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The absorption laws for the generalized inverses ☆

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ABSTRACT

In this paper, we give necessary and sufficient conditions for the absorption laws in terms of $\{1\},\{1,2\},\{1,3\}$ and $\{1,4\}$-inverses. Also, we consider the various types of mixed absorption law for the generalized inverses.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For a given $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively.

Recall that $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ has the Moore–Penrose inverse if there exists an operator $X\in\mathcal{B}(\mathcal{K},\mathcal{H})$ such that

$$(1)\ AXA=A\quad (2)\ XAX=X\quad (3)\ (AX)^*=AX\quad (4)\ (XA)^*=XA. \tag{1.1}$$

The Moore–Penrose inverse of an operator $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ exists if and only if A has closed range and in this case it is unique. It is denoted by $A^\dagger$.

If $\eta\subseteq\{1,2,3,4\}$ is arbitrary we shall say that $B\in\mathcal{B}(\mathcal{K},\mathcal{H})$ is a *η-inverse* of $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ if B satisfies the Penrose equation (j) for each $j\in\eta$. We shall write $A\eta$ for the collection of all η-inverses of A, and A^η for an unspecified element $X\in A\eta$. Evidently, $A\{1,2,3,4\}=\{A^\dagger\}$, when A has closed range.

The various types of generalized inverse have been widely used in practise. The reverse order law for generalized inverse plays an important role in theoretical research and numerical computations in many areas, including the singular matrix problem, ill-posed problems, optimization problems, and statics problems (see for instance [1,15,5,11,12,14,16]). These problems have attracted considerable attention since the middle 1960s, and many interesting results for generalized inverses of products of matrices or operators have been obtained (see [3,4,7,8,13,15]). Greville [6] first proved that $(AB)^\dagger=B^\dagger A^\dagger$ if and only if $\mathcal{R}(A^*AB)\subseteq\mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*)\subseteq\mathcal{R}(A^*)$, for matrices A and B. This result was extended to linear bounded operators on Hilbert spaces in [8]. Later, the reverse order law for the Moore–Penrose inverse was considered in rings with involution (see [9]).

The "absorption law" for the two-sided inverse, in a ring A or additive category, says that if $a,b\in A$ are both invertible then

$$a^{-1}+b^{-1}=a^{-1}(a+b)b^{-1}$$

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and it is a natural question whether this has extension to generalized inverses. The question certainly makes sense for specific inverses such as the Moore–Penrose or the Drazin inverse, although can also be formulated for arbitrary generalized inverses.

Let $A, B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. If for every $E \in A\eta$ and every $F \in B\eta$,

$$E + F = E(A + B)F, \tag{1.2}$$

then we say that the absorption law for η-inverses of A and B is satisfied. Moreover, if (1.2) holds for every $E \in A\eta$ and every $F \in B\mu$, then we say that the mixed absorption law for A and B is satisfied.

The absorption laws for matrices were studied by Chen et al. [2] and Lin and Gao [10]. In [2], the maximal and the minimal ranks of $G + H - G(A + B)H$ are described using the rank of the generalized Schur complement in the case when G and H are generalized inverse of A and B, respectively. Based on that, the rank conditions for the absorption laws for $\{1,3\}$-inverses and $\{1,4\}$-inverses were obtained. In [10], the authors considered the mixed first and second absorption laws for $\{1,2\}$ and $\{1,3\}$-inverses using the matrix rank method, the generalized Schur complement and singular value decompositions. In this paper, we will give the equivalent conditions for the absorption laws for the various generalized inverses of operators on Hilbert spaces. Also, we consider the mixed absorption laws.

First, we will state some auxiliary lemmas:

Lemma 1.1 [2]. *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has closed range. Then*

$$A\{1\} = \{A^\dagger + X - A^\dagger AXAA^\dagger : X \in \mathcal{B}(\mathcal{K}, \mathcal{H})\}.$$

Lemma 1.2. *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has closed range. Then*

$$A\{1,2\} = \left\{ \left(A^\dagger + (I - A^\dagger A)X\right)A\left(A^\dagger + Y(I - AA^\dagger)\right) : X, Y \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \right\}.$$

Lemma 1.3 [2]. *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has closed range and let $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Then the following statements are equivalent:*

(a) $ABA = A$ and $(AB)^* = AB$;
(b) $B = A^\dagger + (I - A^\dagger A)X$, for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

