On Canonical Consistency of 2-path Signed Graphs

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Declaration

I do hereby declare that the work contained in this dissertation entitled "**On Canonical Consistency of 2-path Signed Graphs**" has done by me, under the supervision of Dr. Biswajit Deb, Assistant Professor, Department of Mathematics, SMIT in partial fulfillment for the award of the degree of Master of Science and this work has not been submitted elsewhere for a degree or diploma.

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Certificate

This is to certify that the dissertation entitled "**On Canonical Consistency of 2-path Signed Graphs**" by ANJAN GAUTAM, a student of Department of Mathematics, SMIT, Sikkim in partial fulfillment for the award of the degree of Master of Science has been carried out under the supervision of Dr. Biswajit Deb and that this work has not been submitted elsewhere for a degree or diploma.

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Abstract

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A marked graph is a graph with a $+$ or $-$ sign on each vertex and is called consistent if each cycle has an even number of signs. This concept is motivated by problems of communication networks and social networks. Formally, a marked graph is a graph $G(V, E)$ together with a function $\mu : V \to \{-1, 1\}$. The function μ is known as the marking of the marked graph and for any $v \in V$, we call $\mu(v)$ to be the mark of v. By mark of a cycle in a marked graph we mean the product of the mark of the vertices in the cycle. A marked graph in which every cycle has a positive mark is called a consistent marked graph. A signed graph $\Sigma = (G, \sigma)$ is a graph $G(V, E)$ together with a function $\sigma : E \to \{-1, 1\}$. For any signed graph $\Sigma = (G, \sigma)$, the marking $\mu : V \to \{-1, 1\}$ defined by

$$\mu(v) = \begin{cases} +1, & \text{if } v \text{ is isolated;} \\ \prod\limits_{u \in N(v)} \sigma(uv), & \text{otherwise;} \end{cases}$$

where $N(v)$ is the open neighborhood of v in G is said to be the canonical marking. In this dissertation we have investigated the relationship of balancedness and canonical consistency of a signed graph to its 2-path signed graphs.

Contents

List of Symbols

Important symbols are listed in the following table. Within the table G and v denotes a graph and a vertex in G, respectively.

$V(G)$	:	vertex set of G
$E(G)$	:	edge set of G
C_v^u	:	a configuration with the robot at u and the hole at v.
$D(G)$	:	the diameter of G
$\delta(G)$	:	minimum degree of G
$\Delta(G)$	:	maximum degree of G
$N_G(v)$ or $N(v)$	:	neighborhood of v
$\overline{N(v)}$	:	the set $N(v) \cup \{v\}$
$G - v$	:	the induced subgraph of G with vertex set $V(G) - \{v\}$
P_n	:	the path of order n
C_n	:	the cycle of order n
K_n	:	the complete graph on n vertices
$v \overset{r}{\leftarrow} u$	:	simple robot move from u to v
$v \overset{o}{\leftarrow} u$	:	simple obstacle move from u to v
$v \xleftarrow[m]{r} u$	:	mRJ-move of the robot from u to v
$\|A\|$	:	cardinality of the set A
$\mathbb{W}$	:	the set of whole numbers
$\mathbb{Z}$	:	the set of integers
$\mathbb{N}$	:	the set of natural numbers

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Chapter 1

Introduction

1.1 Definitions

Conceptually, a graph is a mathematical model that represents the binary relationship among a collection of well defined objects. Formally, a *graph G* consists of two finite non-empty sets V and E. The elements of V are known as the vertices of the graph G. The set E consists of unordered pairs of distinct vertices and are known as the edges of G. We use $G(V, E)$ or G to represent a graph G with vertex set V and edge set E. Two vertices u, v in G are said to be adjacent if $\{u, v\} \in E(G)$.

