Dot Product, Angle of two vectors
Inner Product Spaces

- Length and Dot Product in R^n
- Inner Product Spaces
- Orthonormal Bases: Gram-Schmidt Process
- Least Square Analysis

Length and Dot Product in R^n

- Length :

The length of a vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ in R^n is given by

$$\|\mathbf{v}\| = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_n}^2} \quad (\|\mathbf{v}\| \text{ is a real number})$$

- Notes: The length of a vector is also called its norm

- Properties of length (or norm)

(1) $\|v\| \geq 0$

(2) $\|v\| = 1 \Rightarrow v$ is called a unit vector

(3) $\|v\| = 0$ if and only if $\mathbf{v}=0$

(4) $\|cv\| = |c| \; \|v\|$

- Ex 1:

(a) In R^5, the length of $\mathbf{v}=(0\,,-2\,,1\,,4\,,-2)$ is given by

$$\|\mathbf{v}\|=\sqrt{0^2+(-2)^2+1^2+4^2+(-2)^2}=\sqrt{25}=5$$

(b) In R^3, the length of $\mathbf{v}=\left(\frac{2}{\sqrt{17}}\,,\frac{-2}{\sqrt{17}}\,,\frac{3}{\sqrt{17}}\right)$ is given by

$$\|\mathbf{v}\|=\sqrt{\left(\frac{2}{\sqrt{17}}\right)^2+\left(\frac{-2}{\sqrt{17}}\right)^2+\left(\frac{3}{\sqrt{17}}\right)^2}=\sqrt{\frac{17}{17}}=1$$

(If the length of **v** is 1, then **v** is a unit vector)

- A standard unit vector in R^n: only one component of the vector is 1 and the others are 0 (thus the length of this vector must be 1)

$$R^2:\{\mathbf{e}_1,\mathbf{e}_2\}=\{(1,0),(0,1)\}$$

$$R^3:\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}=\{(1,0,0),(0,1,0),(0,0,1)\}$$

$$R^n:\{\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n\}=\{(1,0,\cdots,0),(0,1,\cdots,0),\cdots,(0,0,\cdots,1)\}$$

- Notes: Two nonzero vectors are parallel if $\mathbf{u}=c\mathbf{v}$

(1) $c>0 \;\Rightarrow\;$ $\mathbf{u}$ and $\mathbf{v}$ have the same direction

(2) $c<0 \;\Rightarrow\;$ $\mathbf{u}$ and $\mathbf{v}$ have the opposite directions

- Theorem 1.1: Length of a scalar multiple

Let $\mathbf{v}$ be a vector in R^n and c be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

Pf:

$$\mathbf{v} = (v_1, v_2, \cdots, v_n)$$

$$\Rightarrow c\mathbf{v} = (cv_1, cv_2, \cdots, cv_n)$$

$$\|c\mathbf{v}\| = \|(cv_1, cv_2, \cdots, cv_n)\|$$

$$= \sqrt{(cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2}$$

$$= \sqrt{c^2(v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2)}$$

$$= |c|\sqrt{v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2}$$

$$= |c|\,\|\mathbf{v}\|$$

- Theorem 1.2: How to find the unit vector in the direction of $\mathbf{v}$

  If $\mathbf{v}$ is a nonzero vector in R^n, then the vector $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ has length 1 and has the same direction as $\mathbf{v}$. This vector $\mathbf{u}$ is called the unit vector in the direction of $\mathbf{v}$

Pf:

$$\mathbf{v} \text{ is nonzero} \Rightarrow \|\mathbf{v}\| > 0 \Rightarrow \frac{1}{\|\mathbf{v}\|} > 0$$

$$\text{If } \mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} \quad (\mathbf{u} \text{ has the same direction as } \mathbf{v})$$

$$\|\mathbf{u}\| = \left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| \overset{\|c\mathbf{v}\| = |c|\|\mathbf{v}\|}{=} \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1 \quad (\mathbf{u} \text{ has length } 1)$$

- Notes:

  (1) The vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is called the unit vector in the direction of $\mathbf{v}$

  (2) The process of finding the unit vector in the direction of $\mathbf{v}$ is called **normalizing** the vector $\mathbf{v}$