Lemma 1.4 [2]. *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has closed range and let $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Then the following statements are equivalent:*

(a) $ABA = A$ and $(BA)^* = BA$;
(b) $B = A^\dagger + Y(I - AA^\dagger)$, for some $Y \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

Lemma 1.5 [3]. *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has closed range. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be closed and mutually orthogonal subspaces of $\mathcal{H}$, such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and let $\mathcal{K}_1$ and $\mathcal{K}_2$ be closed and mutually orthogonal subspaces of $\mathcal{K}$, such that $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. Then the operator A has the following matrix representations with respect to the orthogonal sums of subspaces $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 = R(A^*) \oplus N(A)$ and $\mathcal{K} = R(A) \oplus N(A^*) = \mathcal{K}_1 \oplus \mathcal{K}_2$:*

(a)

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{pmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$: $\mathcal{R}(A) \to \mathcal{R}(A)$ and $D > 0$. Also,

$$A^\dagger = \begin{pmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{pmatrix}.$$

(b)

$$A = \begin{pmatrix} A_1 & 0 \\ A_2 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{pmatrix} \to \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{pmatrix},$$

where $E = A_1^*A_1 + A_2^*A_2$: $\mathcal{R}(A^*) \to \mathcal{R}(A^*)$ and $E > 0$. Also,

$$A^\dagger = \begin{pmatrix} E^{-1}A_1^* & E^{-1}A_2^* \\ 0 & 0 \end{pmatrix}.$$

Here A_i denotes different operators in any of these two cases.

2. The absorption laws for Moore–Penrose inverse, $\{1\}, \{1,2\}, \{1,3\}$ and $\{1,4\}$-inverses

In this section we will present necessary and sufficient conditions for the absorption laws of the Moore–Penrose inverse, $\{1\}, \{1,2\}, \{1,3\}$ and $\{1,4\}$-inverses.

Throughout the paper we will assume that A and B are given by

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{pmatrix}, \tag{2.1}$$

and

$$B = \begin{pmatrix} B_1 & 0 \\ B_2 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{pmatrix}. \tag{2.2}$$

Also,

$$A^\dagger = \begin{pmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{pmatrix} \quad \text{and} \quad B^\dagger = \begin{pmatrix} E^{-1}B_1^* & E^{-1}B_2^* \\ 0 & 0 \end{pmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ and $E = B_1^*B_1 + B_2^*B_2$.

In the following theorem, we proved that $A^\dagger + B^\dagger = A^\dagger(A+B)B^\dagger$ if and only if $BA^\dagger A = B$ and $BB^\dagger A = A$, i.e., $R(B^*) \subseteq R(A^*)$ and $R(A) \subseteq R(B)$.

Theorem 2.1. *Let $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be such that $R(A), R(B)$ are closed. The following conditions are equivalent:*

(i) $A^\dagger + B^\dagger = A^\dagger(A+B)B^\dagger$,
(ii) $R(B^*) \subseteq R(A^*)$ and $R(A) \subseteq R(B)$.

Proof. $(i) \Rightarrow (ii)$: From $A^\dagger(A+B)B^\dagger = A^\dagger + B^\dagger$, we get that the following four equalities hold:

$$A_1^*D^{-1}A_1E^{-1}B_1^* + A_1^*D^{-1}B_1E^{-1}B_1^* = A_1^*D^{-1} + E^{-1}B_1^*, \tag{2.3}$$
$$A_1^*D^{-1}A_1E^{-1}B_2^* + A_1^*D^{-1}B_1E^{-1}B_2^* = E^{-1}B_2^*, \tag{2.4}$$
$$A_2^*D^{-1}A_1E^{-1}B_1^* + A_2^*D^{-1}B_1E^{-1}B_1^* = A_2^*D^{-1}, \tag{2.5}$$
$$A_2^*D^{-1}A_1E^{-1}B_2^* + A_2^*D^{-1}B_1E^{-1}B_2^* = 0. \tag{2.6}$$

Now, from (2.3) $\times B_1$ + (2.4) $\times B_2$, we get

$$A_1^*D^{-1}A_1 + A_1^*D^{-1}B_1 = A_1^*D^{-1}B_1 + I, \quad \text{i.e., } A_1^*D^{-1}A_1 = I.$$

