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Outer inverses: Characterization and applications

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We characterize the elements with outer inverse in a semigroup S, and provide explicit expressions for the class of outer inverses b of an element a such that $bS \subseteq yS$ and $Sb \subseteq Sx$, where x, y are any arbitrary elements of S. We apply this result to characterize pairs of outer inverses of given elements from an associative ring R, satisfying absorption laws extended for the outer inverses. We extend the result on right–left symmetry of $aR \oplus bR = (a+b)R$ (Jain–Prasad, 1998) to the general case of an associative ring. We conjecture that 'given an outer inverse x of a regular element a in a semigroup S, there exists a reflexive generalized inverse y of a such that $x \leq^- y$' and prove the conjecture when S is an associative ring.

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Absorption law
Associative ring

1. Preliminaries

In this article, we make use of simple but interesting characterizations of regular elements, and elements with outer inverse in a semigroup, to obtain explicit expressions for the class of outer inverses with the property like range inclusion in the context of matrices. Throughout this article, S denotes a semigroup and R denotes an associative ring. The semigroup S and associative ring R need not have multiplicative identities.

Definition 1. An element a in a semigroup S is said to be *regular* (or von Neumann regular) if there exists an element b in S satisfying the equation

$$aba = a, \tag{1}$$

in which case b is said to be a *generalized inverse* of a. If an element c satisfies the equation

$$cac = c, \tag{2}$$

then c is called an *outer inverse* of a. Further, b is said to be a *reflexive generalized inverse* of a if $a = aba$ and $b = bab$.

An arbitrary outer inverse of a is denoted by $a^=$, a generalized inverse of a by a^- and a reflexive generalized inverse of a by a_r^-. Readers are referred to [1–3] for definitions and properties of different types of outer inverses (pseudo inverses). We refer to [4] and [5] for basic notions in the theory of generalized inverse of matrices. Our discussion is confined to results associated with generalized inverses and outer inverses. They can be extended to other generalized inverses (e.g., Moore–Penrose inverse and core–EP generalized inverse).

The absorption law "$a^{-1}(a+b)b^{-1} = a^{-1} + b^{-1}$" has been extended to singular elements of an associative ring by several authors, e.g., [6,7]. They considered the problem of finding equivalent conditions for the absorption laws in terms of the Moore–Penrose, group, core inverse, core inverse dual, and $\{1\}$, $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$ inverses in rings with identity. In [6,7], the authors characterized the elements a, b satisfying $a^-(a+b)b^- = a^- + b^-$ for all a^- and b^- belonging to a certain class of generalized inverse. In Section 3, we readdress the absorption law by considering $a^=(a+b)b^= = a^= + b^=$ for some $a^=$ and $b^=$. This modification is inspired by the observation that $e_1(e_1+e_2)e_2 = e_1 + e_2 \Leftrightarrow e_1 = e_2$, where e_1 and e_2 are any real or complex idempotent matrices. We characterize every pair of outer inverses $a^=$ and $b^=$ that satisfy the modified absorption law.

An *idempotent* element e in S is a regular element in S, as $e^2 = e$. We say that two elements a, b of a semigroup are *space equivalent* if

$$aS = bS \quad \text{and} \quad Sa = Sb. \tag{3}$$

If b, c are outer inverses of an element a in a semigroup and are space equivalent, then $b = c$. The following simple lemmas will be useful in our discussion.

Lemma 1. *Given an element a in a semigroup S, the following statements are equivalent.*

(i) *a has an outer inverse in S.*
(ii) *There exists an idempotent element $e \in S$ such that $eS \subseteq aS$.*
(iii) *There exists an idempotent element $f \in S$ such that $Sf \subseteq Sa$.*

In fact, e in (ii) *is given by ax for some $x \in \{a^=\}$ and f in* (iii) *is given by ya for some $y \in \{a^=\}$.*

Proof. Let c be an arbitrary outer inverse of a. Now for $e = ac$, we observe that $eS \subseteq aS$. This proves (i) $\Rightarrow$ (ii).

