LECTURE NOTES ON

ORDINARY DIFFERENTIAL EQUATION AND APPLICATIONS

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NOTICE: this material must not be treated as a substitute for the lectures.

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1 Introduction and Motivation.

Let us begin with an example from elctrical engineering. Consider the electrical circuit shown in the Figure 1. It contains an electromotive force E (supplied by a battery or generator), a resistor

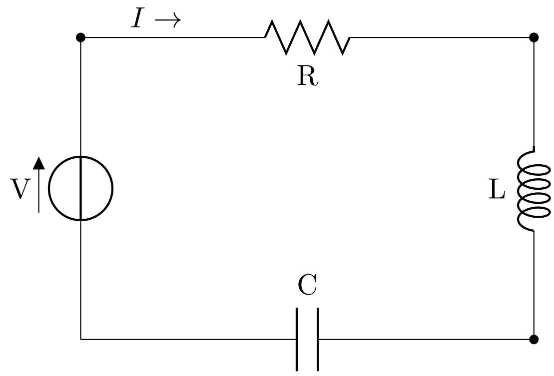

Figure 1: LCR circuit.

R, an inductor L, and a capacitor C, in series. If the charge on the capacitor at time t is $Q = Q(t)$ then the current is the rate of change of Q with respect to t : $I = \frac{dQ}{dt}$. It is known from physics that the voltage drops across the resistor, inductor, and capacitor are

$$IR, \quad L\frac{dI}{dt}, \quad \frac{Q}{C}$$

respectively. Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage:

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t). \tag{1}$$

Such a relation between a function $I(t)$ and its derivatives is called a **differential equation**. Since $I = \frac{dQ}{dt}$, this equation becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t). \tag{2}$$

which is a **second-order linear differential equation** with constant coefficients. Equation (2) is of second order since the highest derivative is of second degree. If the charge Q_0 and the current I_0 are known at time $t = 0$, then we have the initial conditions

$$Q(0) = Q_0, \quad Q'(0) = I(0) = I_0 \tag{3}$$

The differential Equation 2 together with the initial conditions 3 is called an initial-value problem.

A differential equation for the current can be obtained by differentiating Equation 1 with respect to t and remembering that $I = \frac{dQ}{dt}$:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t). \tag{4}$$

Solving the differential equations 3 and 4 one can obtain the charge and current at time t in the circuit of Figure 1. I am sure at the end of this course you will gather sufficient knowledge to solve such type of practical problems.

1.1 Definitions and Classification.

A differential equation is an equation involving differential (or differential coefficients) with or without the variables from which these differentials (or differential coefficients) are derived.

Example 1.1. *Following are some examples of differential equations:*

1. $\dfrac{dy}{dx} = e^x.$

2. $\left(\dfrac{dy}{dx}\right)^2 = ax^2 + bx + c.$

3. $\dfrac{d^2y}{dx^2} = 0.$

4. $\left(\dfrac{d^3y}{dx^3}\right)^2 = x^2\left(\dfrac{dy}{dx}\right)^5.$

5. $\dfrac{d^2y}{dx^2} = \dfrac{\frac{dy}{dx}}{\sqrt{1 + \frac{d^2y}{dx^2}}}.$

6. $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u.$

7. $\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial y^2} = 0.$

Differential equations can be classified into two classes viz. (i) Ordinary differential equations and (ii) Partial differential equations.

An *ordinary differential equation(ODE)* is one in which all the differentials (or derivatives) involved have reference to a single independent variable.

A *partial differential equation(PDE)* is one which contains partial differentials (or partial derivatives) and as such involves two or more independent variables.

The *order* of a differential equation is the order of the highest order derivative (or differential) in that equation.

The *degree* of a differential equation is the degree of the derivative(or differential) of the highest order in the equation after the equation is freed from radicals and fractions in its derivatives.

Problem 1.1. Classify the differential equations in the Example 1.1 for ODE and PDE. Also write their degree and order.

1.2 Formation of Ordinary Differential Equations.

If we have an equation

$$f(x, y, c_1, c_2, \ldots, c_n) \tag{5}$$

containing n arbitrary constants $c_1, c_2, \ldots, c_n$, then by differentiating this n-times we shall get n-equation.

Now between these n-equations and the Equation (5), in all $(n+1)$-equations, if the n arbitrary constants $c_1, c_2, \ldots, c_n$ be eliminated, we shall evidently get a differential equation of the n^{th} order, for there being n differentiations the resulting equation must contain a derivative of the n^{th} order. We will use

$$f(y_1, y_2, \ldots y_n, y, x) = 0$$

represent a typical differential equation of n^{th} order.

Example 1.2.

Form the differential equation for the family of curves $y = ae^{2x} + be^{-3x}$, where a, b are arbitrary constants.

Solution: The given equation is

$$y = ae^{2x} + be^{-3x} \tag{6}$$

Differentiating (6) w.r.t. x, we get

$$\frac{dy}{dx} = 2ae^{2x} - 3be^{-3x} \tag{7}$$

Again, differentiating (7) w.r.t. x we get

$$\frac{d^2y}{dx^2} = 4ae^{2x} + 9be^{-3x} \tag{8}$$

Now $3 \times (7) + (8) \Rightarrow$

$$3\frac{dy}{dx} + \frac{d^2y}{dx^2} = 10ae^{2x} \tag{9}$$

and $2 \times (7) - (8) \Rightarrow$

$$2\frac{dy}{dx} - \frac{d^2y}{dx^2} = -15be^{-3x} \tag{10}$$

Finally, $3 \times (9) - 2 \times (10)$ we get

$$5\frac{dy}{dx} + 5\frac{d^2y}{dx^2} = 30y$$
$$\Rightarrow \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

which is the required differential equation. ⋈

Example 1.3.

Form the differential equation of

$$v = \frac{\alpha}{r} + \beta,$$

where α, β are arbitrary constants.

Solution: Given

$$v = \frac{\alpha}{r} + \beta$$

Differentiating w.r.t. r, we get

$$\frac{dv}{dr} = -\frac{\alpha}{r^2}$$
$$\Rightarrow \quad r^2\frac{dv}{dr} = -\alpha$$

Again, differentiating w.r.t. r, we get

$$r^2\frac{d^2v}{dr^2} + 2r\frac{dv}{dr} = 0$$
$$\Rightarrow \quad \frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$$

which is the required differential equation. ⋈

1.3 Solution of a Differential Equation.

Consider the n^{th} order differential equation

$$f(y_1, y_2, \dots y_n, y, x) = 0. \tag{11}$$

Any function

$$y = \phi(x, a_1, a_2, \dots a_r), \tag{12}$$

where $a_1, a_2, \dots a_r$ linearly independent arbitrary constants, that satisfies the differential equation (11) is said to be a solution of it.

In the solution (12) of the differential equation (11), if $r = n$ i.e., the number of independent arbitrary constants is equal to the order of the differential equation then it is said to be the *general solution* or *complete solution* or *complete primitive* of the the differential equation (11).

The solution obtained by assigning particular values to the arbitrary constants involved in the general solution of a differential equation is called a *particular solution* of the differential equation.

For example, $y = ax$ is the general solution of the differential equation $xy_1 - y = 0$ but $y = 2x$ is a particular solution of it.

1.4 Exercise.

Short Answer Type Questions:

1. The number of arbitrary constants present in the general solution of a third order second degree ordinary differential equation is 3.(True / False)
2. What is the degree and order of $\left(\dfrac{d^3y}{dx^3}\right)^5 = x^2\left(\dfrac{d^2y}{dx^2}\right)^3$?
3. The number of arbitrary constants present in the general solution of a third order second degree ordinary differential equation is 5.(True / False)
4. What is the degree and order of $\left(\dfrac{d^3y}{dx^3}\right)^5 = x^2\left(\dfrac{d^4y}{dx^4}\right)$?
5. The function e^{2x} is a particular solution of $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$. (True / False)

Long Answer Type Questions:

1. Find the differential equation for the following family of functions;
   (a) $y = \alpha e^x + \beta e^{-x}$.
   (b) $y = a \log x + b$.
   (c) $y = A\cos x + B\sin x + x\sin x$.
   (d) $y = (A\cos x + B\sin x)e^x$.
   (e) $y = Ae^x + Be^{-x}$.
   (f) $r = a + b\cos\theta$.
   (g) $ax^2 + by^2 = 1$.
   (h) $x = \alpha\sin(\omega t + \beta)$.
   (i) $xy = c^2$.
   (j) $y = A\cos x + B\sin x + C\cosh x + D\sinh x$.
   (k) $y = a\tan^{-1} x + b$.

2. Obtain the differential equation of all parabolas each of which has latus rectum $4a$ and whose axes are parallel to the x-axis.
3. Find the differential equation of all straight lines passing through the origin.
4. Show that the differential equation of the system of concentric circles having centers at the origin is $xdx + ydy = 0$.
5. Prove that the differential equation of the family of circles touching the x-axis at the origin is $(x^2 - y^2)dy - 2xydx = 0$.
6. Show that the differential equation of the family of parabolas
   (a) having their axes parallel to the y-axis is $y_3 = 0$.
   (b) with foci at the origin axes along the x-axis is $yy_1^2 + 2xyy_1 - y = 0$.
7. Find the differential equation of all circles with radius 5 and center at (a, b).
8. Find the differential equation of all parabolas with axis as the x-axis and focus at $(a, 0)$.

2 First order and first degree differential equations.

Consider the first order first degree differential equation

$$xdx - ydy = \sqrt{x^2 + y^2}dx.$$

The above equation can also be written as

$$(\sqrt{x^2 + y^2} - x)dx + ydy.$$

In fact, any differential equation of the first order and first degree can be put in the form

$$Mdx + Ndy = 0$$

where M and N are functions of x and y or constants not involving the derivatives.

The general solution of this type of equation contains only one arbitrary constant. In this section we will discuss some specific methods of solving first order first degree differential equations.

2.1 Method of Separation of Variables.

If the differential equation

$$Mdx + Ndy = 0,$$

can be put in the form

$$f(x)dx + g(y)dy = 0 \tag{13}$$

then its general solution is given by

$$\int f(x)dx + \int g(x)dy = c,$$

where c is an arbitrary constant.

Example 2.1. *Solve:* $(1 + y^2)dx + (1 + x^2)dy = 0.$

Solution: Dividing the given equation by $(1+y^2)(1+x^2)$ we get

$$\frac{1}{1+x^2}dx + \frac{1}{1+y^2}dy = 0$$

$\therefore$ Integrating both sides, we get

$$\begin{aligned}
&\int \frac{1}{1+x^2}dx + \int \frac{1}{1+y^2}dy = \tan^{-1} c \\
\Rightarrow\quad & \tan^{-1} x + \tan^{-1} y = \tan^{-1} c \\
\Rightarrow\quad & x + y = c(1-xy)
\end{aligned}$$

⋈

Example 2.2. *Find the general solution of* $y\,dx + (1+x^2)\tan^{-1} x\,dy = 0$.

Solution: The given equation is

$$\begin{aligned}
& y\,dx + (1+x^2)\tan^{-1} x\,dy = 0 \\
\Rightarrow\quad & \frac{1}{(1+x^2)\tan^{-1} x}\,dx + \frac{1}{y}dy = 0 \\
\Rightarrow\quad & \frac{1}{\tan^{-1} x}\,d(\tan^{-1} x) + \frac{1}{y}dy = 0 \\
\Rightarrow\quad & \int \frac{1}{\tan^{-1} x}\,d(\tan^{-1} x) + \int \frac{1}{y}\,dy = \log c \\
\Rightarrow\quad & \log(\tan^{-1} x) + \log y = \log c \\
\Rightarrow\quad & \log(y\tan^{-1} x) = \log c \\
\Rightarrow\quad & y\tan^{-1} x = c
\end{aligned}$$

which is the required solution and where c is an arbitrary constant. ⋈

Note: If $x = r\cos\theta$ and $y = r\sin\theta$ then

$$\text{(i) } xdx + ydy = rdr. \text{ (i) } dx^2 + dy^2 = dr^2 + r^2 d\theta^2, \text{(iii) } xdx - ydy = r^2 d\theta.$$

Exercise.