From (2.5) $\times B_1$ + (2.6) $\times B_2$, we get

$$A_2^*D^{-1}A_1 + A_2^*D^{-1}B_1 = A_2^*D^{-1}B_1, \quad \text{i.e., } A_2^*D^{-1}A_1 = 0.$$

From $A_1 \times$ (2.3) + $A_2 \times$ (2.5), we get

$$A_1E^{-1}B_1^* + B_1E^{-1}B_1^* = I + A_1E^{-1}B_1^*, \quad \text{i.e., } B_1E^{-1}B_1^* = I.$$

From $A_1 \times$ (2.4) + $A_2 \times$ (2.6), we get

$$A_1E^{-1}B_2^* + B_1E^{-1}B_2^* = A_1E^{-1}B_1^*, \quad \text{i.e., } B_1E^{-1}B_2^* = 0.$$

Now, it is easy to check that $B^\dagger BA^\dagger A = B^\dagger B$ and $AA^\dagger BB^\dagger = AA^\dagger$ which is equivalent to $BA^\dagger A = B$ and $BB^\dagger A = A$.

$(ii) \Rightarrow (i)$: Suppose that $R(B^*) \subseteq R(A^*)$ and $R(A) \subseteq R(B)$. Then, $B^\dagger BA^\dagger A = B^\dagger B$ and $AA^\dagger BB^\dagger = AA^\dagger$, which imply that $A_1^*D^{-1}A_1 = I, A_2^*D^{-1}A_1 = 0, B_1E^{-1}B_1^* = I$ and $B_1E^{-1}B_2^* = 0$. Now, by computation we get that (i) holds. $\square$

Now, we consider the absorption law for $\{1\}$-inverses:

Theorem 2.2. *Let $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be such that $R(A), R(B)$ are closed. The following conditions are equivalent:*

(i) $A^{(1)} + B^{(1)} = A^{(1)}(A+B)B^{(1)}$, *for any* $A^{(1)} \in A\{1\}$ *and any* $B^{(1)} \in B\{1\}$,
(ii) $A^{\dagger}A = I, BB^{\dagger} = I$.

Proof. By Lemma 1.1, arbitrary $A^{(1)}$ and $B^{(1)}$ have the following matrix forms:

$$A^{(1)} = A^{\dagger} + X - A^{\dagger}AXAA^{\dagger} = \begin{pmatrix} A_1^*D^{-1} + X_{11} - A_1^*D^{-1}A_1X_{11} - A_1^*D^{-1}A_2X_{21} & X_{12} \\ A_2^*D^{-1} + X_{21} - A_2^*D^{-1}A_1X_{11} - A_2^*D^{-1}A_2X_{21} & X_{22} \end{pmatrix}$$

and

$$B^{(1)} = B^{\dagger} + Y - B^{\dagger}BYBB^{\dagger} = \begin{pmatrix} E^{-1}B_1^* + Y_{11} - (Y_{11}B_1 + Y_{12}B_2)E^{-1}B_1^* & E^{-1}B_2^* + Y_{12} - (Y_{11}B_1 + Y_{12}B_2)E^{-1}B_2^* \\ Y_{21} & Y_{22} \end{pmatrix}.$$

A simple computation shows

$$A^{(1)}(A+B)B^{(1)} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},$$

where

$$\begin{aligned} G_{11} = {}& \left(A_1^*D^{-1} + X_{11} - A_1^*D^{-1}A_1X_{11} - A_1^*D^{-1}A_2X_{21}\right)(A_1+B_1)\left(E^{-1}B_1^* + Y_{11} - Y_{11}B_1E^{-1}B_1^* - Y_{12}B_2E^{-1}B_1^*\right) \\ & + X_{12}B_2\left(E^{-1}B_1^* + Y_{11} - Y_{11}B_1E^{-1}B_1^* - Y_{12}B_2E^{-1}B_1^*\right) + \left(A_1^*D^{-1} + X_{11} - A_1^*D^{-1}A_1X_{11} - A_1^*D^{-1}A_2X_{12}\right)A_2Y_{21}, \end{aligned}$$