The graph G with vertex set $V = \{1, 2, 3, 4, 5, 6\}$ and edge set

$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{1, 6\}, \{2, 5\}\}$$

is shown in the Figure **??** A walk in a graph G is a sequence of vertices $v_1, v_2, ..., v_n$ in which v_i is adjacent to v_{i+1}, for $i = 1, 2, \cdots, n-1$. By a *path* in a graph G we mean a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ in which v_i is adjacent to v_{i+1}, for $i = 1, 2 \ldots n-1$. A path in which end vertex and the start vertex are same is called a *cycle*. A graph that has a cycle is known as a *cyclic* graph; otherwise, it is said to be acyclic. A graph G is said to be *connected* if each pair of vertices in a graph G are connected by a path. A *tree*

Figure 1.1: The graph G.

is a graph that is connected and acyclic. For basic terminologies used in this dissertation we refer [?]. The walk $1-2-3-4-5-2-1$ in the Figure **??** is

Figure 1.2: The graph G.

represented by the red edges. The green line in the Figure **??** represents the path $1-2-5-3-4$. The blue line in the Figure **??** represents the cycle $2-5-3-2$.

Figure 1.3:

Figure 1.4:

1.2 Basic Results

Theorem 1.1. *Total degree of the vertices of any graph G is twice the number of edges in it.*

Theorem 1.2. *The number of odd vertices in any graph G is even.*

Theorem 1.3. *A connected graph on p-vertices and q-edges with $q \geq p$ contains a cycle of length at least $\delta(G) - 1$, where $\delta(G)$ is the minimum degree of the graph.*

Theorem 1.4. *Let G be a graph with p vertices and q edges. The following are equivalent:*

(a). G is a tree.

(b). G is connected and $q = p - 1$.

(c). G is acyclic and $q = p - 1$.

A graph that has exactly one cycle is known as an *uni-cyclic graph.*

Theorem 1.5. *A connected (p, q)-graph is unicyclic if and only if $q = p$.*

Proof. Let $G = (p, q)$ be a connected graph. First suppose that $G = (p, q)$ is unicyclic. Then for any cyclic edge e of G, the graph $G - e$ is a tree and so $q(G - e) = p(G - e) - 1$ i.e. $q - 1 = p - 1$ i.e. $q = p$.

Next assume that $q = p$. We need to show that G is unicyclic. Let $G = (p, q)$ be acyclic. Since $G = (p, q)$ is connected and acyclic, so it is a tree and hene $q = p - 1$, which is a contradiction. Therefore G is unicyclic. Also, for any cyclic edge e of G, the graph $G-e$ is connected and $q = p-1$ i.e. $G-e$ is acyclic. Hence, removal of any cyclic edge makes the graph G acyclic. Therefore G has exactly one cycle or G is unicyclic. $\square$

1.3 Brief Review of the Literature

The concept of signed graphs and balance signed graphs were introduced by Frank Harary to treat a question in social psychology [?]. It is remarkable that years before Harary, Konig [?] had the idea of a graph with a distinguished subset of edges, in a way we now recognize as equivalent to signed graphs and even proved Harary's balance theorem and defined switching in the form of taking the sum of a cut-set with negative edges. Data in the social sciences can often be modeled using a signed graph, a graph where every edge has a sign $+$ or $-$,or a marked graph, a graph where every vertex has a sign $+$ or $-$. A marked graph is called consistent if every cycle has an even number of vertices with $-$ sign. The concept of consistency is analogous to the concept of balance in signed graphs: A signed graph is called balanced if every cycle has an even number of edges with $-$ sign.

Given signed graph G, a marking of G induces naturally as follows: sign of a vertex $v \in V(G)$ is the product of the sign of the edges incident with it. This marking of a signed graph G is known as canonical marking. A signed graph that is consistent with respect to its canonical marking is known as a canonically consistent signed graph.

Many results related to characterization of consistent marked graphs are available in the literature. Beineke and Harary [**?**, **?**] were the first to pose the problem of characterizing consistent marked graphs, which was eventually settled independently by Acharya in [**?**, **?**], Hoede in [**?**] and Rao in [**?**]. In [**?**] it was shown that a marked graph is consistent if and only, for any spanning tree T all fundamental cycles are positive and all common paths of pairs of fundamental cycles have end points with the same marking. Further characterizations of consistent marked graphs have been obtained by Roberts and Xu in [**?**, **?**]. A characterization of canonically consistent total signed graphs is discussed in [**?**].

