{1}{u-a} - \frac{u}{u^2+b^2}\right] du.\\ &= -\log\frac{s-a}{(s^2+b^2)^{1/2}}.\end{aligned}$$

⋈

Problem 3.9. *Find Laplace Transform of the following functions:*

(a). $f(t) = \dfrac{1-e^t}{t}$.

(b). $f(t) = \dfrac{\cos at - \cos bt}{t}$.

(c). $f(t) = \dfrac{e^{-at} - e^{-bt}}{t}$.

(d). $f(t) = \dfrac{\sin at}{t}$.

(e). $f(t) = \dfrac{1-\cos 2t}{t}$.

(f). $f(t) = \dfrac{e^{-t}\sin t}{t}$.

Problem 3.10. *Evaluate* $\displaystyle\int_0^\infty te^{-2t}\sin t dt$.

Solution: We have $\mathcal{L}(\sin t) = \dfrac{1}{s^2+1}$. Therefore,

$$\begin{aligned}
\mathcal{L}(t\sin t) &= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) \\
\Rightarrow \int_0^\infty e^{-st} t\sin t dt &= \frac{2s}{(s^2+1)^2} \\
\Rightarrow \int_0^\infty te^{-2t}\sin t dt &= \frac{4}{25}.
\end{aligned}$$

Problem 3.11. *Evaluate the following integrals:*

1. $\displaystyle\int_0^\infty te^{-3t}\sin t dt$.

2. $\displaystyle\int_0^\infty te^{-2t}\cos t dt$.

3. $\displaystyle\int_0^\infty te^{-t}\sin^4 t dt$.

4. $\displaystyle\int_0^\infty \frac{\cos at - \cos bt}{t} dt$.

5. $\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\, dt.$

Problem 3.12. *Show that* $\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}.$

3.4 Laplace Transform of an Integral

Theorem 3.8. *If* $F(s) = \mathcal{L}(f(t))$ *then* $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}.$

Proof. By definition,

$$\mathcal{L}\left(\int_0^t f(\tau)\, d\tau\right) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\, d\tau\right)\, dt = \int_0^\infty \int_0^t e^{-st} f(\tau)\, d\tau dt.$$

We don't go into the details of the proof of the change in the order of integration. We assume that the order of the integrations can be changed and therefore

$$\int_0^\infty \int_0^t e^{-st} f(\tau)\, d\tau dt = \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\, dt\, d\tau.$$

Thus,

$$\begin{aligned}
\mathcal{L}\left(\int_0^t f(\tau)\, d\tau\right) &= \int_0^\infty \int_0^t e^{-st} f(\tau)\, d\tau dt \\
&= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\, dt\, d\tau \\
&= \int_0^\infty \int_\tau^\infty e^{-s(t-\tau)-s\tau} f(\tau)\, dt\, d\tau \\
&= \int_0^\infty e^{-s\tau} f(\tau) d\tau \left(\int_\tau^\infty e^{-s(t-\tau)} dt\right) \\
&= \int_0^\infty e^{-s\tau} f(\tau) d\tau \left(\int_0^\infty e^{-sz} dz\right) \\
&= F(s)\frac{1}{s}.
\end{aligned}$$

□

Example 3.3. *Find* $\mathcal{L}\left(\int_0^t \sin(az)dz\right).$

Solution: We know $\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}$. Hence

$$\mathcal{L}\left(\int_0^t \sin(az)dz\right) = \frac{1}{s}\cdot\frac{a}{(s^2+a^2)} = \frac{a}{s(s^2+a^2)}.$$

⋈

Problem 3.13. *Find* $\mathcal{L}\left(\int_0^t \tau^2 d\tau\right)$.

Solution: We know that $\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{F(s)}{s}$. Therefore,

$$\mathcal{L}\left(\int_0^t \tau^2 d\tau\right) = \frac{\mathcal{L}\left(t^2\right)}{s} = \frac{1}{s}\cdot\frac{2!}{s^3} = \frac{2}{s^4}.$$

⋈

Problem 3.14. *Find the function* $f(t)$ *such that* $F(s) = \frac{4}{s(s-1)}$.