$$\begin{aligned} G_{12} = {}& \left(A_1^*D^{-1} + X_{11} - A_1^*D^{-1}A_1X_{11} - A_1^*D^{-1}A_2X_{21}\right)(A_1+B_1)\left(E^{-1}B_2^* + Y_{12} - Y_{11}B_1E^{-1}B_2^* - Y_{12}B_2E^{-1}B_2^*\right) \\ & + X_{12}B_2\left(E^{-1}B_2^* + Y_{12} - Y_{11}B_1E^{-1}B_2^* - Y_{12}B_2E^{-1}B_2^*\right) + \left(A_1^*D^{-1} + X_{11} - A_1^*D^{-1}A_1X_{11} - A_1^*D^{-1}A_2X_{12}\right)A_2Y_{22}, \end{aligned}$$

$$\begin{aligned} G_{21} = {}& \left(A_2^*D^{-1} + X_{21} - A_2^*D^{-1}A_1X_{11} - A_2^*D^{-1}A_2X_{21}\right)(A_1+B_1)\left(E^{-1}B_1^* + Y_{11} - Y_{11}B_1E^{-1}B_1^* - Y_{12}B_2E^{-1}B_1^*\right) \\ & + X_{22}B_2\left(E^{-1}B_1^* + Y_{11} - Y_{11}B_1E^{-1}B_1^* - Y_{12}B_2E^{-1}B_1^*\right) + \left(A_2^*D^{-1} + X_{21} - A_2^*D^{-1}A_1X_{11} - A_2^*D^{-1}A_2X_{21}\right)A_2Y_{21}, \end{aligned}$$

$$\begin{aligned} G_{22} = {}& \left(A_2^*D^{-1} + X_{21} - A_2^*D^{-1}A_1X_{11} - A_2^*D^{-1}A_2X_{21}\right)(A_1+B_1)\left(E^{-1}B_2^* + Y_{12} - Y_{11}B_1E^{-1}B_2^* - Y_{12}B_2E^{-1}B_2^*\right) \\ & + X_{22}B_2\left(E^{-1}B_2^* + Y_{12} - Y_{11}B_1E^{-1}B_2^* - Y_{12}B_2E^{-1}B_2^*\right) + \left(A_2^*D^{-1} + X_{21} - A_2^*D^{-1}A_1X_{11} - A_2^*D^{-1}A_2X_{21}\right)A_2Y_{22}. \end{aligned}$$

Similarly,

$$A^{(1)} + B^{(1)} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

where

$$\begin{aligned} F_{11} &= A_1^*D^{-1} + X_{11} - A_1^*D^{-1}A_1X_{11} - A_1^*D^{-1}A_2X_{12} \\ &\quad + E^{-1}B_1^* + Y_{11} - Y_{11}B_1E^{-1}B_1^* - Y_{12}B_2E^{-1}B_1^*, \\ F_{12} &= X_{12} + E^{-1}B_2^* + Y_{12} - Y_{11}B_1E^{-1}B_2^* - Y_{12}B_2E^{-1}B_2^*, \\ F_{21} &= A_2^*D^{-1} + X_{21} - A_2^*D^{-1}A_1X_{11} - A_2^*D^{-1}A_2X_{21} + Y_{21}, \\ F_{22} &= X_{22} + Y_{22}. \end{aligned}$$

$(ii) \Rightarrow (i)$: Since $A^{\dagger}A = I$ and $BB^{\dagger} = I$, we get

$$A_1^*D^{-1}A_1 = I,\ A_2^*D^{-1}A_1 = 0,\ A_2^*D^{-1}A_2 = I \tag{2.7}$$

and

$$B_1E^{-1}B_1^* = I,\ B_1E^{-1}B_2^* = 0,\ B_2E^{-1}B_2^* = I. \tag{2.8}$$

Substituting (2.7) and (2.8) in the expressions of $A^{(1)}(A+B)B^{(1)}$ and $A^{(1)} + B^{(1)}$, we have

$$A^{(1)}(A+B)B^{(1)} = \begin{pmatrix} A_1^*D^{-1} + E^{-1}B_1^* & X_{12} + E^{-1}B_2^* \\ A_2^*D^{-1} + Y_{21} & X_{22} + Y_{22} \end{pmatrix} = A^{(1)} + B^{(1)}.$$

