Backpropagation Algorithms
Basics

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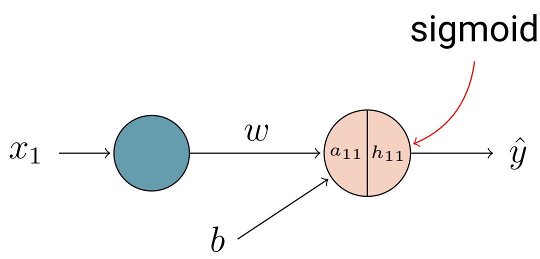
[1]:
```
import numpy as np
```

[281]:
```
import pandas as pd
import seaborn as sns
import matplotlib.pyplot as plt
sns.set()
```

Consider a single neuron with one output layer and no hidden layers. Assume that the output neuron uses the sigmoid activation function.



Write code to run Gradient descent algorithm (Batch, mini-batch, stochastic) for the following data

[7]:
```
X = np.array([3.5, 0.35, 3.2, -2.0, 1.5, -0.5])
Y = np.array([0.5, 0.50, 0.5,  0.5, 0.1,  0.3])
```

Lets first calculate some feed forwards.

[29]:
```
w_init = -1
b_init = 2

def sigmoid(a):
    return 1./(1+np.exp(-a))

## calculating a_11
for x in X:
    a_11 = w_init*x + b_init
    h_11 = sigmoid(a_11)
    print(h_11)
```

```
0.18242552380635635
0.8388910504234147
0.23147521650098232
0.9820137900379085
0.6224593312018546
0.9241418199787566
```

[32]:

```
## What is the error now?

for x,y in zip(X,Y):
    a_11 = w_init*x + b_init
    h_11 = sigmoid(a_11)
    err = h_11 - y
    print(err)
```

```
-0.31757447619364365
0.3388910504234147
-0.26852478349901765
0.48201379003790845
0.5224593312018546
0.6241418199787565
```

[33]:

```
for x,y in zip(X,Y):
    a_11 = w_init*x + b_init
    h_11 = sigmoid(a_11)
    err = (h_11 - y)**2
    print(err)
```

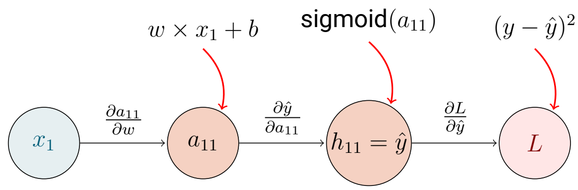
```
0.10085354792966714
0.11484714405708542
0.0721055593531943
0.2323372937867089
0.2729637527598892
0.3895530114463945
```

[34]:

```
err = 0
for x,y in zip(X,Y):
    a_11 = w_init*x + b_init
    h_11 = sigmoid(a_11)
    err += (h_11 - y)**2
print(err)
```

```
1.1826603093329395
```

1 Stochastic Gradient Descent



Loss Function: $L(w, b) = (y - \widehat{y})^2$.

Therefore,

$$\frac{\partial L}{\partial w} = \frac{\partial L}{\partial \widehat{y}} \times \frac{\partial \widehat{y}}{\partial a_{11}} \times \frac{\partial a_{11}}{\partial w}$$

Note that,

$$\begin{aligned}
\frac{\partial a_{11}}{\partial w} &= x_1 \\
\frac{\partial \widehat{y}}{\partial a_{11}} &= \frac{\partial}{\partial a_{11}} \left(sigmoid(a_{11})\right) \\
&= \frac{\partial}{\partial a_{11}} \left(\frac{1}{1 + e^{-a_{11}}}\right) \\
&= \frac{-1}{\left(1 + e^{-a_{11}}\right)^2} \times e^{-a_{11}} \times -1 \\
&= \frac{e^{-a_{11}}}{\left(1 + e^{-a_{11}}\right)^2} \\
&= \widehat{y}(1 - \widehat{y}) \quad (why?) \\
\frac{\partial L}{\partial \widehat{y}} &= 2(y - \widehat{y})(-1) = -2(y - \widehat{y})
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\partial L}{\partial w} &= \frac{\partial L}{\partial \widehat{y}} \times \frac{\partial \widehat{y}}{\partial a_{11}} \times \frac{\partial a_{11}}{\partial w} \\
&= -2(y - \widehat{y}) \times \widehat{y}(1 - \widehat{y}) \times x_1
\end{aligned}$$