Solution: We know that $\mathcal{L}(e^t) = \frac{1}{s-1}$. So,

$$\mathcal{L}^{-1}\left(\frac{4}{s(s-1)}\right) = 4\mathcal{L}^{-1}\left(\frac{1}{s}\frac{1}{s-1}\right) = 4\int_0^t e^\tau d\tau = 4(e^t-1).$$

⋈

4 Periodic Functions and Laplace Transforms

A function f is said to be *periodic* with period T (T being a nonzero constant) if we have

$$f(x+T) = f(x)$$

for every x in the domain of f. If there exists a least positive constant T with this property, it is called the fundamental period. A function with period T will repeat on intervals of length T, and these intervals are referred to as periods. Examples: $\sin x$, $\cos x$ etc.

A function that is not periodic is known as *aperiodic* function.

Theorem 4.1. *Let* $f(t)$ *be a periodic function having period* T *so that* $f(t+T) = f(t)$. *Then*

$$\mathcal{L}(f(t)) = \frac{\displaystyle\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

Example 4.1. *Find* $\mathcal{L}\left(f(t)\right)$, *where* $f(t+2) = f(t)$ *and*

$$f(t) = \begin{cases} t^2, & 0 < t \le 1 \\ t, & 1 \le t < 2 \end{cases}$$

Solution: Here $f(t)$ is a periodic function with period $T = 2$.

$$\begin{aligned}
\therefore \mathcal{L}(f(t)) &= \frac{\displaystyle\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}} \\
&= \frac{\displaystyle\int_0^2 e^{-st} f(t)\,dt}{1-e^{-4s}} \\
&= \frac{\displaystyle\int_0^1 e^{-st} t^2\,dt + \int_1^2 e^{-st} t\,dt}{1-e^{-4s}}
\end{aligned}$$

$\bowtie$

Problem 4.1. *Trace the periodic function*

$$f(t) = \begin{cases} t, & 0 < t \leq 2 \\ 4 - t, & 2 \leq t < 4 \end{cases}$$

and $f(t+2) = f(t)$*. Also obtain* $\mathcal{L}(f(t))$*.*

Problem 4.2. *Trace the periodic function*

$$f(t) = \begin{cases} 1 - t, & 0 < t \leq 1 \\ t - 1, & 1 \leq t < 2 \end{cases}$$

and $f(t+2) = f(t)$*. Also obtain* $\mathcal{L}(f(t))$*.*

5 Inverse Laplace Transforms

Definition 5.1. (**Inverse Laplace Transform**) *Let $\mathcal{L}(f(t)) = F(s)$. That is, $F(s)$ is the Laplace transform of the function $f(t)$. Then $f(t)$ is called the inverse Laplace transform of $F(s)$. In that case, we write $f(t) = \mathcal{L}^{-1}\{F(s)\}$.*

The following table lists the Inverse Laplace Transforms of various elementary functions.

Sl. No.	$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
1.	$\frac{1}{s}$, $s > 0$	1
2.	$\frac{1}{s^2}$, $s > 0$	t
3.	$\frac{1}{s^{n+1}}$, $s > 0$	$\frac{t^n}{n!}$, $n = 0, 1, 2\ldots$
4.	$\frac{1}{s-a}$, $s > a$	e^{at}
5.	$\frac{1}{s^2+a^2}$, $s > 0$	$\frac{\sin at}{a}$
6.	$\frac{s}{s^2+a^2}$, $s > 0$	$\cos at$
7.	$\frac{1}{s^2-a^2}$, $s > \lvert a\rvert$	$\frac{\sinh at}{a}$
8.	$\frac{s}{s^2-a^2}$, $s > \lvert a\rvert$	$\cosh at$

6 Properties of Inverse Laplace Transforms

In this section we discuss some important properties of Laplace transforms.