Solve the following differential equations:

1. $x(y^2+1)dx + y(x^2+1)dy = 0$
2. $\sin 2y dx + (1+2e^{-x})\cos 2y dy = 0$ given that $y(0) = \dfrac{\pi}{4}$.
3. $(2a^2+r^2) = r^2\cos\theta\dfrac{d\theta}{dr},\ r(0) = a$.
4. $xe^{x+y} = \dfrac{dy}{dx},\ \ y(0) = 0$.

Example 2.3. $\dfrac{dy}{dx} = \cos(x+y)$.

Solution: Put $v = x + y$, so that $\dfrac{dv}{dx} = 1 + \dfrac{dy}{dx}$.
Then, the given equation reduces to

$$\begin{aligned} &\frac{dv}{dx} - 1 = \cos v \\ \Rightarrow\quad &\frac{1}{1+\cos v}\, dv = dx \\ \Rightarrow\quad &\int \frac{1}{1+\cos v}\, dv = \int dx + c \\ \Rightarrow\quad &\tan\frac{v}{2} = x + c \\ \Rightarrow\quad &\tan\frac{x+y}{2} = x + c \end{aligned}$$

which is the required solution and where c is an arbitrary constant.

2.2 Homogeneous Equations.

A function $f(x_1, x_2, \cdots, x_n)$ on n-variable $x_1, x_2, \cdots, x_n$ is said to be a homogeneous function of degree r if it satisfies any of the following equivalent conditions:

1. $f(vx_1, vx_2, \cdots, vx_n) = v^r f(x_1, x_2, \cdots, x_n)$.
2. $f(x_1, x_2, \cdots, x_n) = x_1^r \phi\left(\dfrac{x_2}{x_1}, \dfrac{x_3}{x_1}, \cdots, \dfrac{x_n}{x_1}\right)$.

Example 2.4. Some homogeneous functions and their degrees are given below:

1. $f(x, y) = 2xy + y^2$ of degree 2.
2. $u(x, y) = x^3 y^2 \sin^{-1}\left(\dfrac{y}{x}\right)$ of degree 5.
3. $u(x, y, z) = 4x^5 z + x^3 y^2 z + 3xy^2 z^3$ of degree 6.

$\bowtie$

There are many equations which are not of the form (13), but by a suitable substitution, they can be reduced to the separable form. We have already encountered such differential equations in the previous section. One such class of equation is

$$\frac{dy}{dx} = \frac{g(x,y)}{h(x,y)} \quad \text{or equivalently} \quad \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

where g and h are homogeneous functions of the same degree in x and y, and f is a continuous function. In this case, we use the substitution,

$$y = vx \qquad \text{to get} \quad \frac{dy}{dx} = x\frac{dv}{dx} + v.$$

Thus, the above equation after substitution becomes

$$x\frac{dv}{dx} + v = f(v),$$

which is a separable equation in v. For illustration, we consider some examples.

Example 2.5. *Find the general solution of* $(y^2 + x^2)dx - 2xydy = 0$.

Solution: The given equation can be written as Let I be any interval not containing 0. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2+y^2}{2xy} \\ OR \quad \frac{dy}{dx} &= f(x,y) \end{aligned}$$

where $f(x,y) = \dfrac{x^2+y^2}{2xy}$. Clearly

$$f(vx, vy) = v^0 f(x,y)$$

and so f is a homogeneous function of degree 0. Let $y = vx$, so that $\dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$. Thus, the above equation after substitution becomes

$$\begin{aligned} x\frac{dv}{dx} + v &= \frac{x^2+v^2x^2}{2xvx} \\ \Rightarrow \quad x\frac{dv}{dx} + v &= \frac{1+v^2}{2v} \end{aligned}$$

$$\Rightarrow \quad \frac{2v}{1-v^2}dv = \frac{1}{x}dx$$

On integration, we get

$$\begin{aligned} & \int \frac{2v}{1-v^2}dv = \int \frac{1}{x}dx + k \\ \Rightarrow \quad & -\log(1-v^2) = \log x + \log c, \qquad\qquad \text{where } k = \log c \\ \Rightarrow \quad & x = c(x^2-y^2) \end{aligned}$$

⋈

A Special Form:

The equations of the form

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2} \qquad \text{where } \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

can easily reduced to an equation where the variables will be separable. To achieve this, we put $x = X + h$ and $y = Y + k$, where h and k are arbitrary constants to be determined. Then

$$dx = dX \quad \text{and} \quad dy = dY.$$

and the above equation reduces to

$$\frac{dY}{dX} = \frac{a_1X+b_1Y+a_1h+b_1k+c_1}{a_2X+b_2Y+a_2h+b_2k+c_2}.$$

Now we choose h and k in such a way that

$$a_1h+b_1k+c_1 = 0 \quad \text{and} \quad a_2h+b_2k+c_2 = 0.$$

For, now the equation reduces to the form

$$\frac{dY}{dX} = \frac{a_1X+b_1Y}{a_2X+b_2Y}.$$

which is a homogeneous equation in X and Y and hence can be reduced to equation of separable variable using the substitution $Y = vX$. The following example illustrates the above discussions.

Example 2.6. *Solve:* $\dfrac{dy}{dx} = \dfrac{2x-y+1}{6x-5y+4}$.

Solution: Consider the transformation $x = X + h$ and $y = Y + k$ in such a way that

$$\begin{aligned} 2h - k + 1 &= 0 \qquad (14)\\ 6h - 5k + 4 &= 0 \qquad (15)\end{aligned}$$

Solving (1) and (2) we get $h = -\dfrac{1}{4}$ and $k = \dfrac{1}{2}$. Also, the given differential equation reduces to

$$\frac{dY}{dX} = \frac{2X - Y}{6X - 5Y}. \qquad (16)$$

The equation (16) is homogeneous. Consider the transformation $Y = vX$, so that

$$\frac{dY}{dX} = v + X\frac{dv}{dX}.$$

Therefore, the equation (16) reduces to

$$\begin{aligned}
& v + X\frac{dv}{dX} = \frac{2-v}{6-5v}.\\
\Rightarrow\quad & \frac{6-5v}{5v^2-7v+2}dv = \frac{dX}{X}.\\
\Rightarrow\quad & \frac{6-5v}{(5v-2)(v-1)}dv = \frac{dX}{X}.\\
\Rightarrow\quad & \frac{1}{3}\frac{1}{v-1}dv - \frac{20}{3}\frac{1}{5v-2}dv = \frac{dX}{X}.\\
\Rightarrow\quad & \int \frac{1}{v-1}\,dv - 20\int \frac{1}{5v-2}\,dv = 3\int \frac{1}{X}\,dX + \log c.\\
\Rightarrow\quad & \log(v-1) - 4\log(5v-2) = 3\log X + \log c\\
\Rightarrow\quad & \frac{v-1}{(5v-2)^4} = cX^3.\\
\Rightarrow\quad & Y - X = c(2X-5Y)^4.\\
\Rightarrow\quad & y - x - \frac{3}{4} = c(2x-5y+3)^4,
\end{aligned}$$

which is the required solution and where c is an arbitrary constant.

Example 2.7. *Solve:* $\dfrac{dy}{dx} = \dfrac{6x-2y-7}{2x+3y-6}$.

Remark 2.1. *If* $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = m$*, say, then the equation*

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

reduces to

$$\frac{dy}{dx} = \frac{m(a_2x + b_2y) + c_1}{a_2x + b_2y + c_2}$$

and by putting $v = a_2x + b_2y$ *the variables can be separated easily.*

Exercise.

1. $xdx - ydy = \sqrt{x^2 + y^2}dx$.
2. $(x^2 - y^2)dx = 2xydy$.
3. $xdx - ydy = \sqrt{x^2 + y^2}dx$.
4. $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}$.
5. $\dfrac{dy}{dx} = \dfrac{y(y+x)}{x(y-x)}$.
6. $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right)$
7. $x + y\frac{dy}{dx} = 2y$.
8. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$.
9. $(x^2 - 2xy)dy + (x^2 - 3xy + 2y^2)dx = 0$.
10. $x^3dx - y^3dy = 3xy(ydx - xdy)$.
11. $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)dy = 0$.
12. $y - x\text{cosec}\left(\dfrac{y}{x}\right) = x\dfrac{dy}{dx}$.
13. $xy' = y + \sqrt{x^2 + y^2}$.
14. $\dfrac{dy}{dx} = (4x + y - 1)^2$.
15. Solve $(x + 2y - 3)dx = (2x - 4y + 6)dy$.
16. Solve $(x - y - 4)dx = (x + y + 2)dy$.
17. Obtain the general solutions of the following: $\dfrac{dy}{dx} = \dfrac{x - y + 2}{-x + y + 2}$.
18. Solve the initial value problem
19. $\dfrac{dy}{dx} = (1 + yy^2)\tan x$, given that $y = \sqrt{3}$ when $x = 0$.
20. $(x^2 + y^2 - a^2)xdx + (x^2 - y^2 - b^2)dy = 0$.
21. $xe^{xy}dx + (xe^{xy} + 2y)dy = 0$.
22. $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$.
23. $(2xy + y - \tan y)dx + (x^2 - x\tan y + sec^2 y)dy = 0$.
24. $\left[y\left(1 + \dfrac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\sin y)dy = 0$.
25. $(2x^2 + y^2 + x)dx + xydy = 0$.
26. $(2x^3 + 4y)dx + (4x + y - 1)dy = 0$.
27. $(\cos x\tan y + \cos(x + y))dx + (\sin x\sec^2 y + \cos(x + y))dy = 0$.
28. $(3x^2e^y - x^2)dx + (x^3e^y + y^2)dy = 0$.

2.3 Exact Equations.

The differential equation

$$Mdx + Ndy = 0 \tag{17}$$

where M and N are functions of x and y or constants not involving the derivatives, is said to be exact if there exists a real valued twice continuously differentiable function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (or the domain is an open subset of $\mathbb{R}^2$) such that

$$\frac{\partial u}{\partial x} = M \text{ and } \frac{\partial u}{\partial y} = N.$$

Remark 2.2. *If the equation (17) is exact, then*

$$Mdx + Ndy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du = 0.$$

This implies that $u(x, y) = c$ (where c is an arbitrary constant) is an implicit solution of (17). In other words, the left side $Mdx + Ndy$ of (17) is an exact differential.

Theorem 2.1. *Let M and N be twice continuously differentiable function in a region D. A necessary and sufficient condition for the equation*

$$Mdx + Ndy = 0$$

to be exact is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Proof. First assume that the differential equation

$$Mdx + Ndy = 0$$

is exact. Then, there exist a function $u(x, y)$ such that

$$Mdx + Ndy = du.$$

Since u is a function of the two variables x and y, so

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Thus,

$$Mdx + Ndy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Now equating the coefficients dx and dy from both sides we get

$$M = \frac{\partial u}{\partial x}, \quad N = \frac{\partial u}{\partial y}$$

Therefore, $\dfrac{\partial M}{\partial y} = \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial N}{\partial x}$. Conversely, assume that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. Now, let

$$\int_{y\ constant} Mdx = \phi(x, y),$$

where the integration is taken w.r.t x treating y as a constant. Then,

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\int M dx\right) &= \frac{\partial \phi}{\partial x} \\
\Rightarrow \quad M &= \frac{\partial \phi}{\partial x} \\
\Rightarrow \quad \frac{\partial M}{\partial y} &= \frac{\partial^2 \phi}{\partial y \partial x} \\
\Rightarrow \quad \frac{\partial N}{\partial x} &= \frac{\partial^2 \phi}{\partial x \partial y}
\end{aligned}
$$

Now, integrating both sides w.r.t x(treating y as constant), we get

$$N = \frac{\partial \phi}{\partial y} + h(y)$$

where $h(y)$ is a function of y alone or constant.