To prove the converse, let e be an idempotent such that $eS \subseteq aS$. Since e is an idempotent, it follows that $e \in eS \subseteq aS$ and therefore $e = az$ for some $z \in S$. By taking $x = ze$, observe that $xax = x$ and $ax = e$ hold by direct substitution. So, we have (ii) $\Rightarrow$ (i).

The proof of (i) $\Leftrightarrow$ (iii) is similar to that of (i) $\Leftrightarrow$ (ii). $\square$

If $eS = aS$ and $a \in aS$ in (ii) of Lemma 1, one verifies that $a = ea = axa$, where $e = ax$ for some x. Thus, we have the following lemma.

Lemma 2. *Given an element a in a semigroup S, the following statements are equivalent.*

(i) *a is regular.*
(ii) *There exists an idempotent element $e \in S$ such that $a \in eS = aS$.*
(iii) *There exists an idempotent element $f \in S$ such that $a \in Sf = Sa$.*

The following corollary is immediate from the above result.

Corollary 3. *Let $a, b \in S$ such that $a, b \in aS = bS$. Then a is regular if and only if b is also regular.*

The above characterization of regular elements helps us to generalize Theorem 1 of [8] on right–left symmetry of $aR \oplus bR = (a + b)R$ to the case of an associative ring that need not have a multiplicative identity (unity). We state this result from [8] in the following without proof.

Theorem 4 *(Theorem 1, [8]). Let R be a ring with multiplicative identity and let $a, b \in R$ be such that $a + b$ is von Neumann regular. Then the following statements are equivalent.*

(i) $aR \oplus bR = (a+b)R$.
(ii) $Ra \oplus Rb = R(a+b)$.
(iii) $aR \cap bR = (0)$ *and* $Ra \cap Rb = (0)$.

2. Explicit expression for outer inverses

In this section, we provide explicit expressions for outer inverses with certain range inclusion property.

Theorem 5. *For $a \in S$ and regular elements $x, y \in S$, the following statements are equivalent.*

(i) *There exists an outer inverse $b \in \{a^=\}$ such that $bS = yS$ and $Sb = Sx$.*
(ii) *xay is regular and*

$$xS = (xay)S \quad \text{and} \quad Sy = S(xay). \tag{4}$$

Proof. ($\Rightarrow$). Suppose $b \in \{a^=\}$ is an outer inverse such that $bS = yS$ and $Sb = Sx$. Since x, y are regular, we have $x \in Sx$ and $y \in yS$, and therefore $y = bay$ and $x = xab$. So, we have

$$xS = xabS = xayS.$$

$Sy = Sxay$ can be proved similarly. The inclusion $xay \in xS$ is trivial and therefore Corollary 3 ensures that $xS = xayS$ implies that xay is regular.

($\Leftarrow$). Suppose x, y are regular elements that satisfy (ii). Define

$$b = ygx, \tag{5}$$

where $g \in \{(xay)_r^-\}$ is an arbitrary reflexive generalized inverse of xay. Note that

$$\begin{aligned} bab &= ygxaygx \\ &= ygx, \end{aligned}$$

which proves that b is an outer inverse of a. Further, from the definition of b we have

$$yS \supseteq ygxS(= bS) \supseteq yg(xay)S = yS,$$

because $yg(xay) = y$, since x, a, y satisfy (4). Therefore $bS = yS$. The equality $Sb = Sx$ can be proved in a similar fashion. □

The condition that x, y are regular, in Theorem 5, could be dropped if statement (ii) is modified to say 'xay is regular, $x \in xS = (xay)S$, and $y \in Sy = S(xay)$'.

The following corollary can be proved in the same way that Theorem 5 was proved.