Theorem 6.1. *(Linear Property) If $\mathcal{L}^{-1}\{F_1(s)\} = f_1(t)$ and $\mathcal{L}^{-1}\{F_2(s)\} = f_2(t)$, then for any two constants a, b,*

$$\mathcal{L}^{-1}\{aF_1(s) + bF_2(s)\} = af_1(t) + bf_2(t) = a\mathcal{L}^{-1}\{F_1(s)\} + b\mathcal{L}^{-1}\{F_2(s)\}.$$

Example 6.1. *Find the Inverse Laplace Transform of the following functions:*

1. $\dfrac{1}{(s-1)(s-2)}$.
2. $\dfrac{1}{(s^2-1)(s^2+2)}$.
3. $\dfrac{1}{s^3+3s}$.

Theorem 6.2. *(First Translation or Shifting Property) If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then*

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t).$$

Theorem 6.3. *(Second Translation or Shifting Property) If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then for any constant a,*

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = \begin{cases} f(t-a), & \text{if } t > a \\ 0, & \text{if } t < a \end{cases},.$$

Example 6.2. *Find* $\mathcal{L}^{-1}\{F(s)\}$, *where* $F(s) = \dfrac{a}{(s-b)^2 + a^2}, \quad s > b.$

Problem 6.1. *Find* $\mathcal{L}^{-1}\{F(s)\}$, *where* $F(s) = \dfrac{e^{-as}}{s^2}, \quad s > 0.$

In fact, using the First Shifting Property, we can easily derive the results listed in the following table.

Sl. No.	$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
1.	$\dfrac{1}{s-a}$,	e^{at}
2.	$\dfrac{1}{(s-a)^2}$,	te^{at}
3.	$\dfrac{1}{(s-a)^{n+1}}$,	$\dfrac{t^n}{n!}e^{at}$, $n = 0, 1, 2\ldots$
5.	$\dfrac{1}{(s-a)^2 + b^2}$,	$\dfrac{e^{at}\sin bt}{b}$
6.	$\dfrac{s-a}{(s-a)^2 + b^2}$,	$e^{at}\cos bt$
7.	$\dfrac{1}{(s-a)^2 - b^2}$,	$\dfrac{e^{at}\sinh bt}{b}$
8.	$\dfrac{s-a}{(s-a)^2 - b^2}$,	$e^{at}\cosh bt$

Problem 6.2. *Find the Inverse Laplace Transform of the following functions*

1. $\dfrac{s-2}{s^2 - 4s + 13}$.
2. $\dfrac{120}{s^2 - 6s + 13}$.
3. $\dfrac{(s-2)e^{2s}}{s^2 - 4s + 13}$.

Theorem 6.4. *(Change of Scale Property) If* $\mathcal{L}^{-1}\{F(s)\} = f(t)$ *then*

$$\mathcal{L}^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right).$$

Proof. Exercise. □

6.1 Inverse Laplace Transform of Derivatives

Theorem 6.5. *If* $\mathcal{L}^{-1}\{F(s)\} = f(t)$ *then* $\mathcal{L}^{-1}\left\{\dfrac{d^n}{ds^n}F(s)\right\} = (-1)^n t^n \mathcal{L}^{-1}\{F(s)\}$.

Problem 6.3. *Find the function* $f(t)$ *such that* $F(s) = \dfrac{4}{(s-1)^3}$.

Solution: We know $\mathcal{L}(e^t) = \dfrac{1}{s-1}$ and

$$\frac{4}{(s-1)^3} = 2\frac{d}{ds}\left(-\frac{1}{(s-1)^2}\right) = 2\frac{d^2}{ds^2}\left(\frac{1}{s-1}\right).$$

Now,

$$\begin{aligned}
&(-1)^n \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right) = \mathcal{L}(t^2e^t)\\
\Rightarrow\quad & \frac{4}{(s-1)^3} = 2\mathcal{L}(t^2e^t)\\
\Rightarrow\quad & F(s) = \mathcal{L}(2t^2e^t)
\end{aligned}$$

Thus we get $f(t) = 2t^2e^t$. ⋈

Problem 6.4. *Find the inverse Laplace transform of* $\frac{s}{(s^2+4)^2}$.