Therefore,

$$
\begin{aligned}
Mdx + Ndy &= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + h(y)dy \\
&= d\phi + h(y)dy \\
&= d\left(\phi + \int h(y)dy\right) \\
&= du, \qquad \text{where } \ u = \phi + \int h(y)dy.
\end{aligned}
$$

which shows that, the equation $Mdx + Ndy = 0$ is exact. □

Steps for solving exact differential equations:

Method A.

1. *Write the given equation in the form $Mdx + Ndy = 0$.*
2. *Verify whether M and N satisfies the equation*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

   If yes, go to next step else stop with the comment that the given differential equation is not exact.

3. *Let $u(x, y)$ be a function of x and y such that $\dfrac{\partial u}{\partial x} = M$ and $\dfrac{\partial u}{\partial y} = N$*
4. *Integrate both side of*

$$\frac{\partial u}{\partial x} = M$$

   w.r.t x(treating y as constant) to get

$$u = \int Mdx + h(y) = \phi(x, y) + h(y), \tag{18}$$

   where $\phi(x, y) = \int Mdx$ and $h(y)$ is the constant of integration (independent of x).

5. *Differentiate the equation (18) partially w.r.t y to get*

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial \phi(x,y)}{\partial y} + h'(y)\\ \Rightarrow \quad N &= \frac{\partial \phi(x,y)}{\partial y} + h'(y)\\ \Rightarrow \quad h'(y) &= N - \frac{\partial \phi(x,y)}{\partial y}\end{aligned}$$

Now, integrating both sides w.r.t y(treating x as constant) we get

$$\Rightarrow \quad h(y) = \int \left(N - \frac{\partial \phi(x,y)}{\partial y} \right) dy. \tag{19}$$

6. *Equations (18) and (19) completely determines $u(x,y)$ and hence complete solution of the given equation is given by*

$$u(x,y) = c,$$

where c is an arbitrary constant.

Example 2.8. $2xe^y + (x^2e^y + \cos y\ )\dfrac{dy}{dx} = 0.$

Solution: *The given equation can be written as*

$$2xe^y dx + (x^2e^y + \cos y\ )dy = 0.$$

Therefore, we have

$$M = 2xe^y,\ N = x^2e^y + \cos y,\ \frac{\partial M}{\partial y} = 2xe^y \quad \textit{and} \quad \frac{\partial N}{\partial x} = 2xe^y.$$

Hence, the given equation is exact and so there exists a function $u(x,y)$ such that

$$\frac{\partial u(x,y)}{\partial x} = 2xe^y \quad \textit{and} \quad \frac{\partial u(x,y)}{\partial y} = x^2e^y + \cos y.$$

Now, integrating both side of $\dfrac{\partial u(x,y)}{\partial x} = 2xe^y$ with respect to x (treating y as a constant) we get,

$$u(x,y) = x^2e^y + h(y).$$

But then

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial (x^2e^y + h(y))}{\partial y}\\ \Rightarrow \quad x^2e^y + \cos y &= x^2e^y + h'(y)\\ \Rightarrow \quad h'(y) &= \cos y\\ \Rightarrow \quad h(y) &= \sin y + \alpha\end{aligned}$$

where α is an arbitrary constant. Thus

$$u(x,y) = x^2e^y + \sin y + \alpha$$

and the general solution of the given equation is

$$x^2e^y + \sin y = c,$$

where c is an arbitrary constant. The solution in this case is in implicit form. ⋈

Method B.

1. *Write the given equation in the form* $Mdx + Ndy = 0$.
2. *Verify whether* M *and* N *satisfies the equation*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

   If yes, go to next step else stop with the comment that the given differential equation is not exact.

3. *Set* $\phi(x,y) = \int M dx$, *where integration is taken w.r.t* x *(treating* y *as constant).*
4. *The general solution of the given equation is given by*

$$\phi(x,y) + \int \left(N - \frac{\partial \phi(x,y)}{\partial y} \right) dy = c,$$

   where c *is an arbitrary constant.*

Example 2.9. *Solve:* $y \sin 2x dx - (y^2 + \cos^2 x) dy = 0$.

Solution: *Comparing the given equation with*

$$Mdx + Ndy = 0$$

we get

$$M = y \sin 2x, \text{ and } N = -y^2 - \cos^2 x).$$

Let $h(x,y) = \int\limits_{y \text{ constant}} M dx = -\frac{1}{2} y \cos 2x$. *Therefore, the general solution of the given equation is given by*

$$\begin{aligned}
& h(x,y) + \int \left(N - \frac{\partial h(x,y)}{\partial y} \right) dy = c \\
\Rightarrow \quad & -\frac{1}{2} y \cos 2x + \int \left(-y^2 - \cos^2 x + \frac{1}{2} \cos 2x \right) dy = c \\
\Rightarrow \quad & -\frac{1}{2} y \cos 2x + \int \left(-y^2 - \frac{1}{2}(1 + \cos 2x) + \frac{1}{2} \cos 2x \right) dy = c \\
\Rightarrow \quad & -\frac{1}{2} y \cos 2x + \int \left(-y^2 - \frac{1}{2} \right) dy = c \\
\Rightarrow \quad & -\frac{1}{2} y \cos 2x + \left(-\frac{y^3}{3} - \frac{1}{2} y \right) = c \\
\Rightarrow \quad & \frac{1}{2} y \cos 2x + \frac{y^3}{3} + \frac{1}{2} y + c = 0,
\end{aligned}$$

where c *is an arbitrary constant.* $\bowtie$

Exercise.

Short Answer Type Questions:

1. If the equation $f(x,y)dx = g(x,y)dy$ is exact, then $\dfrac{\partial f}{\partial y} + \dfrac{\partial g}{\partial x} = 0$. (True / False)
2. Write the solution of the exact differential equation $y^2\, dx + 2xy\, dy = 0$.

3. If the equation $f(x,y)dx = g(x,y)dy$ is exact, then $\dfrac{\partial f}{\partial y} + \dfrac{\partial g}{\partial x} = 0$. (True / False)

4. Write the solution of the exact differential equation $(2x^2 + y^2 + x)\,dx + 2xy\,dy = 0$.

Long Answer Type Questions:

1. Solve the following:

   (a) $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$.

   (b) $(x^2 + y^2 - a^2)xdx + (x^2 - y^2 - b^2)dy = 0$.

   (c) $xe^{xy}dx + (xe^{xy} + 2y)dy = 0$.

   (d) $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$.

   (e) $(2xy + y - \tan y)dx + (x^2 - x\tan y + sec^2 y)dy = 0$.

   (f) $\left[y\left(1 + \dfrac{1}{x}\right) + \cos y\right]dx + (x + \log x - x\sin y)dy = 0$.

   (g) $(2x^2 + y^2 + x)dx + xydy = 0$.

   (h) $(2x^3 + 4y)dx + (4x + y - 1)dy = 0$.

   (i) $(\cos x\tan y + \cos(x + y))dx + (\sin x\sec^2 y + \cos(x + y))dy = 0$.

   (j) $(3x^2e^y - x^2)dx + (x^3e^y + y^2)dy = 0$.

2. Solve: $(3x^2 + 2xy^2)dx + 2x^2ydy = 0$, given that $y(2) = -3$

3. Solve the differential equation $(x^2 - ay)dx = (ax - y^2)dy$ subject to the condition $y(0) = -3$.

4. Find values of ℓ and m such that the equation

$$\ell y^2 + mxy\frac{dy}{dx} = 0$$

   is exact. Also, find its general solution.

2.4 Equations Reducible to Exact Equations.

On many occasion the differential equation

$$Mdx + Ndy = 0 \tag{20}$$

may not be exact. For example, consider the differential equation

$$xy\log ydx + \frac{x^2}{2}dy = 0.$$

Clearly, it is not an exact differential equation. But, on multiplying the above equation by $\dfrac{1}{y}$ it reduces to

$$x\log ydx + \frac{x^2}{2y}dy = 0$$

and which is an exact differential equation. Such a factor (in this case, $\dfrac{1}{y}$) is called an **integrating factor** for the given equation. Formally,

Definition 2.1 (Integrating Factor)**.** A function $f(x,y)$ is called an *integrating factor* (IF) for the differential equation (20) if the equation

$$f(x,y)Mdx + f(x,y)Ndy = 0$$

is exact.

Remark 2.3. *1. If (20) has a general solution, then it can be shown that (20) admits an integrating factor.*

2. If (20) has an integrating factor, then it has many (in fact infinitely many) integrating factors.

3. Given (20)), whether or not it has an integrating factor, is a tough question to settle.

Rules for finding IF

In some cases, we use the following rules to find the integrating factors.

1. *The equation $Mdx+Ndy=0$ has $e^{\int f(x)dx}$ as an integrating factor, if $f(x)=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x alone.*
2. *The equation $Mdx+Ndy=0$ has $e^{-\int g(y)dy}$ as an integrating factor, if $g(y)=\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of y alone.*
3. *If M and N are homogeneous functions of equal degree and $Mx+Ny\neq 0$ then $\frac{1}{Mx+Ny}$ is an Integrating Factor of the equation $Mdx+Ndy=0$.*
4. *If the equation $Mdx+Ndy=0$ is of the form*

$$yf(xy)dx+xg(xy)dy=0$$

   with $Mx-Ny\neq 0$, then the function $\frac{1}{Mx-Ny}$ is an integrating factor.
5. *If the functions $M(x,y)$ and $N(x,y)$ are polynomial functions in x,y; then $x^{\alpha}y^{\beta}$ works as an integrating factor for some appropriate values of α and β.*

Exercise.

Short Answer Type Questions:

1. An integrating factor for $xdy-ydx=0$ is ....
2. A differential equation may have infinitely many integrating factors.(T/F)
3. Write an integrating factor for the differential equation $(x+x^2)\,dx-y\,dx=0$.
4. Write the solution of the exact differential equation $(x^2-ay)\,dx=(ax-y^2)\,dy$.
5. Write the solution of the exact differential equation $(2x^3+4y)\,dx+(4x+10y-1)\,dy=0$.
6. Write the solution of the exact differential equation $(2x^2+y^2+x)\,dx+2xy\,dy=0$.
7. Write an integrating factor for the differential equation $y\log y\,dx+(x-\log y)\,dy=0$.
8. Write an integrating factor for the differential equation $y\log y\,dx+(x-(\log y)^2)\,dy=0$.

Long Answer Type Questions:

1. Solve the following:
   (a) $(2x^2+y^2+x)dx+xydy=0$.
   (b) $(x^3+y^3)dx-xy^2dy=0$.
   (c) $(y+x)dx+xdy=0$.

(d) $y(1+xy)dx - xdy = 0$.

(e) $y(x^3y^3 + x^2y^2 + xy + 1)dx + x(x^3y^3 - x^2y^2 - xy + 1)dy = 0$.

(f) $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$.

(g) $y(1+xy)dx + x(1-xy)dy = 0$.

(h) $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$.

(i) $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2+y^2}$.

(j) $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$.

(k) $\dfrac{dy}{dx} = \dfrac{x^3+y^3}{xy^2}$.

(l) $(x+y)dx + (y-x)dy = 0$.

(m) $2xydx - (x^2 - y^2)dy = 0$.

(n) $(xy^2 + 3e^{x^{-3}})dx - x^2ydy = 0$.

(o) $\dfrac{dy}{dx}\sin x - y\cos x + y^2 = 0$.

(p) $x\cos(y/x)(ydx + xdy) = y\sin(y/x)(xdy - ydx)$.

(q) $(x^2y^2 + xy)ydx + (x^2y^2 - 1)xdy = 0$.

2. Find conditions on the function $g(x, y)$ so that the equation

$$(x^2 + xy^2) + \{ax^2y^2 + g(x, y)\}\frac{dy}{dx} = 0$$

is exact.

3. What are the conditions on $f(x)$, $g(y)$, $\phi(x)$, and $\psi(y)$ so that the equation

$$(\phi(x) + \psi(y)) + (f(x) + g(y))\frac{dy}{dx} = 0$$

is exact.