Corollary 6. *Let $a \in S$ and let x, y in S be regular.*

(i) *There exists $a^=$ such that $a^= S = yS$ and $Sa^= \subseteq Sx$ if and only if xay is regular and $xS = (xay)S$.*
(ii) *There exists $a^=$ such that $Sa^= = Sx$ and $a^= S \subseteq yS$ if and only if xay is regular and $Sy = S(xay)$.*

The classes of outer inverses in both cases (i) *and* (ii) *are given by*

$$\{y(xay)_r^- x\}.$$

The outer inverse discussed in Theorem 5 is exactly the same as the (y, x)-inverse introduced by Drazin in [3]. In fact, the expression on the right-hand side of (5) is uniquely determined and we may replace it by $y(xay)^- x$ for any generalized inverse $(xay)^-$ of xay. This might not be the case in Corollary 6, as $y(xay)_r^- x$ is not uniquely determined. Now, we relax the conditions given by Drazin [3] and in Theorem 5, and define a (y, x)-outer inverse.

Definition 2 *((y, x)-outer inverse).* Given elements a, x, y from a semigroup S, an outer inverse b of a is said to be a (y, x)-*outer inverse* if

$$bS \subseteq yS \quad \text{and} \quad Sb \subseteq Sx. \tag{6}$$

Theorem 7. *Let a, x, y be any elements in a semigroup S. Then the following statements are equivalent.*

(i) *a has a (y, x)-outer inverse.*
(ii) *xay has an outer inverse.*

In both cases (i) *and* (ii)*, the classes of (y, x)-outer inverses are given by*

$$\{y(xay)^= x\}, \tag{7}$$

for all choices of outer inverses $(xay)^=$.

Proof. (ii) $\Rightarrow$ (i) is proved by verifying $b = y(xay)^= x$ is an outer inverse of a for every choice of $(xay)^=$.

To prove (i) $\Rightarrow$ (ii), consider an outer inverse b of a such that $bS \subseteq yS$ and $Sb \subseteq Sx$. Since ba is an idempotent in yS, Lemma 1 ensures that there exists an outer inverse c of y such that $yc = ba$. Similarly, there exists $d \in \{x^=\}$ such that $dx = ab$. Since $bab = b$, one verifies that $b = yzx$ for some z such that $cyzxd = z$. Now substitute $b = yzx$ in $bab = b$ and then multiply c from the left and d from the left to get $z(xay)z = z$. So, xay has an outer inverse and $z \in \{(xay)^=\}$. $\square$

3. Absorption law

In this section, we consider the problem of finding necessary and sufficient conditions on the pair of outer inverses of the elements in order that they satisfy the extended absorption law. Throughout this section, R is an associative ring that need not have a multiplicative identity.

Definition 3. Elements $a, b \in R$ with outer inverses $c \in \{a^=\}$ and $d \in \{b^=\}$ are said to satisfy the *absorption (extended) law* if

$$c(a+b)d = c+d. \tag{8}$$

The following theorem characterizes pairs of all outer inverses from $\{a^=\}$ and $\{b^=\}$ that satisfy the absorption law.

Theorem 8. *Let a, b be any elements in R. If c and d are any outer inverses of a and b, respectively, then the following statements are equivalent.*

(i) *The outer inverses satisfy the absorption law (8).*
(ii) *There exist regular elements x and y in R such that*

$$xR = (xby)R, \tag{9}$$

$$Ry = R(xay). \tag{10}$$

(iii)

$$cR \supseteq dR \quad \textit{and} \quad Rc \subseteq Rd. \tag{11}$$

Further, if $x, y \in S$ satisfy (9) and (10), then corresponding pairs of $c \in \{a^=\}$ and $d \in \{b^=\}$ with the property $yR = cR \supseteq dR$ and $Rx = Rd \supseteq Rc$ are given by

$$c = y(xay)_r^- x, \quad d = y(xby)_r^- x, \tag{12}$$

for different choices of $(xay)_r^-$ and $(xby)_r^-$.