Solution: Let $F(s) = \frac{1}{s^2+4}$ We know that $f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{\sin 2t}{2}$. Now

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\} = (-1)t\mathcal{L}^{-1}\{F(s)\}\\
\Rightarrow\quad & \mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+4)^2}\right\} = -t\frac{\sin 2t}{2}\}\\
\Rightarrow\quad & \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = t\frac{\sin 2t}{4}\}
\end{aligned}$$

⋈

Problem 6.5. *Find* $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$.

Problem 6.6. *Find the Inverse Laplace transform of* $F(s) = \frac{s^2+2}{s(s^2+4)}$.

Solution: ⋈

6.2 Multiplication by s^n

Theorem 6.6. *If* $\mathcal{L}^{-1}\{F(s)\} = f(t)$ *and* $f(0) = f'(0) = \cdots f^{(n-1)}(0) = 0$ *then*

$$\mathcal{L}^{-1}\{s^nF(s)\} = \frac{d^n}{dt^n}f(t).$$

Problem 6.7. *Find the function* $f(t)$ *such that* $F(s) = \frac{4}{(s-1)^3}$.

Solution: We know $\mathcal{L}(e^t) = \frac{1}{s-1}$ and

$$\frac{4}{(s-1)^3} = 2\frac{d}{ds}\left(-\frac{1}{(s-1)^2}\right) = 2\frac{d^2}{ds^2}\left(\frac{1}{s-1}\right).$$

Now,

$$\begin{aligned}
&(-1)^n \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right) = \mathcal{L}(t^2e^t) \\
\Rightarrow \quad & \frac{4}{(s-1)^3} = 2\mathcal{L}(t^2e^t) \\
\Rightarrow \quad & F(s) = \mathcal{L}(2t^2e^t)
\end{aligned}$$

Thus we get $f(t) = 2t^2e^t$. $\bowtie$

6.3 Division by s

Theorem 6.7. *If* $\mathcal{L}^{-1}\{F(s)\} = f(t)$ *then*

$$\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(u)du.$$

6.4 Inverse Laplace Transform of an Integral

Theorem 6.8. *If* $\mathcal{L}^{-1}\{F(s)\} = f(t)$ *then*

$$\mathcal{L}^{-1}\left(\int_s^{\infty} F(u)du\right) = \frac{f(t)}{t}.$$

Problem 6.8. *Find the Inverse Laplace Transform of the following functions:*

1. $F(s) = \dfrac{3s+7}{s^2-2s-3}$
2. $F(s) = \dfrac{s}{s^2+3s+2}$
3. $F(s) = \dfrac{s+2}{s^2+7s+12}$
4. $F(s) = \dfrac{s+3}{s^2+6s+13}$
5. $F(s) = \dfrac{1}{s^2(s^2+1)}$
6. $F(s) = \dfrac{s+3}{s^2+6s+13}$
7. $F(s) = \ln\left\{\dfrac{s+2}{s+1}\right\}$

Problem 6.9. *Find the inverse Laplace transform of* $\dfrac{s}{s^2+1}$.

Solution: Let $F(s) = \dfrac{1}{s^2+1}$ We know that $f(t) = \mathcal{L}^{-1}(F(s)) = \sin t$. Then $f(0) = \sin(0) = 0$ and therefore using corollary 3.1, we have

$$\begin{aligned} \mathcal{L}^{-1}(sF(s)) &= \cos t. \\ i.e.,\ \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) &= \cos t. \end{aligned}$$

⋈

7 Convolution Theorem

Theorem 7.1. *If $\mathcal{L}^{-1}(F(S)) = f(t)$ and $\mathcal{L}^{-1}(G(S)) = g(t)$. Then*

$$\mathcal{L}^{-1}\left(F(s)G(s)\right) = f \star g.$$

where $f \star g = \int_0^t f(u)g(t-u)\,du$, and it is called the convolution of the functions $f(t)$ and $g(t)$.