In this dissertation, we have discussed the characterization of canonically consistent 2-path signed graphs and 2-path product signed graphs. is discussed in We also tried to address the following questions:

Let G be any signed graph and $G\#G$ be the 2-path signed graph of G. Define

$$S = \{G \mid G \text{ is balanced and canonically consistent}\}$$

$$S_1 = \{G \in S \mid G\#G \text{ is balanced and canonically consistent}\}$$

$$S_2 = \{G \in S \mid G\#G \text{ is balanced but not canonically consistent}\}$$

$$S_3 = \{G \in S \mid G\#G \text{ is canonically consistent but not balanced}\}$$

$$S_4 = \{G \in S \mid G\#G \text{ is neither canonically consistent nor balanced}\}$$

1. Is $S_1 = \Phi$?

2. Is $S_2 = \Phi$?

3. Is $S_3 = \Phi$?

4. Is $S_4 = \Phi$?

5. If $S_1 \neq \Phi$ then under what conditions $G \in S \Rightarrow G \in S_1$?

6. If $S_2 \neq \Phi$ then under what condition $G \in S \Rightarrow G \in S_2$?

7. If $S_3 \neq \Phi$ then under what condition $G \in S \Rightarrow G \in S_3$?

8. If $S_4 \neq \Phi$ then under what condition $G \in S \Rightarrow G \in S_4$?

Chapter 2

Signed and Marked Graphs

2.1 Signed Graphs

Frank Harary first introduce the concept of signed graphs and balance signed graphs to treat a question in social psychology [**?**]. A graph $G(V, E)$ together with a function $\sigma : E \longrightarrow \{-1, 1\}$ is known as a *signed graph*. The function σ is known as the *sign function* or *signature* of the signed graph. A singed graph is denoted by $\Sigma = (G, \sigma)$, where G is the underlying graph and σ is the sign function. To each non-empty subgraph Σ' of Σ we assign a sign $\sigma(\Sigma')$ which is the product of the signs of the edges in Σ'.

Figure 2.1: A Signed Graph $\sum = (G, \sigma)$.

Figure 2.2: A subgraph Σ' of the Graph $\Sigma = (G, \sigma)$.

Consider the subgraph Σ' in the Figure **??** of the signed graph $\Sigma = (G, \sigma)$ in the Figure **??**. The sign of Σ' is given by

$$\sigma(\Sigma') = (-1)(+1)(+1)(-1)(-1) = -1$$

A graph $G(V, E)$ together with a function $\mu : V \longrightarrow \{-1, 1\}$ is known as a *marked graph*. The function μ is known as the *marking function* of the marked graph. To each non-empty subgraph Σ'_μ of a marked graph Σ we assign a mark $\mu(\Sigma'_\mu)$ which is the product of the markings of the vertices in Σ'_μ.

Figure 2.3: A Marked Graph.

A *marked signed graph* $\Sigma_\mu = (\Sigma, \mu)$ is a marked graph whose underlying graph $\Sigma = (G, \sigma)$ is a signed graph.

Figure 2.4: A Marked Signed Graph.

2.2 Balanced Signed Graphs

A signed graph $\Sigma = (G, \sigma)$ is said to be *balanced* if each cycle in G has positive sign. The signed graph in the Figure **??** is balanced but the signed graph in the Figure **??** is not balanced.

Figure 2.5: A Balanced Signed Graph.

Two important signatures associated with a graph G are the all-positive one, denoted by $+G = (G, +)$, and the all- negative one, denoted by $-G = (G, -)$, where every edge has the same sign. In most ways an unsigned graph G behaves like $+G$, while $-G$ acts rather like a generalization of a bipartite graph. In particular, in $+G$ every cycle is positive. In $-G$ the even cycles are positive while the odd ones are negative, so $-G$ is balanced if and only if G bipartite. The following fundamental result introduced by Frank Harary in [3] gives a characterization of the balanced signed graphs.