4. Find an integrating factor and hence solve the differential equation

$$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$$

5. Find the solution of

(a) $(x^2y + 2xy^2) + 2(x^3 + 3x^2y)\dfrac{dy}{dx} = 0$ with $y(1) = 0$.

(b) $y(xy + 2x^2y^2) + x(xy - x^2y^2)\dfrac{dy}{dx} = 0$ with $y(1) = 1$.

2.5 (Leibnitz's) Linear Equations.

An equation of the form

$$\frac{dy}{dx} + Py = Q \tag{21}$$

where P and Q are functions of x, is called a linear differential equation on y.

Theorem 2.2. *An integrating factor for the linear differential equation (21) is* $e^{\int P dx}$.

Proof. Hints: It is enough to show that the differential equation

$$\frac{dy}{dx}e^{\int Pdx} + Pye^{\int Pdx} = Qe^{\int Pdx} \tag{22}$$

is exact.

□

Remark 2.4. *An integrating factor for the linear differential equation (21) is given by*

$$IF = e^{\int Pdx}$$

and its general solution solution is given by

$$y \cdot (IF) = \int Q \cdot (IF)dx + c,$$

where c is an arbitrary constant.

Example 2.10. *Find the particular solution of the differential equation*

$$\sin x\frac{dy}{dx} + y\cos x = 4x,$$

given that $y = 0$ when $x = \frac{\pi}{2}$.

Solution: *The given equation can be written as*

$$\frac{dy}{dx} + y\,cot\,x = 4x\,cosec\,x,$$

which is a linear differential equation x and y. Here, $P = cot\,x$ and $Q = 4x\,cosec\,x$. So,

$$IF = e^{\int P\,dx} = e^{\int cot\,x\,dx} = e^{\log\sin x} = \sin x,$$

and its solution is given by

$$\begin{aligned} & y \cdot (IF) = \int Q \cdot (IF)\,dx + c \\ \Rightarrow\quad & y \cdot \sin x = \int 4x\,cosec\,x \cdot (IF)\,dx + c \\ \Rightarrow\quad & y \cdot \sin x = 2x^2 + c \end{aligned}$$

Given $y = 0$ when $x = \frac{\pi}{2}$. Therefore,

$$0 \cdot \sin\frac{\pi}{2} = 2(\frac{\pi}{2})^2 + c$$

or

$$c = -\frac{\pi^2}{2}$$

Hence the required particular solution is

$$y \cdot \sin x = 2x^2 - \frac{\pi^2}{2}.$$

2.6 Bernoulli's Equations

An equation of the form

$$\frac{dy}{dx} + Py = Qy^n \tag{23}$$

where P and Q are functions of x, is called a Bernoulli's Equations.

To solve (23), we divide it by y^n so that

$$\frac{1}{y^n}\frac{dy}{dx} + P\frac{1}{y^{n-1}} = Q \tag{24}$$

Now, let $v = \dfrac{1}{y^{n-1}}$ so that $\dfrac{1}{1-n}\dfrac{dv}{dx} = \dfrac{1}{y^n}\dfrac{dy}{dx}$. Then, the equation (24) reduces to

$$\frac{1}{1-n}\frac{dv}{dx} + Pv = Q$$

$$or, \quad \frac{dv}{dx} + (1-n)Pv = (1-n)Q$$

which is linear on v and hence can be solved by the method discussed in the previous section.

Remark 2.5. *General equation reducible to linear form is*

$$f'(x)\frac{dy}{dx} + Pf(y) = Q \tag{25}$$

where P and Q are functions of x. This equation is known as the generalized form of the Bernoulli's Equations. We use the substitution $v = f(y)$ to reduce this equation to linear form.

Exercise.

Long Answer Type Questions:

1. (a) $\dfrac{dy}{dx} + (1+2x)y = xe^{-x^2}$.

   (b) $x\dfrac{dy}{dx} + y = x^3y^6$.

   (c) $\dfrac{dy}{dx} + y\tan x = y^3\sec x$.

   (d) $r\sin\theta - \dfrac{dr}{d\theta}\cos\theta = r^2$.

   (e) $(x^3y^2 + xy)dx = dy$.

   (f) $xy(1+xy^2)\dfrac{dy}{dx} = 1$.

   (g) $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y$.

   (h) $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$.

   (i) $\sec^2 y\dfrac{dy}{dx} + x\tan y = x^3$.

   (j) $\sec^2 y\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = x^3$.

   (k) $\dfrac{dz}{dx} + \dfrac{z\log z}{x} = \dfrac{z}{x^2}(\log z)^2$.

   (l) $\dfrac{dy}{dx} + xy = xy^3$.

2. Find the particular solution of $\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$ given that $y = 0$ when $x = \frac{\pi}{2}$.

3. Find the particular solution of the differential equation

$$(x+1)\frac{dy}{dx} - y = 3^{3x}(x+1)^2,$$

given that $y = 1$ when $x = 0$.

3 Linear Differential Equations of Higher Order.

3.1 Introduction.

Consider the second order linear differential equation

$$\frac{d^2y}{dx^2} - 4y = 0. \tag{26}$$

Notice that, if we take $y = e^{2x}$ then

$$\frac{d^2y}{dx^2} - 4y = 4e^{2x} - 4e^{2x} = 0.$$

Hence $y = e^{2x}$ is a particular solution of (26). Also $y = e^{-2x}$ is another particular solution of (26). For any two arbitrary real constants c_1, c_2, in fact $y = c_1 e^{2x} + c_2 e^{-2x}$ is also a solution of (26). Since (26) is a second order differential equation, so its general solution will involve two arbitrary constants. Therefore,

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

is the general solution of (26).

In the above example, first we obtained two **linearly independent** particular solutions and then obtained the general solution by taking **linear combination** of these two particular solutions. In fact, we will adopt the same procedure to obtain the general solution of second and higher order linear differential equations. But, before we proceed further we need to clear the following concepts.

Definition 3.1 (Linear Combination)**.** *A function $y(x)$ is said to be a linear combination of the functions $y_1(x), y_2(x), \dots, y_r(x)$, if*

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \dots c_r y_r(x),$$

for some scalars (real numbers) $c_1, c_2, \dots, c_r$.

For example, $x^2 - 1$ is a linear combination of $2x + 1, x^2 + 4x, x$, because

$$x^2 - 1 = (-1) \cdot (2x + 1) + 1 \cdot (x^2 + 4x) + (-2) \cdot x.$$

Definition 3.2 (Linear Dependence and Linear Independence)**.** *A collection of functions*

$$\{y_1(x), y_2(x), \dots, y_r(x)\}$$

is said to be linearly dependent if one of them can be expressed as a linear combination of the remaining. Otherwise, it is said to be linearly independent.

For example, $\{2x + 1, x^2 + 4x, x^2 - 1, x\}$ is a linearly dependent set but $\{1, 1 + x, x + x^2\}$ is linearly independent.

Problem 3.1. *Which of the following are linearly dependent sets:*

(a) $\{1, x, x^2, x^3, x^4\}$.

(b) $\{e^x, e^{2x}, e^{4x}, e^{8x}\}$.

(c) $\{e^x, e^{-x}, 2e^x - 3e^{-x}\}$.

(d) $\{1, x, x^2, 1 + x(x^2 + 1), x^3\}$.

Definition 3.3. *An equation of the form*

$$\frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = X \tag{27}$$

where $a_1, a_2, \cdots, a_n$ are constants and X is a function of x, is called a linear differential equation of order n.

When $X \equiv 0$, the equation (27) reduces to

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \tag{28}$$

and it is called the *Homogeneous linear differential equation* of order n.

Let $y_1, y_2, \cdots, y_n$ be n linearly independent solutions of (28), then

$$u = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

is also a solution of (28) and hence it is the **complete or general solution** of (28). Further, for any particular solution v of (27) it is easy to see that

$$y = u + v$$

is also a solution of (27) and it contains n arbitrary constants. This shows that $y = u + v$ is the **complete or general solution** of (27).

The part u is called the *complementary function*(CF) and the part v is called the *particular integral*(PI) of (27). Therefore, the general or complete solution of (27) is given by

$$y = CF + PI.$$

Remark 3.1. In order to obtain the complete solution of (27), we have to first find the CF of (27) i.e., the complete solution of (27) and then the PI i.e., a particular solution of (27).

3.2 The Operator D

Denoting $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3} \cdots$ by $D, D^2, D^3 \cdots$ respectively so that

$$\frac{dy}{dx} = Dy, \qquad \frac{d^2 y}{dx^2} = D^2 y, \qquad \frac{d^3 y}{dx^3} = D^3 y \cdots$$

the equation (27) can be written as

$$\begin{aligned} &(D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y = X \\ i.e., \quad &f(D) y = X \end{aligned}$$

where $f(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$. Now $f(D)$ can be factorized by ordinary rules of algebra and the factors may be taken in any order. For example, using the operator D the differential equation

$$\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} - 3y = \cos^2 x$$

can be written as

$$\begin{aligned} &(D^2 + 5D - 3) y = \cos^2 x \\ i.e., \quad &f(D) y = X \end{aligned}$$

where $f(D) = D^2 + 5D - 3$ and $X = \cos^2 x$.

3.3 Rules for finding CF.

To procedure to obtain the complete solution of the differential equation (27) is described in the following steps:

Step 1 Using the operator $D \equiv \frac{d}{dx}$ reduce the given equation to the form

$$f(D)y = X,$$

where $f(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n$ is a polynomial function of degree n on D.

Step 2 Equate $f(m)$ to zero to obtain the algebraic equation of degree n, i.e.,

$$f(m) = 0 \tag{29}$$

The equation (29) is called the Auxiliary Equation(AE) of (27). Let $m_1, m_2, \cdots m_n$ are the roots of the auxiliary equation (40).

Step 3 Cosider the following cases to obtain the CF of (27):

Case A. If all the roots $m_1, m_2, \cdots m_n$ are real and distinct, then

$$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

Case B. If some of the roots (say first r) are equal and the rest are distinct i.e., if $m_1 = m_2 = \cdots = m_r = m$(say) and $m_{r+1}, \cdots m_n$ are distinct then

$$CF = (c_1 + c_2 x + \cdots + c_r x^{r-1})e^{mx} + C_{r+1} e^{m_{r+1} x} + \cdots + c_n e^{m_n x}.$$

Case C. If one pair of roots be imaginary i.e., $m_1 = \alpha + i\beta, \quad m_2 = \alpha - i\beta$ then

$$CF = (c_1 \cos \beta x + c_2 \sin \beta x)e^{\alpha x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$

Case D. If two pairs of imaginary roots be equal i.e., $m_1 = m_2 = \alpha + i\beta, \quad m_3 = m_4 = \alpha - i\beta$ then

$$CF = [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]e^{\alpha x} + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}.$$

At this point it is better recall that the CF of (27) is nothing but the complete solution of (28). So the above three steps also describe the process of solving the homogeneous equation (28).