Proof. Let $c \in \{a^=\}$ and $d \in \{b^=\}$ be outer inverses of a and b, respectively, satisfying (8). Multiply ca on the both sides of (8) from the left and multiply bd on the both sides of (8) from the right, to get

$$c = cbd \in Rd \quad \text{and} \quad d = cad \in cR. \tag{13}$$

So, (ii) is proved by taking $x = c$ and $y = d$.

To prove (ii) ⇒ (iii), consider any regular elements x, y satisfying (ii) of the theorem. Since x is regular, we have $x \in xR$. Now (9) and Corollary 3 imply that xby is regular. Similarly, regularity of y together with equation (10) and Corollary 3 implies that xay is regular. Let $(xay)^-_r$ and $(xby)^-_r$ be reflexive generalized inverses of xay and xby respectively. Now let

$$c = y(xay)^-_r x \quad \text{and} \quad d = y(xby)^-_r x. \tag{14}$$

Since y is regular and therefore $y \in Ry$, (10) implies that $y(xay)^-_r(xay) = y$ and we get

$$cR = y(xay)^-_r xR \supseteq y(xay)^-_r(xay)R = yR \supseteq dR. \tag{15}$$

Similarly, using regularity of x and (9), we get

$$Rd = Ry(xby)^-_r x \supseteq R(xby)(xby)^-_r x = Rx \supseteq Rc. \tag{16}$$

Hence (iii) is proved.

(iii) ⇒ (i) is proved by observing that (iii) is equivalent to the condition given in (13), from which the absorption law follows.

The rest of the theorem follows immediately from Corollary 6. □

The following corollary follows from equality of the row rank and column rank of a matrix.

Corollary 9. *Let a, b be $n \times n$ matrices over the complex field. Then $c \in \{a^=\}$ and $d \in \{b^=\}$ are outer inverses for which the absorption law holds if and only if c and d are space equivalent (i.e., the column spaces of c and d are same and similarly with row spaces). In this case the condition (ii) of Theorem 8 is*

$$rank(x) = rank(xay) = rank(xby) = rank(y).$$

Further, an outer inverse $c \in \{a^=\}$ is paired with at most one $d \in \{b^=\}$ satisfying the absorption law.

Many results related to the absorption law with $c \in \{a^=\}$ and $d \in \{b^=\}$ belonging to certain classes of outer inverses (discussed in [6,7]) can be derived from Theorem 8. As an illustration we consider the case of the Drazin inverse.

A reflexive generalized inverse $g \in S$ of $a \in S$ is a *group inverse* if $ag = ga$. The group inverse, whenever exists, is unique and is denoted by $a^\#$. It is known that the group inverse g of a, whenever it exists, is a generalized inverse satisfying $gS = aS$ and $Sg = Sa$. Similarly, the *Drazin inverse* of a, denoted by a^D, whenever it exists, is a commuting outer inverse satisfying the condition $a^D a^{k+1} = a^k$ for some positive integer k. The smallest integer satisfying $a^D a^{k+1} = a^k$ is known as the *Drazin index* (or simply index) of a and the Drazin inverse is an outer inverse with the property $a^D R = a^k R$ and $Ra^D = Ra^k$. The Drazin inverse of a is unique whenever it exists; if the Drazin index of a is one, then $a^D = a^\#$. The following corollary is immediate from Theorem 8.

Corollary 10. *Let a and b be elements in an associative ring R with indices k_1 and k_2, respectively. Then $a^D(a+b)b^D = a^D + b^D$ if and only if $a^{k_1}R \supseteq b^{k_2}R$ and $Ra^{k_1} \subseteq Rb^{k_2}$.*

4. Minus partial order and right–left symmetry of $aR \oplus bR = (a+b)R$

In this section, we revisit the right–left symmetric property of $aR \oplus bR = (a+b)R$ considered earlier by Jain–Prasad [8], in which the associative (regular) ring has a multiplicative identity. We use the characterization of regular elements discussed in Section 1 to discuss the right–left symmetric property to the extent possible. Also, the characterizations of regular elements and outer inverses given in Section 1 help us to reveal structural properties such as the relation between outer inverses and reflexive generalized inverses of a regular element. Throughout this section, R is an associative ring that need not have a multiplicative identity. Further, we consider the *minus partial order* on regular elements introduced independently by Hartwig [9] and Nambooripad [10]. Mitra in [11] explored several interesting properties of the minus partial order in the context of a class of rectangular matrices, where the entries of the matrices are from the real or complex field. Our discussion of the minus partial order helps to explain several interesting properties of regular elements.