$(i) \Rightarrow (ii)$: From $A^\dagger(A+B)B^\dagger = A^\dagger + B^\dagger$ by Theorem 2.1, we have $A_1^*D^{-1}A_1 = I, A_2^*D^{-1}A_1 = 0, B_1E^{-1}B_1^* = I, B_1E^{-1}B_2^* = 0$. Now, $M = \begin{pmatrix} A_1^*D^{-1} & X_{12} \\ A_2^*D^{-1} & 0 \end{pmatrix} \in A\{1\}$ and $N = \begin{pmatrix} E^{-1}B_1^* & E^{-1}B_2^* \\ Y_{21} & 0 \end{pmatrix} \in B\{1\}$, for arbitrary operators X_{12} and Y_{21} on appropriate subspaces. From $M(A+B)B^\dagger = M + B^\dagger$, i.e., from the part $(M(A+B)B^\dagger)_{12} = (M+B^\dagger)_{12}$, we obtain

$$A_1^*D^{-1}A_1E^{-1}B_2^* + A_1^*D^{-1}B_1E^{-1}B_2^* + X_{12}B_2E^{-1}B_2^* = E^{-1}B_2^* + X_{12}$$

which is by $A_1^*D^{-1}A_1 = I$ and $B_1E^{-1}B_2^* = 0$, equivalent to

$$X_{12}B_2E^{-1}B_2^* = X_{12}.$$

Since X_{12} is arbitrary, we get $B_2E^{-1}B_2^* = I$. Similarly, by $A^\dagger(A+B)N = A^\dagger + N$, we get $A_2^*D^{-1}A_2 = I$. Now, it is easy to verify that $A_1^*D^{-1}A_1 = I, A_2^*D^{-1}A_1 = 0, A_2^*D^{-1}A_2 = I$ imply $A^\dagger A = I$ while $B_1E^{-1}B_1^* = I, B_1E^{-1}B_2^* = 0, B_2E^{-1}B_2^* = I$ imply $BB^\dagger = I$. □

In the following theorem we proved that the absorption law for $\{1,2\}$-inverses holds under the same conditions as in the case of $\{1\}$-inverses.

Theorem 2.3. *Let $A, B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be such that $R(A), R(B)$ are closed. The following conditions are equivalent:*

(i) $A^{(1,2)} + B^{(1,2)} = A^{(1,2)}(A+B)B^{(1,2)}$, for any $A^{(1,2)} \in A\{1\}$ and any $B^{(1,2)} \in B\{1\}$,
(ii) $A^\dagger A = I, BB^\dagger = I$.

Proof. By Lemma 1.2, $A^{(1,2)} = \left(A^\dagger + (I - A^\dagger A)X\right)A\left(A^\dagger + Y(I - AA^\dagger)\right)$, where $X, Y \in \mathcal{B}(\mathcal{K},\mathcal{H})$ are arbitrary operators. Similarly, $B^{(1,2)} = (B^\dagger + (I - B^\dagger B)U)B(B^\dagger + V(I - BB^\dagger))$, where $U, V \in \mathcal{B}(\mathcal{K},\mathcal{H})$ are arbitrary operators. Now, we have

$$A^{(1,2)} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where

$$\begin{aligned}
H_{11} &= A_1^*D^{-1} + (I - A_1^*D^{-1}A_1)X_{11} - A_1^*D^{-1}A_2X_{21}, \\
H_{12} &= \left(A_1^*D^{-1} + (I - A_1^*D^{-1}A_1)X_{11} - A_1^*D^{-1}A_2X_{21}\right)(A_1Y_{12} + A_2Y_{22}), \\
H_{21} &= A_2^*D^{-1} - A_2^*D^{-1}A_1X_{11} + (I - A_2^*D^{-1}A_2)X_{21}, \\
H_{22} &= \left(A_2^*D^{-1} - A_2^*D^{-1}A_1X_{11} + (I - A_2^*D^{-1}A_2)X_{21}\right)(A_1Y_{12} + A_2Y_{22}),
\end{aligned}$$

and $B^{(1,2)} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$,
where

$$\begin{aligned}
J_{11} &= E^{-1}B_1^* + V_{11}(I - B_1E^{-1}B_1^*) - V_{12}B_2E^{-1}B_1^*, \\
J_{12} &= E^{-1}B_2^* - V_{11}B_1E^{-1}B_2^* + V_{12}(I - B_2E^{-1}B_2^*), \\
J_{21} &= (U_{21}B_1 + U_{22}B_2)(E^{-1}B_1^* + V_{21}(I - B_1E^{-1}B_1^*) - V_{22}B_2E^{-1}B_1^*), \\
J_{22} &= (U_{21}B_1 + U_{22}B_2)(E^{-1}B_2^* - V_{21}B_1E^{-1}B_2^* + V_{22}(I - B_2E^{-1}B_2^*)).
\end{aligned}$$