Theorem 2.1. (Harary's Balance Theorem). A signed graph Σ is balanced if and only if there is a bipartition of its vertex set, $V = X \cup Y$, such that every positive edge is induced by X or Y while every negative edge has one endpoint in X and one in Y. Also, if and only if for any two vertices v, w, every path between them has the same sign.

2.3 Consistent Signed Graphs

Beineke and Harary [**?**] raised the problem of characterizing consistent marked graphs, a marked graph in which mark of every cycle has positive mark. The mark $\mu(\Sigma')$ of a nonempty subgraph Σ' of Σ_μ is then defined as the product of the marks of the vertices in Σ'. A cycle Z in Σ_μ is said to be consistent if $\mu(Z) = +1$; otherwise, it is said to be inconsistent. Further, Σ_μ is said to be consistent if every cycle in it is consistent. Otherwise it is said to be inconsistent.

Figure 2.6: A Consistent Marked Graph $\sum = (G, \sigma)$.

The following characterization of consistent marked graphs is given by Hoede [**?**].

Theorem 2.2. *A marked graph $\sum = (G, \sigma)$ is consistent if and only if, for any span- ning tree T of G the following holds:*

(i) all fundamental cycles are consistent.

(ii) all common paths of pairs of fundamental cycles have end points with the same marking.

Figure 2.7: An Inconsistent Marked Graph

Roberts and Shaoji gave further characterization of consistent marked graphs in [**?**].

2.4 Canonically Consistent Signed Graphs

Given a signed graph $\sum = (G, \sigma)$ we can associate a natural marking

$$\mu \ : \ V(G) \rightarrow \{-1, 1\}$$

as follows: For any vertex $v \in V(G)$

$$\mu(v) = \begin{cases} +1, & \text{if } v \text{ is isolated;} \\ \prod\limits_{u \in N(v)} \sigma(uv), & \text{otherwise;} \end{cases}$$

where $N(v)$ is the open neighborhood of v in G. This marking μ is known as the *canonical marking* of the signed graph $\sum$. Further, the signed graph $\sum$ is said to be *canonically consistent* if it is consistent with respect to the canonical marking.

Figure 2.8:

The signed graph in the Figure **??** represents a canonically consistent signed graph. But the signed graph in the Figure **??** is not canonically consistent. In this chapter, we tried to characterize certain classes of signed graphs that are canonically consistent.

Figure 2.9:

Proposition 2.1. *Every signed cycle is canonically consistent.*

Proof. Let $\Sigma = (C, \sigma)$ be any signed graph, where C is a cycle. We need to prove that Σ is canonically consistent. Let $\Sigma_\mu = (\Sigma, \mu)$ be the canonical marked signed graph of Σ. If $R(\sigma) = \{+\}$ or $R(\sigma) = \{-\}$, then every vertex of Σ will get mark $+$ with respect to canonical marking and so it must be canonically consistent.

So, let Σ has both positive and negative edges. We apply induction on the number of negative edges to prove this result. Let Σ has n negative edges.

Base Step: Let $n = 1$ and $e = uv$ be the negative edge in G. Then according to canonical marking every vertex will get mark $+$ except the vertices u, v in Σ. Hence the number of vertices with negative marking in Σ_μ is even and so it is consistent. Hence Σ is canonically consistent.

Induction Step: As an induction hypothesis, assume that the result holds for fewer than n negative edges in Σ and $e = uv$ be any edge in Σ with negative sign. Let Σ' be the signed graph obtained from Σ by switching the sign of e to $+$. Thus Σ' has $n-1$ negative edges and so by induction hypothesis it is canonically consistent. But mark of the vertices in Σ_μ and Σ'_μ differs only at the vertices u and v. So, the number of negative vertices in Σ_μ is even as it is so in Σ'_μ. Hence Σ_μ is consistent and which in turn implies that Σ is canonically consistent.