Definition 4 *(Minus partial order).* Let $a, b \in S$. We say that $a \leq^- b$ if there exists a generalized inverse $g_b \in \{b^-\}$ of b such that

$$bg_b = ag_b, \quad g_b b = g_b a. \tag{17}$$

The relation $\leq^-$ is known to be a partial order on the class of regular elements of S; it is called the 'minus partial order' on S.

The general difficulty in extending known results about the minus partial order to a semigroup or an associative ring is due to the absence of a multiplicative identity and particularly for the reason that $x \in xS$ or $x \in xR$ may not hold. The properties derived for regular elements and outer inverses in the earlier sections can be used to generalize results such as right–left symmetry and relations between outer inverses and reflexive generalized inverses.

The following lemma explains the association between outer inverses of an element a from a semigroup and the elements b such that $b \leq^- a$.

Lemma 11. *Let S be a semigroup and let $a \in S$.*

(i) *If c is an outer inverse of a, then $b = aca$ is such that $b \leq^- a$ and $c \in \{b_r^-\}$ satisfies (17).*

(ii) *If $b \leq^- a$ and $g_b \in \{b^-\}$ satisfies (17), then $g_b b g_b \in \{a^=\}$ (in fact, a reflexive generalized inverse of b satisfying (17) is in $\{a^=\}$).*

(iii) *The relation $\sim$ defined on $\{a^=\}$ by*

$$x \sim y \quad \text{if} \quad axa = aya \tag{18}$$

for $x, y \in \{a^=\}$, is an equivalence relation and each of equivalence class is associated with a distinct $b \leq^- a$.

Proof. Let c be an outer inverse of a. For $b = aca$, one verifies that $c \in \{b_r^-\}$. Also, $bc = (aca)c = ac$ and $cb = c(aca) = ca$, and therefore $g_b = c$ satisfies (17). This proves part (i).

To prove (ii), consider $b \leq^- a$ and a $g_b \in \{b^-\}$ that satisfy (17). Then $h = g_b b g_b$ is a reflexive generalized inverse and

$$hah = (g_b b g_b)a(g_b b g_b) = (g_b b)(g_b a)(g_b b g_b) = (g_b b)(g_b b)(g_b b g_b) = g_b b g_b = h.$$

This proves (ii).

(iii) follows immediately from the definition of $\sim$ and (i). $\square$

Theorem 12. *Let R be an associative ring and let a be a regular element in R. If $a = b+c$ for some $b, c \in R$, then the following statements are equivalent.*

(i) *b is regular and $b \leq^- a$.*

(ii) *$b \in aR \cap Ra$ and $\{a^-\} \subseteq \{b^-\}$.*

(iii) *c is regular and $c \leq^- a$.*

(iv) *$c \in aR \cap Ra$ and $\{a^-\} \subseteq \{c^-\}$.*

Proof. Let g_b be generalized inverse of b such that $ag_b = bg_b$ and $g_b a = g_b b$. So, we get $ag_b b = bg_b b = b$ and $bg_b a = bg_b b = b$, and therefore $b \in aR \cap Ra$. Now for any generalized inverse a^- of a, note that $ba^-b = (bb^-a)a^-(ab^-b) = bb^-ab^-b = bb^-bb^-b = b$. Hence (i) $\Rightarrow$ (ii).