[69]:
```
def grad_w(x,y,w,b):
    a_11 = w*x + b
    y_hat = sigmoid(a_11)
    grad = -2 * (y - y_hat) * y_hat * (1-y_hat) * x
    return grad
```

[48]:
```
for x,y in zip(X,Y):
    grad_w(x,y,w = 2, b=1)
```

```
0.001172544893670911
0.03159017550957496
0.001951232942678785
0.08178314932188695
0.046736251348533014
-0.05
```

1.1 Calculating $\frac{\partial L}{\partial b}$

Similarly,

$$\begin{aligned}\frac{\partial L}{\partial b} &= \frac{\partial L}{\partial \hat{y}} \times \frac{\partial \hat{y}}{\partial a_{11}} \times \frac{\partial a_{11}}{\partial b} \\ &= -2(y-\hat{y}) \times \hat{y}(1-\hat{y}) \times 1\end{aligned}$$

[68]:
```
def grad_b(x,y,w,b):
    a_11 = w*x + b
    y_hat = sigmoid(a_11)
    grad = -2 * (y - y_hat) * y_hat * (1-y_hat)
    return grad
```

[ ]:
```
x = 3.5
```

At the forward pass, we get

[49]:
```
w = -2; b = 1

a_11 = w*x + b
y_hat = sigmoid(a_11)
y_hat
```

[49]:
```
0.8807970779778823
```

But, the true y value is 0.5

So, what is the squared error loss?

[50]:
```
y = 0.5
err = (y-y_hat)**2
err
```

[50]:
```
0.14500641459649338
```

Let us now, update the w and b:

[70]:
```
eta = 0.9
print('grad w = ', grad_w(x=3.5,y=0.5,w=-2,b=1))
print('grad w = ', grad_b(x=3.5,y=0.5,w=-2,b=1))
```

```
grad w =  -0.008590091283831066
grad w =  -0.0024543117953803044
```

[73]:
```
w_new = w - eta * grad_w(x=3.5,y=0.5,w=-2,b=1)
b_new = b - eta * grad_b(x=3.5,y=0.5,w=-2,b=1)
print("w_new : ", w_new)
print("b_new : ", b_new)
```

```
w_new :  -1.992268917844552
b_new :  1.0022088806158422
```

Now, for the next pass, we will use the next row, and so on...

[76]:
```
w = -2; b = 1; eta = 0.9
for (x,y) in zip(X,Y):
    dw = grad_w(x=x,y=y,w=w,b=b)
    db = grad_b(x=x,y=y,w=w,b=b)
    w = w - eta * dw
    b = b - eta * db
```

[79]:
```
print(w,'\t',b) # updated w and b
```

```
-1.9284102929220686      0.8504362751346908
```

1.2 Running for many times to improve this

[115]:
```
epochs = 10
w = -2; b = 1; eta = 0.9
for i in range(epochs):
    for (x,y) in zip(X,Y):
        dw = grad_w(x=x,y=y,w=w,b=b)
        db = grad_b(x=x,y=y,w=w,b=b)
        w = w - eta * dw
        b = b - eta * db
```

[100]:
```
b
```

[100]:
```
-0.384384524671664
```

[101]:
```
y_hat = sigmoid(w*3.5 + b)
y_hat
```

[101]:
```
0.24528950816097767
```

[102]:
```
y = 0.5
err = (y-y_hat)**2
err
```

[102]:
```
0.06487743465287667
```

We will do a small adjustments to the code to keep track of the loss, w and b values at every epochs and plot them to understand what is going on.