Example 3.1. *Solve:* $\frac{d^3 y}{dx^3} + 3\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} = 0.$

Solution: Let $D \equiv \frac{d}{dx}$. Then the given equation reduces to

$$(D^3 + 3D^2 - 4D)y = 0 \quad \text{Or} \quad f(D)y = 0$$

where $f(D) = D^3 + 3D^2 - 4D$. The AE is

$$\begin{aligned} & f(m) = 0 \\ \Rightarrow\ & m^3 + 3m - 4m = 0 \\ \Rightarrow\ & m(m^2 + 3m - 4) = 0 \\ \Rightarrow\ & m = 0 \text{ or } \ m^2 + 3m - 4 = 0 \\ \Rightarrow\ & m = -1 \text{ or } \ m = 1, -4 \end{aligned}$$

Therefore, the complete solution of the given equation is

$$y = c_1 e^{0 \cdot x} + c_2 e^x + c_3 e^{-4x},$$

$$\textit{i.e.,} \quad y = c_1 + c_2 e^x + c_3 e^{-4x},$$

where c_1, c_2, c_3 are arbitrary constants. ⋈

Example 3.2. *Solve:* $\dfrac{d^3y}{dx^3} + y = 0.$

Solution: Let $D \equiv \dfrac{d}{dx}$. Then the given equation reduces to

$$(D^3 + 1)y = 0 \quad \text{Or} \quad f(D)y = 0$$

where $f(D) = D^3 + 1$. The AE is

$$\begin{aligned} & f(m) = 0 \\ \Rightarrow \quad & m^3 + 1 = 0 \\ \Rightarrow \quad & (m+1)(m^2 - m + 1) = 0 \\ \Rightarrow \quad & m = -1 \text{ or } m^2 - m + 1 = 0 \\ \Rightarrow \quad & m = -1 \text{ or } m = \frac{1 \pm i\sqrt{3}}{2} \end{aligned}$$

Therefore, the complete solution of the given equation is

$$y = c_1 e^{-x} + (c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2}) e^{\cos \frac{\sqrt{1}x}{2}},$$

where c_1, c_2, c_3 are arbitrary constants. ⋈

Example 3.3. *Solve:* $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0.$

Solution: Let $D \equiv \dfrac{d}{dt}$. Then the given equation reduces to

$$(D^2 + 6D + 9)x = 0 \quad \text{Or} \quad f(D)x = 0$$

where $f(D) = D^2 + 6D + 9$. The AE is

$$\begin{aligned} & f(m) = 0 \\ \Rightarrow \quad & m^2 + 6m + 9 = 0 \\ \Rightarrow \quad & m = -3, -3. \end{aligned}$$

Therefore, the complete solution of the given equation is

$$x = (c_1 + c_2 t)e^{-3t},$$

where c_1, c_2 are arbitrary constants. ⋈

3.3.1 Exercise

Short Answer Type Questions:

1. If the differential equation $f(D)y = \cos^2 x$ has $CF = c_1 e^{-3x} + c_2 e^{2x}$, then $f(D) =$ ....
2. The number of arbitrary constants present in the general solution of a third order second degree ordinary differential equation is 3.(True / False)
3. The number of arbitrary constants present in the general solution of a third order second degree ordinary differential equation is 5.(True / False)
4. The complete solution of $(D^2 + 1)y = 0$ is $y =$.............

5. Write the homogeneous linear differential equation the roots of whose auxiliary equation are 1,2,-2.

Long Answer Type Questions:

1. Solve: $\dfrac{d^2y}{dx^2} + 4y = 0$.
2. Solve: $\dfrac{d^4x}{dt^4} + 4x = 0$.
3. Solve: $\dfrac{d^2x}{dt^2} + 3a\dfrac{dx}{dt} - 4a^2x = 0$.
4. Solve: $\dfrac{d^4x}{dt^4} = m^4x$.
5. Solve: $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0$.
6. Solve: $\dfrac{d^4y}{dx^4} + a^4y = 0$.
7. Find the particular solution of $y'' - 2y' + 10y = 0$, given that $y(0) = 4$ and $y'(0) = 1$.
8. Solve: $\dfrac{d^2y}{dx^2} + 2p\dfrac{dy}{dx} + (p^2 + q^2)y = 0$.
9. Solve: $D^3 + D^2 + 4D + 4)y = 0$ where $D \equiv \dfrac{d}{dx}$.
10. Solve: $(D^2 + D + 1)^2y = 0$ where $D \equiv \dfrac{d}{dx}$.
11. Solve: $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = 0$ where $D \equiv \dfrac{d}{dx}$.

3.4 Inverse Operator.

Definition 3.4. *We use $\dfrac{1}{f(D)}X$ to represent that function of x not containing arbitrary constants which when operated by $f(D)$ gives X.*

For example, if $f(D) = D^2 + 1$ and $X = \sin 2x$ then

$$\begin{aligned} f(D)(-\frac{\sin 2x}{3}) &= (D^2+1)(-\frac{\sin 2x}{3}) \\ &= -\frac{1}{3}\frac{d^2 \sin 2x}{dx^2} - \frac{\sin 2x}{3} \\ &= \frac{4\sin 2x}{3} - -\frac{\sin 2x}{3} \\ &= \sin 2x \end{aligned}$$

Therefore, if $f(D) = D^2 + 1$ then $\dfrac{1}{f(D)}X = -\dfrac{\sin 2x}{3}$.

Thus, $\dfrac{1}{f(D)}X$ satisfies the differential equation

$$f(D)y = X$$

and is, therefore, its particular solution. Obviously, $\dfrac{1}{f(D)}$ and $f(D)$ are inverse operators.

Proposition 3.1. $\frac{1}{D}X = \int X dx$.

Proof. Let $\frac{1}{D}X = y$. Then, by definition

$$\begin{aligned}
& Dy = X \\
\Rightarrow\ & \frac{dy}{dx} = X \\
\Rightarrow\ & dy = X dx \\
\Rightarrow\ & \int dy = \int X dx \\
\Rightarrow\ & y = \int X dx \\
\Rightarrow\ & \frac{1}{D}X = \int X dx
\end{aligned}$$

□

Proposition 3.2. $\frac{1}{D-a}X = e^{ax}\int X \cdot e^{-ax} dx$.

Proof. Let $\frac{1}{D-a}X = y$. Then, by definition

$$\begin{aligned}
& (D-a)y = X \\
\Rightarrow\ & \frac{dy}{dx} - ay = X
\end{aligned}$$

which is a linear differential equation in x and y with $P = -a$ and $Q = X$. Its

$$IF = e^{\int P dx} = e^{-\int a dx} = e^{-ax}.$$

and its solution is

$$\begin{aligned}
& y \cdot (IF) = \int Q \cdot (IF) dx \\
or,\ & y \cdot e^{-ax} = \int X \cdot e^{-ax} dx \\
or,\ & y = e^{ax}\int X \cdot e^{-ax} dx \\
or,\ & \frac{1}{D-a}X = e^{ax}\int X \cdot e^{-ax} dx
\end{aligned}$$

□

Example 3.4. *Solve:* $(D^3 - 9D)y = e^x$ *where* $D \equiv \frac{d}{dx}$.

Solution: The given equation is

$$f(D)y = X$$

where $f(D) = D^3 - 9D$ and $X = e^x$. The AE is

$$f(m) = 0 \Rightarrow m^3 - 9m = 0 \Rightarrow m = 0, 3, -3.$$

Therefore,

$$CF = c_1 + c_2 e^{3x} + c_3 e^{-3x}.$$

Now

$$\begin{aligned}
PI &= \frac{1}{f(D)}X \\
&= \frac{1}{D^3 - 9D}e^x \\
&= \frac{1}{D^2 - 9}\frac{1}{D}e^x \\
&= \frac{1}{D^2 - 9}\int e^x dx \\
&= \frac{1}{(D+3)(D-3)}e^x \\
&= \frac{1}{6}\left[\frac{1}{D-3} - \frac{1}{D+3}\right]e^x \\
&= \frac{1}{6}\left[\frac{1}{D-3}e^x - \frac{1}{D+3}e^x\right] \\
&= \frac{1}{6}\left[e^{3x}\int e^x \cdot e^{-3x}dx - e^{-3x}\int e^x \cdot e^{3x}dx\right] \\
&= \frac{1}{6}\left[\frac{e^x}{-2} - \frac{e^x}{4}\right] \\
&= -\frac{1}{8}e^x
\end{aligned}$$

Hence the complete solution of the given equation is

$$\begin{aligned}
& y = CF + PI \\
i.e., \quad & y = c_1 + c_2e^{3x} + c_3e^{-3x} - \frac{1}{8}e^x
\end{aligned}$$

where c_1, c_2, c_3 arbitrary constants. ⋈

3.5 Rules for Finding PI.

In this section we will learn some shortcut ways to find the PI of the differential equation

$$f(D)y = X.$$

All these rules depends upon the function X. Recall that,

$$PI = \frac{1}{f(D)}X.$$

Rule A. When $X = e^{ax}$.

$$PI = \frac{1}{f(D)}X = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}, \quad \text{provided } f(a) \neq 0.$$

$$\text{If } f(a) = 0, \quad \text{then} \quad PI = x\left[\frac{1}{f'(D)}e^{ax}\right].$$

Example 3.5. *Solve:* $D^2(D^2 - 4)y = e^{3x}$ *where* $D \equiv \frac{d}{dx}$.

Solution: The given equation is

$$f(D)y = X$$

where $f(D) = D^2(D^2 - 4)$ and $X = e^{3x}$. Therefore,

$$CF = c_1 + c_2x + c_3e^{2x} + c_4e^{-2x}.$$

Since $f(3) = 45 \neq 0$, so

$$\begin{aligned} PI &= \frac{1}{f(D)}X \\ &= \frac{1}{f(D)}e^{3x} \\ &= \frac{1}{f(3)}e^{3x} \\ &= \frac{1}{45}e^{3x} \end{aligned}$$

Hence the complete solution of the given equation is

$$\begin{aligned} & y = CF + PI \\ i.e., \quad & y = c_1 + c_2x + c_3e^{2x} + c_4e^{-2x} + \frac{1}{45}e^{3x} \end{aligned}$$

where c_1, c_2, c_3, c_4 are arbitrary constants. $\bowtie$

Example 3.6. *Solve:* $D^2(D^2 - 4)y = e^{-2x}$ *where* $D \equiv \frac{d}{dx}$.

Solution: The given equation is

$$f(D)y = X$$

where $f(D) = D^2(D^2 - 4)$ and $X = e^{-2x}$. Therefore,

$$CF = c_1 + c_2x + c_3e^{2x} + c_4e^{-2x}.$$

Since $f(-2) = 0 \neq 0$ and $f'(D) = 4D(D^2 - 2)$, so

$$\begin{aligned} PI &= \frac{1}{f(D)}X \\ &= \frac{1}{f(D)}e^{-2x} \\ &= x\left[\frac{1}{f'(D)}e^{-2x}\right], \quad [\because f(-2) = 0] \\ &= x\left[\frac{1}{f'(-2)}e^{-2x}\right], \quad [\because f'(-2) = -16 \neq 0] \\ &= -\frac{1}{16}xe^{-2x} \end{aligned}$$

Hence the complete solution of the given equation is

$$\begin{aligned} & y = CF + PI \\ i.e., \quad & y = c_1 + c_2x + c_3e^{2x} + c_4e^{-2x} - \frac{1}{16}xe^{-2x} \end{aligned}$$

where c_1, c_2, c_3, c_4 are arbitrary constants. $\bowtie$

Rule B. When $X = \cos(ax + b)$ or $\sin(ax + b)$.
Let $f(D) = \phi(D^2, D)$.

$$\begin{aligned} \text{Then } PI &= \frac{1}{f(D)}X \\ &= \frac{1}{\phi(D^2, D)}\cos(ax + b) \\ &= \frac{1}{\phi(-a^2, D)}\cos(ax + b), \quad [\text{ provided } \phi(-a^2, D) \neq 0] \end{aligned}$$

If $\phi(-a^2, D) = 0$ then $PI = x\left[\frac{1}{f'(D)}\cos(ax+b)\right]$.

Example 3.7. *Solve:* $(D^3+1)y = \cos(2x+1)$ *where* $D \equiv \frac{d}{dx}$.