$(ii) \Rightarrow (i)$: Since $A^\dagger A = I$ and $BB^\dagger = I$, we get

$$A_1^*D^{-1}A_1 = I, \quad A_2^*D^{-1}A_1 = 0, \quad A_2^*D^{-1}A_2 = I \tag{2.9}$$

and

$$B_1E^{-1}B_1^* = I, \quad B_1E^{-1}B_2^* = 0, \quad B_2E^{-1}B_2^* = I. \tag{2.10}$$

Now, we have that arbitrary $A^{(1,2)}$ is given by $A^{(1,2)} = \begin{pmatrix} A_1^*D^{-1} & Y_{12} \\ A_2^*D^{-1} & Y_{22} \end{pmatrix}$ while arbitrary $B^{(1,2)}$ is given by $B^{(1,2)} = \begin{pmatrix} E^{-1}B_1^* & E^{-1}B_2^* \\ U_{21} & U_{22} \end{pmatrix}$, for some operators Y_{12}, Y_{22}, U_{21} and U_{22}. Now, by (2.9) and (2.10), we get that

$$A^{(1,2)}(A+B)B^{(1,2)} = \begin{pmatrix} A_1^*D^{-1} + E^{-1}B_1^* & Y_{12} + E^{-1}B_2^* \\ A_2^*D^{-1} + U_{21} & Y_{22} + U_{22} \end{pmatrix} = A^{(1,2)} + B^{(1,2)}.$$

$(i) \Rightarrow (ii)$: From $A^\dagger(A+B)B^\dagger = A^\dagger + B^\dagger$, by Theorem 2.1, we have that $A_1^*D^{-1}A_1 = I, A_2^*D^{-1}A_1 = 0, B_1E^{-1}B_1^* = I$ and $B_1E^{-1}B_2^* = 0$. Now, using obtained conditions, we have that $M = \begin{pmatrix} A_1^*D^{-1} & X_{12} \\ A_2^*D^{-1} & 0 \end{pmatrix} \in A\{1,2\}$ and $N = \begin{pmatrix} E^{-1}B_1^* & E^{-1}B_2^* \\ Y_{21} & 0 \end{pmatrix} \in B\{1,2\}$, for arbitrary operators X_{12} and Y_{21} on appropriate subspaces. From $A^\dagger(A+B)N = A^\dagger + N$, we get: $A_2^*D^{-1}A_2 = I$. Similarly, from $M(A+B)B^\dagger = M + B^\dagger$, i.e., from the part $(M(A+B)B^\dagger)_{12} = (M+B^\dagger)_{12}$, we obtain

$$E^{-1}B_2^* + X_{12}B_2E^{-1}B_2^* = E^{-1}B_2^* + X_{12}$$

which is equivalent to

$$X_{12}B_2E^{-1}B_2^* = X_{12}.$$

Since X_{12} is arbitrary, we get $B_2E^{-1}B_2^* = I$. Now, it is easy to verify that $A_1^*D^{-1}A_1 = I, A_2^*D^{-1}A_1 = 0, A_2^*D^{-1}A_2 = I$ imply $A^\dagger A = I$ while $B_1E^{-1}B_1^* = I, B_1E^{-1}B_2^* = 0, B_2E^{-1}B_2^* = I$ imply $BB^\dagger = I$. $\square$

Corollary 2.1. *Let $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be such that $R(A), R(B)$ are closed. The following conditions are equivalent:*

(a) $A^{(1,2)} + B^{(1,2)} = A^{(1,2)}(A+B)B^{(1,2)}$, for any $A^{(1,2)} \in A\{1\}$ and any $B^{(1,2)} \in B\{1\}$,
(b) $A^{(1)} + B^{(1)} = A^{(1)}(A+B)B^{(1)}$, for any $A^{(1)} \in A\{1\}$ and any $B^{(1)} \in B\{1\}$,
(c) $A^\dagger A = I, BB^\dagger = I$.