[301]:
```
epochs = 10
eta = 0.9
# w = -7; b = 4;
w = 2; b = 1;

J = []

for i in range(epochs):
    err = 0
    for (x,y) in zip(X,Y):
        dw = grad_w(x=x,y=y,w=w,b=b)
        db = grad_b(x=x,y=y,w=w,b=b)
        w = w - eta * dw
        b = b - eta * db
        err += (y-sigmoid(w*x+b))**2
    J.append([w.round(4),b.round(4),(err/X.shape[0]).round(4)])
```

[302]:
```
journal = pd.DataFrame(J,columns=["w","b","Loss"])#.reset_index()
journal.index.name = "epochs"
journal
```

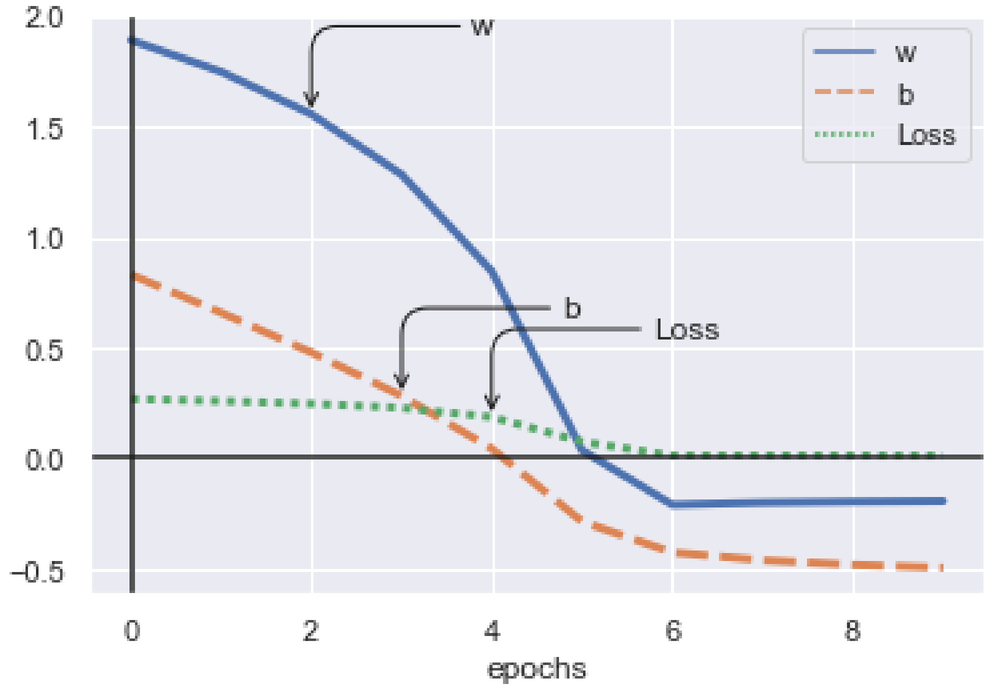
[302]:

epochs	w	b	Loss
0	1.8900	0.8304	0.2673
1	1.7471	0.6583	0.2587
2	1.5557	0.4788	0.2470
3	1.2830	0.2819	0.2277
4	0.8453	0.0430	0.1863
5	0.0362	-0.2862	0.0749
6	-0.2105	-0.4255	0.0118
7	-0.2009	-0.4602	0.0116
8	-0.1964	-0.4812	0.0115
9	-0.1935	-0.4938	0.0114

[355]:
```
offset=30
arrowprops = dict(
    arrowstyle="->",
    color='k',
    connectionstyle="angle,angleA=0,angleB=90,rad=10")
fig, ax = plt.subplots()
sns.lineplot(journal,linewidth = 3,ax=ax)
for (i,col) in enumerate(['w','b','Loss']):
    ax.annotate(col,
            xy=[i+2,journal.iloc[i+2][col]],
            xytext=(2*offset, offset),
```

```
             textcoords='offset points',
             arrowprops=arrowprops)

ax.axhline(color='k')
ax.axvline(color='k')
plt.show()
```



[356]:
```
sns.lineplot(journal,x = 'epochs',y='Loss',linewidth = 3,color='g')
```

[356]: <AxesSubplot:xlabel='epochs', ylabel='Loss'>