Let $b \in aR \cap Ra$ and $\{a^-\} \subseteq \{b^-\}$. Since $b \in aR \cap Ra$, we have $aa^-b = b = ba^-a$ for every choice of a^-. Now for any $g_a \in \{a^-\}$, define $h = g_a b g_a$, which is in $\{b_r^-\}$. Observe that $bh = (ag_a b)(g_a b g_a) = ag_a(bg_a b)g_a = a(g_a b g_a) = ah$ and $hb = (g_a b g_a)(bg_a a) = g_a(bg_a b)g_a a = (g_a b g_a)a = ha$. This proves (ii) $\Rightarrow$ (i).

The proof of (iii) $\Leftrightarrow$ (iv) is similar to that of (i) $\Leftrightarrow$ (ii).

Now we prove (ii) $\Rightarrow$ (iv) to complete the proof of the theorem as (iv) $\Rightarrow$ (ii) is symmetrical.

Let $b \in aR \cap Ra$ and $\{a^-\} \subseteq \{b^-\}$. Since $a = b + c$, $b \in aR \cap Ra$ implies that $c \in aR \cap Ra$ and $ca^-a = c$. Now use $ba^-b = b$ in $aa^-b = b$ to get $ca^-b = 0$. Further, substitute this in $c = ca^-a = ca^-(b + c)$ to get $c = ca^-c$. This proves (ii) ⇒ (iv). □

Remark 1. If R is a matrix ring over the real or complex field, then it may be noted that $b \in aR \cap Ra$ in (ii) and $c \in aR \cap Ra$ in (iv) of Theorem 12 are trivial, when $ba^-b = b$ holds for all a^-.

Remark 2. If $[a^=]_b$ denotes the equivalence class of all outer inverses associated with $b \leq^- a$, as described in (iii) of Lemma 11, then from (i) and (ii) of the same lemma, it follows that

$$[a^=]_b = \{g_b \in \{b_r^-\} : bg_b = ag_b, \ \ g_bb = g_ba\}. \tag{19}$$

Remark 3. Let S be a semigroup and let $a \in S$. Note that (i) ⇒ (ii) of Theorem 12 holds for any elements a, b in a semigroup (we do not need to assume that $a = b + c$). If g is any reflexive generalized inverse of a and if $b \leq^- a$, then Theorem 12 ensures that g and $h = gbg$ are generalized inverses of b. In fact, h is a reflexive generalized inverse of b and we have

$$bh = b(gbg) = bg \tag{20}$$

and

$$hb = (gbg)b = gb. \tag{21}$$

Thus, $h \leq^- g$. So, given a reflexive generalized inverse a and an element $b \leq^- a$, we have proved that there exists an outer inverse h associated with b such that $h \leq^- g$. Now a natural question that arises from this observation is 'given an outer inverse (associated with an element $b \leq^- a$), does there exist a reflexive generalized inverse that dominates it?'. We make the following conjecture:

Conjecture 1. *Given an outer inverse x of a regular element a in a semigroup S, there exists a reflexive generalized inverse $y \in \{a_r^-\}$ such that $x \leq^- y$.*

We prove this conjecture in Corollary 14, if S is an associative ring.

The following theorem relates a reflexive generalized inverse of a regular element to its outer inverses, whenever the algebraic structure under discussion is an associative ring.

Theorem 13. *Let R be an associative ring and let $a, b, c \in R$ be such that a is regular and $a = b + c$. If $b, c \leq^- a$, then*

$$\{a_r^-\} = [a^=]_b + [a^=]_c, \tag{22}$$

where

$$[a^{=}]_b + [a^{=}]_c = \{x + y : x \in [a^{=}]_b \ \textit{and}\ y \in [a^{=}]_c\}. \tag{23}$$

Further, for every $g \in \{a_r^-\}$ the decomposition $g = x + y$, where $x \in [a^{=}]_b$ and $y \in [a^{=}]_c$, is unique and

$$x, y \leq^- g. \tag{24}$$

Proof. Consider any two elements $x \in [a^{=}]_b$ and $y \in [a^{=}]_c$. From the definition of $[a^{=}]_b$ and $[a^{=}]_c$ as given in Remark 2 and (19), it follows that x, y are reflexive generalized inverses of b, c, respectively, and further $xc = cx = yb = by = 0$. One verifies that $x + y \in \{a_r^-\}$ and $x, y \leq^- x + y$. Hence $\{a_r^-\} \supseteq [a^{=}]_b + [a^{=}]_c$.