In the following theorem we proved that the absorption law for $\{1,3\}$-inverses holds if and only if $A^\dagger A = I$ and $R(A) \subseteq R(B)$ which is equivalent with $A^\dagger A = I$ and $BB^\dagger A = A$.

Theorem 2.4. *Let $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be such that $R(A), R(B)$ are closed. The following conditions are equivalent:*

(i) $A^{(1,3)} + B^{(1,3)} = A^{(1,3)}(A+B)B^{(1,3)}$, for any $A^{(1,3)} \in A\{1,3\}$ and any $B^{(1,3)} \in B\{1,3\}$,
(ii) $A^\dagger A = I, R(A) \subseteq R(B)$.

Proof. Let $B^{(1,3)} \in B\{1,3\}$ be arbitrary. By Lemma 1.3, it follows that there exists some $Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, such that

$$B^{(1,3)} = B^\dagger + (I - B^\dagger B)Y = \begin{pmatrix} E^{-1}B_1^* & E^{-1}B_2^* \\ Y_{21} & Y_{22} \end{pmatrix},$$

where Y_{21} and Y_{22} are arbitrary bounded linear operators on appropriate subspaces.

$(ii) \Rightarrow (i)$: By $A^\dagger A = I, R(A) \subseteq R(B)$, i.e., $A^\dagger A = I$ and $BB^\dagger A = A$, we get that

$$A_1^*D^{-1}A_1 = I, \quad A_2^*D^{-1}A_1 = 0, \quad A_2^*D^{-1}A_2 = I, \tag{2.11}$$

and

$$B_1E^{-1}B_1^* = I, \quad B_1E^{-1}B_2^* = 0. \tag{2.12}$$

Evidently, $A\{1,3\} = \{A^\dagger\}$, so by (2.11) and (2.12), it follows that

$$A^\dagger(A+B)B^{(1,3)} = A^\dagger + B^{(1,3)} = \begin{pmatrix} A_1^*D^{-1} + E^{-1}B_1^* & E^{-1}B_2^* \\ A_2^*D^{-1} + Y_{21} & Y_{22} \end{pmatrix}.$$

$(i) \Rightarrow (ii)$: By Theorem 2.1, we get that $BA^\dagger A = B$ and $BB^\dagger A = A$, i.e., $A_1^*D^{-1}A_1 = I, A_2^*D^{-1}A_1 = 0, B_1E^{-1}B_1^* = I$ and $B_1E^{-1}B_2^* = 0$. Since $N = \begin{pmatrix} E^{-1}B_1^* & E^{-1}B_2^* \\ Y_{21} & 0 \end{pmatrix} \in B\{1,3\}$, for an arbitrary operator Y_{21} on an appropriate subspace, we have that $A^\dagger(A+B)N = A^\dagger + N$, i.e., $(A^\dagger(A+B)N)_{12} = (A^\dagger + N)_{12}$ implies

$$A_2^*D^{-1}A_1E^{-1}B_1^* + A_2^*D^{-1}B_1E^{-1}B_1^* + A_2^*D^{-1}A_2Y_{21} = A_2^*D^{-1} + Y_{21}$$

Since $A_2^*D^{-1}A_1 = 0$ and $B_1E^{-1}B_1^* = I$, we get $A_2^*D^{-1}A_2Y_{21} = Y_{21}$. As Y_{21} is arbitrary, we have $A_2^*D^{-1}A_2 = I$. Finally, $A_1^*D^{-1}A_1 = I, A_2^*D^{-1}A_1 = 0, A_2^*D^{-1}A_2 = I$ imply $A^\dagger A = I$ while $B_1E^{-1}B_1^* = I, B_1E^{-1}B_2^* = 0$ imply $R(A) \subseteq R(B)$. $\square$

The case $K = \{1,4\}$ is treated completely analogously and the corresponding result follows by taking adjoints, or by reversal of products:

Theorem 2.5. *Let $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be such that $R(A), R(B)$ are closed. The following conditions are equivalent:*

(i) $A^{(1,4)} + B^{(1,4)} = A^{(1,4)}(A+B)B^{(1,4)}$, *for any* $A^{(1,4)} \in A\{1,4\}$ *and any* $B^{(1,4)} \in B\{1,4\}$,
(ii) $BB^\dagger = I, R(B^*) \subseteq R(A^*)$.