$\square$

Chapter 3

2-Path Signed Graphs

3.1 2-Path Product Signed Graphs

Let $G = (V, E, \sigma)$ be a signed graph. The 2-path product signed graph $G\hat{\#}G = (V, E', \sigma')$ is defined as follows: The vertex set is same as the original signed graph G and two vertices $u, v \in V(G\hat{\#}G)$, are adjacent if and only if there exists a uv-path of length two in G. The sign $\sigma'(uv) = \mu(u)\mu(v)$, μ is the canonical marking. The Figure **??** is the 2-path product signed graph of the signed graph $K_{4,1}^{-}$ in the Figure **??**.

Figure 3.1: The signed graph $K_{4,1}^{-}$.

The following result about 2-path product signed graph is cited from [**?**].

Proposition 3.1. *2-path product signed graph of a signed graph S is always balanced.*

Figure 3.2: The 2-path product signed graph of $K_{4,1}^{-}$.

We noticed the following about canonical consistency of the 2-path product signed graph of a signed graph.

Proposition 3.2. *2-path product signed graph of a signed graph S is always canonically consistent.*

3.2 2-Path Signed Graphs

Let $G = (V, E, \sigma)$ be a signed graph. The *n-path signed graph* $G\#G = (V, E', \sigma')$ is defined as follows: The vertex set is same as the original signed graph G and two vertices $u, v \in V(G\#G)$, are adjacent if and only if there exists a uv-path of length n in G. The sign $\sigma'(uv)$ is -1 whenever in every uv-path of length n in G all edges are negative.

Let G be any signed graph and $G\#G$ be the 2-path signed graph of G. Define

$$\begin{aligned}
S &= \{G \mid G \text{ is balanced and canonically consistent}\}\\
S_1 &= \{G \in S \mid G\#G \text{ is balanced and canonically consistent}\}\\
S_2 &= \{G \in S \mid G\#G \text{ is balanced but not canonically consistent}\}\\
S_3 &= \{G \in S \mid G\#G \text{ is canonically consistent but not balanced}\}\\
S_4 &= \{G \in S \mid G\#G \text{ is neither canonically consistent nor balanced}\}
\end{aligned}$$

The following proposition shows that $S_1 \neq \Phi$.

Proposition 3.3. *For any graph G, $G^+ \# G^+$ is balanced and canonically consistent.*

Proof. For any graph G, G^+ denotes the corresponding signed graph with all edges positive and so $G \in S$. So, all edges in $G^+ \# G^+$ are positive. Hence the result. $\square$

Remark 3.1. $C^-_{2n+1} \notin S$ *but* $C^-_{2n} \in S$.

Proposition 3.4. $C^-_{2n} \in S_1$ *if and only if n is even.*

Proof. First suppose that $C^-_{2n} \in S_1$. We need to show that n is even. If possible let n be odd. Then $C^-_{2n} \# C^-_{2n}$ will consist of two copies of C^-_n. Since n is odd, so the component C^-_n will have sign -1. Hence $C^-_{2n} \# C^-_{2n}$ is not balanced, a contradiction. So, n must be even.

Conversely, suppose that n is even. We consider the following cases:

CASE 1: $n = 2$.

In this case $C^-_{2n} \# C^-_{2n}$ is acyclic and so it must be balanced and canonically consistent.

CASE 2: $n \geq 4$ i.e. $n = 4, 6, 8, \ldots$ Here for each n, $C^-_{2n} \sharp C^-_{2n}$ consist of two copies of C^-_n. Since for even n, C^-_n is balanced and canonically consistent, so $C^-_{2n} \# C^-_{2n}$ is balanced and canonically consistent. Hence $C^-_{2n} \in S_1$. $\square$

Remark 3.2. *For any positive integer n, let $\Sigma = (K_{n,1}, \sigma)$. Then $\Sigma \# \Sigma$ will consist of two components with underlying graphs as K_n and K_1. Further, the component K_1 will consist of the center of $K_{n,1}$.*

Remark 3.3. *For $n = 1, 2$, if $\Sigma = (K_{n,1}, \sigma)$ then $\Sigma \# \Sigma$ will be acyclic and so $\Sigma \in S_1$.*

Proposition 3.5. *If $\Sigma = (K_{n,1}, \sigma)$ and $n \geq 3$, then $\Sigma \in S_1$ if and only if Σ has at most one edge with negative sign.*

Proof. First suppose that, Σ has at most one edge with negative sign. Then each edge of $\Sigma\#\Sigma$ will be positive and the result follows.