Let g be a reflexive generalized inverse of a, and suppose that $b, c \leq^- a$. Then $g = gbg + gcg$. It follows from (20) and (21) that both gbg and gcg are dominated by g. Now for $h = gbg$, we have

$$\begin{aligned} ha &= (gbg)a \\ &= gb \quad (\because b \in Ra) \\ &= gbgb = hb \quad (\because gb \text{ is idempotent}). \end{aligned}$$

Similarly, we have $ah = bh$. It also follows from (19) that $h \in [a^{=}]_b$. The inclusion $k = gcg \in [a^{=}]_c$ can be proved in a similar fashion. Therefore, $\{a_r^-\} \subseteq [a^{=}]_b + [a^{=}]_c$.

Now for $g \in \{a_r^-\}$, let $g = h + k$ for some $h \in [a^{=}]_b$ and $k \in [a^{=}]_c$. Then $hb = ha$, $bh = ah$ and $h \leq^- g$ as observed in the beginning of the proof. Therefore $h = hbh = (gah)b(hag) = (gbh)b(hgb) = gbg$. One can prove that $k = gcg$ in the same way, so uniqueness of the decomposition follows. $\square$

Corollary 14. *Let a be a regular element in an associative ring R and x be an outer inverse of a. Then there exists a reflexive generalized inverse y of a (i.e., $y \in \{a_r^-\}$) such that $x \leq^- y$.*

Proof. If $x \in \{a^{=}\}$, (ii) of Lemma 11 ensures that $b = axa$ is a regular element such that $b \leq^- a$ and $x \in [a^{=}]_b$. Theorem 12 ensures that $c = a - b$ is regular and $c \leq^- a$. Now for any $y \in [a^{=}]_c$, it follows from Theorem 12 and (24) that $g = x + y \in \{a_r^-\}$ and $x \leq^- g$. $\square$

Before addressing right–left symmetry of the decomposition, we consider the following lemma, in which the statements (i) and (iii) are not trivial. In fact, even the statement '$b, c \in aR$' is not trivial, even though $aR = bR \oplus cR$.

Lemma 15. *Let a be a regular element in a ring R and let b, c be elements in R such that $a = b + c$ and $aR = bR \oplus cR$. Then the following statements are equivalent.*

(i) $b \in bR$.
(ii) *b is regular.*
(iii) $c \in cR$.
(iv) *c is regular.*
(v) $\{a^-\} \subseteq \{b^-\}$.
(vi) $\{a^-\} \subseteq \{c^-\}$.

Proof. The implications (ii) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (iii) follow from Lemma 2, and (v) $\Rightarrow$ (ii), and (vi) $\Rightarrow$ (iv) are trivial. Let $b \in bR \subseteq aR$. Therefore $b = aa^-b$ for every a^- and hence $b = ba^-b + ca^-b$. This in turn gives $b = ba^-b$ and $ca^-b = 0$ as $b \in bR$ and $bR \cap cR = (0)$. Thus (i) $\Rightarrow$ (ii), (v) are proved. This shows the equivalence of (i), (ii), and (v). The equivalence of (iii), (iv) and (vi) can be proved in a similar fashion.

Consider $b+c = ba^-a+ca^-a$. Now the equivalence of (i) and (iii) follows immediately from directness of $aR = bR \oplus cR$. □

If R has a multiplicative identity, then the statements (i) and (iii) of Lemma 15 are trivially true and in fact,

$$b \in bR \subseteq aR \tag{25}$$

is immediate.