3. The mixed absorption law

In this section, we consider the various types of mixed absorption law for the generalized inverses. It is interesting that for any of considered cases the necessary and sufficient conditions are the same.

Theorem 3.1. *Let* $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *be such that* $R(A), R(B)$ *are closed. The following conditions are equivalent:*

(1) $A^{(1,4)} + B^{(1,3)} = A^{(1,4)}(A+B)B^{(1,3)}$, *for any* $A^{(1,4)} \in A\{1,4\}$ *and any* $B^{(1,3)} \in B\{1,3\}$,
(2) $A^{(1,3)} + B^{(1,4)} = A^{(1,3)}(A+B)B^{(1,4)}$, *for any* $A^{(1,3)} \in A\{1,3\}$ *and any* $B^{(1,4)} \in B\{1,4\}$,
(3) $A^{(1,2)} + B^{(1,3)} = A^{(1,2)}(A+B)B^{(1,3)}$, *for any* $A^{(1,2)} \in A\{1,2\}$ *and any* $B^{(1,3)} \in B\{1,3\}$,
(4) $A^{(1,3)} + B^{(1,2)} = A^{(1,3)}(A+B)B^{(1,2)}$, *for any* $A^{(1,3)} \in A\{1,3\}$ *and any* $B^{(1,2)} \in B\{1,2\}$,
(5) $A^{(1,2)} + B^{(1,4)} = A^{(1,2)}(A+B)B^{(1,4)}$, *for any* $A^{(1,2)} \in A\{1,2\}$ *and any* $B^{(1,4)} \in B\{1,4\}$,
(6) $A^{(1,4)} + B^{(1,2)} = A^{(1,4)}(A+B)B^{(1,2)}$, *for any* $A^{(1,4)} \in A\{1,4\}$ *and any* $B^{(1,2)} \in B\{1,2\}$,
(7) $A^\dagger A = I, BB^\dagger = I$.

Proof. By Theorem 2.2, it follows that $(7) \Rightarrow (i)$, for $i \in \{1,2,3,4,5,6\}$. Now, we need to prove that $(i) \Rightarrow (7)$, for $i \in \{1,2,3,4,5,6\}$. It is evident that if we suppose that (i) holds for some $i \in \{1,2,3,4,5,6\}$, then $A^\dagger(A+B)B^\dagger = A^\dagger + B^\dagger$, which by Theorem 2.1 implies that $A_1^* D^{-1} A_1 = I, A_2^* D^{-1} A_1 = 0, B_1 E^{-1} B_1^* = I$ and $B_1 E^{-1} B_2^* = 0$. So, in all the proofs $(i) \Rightarrow (7)$, where $i \in \{1,2,3,4,5,6\}$ we must prove that $A_2^* D^{-1} A_2 = I$ and $B_2 E^{-1} B_2^* = I$.

$(1) \Rightarrow (7)$: Remark that $M = \begin{pmatrix} A_1^* D^{-1} & X_{12} \\ A_2^* D^{-1} & 0 \end{pmatrix} \in A\{1,4\}$ and $N = \begin{pmatrix} E^{-1}B_1^* & E^{-1}B_2^* \\ Y_{21} & 0 \end{pmatrix} \in B\{1,3\}$, for arbitrary operators X_{12} and Y_{21} on appropriate subspaces. From

$$M(A+B)B^\dagger = M + B^\dagger \quad \text{and} \quad A^\dagger(A+B)N = A^\dagger + N, \tag{3.1}$$

as in the proof of Theorem 2.2, we obtain $A_2^* D^{-1} A_2 = I$ and $B_2 E^{-1} B_2^* = I$. Thus, $A^\dagger A = I$ and $BB^\dagger = I$.

$(3) \Rightarrow (7)$: Since $M = \begin{pmatrix} A_1^* D^{-1} & Y_{12} \\ A_2^* D^{-1} & 0 \end{pmatrix} \in A\{1,2\}$ and $N = \begin{pmatrix} E^{-1}B_1^* & E^{-1}B_2^* \\ Z_{21} & 0 \end{pmatrix} \in B\{1,3\}$, for arbitrary operators Y_{12} and Z_{21} on appropriate subspaces, the proofs follows by (3.1). Proofs of the parts $(i) \Rightarrow (7)$, where $i \in \{2,4,5,6\}$ follow analogously. □

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