Remark 7.1. *For any two functions f and g, $f \star g = g \star f$.*

Example 7.1. *Evaluate each of the following by using convolution theorem :*

(a) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+a)(s+b)}\right\}$.

(b) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$.

(c) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s+a)^2}\right\}$.

8 The Unit Step Function

The unit step function is defined by

$$u(t-a) = \begin{cases} 1 & \text{if } t < a \\ 0 & \text{if } t > a \end{cases}$$

Problem 8.1. *Prove that $\mathcal{L}\big(u(t-a)\big) = \dfrac{e^{-as}}{s}$*

Problem 8.2. *Prove that $\mathcal{L}\big(f(t-a)u(t-a)\big) = e^{-as}\mathcal{L}\{f(t)\}$.*

Remark 8.1. *If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$.*

Example 8.1. *Find the Laplace Transform of $f(t) = \begin{cases} t^2, & \text{if } 0 < t < 2 \\ t-1, & \text{if } 2 < t < 3 \\ 7, & \text{if } t > 3 \end{cases}$.*

Solution: *Here*

$$\begin{aligned} f(t) &= t^2 + (t-1-t^2)u(t-2) + (7-t+1)u(t-3) \\ &= t^2 + (t-1-t^2)u(t-2) + (8-t)u(t-3) \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{t^2 + (t-1-t^2)u(t-2) + (8-t)u(t-3)\} \\ &= \mathcal{L}\{t^2\} + \mathcal{L}\{(t-1-t^2)u(t-2)\} + \mathcal{L}\{(8-t)u(t-3)\} \\ &= \frac{2!}{s^3} + \mathcal{L}\{(t-2+1-(t-2+2)^2)u(t-2)\} + \mathcal{L}\{(5-(t-3))u(t-3)\} \\ &= \frac{2!}{s^3} + e^{-2s}\mathcal{L}\{(t+1-(t+2)^2)\} + e^{-3s}\mathcal{L}\{(5-t)\} \\ &= \frac{2!}{s^3} + e^{-2s}\mathcal{L}\{(-t^2-3t-3)\} + e^{-3s}\mathcal{L}\{(5-t)\} \\ &= \frac{2!}{s^3} - e^{-2s}\left\{\frac{2!}{s^3} - 3\frac{1}{s^2} - 3\frac{1}{s}\right\} + e^{-3s}\left\{\frac{5}{s} - \frac{1}{s^2}\right\} \end{aligned}$$

Example 8.2. *Find the Inverse Laplace Transform of* $F(s) = \dfrac{6e^{-2s}}{s^2+9}$.

Solution: *Let* $F(s) = \dfrac{6}{s^2+9}$. *Then* $f(t) = \mathcal{L}^{-1}\{F(s)\} = 2\sin 3t$. *Now*

$$\begin{aligned} \mathcal{L}^{-1}\{e^{-2s}F(s)\} &= f(t-2)u(t-2) \\ \Rightarrow \mathcal{L}^{-1}\left\{\frac{6e^{-2s}}{s^2+9}\right\} &= u(t-2)2\sin 3(t-2) \\ &= \begin{cases} 0, & \text{if } t < 2 \\ 2\sin 3(t-2), & \text{if } t > 2 \end{cases} \end{aligned}$$

⋈

Example 8.3. *Find the Inverse Laplace Transform of* $F(s) = \dfrac{2(s+3)e^{-5s}}{s^2+6s+13}$