Conversely, suppose $\Sigma \in S_1$. If possible let, Σ has more than one negative edge. Let v be the center and u_1, u_2, u_3 be any three pendent vertices. Then the vertices u_1, u_2, u_3 will form a triangle in $\Sigma\#\Sigma$. So, it is enough to consider the following cases:

CASE 1: Σ has two negative edges. In particular, let the edges $\{v, u_1\}$ and $\{v, u_2\}$ be negative. In this case, the triangle $u_1u_2u_3u_1$ in $\Sigma\#\Sigma$ will have one negative edge $\{u_1, u_2\}$ and so $\Sigma\#\Sigma$ is not balanced.

CASE 2: Σ has three negative edges. In particular, let the edges $\{v, u_1\}, \{v, u_2\}$ and $\{v, u_3\}$ be negative. In this case, the triangle $u_1u_2u_3u_1$ in $\Sigma\#\Sigma$ will have all edges negative and so $\Sigma\#\Sigma$ is not balanced.

In both the cases, $\Sigma\#\Sigma$ not balanced, a contradiction. So, Σ can have at most one edge with negative sign. □

Remark 3.4. *2-path signed graph of C_{2n+1} is isomorphic to itself.*

Theorem 3.1. *If there are even number of consecutive negative edges in a cycle other than $\{\ C^-_{2n},\ n$ is even$\}$ and C_4 then it will belong to S_3.*

Proof. Let us consider there is even number of consecutive negative edges in a cycle C_n, where $n = 3, 4, 5, \dots\ \}$

CASE 1: If n is odd. Then 2-path signed graph of C_n is isomorphic to itself. For even number of consecutive negative edges $2k$ (say) its 2-path signed graph will have $2k - 1$ negative edges, i.e. odd number of negative edges in cycle C_n i.e.$C_n\#C_n$ is not balanced but canonically consistent. Hence, $C_n \in S_3$

CASE 2: If n is even.

For $n = 4$ in any case 2-path signed graph of C_4 does not contains any cycle hence it is balanced and canonically consistent.

Again we know that a graph $C_{2n}^- \in S_1$ if and only if n is even.

Now for other cycles C_n, its 2-path signed graph will consists two copies of $C_{n/2}$. If component $C_{n/2}$ is odd cycle then same as CASE 1 we can say that in either of component $C_{n/2}$ there is odd number of negative edges, i.e. $C_n \# C_n$ is not balanced but canonically consistent. Hence, $C_n \in S_3$

If $C_{n/2}$ is even cycle, then it will be less than n. Now with same argument as above either of component $C_{n/2}$ will have odd number of negative edges i.e. $C_n \# C_n$ is not balanced but canonically consistent. Hence, $C_n \in S_3$. $\square$

Proposition 3.6. *$C_{2n}^- \in S_3$ if and only if n is odd.*

Proof. First suppose that $C_{2n}^- \in S_3$. We need to show that n is odd. If possible let n be even. Then $C_{2n}^- \# C_{2n}^-$ will consist of two copies of C_n^-. Since n is even so the component C_n^- will be balanced, a contradiction. So n must be odd.

Conversely, suppose that n is odd. Then $C_{2n}^- \# C_{2n}^-$ will consist of a pair of disjoint cycles C_n^-. Since n is odd C_n^- is not balanced but canonically consistent. So $C_{2n}^- \# C_{2n}^-$ is not balanced but canonically consistent. Hence $C_{2n}^- \in S_3$. $\square$

Remark 3.5. *If $\Sigma = (C_n, \sigma)$, then $n \geq 3$ the underlying graph of $\Sigma \# \Sigma$ will consist of either a cycle or a pair disjoint cycle. So, $\Sigma \# \Sigma$ will be canonically consistent and hence $\Sigma \notin S_4$.*

The 2-path signed graph of the signed graph Σ_1 in the Figure **??** is shown in the Figure **??** and it is evident that $\Sigma_1 \in S_4$. Hence $S_4 \neq \Phi$.