Solution: The given equation is

$$f(D)y = X$$

where $f(D) = D^3+1$ and $X = \cos(2x+1)$. Therefore,

$$CF = Ae^{-x} + e^{\frac{1}{2}x}\left(B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right)$$

Since $f(D) = D^3+1 = D^2 \cdot D + 1 = \phi(D^2, D)$ and $\phi(-2^2, D) = -4D+1 \neq 0$, so

$$\begin{aligned}
PI &= \frac{1}{f(D)}X \\
&= \frac{1}{\phi(D^2, D)}\cos(2x+1) \\
&= \frac{1}{\phi(-2^2, D)}\cos(2x+1), \quad [\because\ \phi(-2^2, D) = -4D+1 \neq 0] \\
&= \frac{1}{1-4D}\cos(2x+1), \\
&= (1+4D)\frac{1}{1-4^2D^2}\cos(2x+1), \\
&= (1+4D)\frac{1}{\phi(D^2, D)}\cos(2x+1), \quad [\textit{where } \phi(D^2, D) = 1-4^2D^2] \\
&= (1+4D)\frac{1}{\phi(-2^2, D)}\cos(2x+1), \quad [\because\ \phi(-2^2, D) = 65 \neq 0] \\
&= (1+4D)\frac{1}{65}\cos(2x+1) \\
&= \frac{1}{65}(\cos(2x+1) + 4D\cos(2x+1)) \\
&= \frac{1}{65}(\cos(2x+1) - 8\sin(2x+1))
\end{aligned}$$

Hence the complete solution of the given equation is

$$y = CF + PI$$

$$i.e., \quad y = Ae^{-x} + e^{\frac{1}{2}x}\left(B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{65}(\cos(2x+1) - 8\sin(2x+1))$$

where A, B, C are arbitrary constants. $\bowtie$

Note: When there is no confusion, the steps for finding the PI in the last example can also written as follows:

$$
\begin{aligned}
PI &= \frac{1}{f(D)}X \\
&= \frac{1}{D^2 \cdot D + 1}\cos(2x+1) \\
&= \frac{1}{-2^2 \cdot D + 1}\cos(2x+1) \\
&= \frac{1}{1-4D}\cos(2x+1), \\
&= (1+4D)\frac{1}{1-4^2D^2}\cos(2x+1), \\
&= (1+4D)\frac{1}{1-4^2(-2^2)}\cos(2x+1) \\
&= (1+4D)\frac{1}{65}\cos(2x+1) \\
&= \frac{1}{65}(\cos(2x+1) - 8\sin(2x+1))
\end{aligned}
$$

Example 3.8. *Find the PI of* $(D^3 + D)y = \sin x$ *where* $D \equiv \frac{d}{dx}$.

Solution: Here $f(D) = D^3 + D = D^2 \cdot D + D = \phi(D^2, D)$ and $\phi(-1^2, D) = 0$, so

$$
\begin{aligned}
PI &= \frac{1}{f(D)}X \\
&= x\left[\frac{1}{f'(D)}\sin x\right] \\
&= x\left[\frac{1}{3D^2+1}\sin x\right] \\
&= x\left[\frac{1}{3(-1^2)+1}\sin x\right] \\
&= -\frac{1}{2}x\sin x
\end{aligned}
$$

⋈

Rule C. When $X = x^m$. In this case,

$$PI = \frac{1}{f(D)}X = [f(D)]^{-1}x^m.$$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term containing D^m and operate on x^m term by term. Since $(m+1)th$ and the higher order derivatives of x^m are zero, we need not consider the terms beyond D^m.

Example 3.9. *Solve:* $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 2x^2 + x + 4.$

Solution: On writing $D \equiv \frac{d}{dx}$, the given equation reduces to

$$
\begin{aligned}
& (D^2 + 4D)y = 2x^2 + x + 4 \\
i.e., \quad & f(D)y = X
\end{aligned}
$$

where $f(D) = D(D+4)$ and $X = 2x^2 + x + 4$. The AE is

$$f(D) = 0 \Rightarrow m(m+4) = 0 \Rightarrow m = 0, -4.$$

Therefore,

$$CF = c_1 + c_2 e^{-4x}.$$

and

$$\begin{aligned}
PI &= \frac{1}{f(D)} X \\
&= \frac{1}{D(D+4)}(2x^2 + x + 4) \\
&= \frac{1}{4D}\left[1 + \frac{D}{4}\right]^{-1}(2x^2 + x + 4) \\
&= \frac{1}{4D}\left[1 + \frac{D}{4}\right]^{-1}(2x^2 + x + 4) \\
&= \frac{1}{4D}\left[1 - \frac{D}{4} + \frac{D^2}{4^2} \cdots\right](2x^2 + x + 4) \\
&= \frac{1}{4D}\left[(2x^2 + x + 4) - \frac{1}{4}(4x + 1) + \frac{1}{16} \cdot 4\right] \\
&= \frac{1}{4D}(2x^2 + 4) \\
&= \frac{1}{4}\int (2x^2 + 4)dx \\
&= \frac{1}{4}\left(\frac{2}{3}x^3 + 4x\right)
\end{aligned}$$

Hence the complete solution of the given equation is

$$\begin{aligned}
& y = CF + PI \\
i.e.,\quad & y = c_1 + c_2 e^{-4x} + \frac{1}{6}x^3 + x
\end{aligned}$$

where c_1, c_2 are arbitrary constants. $\bowtie$

Rule D. When $X = e^{ax}V$, V being a function of x. In this case,

$$PI = \frac{1}{f(D)} X = \frac{1}{f(D)} e^{ax} V = e^{ax}\left[\frac{1}{f(D+a)} V\right].$$

Example 3.10. *Find the PI of* $(D^2 - 2D + 4)y = e^x \sin x$ *where* $D \equiv \frac{d}{dx}$.

Solution: Here $f(D) = D^2 - 2D + 4$ and so

$$\begin{aligned}
PI &= \frac{1}{f(D)}X \\
&= \frac{1}{f(D)}e^x \sin x \\
&= e^x \left[\frac{1}{f(D+1)} \sin x\right] \\
&= e^x \left[\frac{1}{(D+1)^2 - 2(D+1) + 4} \sin x\right] \\
&= e^x \left[\frac{1}{D^2+3} \sin x\right] \\
&= e^x \left[\frac{1}{-1^2+3} \sin x\right] \\
&= \frac{1}{2}e^x \sin x
\end{aligned}$$

$\bowtie$

Rule E. When X is any other function. In this case, we will resolve $\frac{1}{f(D)}$ into partial fractions and then use the result

$$\frac{1}{D-a}X = e^{ax}\int X \cdot e^{-ax}dx.$$

Example 3.11. *Find the PI of* $(D^2+1)y = \tan x$ *where* $D \equiv \frac{d}{dx}$.

Solution: Here $f(D) = D^2 + 1$ and $X = \tan x$, so

$$\begin{aligned}
PI &= \frac{1}{f(D)}X \\
&= \frac{1}{D^2+1} \tan x \\
&= x \left[\frac{1}{(D+i)(D-i)} \sin x\right] \\
&= \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] \tan x \\
&= \frac{1}{2i}\left[e^{ix}\int \tan x \cdot e^{-ix}dx - e^{-ix}\int \tan x \cdot e^{ix}dx\right]
\end{aligned}$$

$\bowtie$

Some special rules:

(a). $\frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax.$

(b). $\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax.$

(c). $\frac{1}{f(D)}(x \cdot V) = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V.$

3.5.1 Exercise

Solve the following:

1. $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = \sin 2x,$
2. $\dfrac{d^2y}{dx^2} - y = \cos x + 2x^3.$
3. $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$
4. $\dfrac{d^2y}{dx^2} - 4y = x \sinh x$
5. $(D^2 - 1)y = x \sin 3x + \cos x$
6. $(D^4 + 2D^2 + 1)y = x^2 \cos x$
7. $(D^2 - 2D + 1)y = xe^x \sin x$
8. $(D^2 + 3D + 2)y = 4\cos^2 x$
9. $(D^3 + 2D^2 + D)y = e^{2x} + \sin 2x$
10. $(D^2 + a^2)y = \sin ax$
11. $(D-1)^2 y = xe^x \sin x$
12. $(D^2 - 2D + 1)y = x \sin x$
13. $(D^2 + 1)y = x \sin^2 x$
14. $(D^2 + 4)y = x \sin x$
15. $(D^2 - 1)y = \cosh x \cos x$
16. $(D^3 - 3D - 2)y = 540x^3 e^{-x}$
17. $(D^4 + D^2 + 1)y = e^{-\frac{x}{2}} \cos(\frac{x\sqrt{3}}{2})$

3.6 Cauchy's Homogeneous Linear Equations.

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \tag{30}$$

where $a_1, a_2, \cdots, a_n$ are constants and X is a function of x, is called a Cauchy's Homogeneous Equations of order n. These type of equation can reduced to linear equation with constant coefficients by using a suitable substitution. First, using $D \equiv \dfrac{d}{dx}$ the given equation can be written as

$$(x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots + a_{n-1} x D + a_n)y = X \tag{31}$$

Let us put

$$x = e^z, \quad \text{so that} \quad z = \log x$$

Then

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x}, \quad \text{or } x\frac{dy}{dx} = \frac{dy}{dz}, \quad \text{or } xDy = \mathcal{D}y$$

where $\mathcal{D} \equiv \dfrac{d}{dz}$. Thus $xD \equiv \mathcal{D}$. Now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}, = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$
$$\text{or } x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}, \quad \text{or } x^2D^2y = (\mathcal{D}^2 - \mathcal{D})y$$

Thus $x^2D^2 \equiv \mathcal{D}(\mathcal{D}-1)$. Continuing in this way, we can see that

$$x^3D^3 \equiv \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2), \quad x^4D^4 \equiv \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)(\mathcal{D}-3), etc.$$

Finally, the equation (31) reduces to

$$(\mathcal{D}(\mathcal{D}-1)\cdots(\mathcal{D}-(n-1)) + a_1\mathcal{D}(\mathcal{D}-1)\cdots(\mathcal{D}-(n-2)) + \cdots + a_{n-1}\mathcal{D} + a_n)y = Z \tag{32}$$

which is a linear equation on y and z.

Example 3.12. *Solve :* $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$

Solution: Let $D \equiv \dfrac{d}{dx}$. Then, the given equation can be written as

$$(x^3D^3 + 2x^2D + 2)y = 10\left(x + \frac{1}{x}\right).$$

Consider the transformation $x = e^z$ so that $z = \log x$. If we write $\mathcal{D} \equiv \dfrac{d}{dz}$ then

$$xD \equiv \mathcal{D}, \quad x^2D^2 \equiv \mathcal{D}(\mathcal{D}-1), \quad x^3D^3 \equiv \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)$$

Therefore, the above equation reduces to

$$\begin{aligned} & [\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2) + 2\mathcal{D}(\mathcal{D}-1) + 2]y = 10(e^z + e^{-z}) \\ \Rightarrow \quad & (\mathcal{D}^3 - \mathcal{D}^2 + 2)y = 10(e^z + e^{-z}) \end{aligned}$$

The AE is

$$\begin{aligned} & m^3 - m + 2 = 0 \\ \Rightarrow \quad & m = -1, 1 = i, 1 - i \\ \therefore \quad & CF = C_1e^{-z} + e^z(c_2\cos z + c_3\sin z) \end{aligned}$$

where C_1, C_2 are arbitrary constants. Also,

$$\begin{aligned} PI &= \frac{1}{\mathcal{D}^3 - \mathcal{D}^2 + 2}10(e^z + e^{-z}) \\ &= 10\left[\frac{1}{\mathcal{D}^3 - \mathcal{D}^2 + 2}e^z + \frac{1}{\mathcal{D}^3 - \mathcal{D}^2 + 2}e^{-z}\right] \\ &= 10\left[\frac{1}{2}e^z + z\frac{1}{3\mathcal{D}^2 - 2\mathcal{D}}e^{-z}\right] \\ &= 10\left[\frac{1}{2}e^z + z\frac{1}{5}e^{-z}\right] \\ &= 5e^z + 2ze^{-z} \end{aligned}$$

Therefore, the complete solution is given by

$$\begin{aligned} y &= CF + PI \\ i.e., \ y &= c_1e^{-z} + e^z(c_2\cos z + c_3\sin z) + 5e^z + 2ze^{-z} \\ i.e., \ y &= c_1\frac{1}{x} + x(c_2\cos(\log x)z + c_3\sin(\log x)) + 5x + 2\frac{\log x}{x} \end{aligned}$$

where c_1, c_2, c_3 are arbitrary constants. ⋈

3.6.1 Exercise

Solve the following:

1. $x^2\dfrac{d^2z}{dx^2} - 4x\dfrac{dz}{dx} + 6z = x^2 + \dfrac{1}{x}$.

2. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = (\log x)^2 + x$.

3. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x \sin(\log x)$.

4. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx}12 - y = x^3(\log x)$.

5. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = (\log x)$.

6. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = x^2\sin(\log x)$.

7. $(x^2D^2 - 4xD + 2)y = x^5$.

8. $(x^2D^2 - 2xD + 6)y = x^2 + \log x$.

9. $(1+2x)^2\dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2$.

10. $(1+x)^2\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + 16y = 4\cos(\log(1+x))$.

3.7 Solution by Method of Variation of Parameters.

This method provides a second way to find a particular integral to a nonhomogeneous equation $f(D)y = X$. In the previous section, calculation of particular integrals/solutions for some special cases have been studied. Recall that the homogeneous part of the equation had constant coefficients. In this section, we deal with a useful technique of finding a particular solution when the coefficients of the homogeneous part are continuous functions and the forcing function X (or the non-homogeneous term) is piecewise continuous.

Definition 3.5 (Wronskian). *Let y_1 and y_2 be two real valued continuously differentiable function on an interval $I \subset \mathbb{R}$. For $x \in I$, define*

$$W = det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1y_2' - y_1'y_2.$$

W is called the Wronskian of y_1 and y_2.

Remark 3.2. If $W(y_1, y_2) \neq 0$ then $\{y_1, y_2\}$ is a linearly independent set.

Example 3.13. *Let $y_1 = \cos x$ and $y_2 = \sin x$, $x \in I \subset \mathbb{R}$. Then*

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} \equiv -1 \quad \textit{for all} \quad x \in I.$$

Hence $\{\cos x, \sin x\}$ is a linearly independent set.

Suppose y_1 and y_2 are two linearly independent solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{33}$$

on I, where $p(x)$ and $q(x)$ are arbitrary continuous functions defined on I. Then we know that

$$y = c_1y_1 + c_2y_2$$

is a solution of (33) for any constants c_1 and c_2. We now "vary" c_1 and c_2 to functions of x, so that

$$y = u(x)y_1 + v(x)y_2, \;\; x \in I \tag{34}$$

is a solution of the equation

$$y'' + q(x)y' + r(x)y = X, \quad \text{on} \;\; I, \tag{35}$$

where X is a piecewise continuous function defined on I. The details are given in the following theorem.

Theorem 3.1 (Method of Variation of Parameters)**.** *Let $q(x)$ and $r(x)$ be continuous functions defined on I and let X be a piecewise continuous function on I. Let y_1 and y_2 be two linearly independent solutions of (33) on I. Then a particular integral of (35) is given by*

$$PI = -y_1 \int \frac{y_2 \cdot X}{W} dx + y_2 \int \frac{y_1 \cdot X}{W} dx, \tag{36}$$

where $W = W(y_1, y_2)$ is the Wronskian of y_1 and y_2. (Note that the integrals in (36) are the indefinite integrals of the respective arguments.)

Example 3.14. *Solve the differential equation*

$$\frac{d^2y}{dx^2} + 4y = \tan 2x$$

by the method variation of parameters.

Solution: On writing $D \equiv \frac{d}{dx}$, the given equation reduces to

$$(D^2 + 4)y = \tan 2x.$$

The Auxiliary Equation is

$$\begin{aligned} & m^2 + 4 = 0 \\ \Rightarrow \;\; & m = \pm 2i. \end{aligned}$$

Therefore, $CF = c_1 \cos 2x + c_2 \sin 2x$.

Here $X = \tan 2x$. Let $y_1 = \cos 2x$ and $y_2 = \sin 2x$. Then,

$$W = det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = 2 \neq 0.$$

and

$$\begin{aligned} PI &= -y_1 \int \frac{y_2 \cdot X}{W} dx + y_2 \int \frac{y_1 \cdot X}{W} dx \\ &= -\cos 2x \int \frac{\sin 2x \cdot \tan 2x}{2} dx + \sin 2x \int \frac{\cos 2x \cdot \tan 2x}{2} dx \\ &= -\cos 2x \frac{1}{4} \log(sec 2x + \tan 2x). \end{aligned}$$

Therefore, the required general solution is

$$y = CF + PI$$

$$i.e., \quad y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \log(sec 2x + \tan 2x),$$

where c_1 and c_2 are arbitrary constants.

3.7.1 Exercise.

Solve by the method of variation of parameters

1. $\dfrac{d^2z}{dx^2} + 4z = 4\sec^2 2x.$
2. $\dfrac{d^2y}{dt^2} - y = \dfrac{2}{1+e^t}.$
3. $y'' + y = 2\sec x$ for all $x \in (0, \frac{\pi}{2})$.
4. $\dfrac{d^2y}{dx^2} + y = cosec\ x.$
5. $y'' - 2y' + 2y = e^x \tan x.$
6. $\dfrac{d^2y}{dx^2} + 4y = \tan 2x.$
7. $y'' + 2y' + y = 4e^{-x}\log x.$

4 Simultaneous Differential Equations.

Example 4.1. *Solve the simultaneous equations :*

$$\frac{dx}{dt} + y = \sin t, \quad \frac{dy}{dt} + x = \cos t,$$

given that $x = 0,\ y = 1$ *when* $t = 0$.

Solution: Let $D \equiv \dfrac{d}{dt}$. Then, the given equations can be written as

$$Dx + y = \sin t \tag{37}$$
$$x + Dy = \cos t \tag{38}$$

Now, operating equation (5) by D we get

$$D^2x + Dy = \cos t \tag{39}$$

Subtracting equation (38) from the equation (39), we get

$$D^2x - x = 0 \Rightarrow (D^2 - 1)x = 0 \tag{40}$$

The AE is

$$\begin{aligned} & m^2 - 1 = 0 \\ \Rightarrow\ & m = 1, -1 \end{aligned}$$

The general solution of (40) is

$$x = c_1e^t + c_2e^{-t}$$

where c_1, c_2 are arbitrary constants. Also, using the equation (5) we get

$$y = \sin t - c_1e^t + c_2e^{-t}$$

where c_1, c_2 are arbitrary constants.

Now, given that $x = 0,\ y = 1$ when $t = 0$. Therefore, $c_1 = -\dfrac{1}{2}$ and $c_2 = \dfrac{1}{2}$. Hence, the required particular solution is

$$x = \frac{1}{2}(e^{-t} - e^t), \quad y = \sin t + \frac{1}{2}(e^t + e^{-t})$$
$$i.e., \quad x = -\sinh t, \quad y = \sin t + \cosh t$$

4.0.1 Exercise.

Solve the following:

1. $\dfrac{dx}{dt} + 4x + 3y = t, \quad \dfrac{dy}{dt} + 2x + 5y = e^t,$
2. $\dfrac{dx}{dt} + 5x - 2y = t, \quad \dfrac{dy}{dt} + 2x + y = \cos t$, given that $x = y = 0$ when $t = 0$.
3. $\dfrac{dx}{dt} + y = \sin t, \quad \dfrac{dy}{dt} + x = \cos t$, given that $x = 2,\ y = 0$ when $t = 0$.
4. $\dfrac{dx}{dt} + 2x + 3y = 0, \quad \dfrac{dy}{dt} + 3x + 2y = 2e^{2t}.$

5 Application of differential equations.

1. [**Meteorology**] The barometric pressure y (in inches of mercury) at an altitude of x miles above sea level decreases at a rate 0.2 times the current pressure. Given that, $y = 29.92$ inches when $x = 0$. Find the barometric pressure (a) at the top of Mt. St. Helens (8364 feet) and (b) at the top of Mt. McKinley (20,320 feet).

2. **(Investment)**A large corporation starts at time $t = 0$ to invest part of its receipts at a rate of P dollars per year in a fund for future corporate expansion. Assume that the fund earns r percent interest per year compounded continuously. So, the rate of growth of the amount A in the fund is given by
$$\frac{dA}{dt} = rA + P,$$
where $A = 0$ when $t = 0$. Solve this differential equation for A as a function of t.

3. [**Population Growth**] When predicting population growth, demographers must consider birth and death rates as well as the net change caused by the difference between the rates of immigration and emigration. Let P be the population at time t and let N be the net increase per unit time due to the difference between immigration and emigration. So, the rate of growth of the population is given by
$$\frac{dP}{dt} = kP + N, \quad N \text{ is constant.}$$
Solve this differential equation to find P as a function of time.

4. [**Sales Growth**] The rate of change in sales S (in thousands of units) of a new product is proportional to the difference between L and S (in thousands of units) at any time t. When $t = 0$, $S = 0$ .Write and solve the differential equation for this sales model. Also write S as a function of t if $L = 500$, $S = 55$ when $t = 0$.

5. [**Population Growth**] The rate of change of the population of a city is proportional to the population P at any time t. In 1998, the population was 400,000, and the constant of proportionality was 0.015. Estimate the population of the city in the year 2005.

6. [**Modeling a Chemical Reaction**] During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of A. When 60 grams of A are present, and after 1 hour only 10 grams of A remain unconverted. How much of A is present after 2 hours?

Sol. Let y be the unconverted amount of substance A at any time t. From the given assumption about the conversion rate, you can write the differential equation as follows.
$$\frac{dy}{dt} = ky^2.$$

7. **[Electrical Circuits (LR circuit)]** It contains an EMF (supplied by a battery or generator) source E, a resistor R and an inductor L in series. If the charge on the capacitor at time t is $q = q(t)$, then the current i is the rate of change of q with respect to t : $i = \frac{dq}{dt}$. It is known from physics that the voltage drops across the resistor, and inductor are

$$Ri, \quad L\frac{di}{dt}$$

respectively. Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage:

$$L\frac{di}{dt} + Ri = E(t)$$

which is a first-order linear differential equation with constant coefficients. If the current i_0 are known at time $t = 0$, then we have the initial conditions

$$q'(0) = i(0) = i_0$$

and the initial-value problem can be solved by the method discussed in the class.

8. A simple electric circuit contains a resistance of $10 ohms$ and an inductance of of 4 henries in series with an induced EMF of $100 \sin 200t$ volts. If the current $i = 0$ when $t = 0$, find the current when $t = 0.01$.

9. An electrical circuit contains a resistance $R = 100\ \Omega$ and an inductance $L = 0.05\ H$ in series with an induced EMF $E(t) = 200 \cos 300t\ V$. Find the current i at time t, given that $i = 0$ when $t = 0$.

10. **[Electrical Circuits (LCR circuit)]** It contains an electromotive force (supplied by a battery or generator), a resistor, an inductor, and a capacitor, in series. If the charge on the capacitor at time t is $q = q(t)$, then the current i is the rate of change of q with respect to t : $i = \frac{dq}{dt}$. It is known from physics that the voltage drops across the resistor, inductor, and capacitor are

$$Ri, \quad L\frac{di}{dt}, \quad \frac{q}{C}$$

respectively. Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage:

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E(t).$$

Since $i = \frac{dq}{dt}$, this equation can be written as

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t).$$

which is a second-order linear differential equation with constant coefficients. If the charge q_0 and the current i_0 are known at time $t = 0$, then we have the initial conditions

$$q(0) = q_0, \quad q'(0) = i(0) = i_0$$

and the initial-value problem can be solved by the method discuss in the class.

11. Find the charge and current at time t in a LCR circuit if $R = 40\Omega$, $L = 1H$, $C = 16 \times 10^{-4}F, E(t) = 100 \cos 10t$, and the initial charge and current are both 0.

12. A series circuit consists of a $R = 20\Omega$, $L = 1H$, $C = 0.002F$ and a 12V battery. If the initial charge and current are both 0, find the charge and current at time t.

13. **[Vibrating Springs]** We consider the motion of an object with mass at the end of a spring that is either vertical or horizontal on a level surface.

    Hooke's Law : It states that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x i.e.,

    $$restoring \ \ force = -kx$$

    where k is a positive constant (called the spring constant).

    If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

    $$m\frac{d^2x}{dt^2} = -kx.$$

    This is a second-order linear differential equation and it can be solved by the method discuss in the class.