Solution: *Let* $F(s) = \dfrac{2(s+3)}{s^2+6s+13}$. *Then* $f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-3t}\sin 2t$. *Now*

$$\begin{aligned} \mathcal{L}^{-1}\{e^{-5s}F(s)\} &= u(t-5)f(t-5) \\ \Rightarrow \mathcal{L}^{-1}\left\{\frac{2(s+3)}{s^2+6s+13}\right\} &= u(t-5)e^{-3(t-5)}\sin 2(t-5) \\ &= \begin{cases} 0, & \text{if } t < 5 \\ e^{-3(t-5)}\sin 2(t-5), & \text{if } t > 5 \end{cases} \end{aligned}$$

Problem 8.3. *Find the Laplace Transform of the following functions:*

1. $f(t) = \begin{cases} t-1, & \text{if } 0 < t < 2 \\ 3-t, & \text{if } 2 < t < 3 \\ 0, & \text{if } t > 3 \end{cases}$.

2. $f(t) = \begin{cases} t^2 - 1, & \textit{if } 0 < t < 1 \\ 3 - 2t, & \textit{if } < t < 3 \\ -t, & \textit{if } t > 3 \end{cases}$.

Problem 8.4. *Find the Inverse Laplace Transform of the following functions:*

1. $F(s) = \dfrac{e^{-bs}}{s^2(s+a)}$
2. $F(s) = \dfrac{6e^{-2s}}{s^2+9}$
3. $F(s) = \dfrac{se^{-2s}}{s^2+3s+2}$
4. $F(s) = \dfrac{(s+2)e^{-2s}}{s^2+4s+5}$

9 Application to Differential Equations

Consider the following example.

Example 9.1. *Solve the following Initial Value Problem:*

$$af''(t) + bf'(t) + cf(t) = g(t) \quad \textit{with} \quad f(0) = f_0,\ f'(0) = f_1.$$

Solution: *Let $\mathcal{L}(g(t)) = G(s)$. Then*

$$G(s) = a(s^2F(s) - sf(0) - f'(0)) + b(sF(s) - f(0)) + cF(s)$$

and the initial conditions imply

$$G(s) = (as^2 + bs + c)F(s) - (as + b)f_0 - af_1.$$

Hence,

$$F(s) = \frac{G(s)}{as^2+bs+c} + \frac{(as+b)f_0}{as^2+bs+c} + \frac{af_1}{as^2+bs+c}.$$

Now, if we know that $G(s)$ is a rational function of s then we can compute $f(t)$ from $F(s)$ by using the method of partial fractions.

Example 9.2. *Solve the IVP*

$$y'' - 4y' - 5y = f(t)$$

where

$$f(t) = \begin{cases} t & \textit{if } 0 \le t \le 5 \\ t+5 & \textit{if } t \ge 5 \end{cases}$$

with $y(0) = 1$ and $y'(0) = 4$.

Solution: *Note that* $f(t) = t + u(t-5)$. *Thus,*

$$\mathcal{L}(f(t)) = \frac{1}{s^2} + \frac{e^{-5s}}{s}.$$

Taking Laplace transform of the above equation, we get

$$\left(s^2Y(s) - sy(0) - y'(0)\right) - 4\left(sY(s) - y(0)\right) - 5Y(s) = \mathcal{L}(f(t)) = \frac{1}{s^2} + \frac{e^{-5s}}{s}.$$

Which gives

$$\begin{aligned} Y(s) &= \frac{s}{(s+1)(s-5)} + \frac{e^{-5s}}{s(s+1)(s-5)} + \frac{1}{s^2(s+1)(s-5)} \\ &= \frac{1}{6}\left[\frac{5}{s-5} + \frac{1}{s+1}\right] + \frac{e^{-5s}}{30}\left[-\frac{6}{s} + \frac{5}{s+1} + \frac{1}{s-5}\right] + \frac{1}{150}\left[-\frac{30}{s^2} + \frac{24}{s} - \frac{25}{s+1} + \frac{1}{s-5}\right]. \end{aligned}$$

$$\textit{Hence, } y(t) = \frac{5e^{5t}}{6} + \frac{e^{-t}}{6} + u(t-5)\left[-\frac{1}{5} + \frac{e^{-(t-5)}}{6} + \frac{e^{5(t-5)}}{30}\right] + \frac{1}{150}\left[-30t + 24 - 25e^{-t} + e^{5t}\right].$$