Figure 3.3: The signed graph Σ_1.

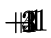

Figure 3.4: The 2-path signed graph of Σ_1.

3.3 Conclusion

In this dissertation we have shown that the sets S_1, S_3, S_4 are non empty. Further we have identified few classes of graphs that belongs to S_1 and S_3. But, we have not been able to get an example of a signed graph that belongs to S_2. We have a strong belief that S_2 is empty.

References

[1] B. Devdas Acharya. A characterization of consistent marked graph. *National Academy Science Letters.*, 6:431–440, 1983.

[2] B. Devdas Acharya. Some further properties of consistent marked graphs. *Indian Journal of Pure and Applied Mathematics*, 15(8):837 – 842, 1984.

[3] Lowell W. Beineke and Frank Harary. Consistency in marked digraphs. *Journal of Mathematical Psychology*, 18:260–269, 1978.

[4] Lowell W. Beineke and Frank Harary. Consistent graphs with signed points. *Rivista di matematica per le scienze economiche e sociali*, 1(2):81–88, Sep 1978.

[5] Deepa Sinha and Pravin Garg. A characterization of canonically consistent total signed graphs. *Notes on Number Theory and Discrete Mathematics.*, 19(3):70 – 77, 2013.

[6] Miklos Bona. *Handbook of Enumerative Combinatorics.* CRC Press, 2015.

[7] D. Cartwright and F. Harary. Structural balance: a generalization of heider's theory. *Psychological Review.*, 63(5):277 – 293, 1956.

[8] Frank Harary. On the notion of balance of a signed graph. *Michigan Math. J.*, 2(2):143–146, 1953.

[9] Frank Harary. *Graph Theory.* Addison-Wesley, Reading, MA, 1969.

[10] Frank Harary and Jerald A. Kabell. A simple algorithm to detect balance in signed graphs. *Mathematical Social Sciences*, 1(1):131 – 136, 1980.

[11] Frank Harary and Jerald A. Kabell. Counting balanced signed graphs using marked graphs. *Proceedings of the Edinburgh Mathematical Society*, 24(2):99–104, 1981.

[12] Cornelis Hoede. A characterization of consistent marked graphs. *Journal of Graph Theory*, 16(1):17–23, 1992.

[13] Denes Konig. *Theorie der endlichen und unendlichen graphen.* Chelsea Publishing Company, 1950.

[14] S.B Rao. Characterization of harmonious marked graphs and consistent nets. *Journal of combinatorics, information and system sciences.*, 9:97 – 112, 1984.

[15] Fred S. Roberts. On the problem of consistent marking of a graph. *Linear Algebra and its Applications*, 217:255 – 263, 1995. Proceedings of a Conference on Graphs and Matrices in Honor of John Maybee.

[16] Fred S Roberts and Shaoji Xu. Characterizations of consistent marked graphs. *Discrete Applied Mathematics*, 127(2):357 – 371, 2003. Ordinal and Symbolic Data Analysis (OSDA '98), Univ. of Massachusetts, Amherst, Sept. 28-30, 1998.

[17] Deepa Sinha and Pravin Garg. Balance and consistency of total signed graphs. *Indian Journal of Mathematics.*, 53(1):71– 81, 2011.

[18] Deepa Sinha and Pravin Garg. Characterization of total signed graph and semi-total signed graphs. *International Journal of Contemporary Mathematical Sciences.*, 6(5):221– 228, 2011.

[19] P. S. K Reddy and M S Subramanya. Note on path signed graphs. *Notes on Number Theory and Discrete Mathematics.*, 4(16), 2009.

[20] Deepa Sinha and Deepakshi Sharma. Characterization of 2-Path Product Signed Graphs with Its Properties. *Computational Intelligence and Neuroscience*, 2017.