14. A spring with a mass of 2 kg has natural length $0.5m$. A force of $25.6N$ is required to maintain it stretched to a length of $0.7m$. If the spring is stretched to a length of $0.7m$ and then released with initial velocity 0, find the position of the mass at any time t.

    Solution: From Hooke's Law, the force required to stretch the spring is

    $$k(0.2) = 25.6$$

    so $k = 128$. The differential equation governing the motion of the spring is given by

    $$2\frac{d^2x}{dt^2} = -128x.$$

    $$i.e., 2\frac{d^2x}{dt^2} + 128x = 0.$$

    The general solution of this equation is

    $$x = c_1 \cos 8t + c_2 \sin 8t. \tag{41}$$

    We are given the initial condition that $x(0) = 0.2$. But, from Equation 41, $x(0) = c_1$ Therefore, $c_1 = 0.2$. Differentiating Equation 41, we get

    $$x' = -8c_1 \sin 8t + 8c_2 \cos 8t. \tag{42}$$

    Since the initial velocity is given as $x'(0) = 0$, we have $c_2 = 0$ and so the solution is

    $$x = 0.2 \cos 8t. \tag{43}$$

    $\bowtie$

15. A spring with a 3-kg mass is held stretched $0.6m$ beyond its natural length by a force of $20N$. If the spring begins at its equilibrium position but a push gives it an initial velocity of $1.2m/s$, find the position of the mass after t seconds.

16. A spring with a 4-kg mass has natural length $1m$ and is maintained stretched to a length of $1.3.m$ by a force of $24.3N$. If the spring is compressed to a length of $0.8m$ and then released with zero velocity, find the position of the mass at any time t.

17. Solve :

(a) $y'' + 4y' = 12t$, given that $y(0) = 0,\ y'(0) = 7$.

(b) $y''(t) - 3y'(t) + 2y(t) = 7t + e^{-t}$, given that $y(0) = 6,\ y'(0) = -1$.

(c) $x''' - x = 3e^t$, given that $y(0) = y'(0) = y''(0) = 0$.

(d) $x' + 2y'' = e^{-t}$, $x' + 2x - y = 1$, given that $x(0) = y(0) = y'(0) = 0$.

18. In an RC circuit (a resistor and a capacitor in series) the applied emf is a constant E. Given that $\dfrac{dq}{dt} = i$, where q is the charge in the capacitor, i the current in the circuit, R the resistance and C the capacitance. The equation for the circuit is

$$Ri + \frac{q}{C} = E.$$

If the initial charge is zero find the charge subsequently.

19. If the voltage in the RC circuit is $E = E_0 \cos \omega t$ find the charge and the current at time t.

20. An object is projected from the Earth's surface. What is the least velocity (the escape velocity) of projection in order to escape the gravitational field, ignoring air resistance. The equation of motion is

$$mv\frac{dv}{dx} = -mg\frac{R^2}{x^2}$$

where the mass of the object is m, its distance from the center of the Earth is x and the radius of the Earth is R.

21. The radial stress p at distance r for the axis of the thick cylinder subjected to internal pressure is given by $p + r\dfrac{dp}{dr} = A - p$, where A is a constant. If $p = p_0$ at the inner wall $r = r_1$ and is negligible ($p = 0$) at the outer wall $r = r_2$ find an expression for p.

22. The equation for an LCR circuit with applied voltage E is

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E.$$

By differentiating this equation find the solution for $q(t)$ and $i(t)$ if $L = 1$, $R = 100$, $C = 10^{-4}$ and $E = 1000$ given that $q = 0$ and $i = 0$ at $t = 0$.

23. Consider the free vibration problem when $m = 1$, $n = 1$ and $k = 2$. (Critical damping) Find the solution for $x(t)$.

24. Repeat problem 23 for the case $m = 1$, $n = 1$ and $k = 1.5$ (light damping)

25. Consider the forced vibration problem with $m = 1$, $n = 25$, $k = 8$, $E = \sin 3t$, $x0 = 0$ with an initial velocity of 3.

Formation of Ordinary Differential Equations.

1. Form the differential equation for the family of curves $y = ae^{2x} + be^{-3x}$, where a, b are arbitrary constants.
2. Form the differential equation of
$$v = \frac{\alpha}{r} + \beta,$$
where α, β are arbitrary constants.
3. Find the differential equation for the following family of functions;
   (a) $y = A\cos x + B\sin x$.
   (b) $r = a + b\cos\theta$.
4. Prove that the differential equation of the family of circles touching the x-axis at the origin is $(x^2 - y^2)dy - 2xydx = 0$.
5. Show that the differential equation of the family of parabolas having their axes parallel to the y-axis is $y_3 = 0$.

Separation of Variables method.

1. Solve: $xdy + ydx = 0$
2. Solve: $4x^3ydx + 2(x - xy^2)dy = 0$
3. Solve: $(1 + y^2)dx + (1 + x^2)dy = 0$.
4. Solve: $\sin 2ydx + (1 + 2e^{-x})\cos 2ydy = 0$ given that $y(0) = \frac{\pi}{4}$.
5. Solve: $\frac{dy}{dx} = \cos(x + y)$.

Homogeneous Equations.

1. Solve: $(x^2 - y^2)dx = 2xydy$.
2. Solve: $\frac{dy}{dx} = \frac{y(y + x)}{x(y - x)}$.
3. Solve: $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$
4. Solve: $(x - y - 4)dx = (x + y + 2)dy$.
5. Obtain the general solutions of the following: $\frac{dy}{dx} = \frac{x - y + 2}{-x + y + 2}$.

Exact Equations.

1. Solve: $3x^2ydx + (x^3 + y^{100})dy = 0$.
2. Solve: $(x^2 + y^2 - a^2)xdx + (x^2 - y^2 - b^2)dy = 0$.
3. Solve: $2xe^y + (x^2e^y + \cos y)\frac{dy}{dx} = 0$.
4. Solve: $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$.
5. Solve: $(2x^3 + 4y)dx + (4x + y - 1)dy = 0$, given that $y = 1$ when $x = 0$.

Equations Reducible to Exact Equations.

1. Solve: $(x^3 + y^3)dx - xy^2 dy = 0$.
2. Solve: $(y + x)dx + xdy = 0$.
3. Solve: $y(1 + xy)dx + x(1 - xy)dy = 0$.
4. Solve: $\dfrac{dy}{dx} = \dfrac{x^3 + y^3}{xy^2}$.
5. Solve: $(x + y)dx + (y - x)dy = 0$.
6. Solve: $(xy^2 + 3e^{x^{-3}})dx - x^2 y dy = 0$.
7. Find the particular solution of $x\cos(y/x)(ydx + xdy) = y\sin(y/x)(xdy - ydx)$ given that $y = 1$ when $x = \dfrac{4}{\pi}$.

Linear Equations.

1. Find the particular solution of $\dfrac{dy}{dx} + y\ \cot x = 4x\ \operatorname{cosec} x$ given that $y = 0$ when $x = \dfrac{\pi}{2}$.
2. Solve: $\dfrac{dy}{dz} - 2\dfrac{y}{z} = z^3 e^z$.
3. Solve the following:
   (a) $x\dfrac{dy}{dx} + y = x^3 y^6$.
   (b) $r\sin\theta - \dfrac{dr}{d\theta}\cos\theta = r^2$.
   (c) $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$.
   (d) $\sec^2 y\dfrac{dy}{dx} + x\tan y = x^3$.

Homogeneous Linear Differential Equations of Higher Order.

1. Which of the following are linearly dependent sets:
   (a) $\{1, 1 + x, 1 + x^2, 1 + x^3\}$.
   (b) $\{2, 1 + e^x, e^{2x}, 3e^x + 2e^{2x}\}$.
2. Solve: $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0$.
3. Solve: $(5D^3 + +D^2 + 2)y = 0$, where $D \equiv \dfrac{d}{dx}$.
4. Solve: $(D^3 + 27)y = 0$, where $D \equiv \dfrac{d}{dx}$.
5. Solve: $(D - 2)^3(D^2 + 1)y = 0$, where $D \equiv \dfrac{d}{dx}$.
6. Solve: $(D + 12)(D^2 + 4)^2 y = 0$, where $D \equiv \dfrac{d}{dx}$.

Non Homogeneous Linear Differential Equations of Higher Order.

1. Solve: $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = e^{2x}$.
2. Solve: $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = e^{-2x}$.
3. Solve: $(D^3 + 9D)y = \cos 3x$, where $D \equiv \dfrac{d}{dx}$.
4. Solve: $(D^3 + 9D)y = \sin 2x$, where $D \equiv \dfrac{d}{dx}$.
5. Solve: $(D - 2)^3(D^2 + 1)y = e^{2x}$, where $D \equiv \dfrac{d}{dx}$.
6. Solve: $(D + 2)(D - 3)y = 5x^3$, where $D \equiv \dfrac{d}{dx}$.
7. $\dfrac{d^2y}{dx^2} - y = \cos x + 2x^3$.
8. $(D - 2)^2 y = 8(e^{3x} + \sin 2x + x^2)$
9. $\dfrac{d^2y}{dx^2} - 4y = x \sinh x$
10. $(D^4 + 2D^2 + 1)y = x^2 \cos x$
11. $(D^2 - 2D + 1)y = xe^x \sin x$
12. $(D^2 + 3D + 2)y = 4\cos^2 x$
13. $(D^2 - 1)y = \cosh x \cos x$
14. $(D^3 - 3D - 2)y = 540x^3 e^{-x}$

Cauchy's Homogeneous Linear Equations.

1. Solve: $x^2\dfrac{d^2z}{dx^2} - 4x\dfrac{dz}{dx} + 6z = x^2 + \dfrac{1}{x}$.
2. Solve: $(x^2D^2 - 2xD + 6)y = x^2 + \log x$.
3. Solve: $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x \sin(\log x)$.
4. Solve: $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = x^2 \sin(\log x)$.
5. Solve: $(1 + x)^2\dfrac{d^2y}{dx^2} + (1 + x)\dfrac{dy}{dx} + 16y = 4\cos(\log(1 + x))$.

Solution by Method of Variation of Parameters.

1. Solve: $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$.
2. Solve: $\dfrac{d^2y}{dt^2} - y = \dfrac{2}{1 + e^t}$.
3. Solve: $y'' + y = 2\sec x$ for all $x \in (0, \frac{\pi}{2})$.
4. Solve: $y'' - 2y' + 2y = e^x \tan x$.

Simultaneous Differential Equations.

1. Solve: $\dfrac{dx}{dt} + 4x + 3y = t, \quad \dfrac{dy}{dt} + 2x + 5y = e^t,$
2. Solve: $\dfrac{dx}{dt} + y = \sin t, \quad \dfrac{dy}{dt} + x = \cos t$, given that $x = 2,\ y = 0$ when $t = 0$.

Application of differential equations.

1. [**Population Growth**] The rate of change of the population of a city is proportional to the population P at any time t. In 1998, the population was 400,000, and the constant of proportionality was 0.015. Estimate the population of the city in the year 2005.
2. [**Sales Growth**] The rate of change in sales S (in thousands of units) of a new product is proportional to the difference between L and S (in thousands of units) at any time t. Write and solve the differential equation for this sales model. Also write S as a function of t if $L = 500,\ S = 55$ when $t = 0$.
3. [**Modeling a Chemical Reaction**] During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of A. When 60 grams of A are present, and after 1 hour only 10 grams of A remain unconverted. How much of A is present after 2 hours?
4. A spring with a mass of 2 kg has natural length $0.5m$. A force of $25.6N$ is required to maintain it stretched to a length of $0.7m$. If the spring is stretched to a length of $0.7m$ and then released with initial velocity 0, find the position of the mass at any time t.
5. A simple electric circuit contains a resistance of 10 ohms and an inductance of of 4 henries in series with an induced EMF of $100 \sin 200t$ volts. If the current $i = 0$ when $t = 0$, find the current when $t = 0.01$.