10 More Exercise for you try

1. *Show that*

   (a) $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}dt = \int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}.$

   (b) $\displaystyle\int_0^\infty \frac{\cos 6t - \cos 4t}{t}dt = ln(2/3).$

   (c) $\displaystyle\int_0^\infty te^{-t}\sin t dt = 0.5.$

2. *Find* $\mathcal{L}^{-1}\{F(s)\}$

   (a) $F(s) = \dfrac{s}{(s+3)^3}$

   (b) $F(s) = \dfrac{s}{(s+3)^{3/2}}$

   (c) $F(s) = \dfrac{s+3}{(s^2+4s+7)}$

   (d) $F(s) = \dfrac{e^{-2s}}{s^3}$

   (e) $F(s) = \dfrac{e^{-2s}}{\sqrt{s+3}}$

   (f) $F(s) = \dfrac{s+3}{(s^2+6s+10)^2}$

3. *Use Convolution theorem to evaluate the following:*

   (a) $\mathcal{L}^{-1}\left[\dfrac{1}{(s+2)(s-3)^4}\right]$

(b) $\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2+4)}\right]$

4. *Solve :*

   (a) $y'' + 4y' = 12t$, *given that* $y(0) = 0,\ y'(0) = 7$.

   (b) $y''(t) - 3y'(t) + 2y(t) = 7t + e^{-t}$, *given that* $y(0) = 6,\ y'(0) = -1$.

   (c) $x''' - x = 3e^t$, *given that* $y(0) = y'(0) = y''(0) = 0$.

   (d) $x' + 2y'' = e^{-t}$, $x' + 2x - y = 1$, *given that* $x(0) = y(0) = y'(0) = 0$.

5. *In an RC circuit (a resistor and a capacitor in series) the applied emf is a constant E. Given that $\frac{dq}{dt} = i$, where q is the charge in the capacitor, i the current in the circuit, R the resistance and C the capacitance. The equation for the circuit is*

$$Ri + \frac{q}{C} = E.$$

   If the initial charge is zero find the charge subsequently.

6. *If the voltage in the RC circuit is $E = E_0 \cos\omega t$ find the charge and the current at time t.*

7. *An object is projected from the Earth's surface. What is the least velocity (the escape velocity) of projection in order to escape the gravitational field, ignoring air resistance. The equation of motion is*

$$mv\frac{dv}{dx} = -mg\frac{R^2}{x^2}$$

   where the mass of the object is m, its distance from the center of the Earth is x and the radius of the Earth is R.

8. *The radial stress p at distance r for the axis of the thick cylinder subjected to internal pressure is given by $p + r\frac{dp}{dr} = A - p$, where A is a constant. If $p = p_0$ at the inner wall $r = r_1$ and is negligible $(p = 0)$ at the outer wall $r = r_2$ find an expression for p.*

9. *The equation for an LCR circuit with applied voltage E is*

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E.$$

   By differentiating this equation find the solution for $q(t)$ and $i(t)$ if $L = 1$, $R = 100$, $C = 0.2 \times 10^{-3}$ and $E = 1000$ given that $q = 0$ and $i = 0$ at $t = 0$.

10. *Consider the free vibration problem when $m = 1$, $n = 1$ and $k = 2$. (Critical damping) Find the solution for $x(t)$.*

11. *Repeat problem 10 for the case $m = 1$, $n = 1$ and $k = 1.5$ (light damping)*

12. *Consider the forced vibration problem with $m = 1$, $n = 25$, $k = 8$, $E = \sin 3t$, $x0 = 0$ with an initial velocity of 3.*