Analogous to Lemma 15, we have the following.

Lemma 16. *Let a be a regular element in the ring R and let b, c be elements in R such that $a = b + c$ and $Ra = Rb \oplus Rc$. Then the following statements are equivalent.*

(i) $b \in Rb$.
(ii) *b is regular.*
(iii) $c \in Rc$.
(iv) *c is regular.*
(v) $\{a^-\} \subseteq \{b^-\}$.
(vi) $\{a^-\} \subseteq \{c^-\}$.

Now we extend the right–left symmetry theorem to the general case.

Theorem 17. *Let a be a regular element in an associative ring R and let $b, c \in R$ be such that $a = b + c$. Then the following statements are equivalent.*

(i) $aR = bR \oplus cR$ *and* $b \in bR$.
(ii) $b \in aR \cap Ra$ *and* $\{a^-\} \subset \{b^-\}$.
(iii) $Ra = Rb \oplus Rc$ *and* $b \in Rb$.
(iv) *b, c are regular and* $bR \cap cR = (0) = Rb \cap Rc$.

Proof. Let $aR = bR \oplus cR$ and $b \in bR$. If a^- is a generalized inverse of a, then $b + c = ba^-a + ca^-a$ and from $aR = bR \oplus cR$ it follows that $b = ba^-a$, that is, $b \in Ra$. Therefore $b \in aR \cap Ra$ and $b = aa^-b = ba^-a$. From the identity $b = aa^-b = ba^-b + ca^-b$, the inclusion $b \in bR \subseteq aR$ ensures that $ca^-b \in bR \cap cR$. So, from the directness of $aR = bR \oplus cR$, it follows that $ca^-b = 0$ and therefore $b = ba^-b$. Thus, (i) $\Rightarrow$ (ii).

Suppose that b is regular. Then $b \in bR$ is immediate from Lemma 2. From (ii) $\Rightarrow$ (i) of Theorem 12, it follows that (ii) implies $b \leq^- a$. Let $ag_b = bg_b$ and $g_b a = g_b b$ for some $g_b \in \{b^-\}$. So, we have $ag_b b = b$ which implies $b \in aR$ and, therefore $c \in aR$ and $aR = bR + cR$. Directness follows from $bg_b b = b$ and $bg_b c = 0$. Hence (ii) $\Rightarrow$ (i) is proved.

The proof of (ii) $\Leftrightarrow$ (iii) is similar to that of (ii) $\Leftrightarrow$ (i).

Having proved the equivalence of (i), (ii), and (iii), the statement (iv) follows trivially whenever any of the first three holds.

Let b and c satisfy (iv). From the regularity of b and c, it follows that $b \in bR \cap Rb$ and $c \in cR \cap Rc$ (by Lemma 2). For any a^-, consider $b + c = a = aa^-a = aa^-b + aa^-c$. Now, $Rb \cap Rc = (0)$ and $b \in Rb$ imply that $b = aa^-b$, and therefore $b \in aR$. Similarly, by taking $b + c = ba^-a + ca^-a$, $bR \cap cR = (0)$ implies that $b = ba^-a$ and therefore $b \in Ra$ is proved. Now substitute $a = b + c$ in $b = ba^-a$ to get $b = ba^-b + ca^-b$. Now, $bR \cap cR = (0)$ implies that $b = ba^-b$ for all a^-. Thus, (iv) $\Rightarrow$ (ii) is proved. □

In Theorem 17 (i), the condition '$b \in bR$' could be replaced by any of the equivalent conditions given in Lemma 15. Similarly, the condition '$b \in Rb$' in (iii) of the theorem could be replaced by any of the equivalent conditions given in Lemma 16. Also, the condition '$b \in aR \cap Ra$' in (ii) of the theorem could be replaced by '$c \in aR \cap Ra$', and 'b is regular' in (iv) could be replaced by 'c is regular'.

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