- Ex 2: Finding a unit vector

Find the unit vector in the direction of **v** = (3, –1, 2), and verify that this vector has length 1

Sol:

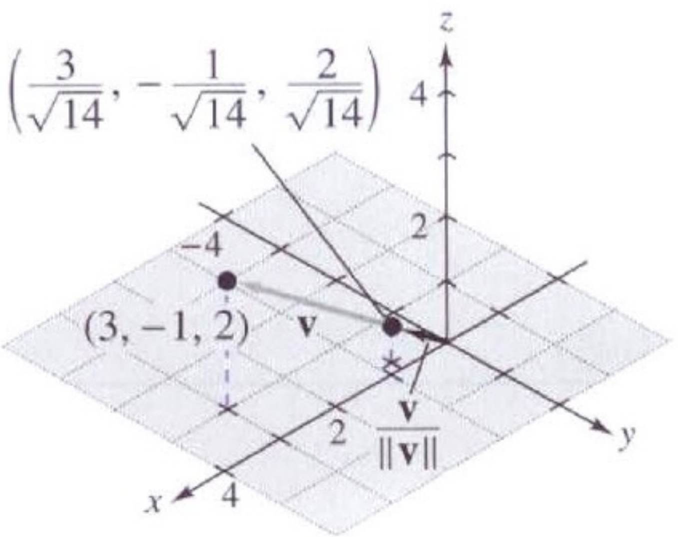
$$\mathbf{v}=(3\,,-1\,,2)\Rightarrow\|\mathbf{v}\|=\sqrt{3^2+(-1)^2+2^2}=\sqrt{14}$$

$$\Rightarrow\frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{(3\,,-1\,,2)}{\sqrt{3^2+(-1)^2+2^2}}=\frac{1}{\sqrt{14}}(3\,,-1\,,2)$$

$$=\left(\frac{3}{\sqrt{14}},\frac{-1}{\sqrt{14}},\frac{2}{\sqrt{14}}\right)$$

$$\because\ \sqrt{\left(\frac{3}{\sqrt{14}}\right)^2+\left(\frac{-1}{\sqrt{14}}\right)^2+\left(\frac{2}{\sqrt{14}}\right)^2}=\sqrt{\frac{14}{14}}=1$$

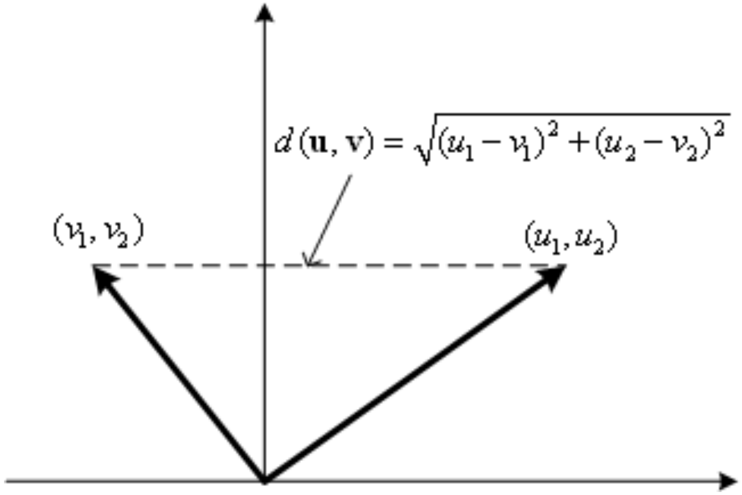
$$\therefore\ \frac{\mathbf{v}}{\|\mathbf{v}\|}\text{ is a unit vector}$$



- Distance between two vectors:

The distance between two vectors **u** and **v** in R^n is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$



- Properties of distance

(1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

(2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

(3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$ (commutative property of the distance function)

- Ex 3: Finding the distance between two vectors

The distance between **u** = (0, 2, 2) and **v** = (2, 0, 1) is

$$d(\mathbf{u}\,,\,\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|(0-2\,,\,2-0\,,\,2-1)\|$$
$$= \sqrt{(-2)^2+2^2+1^2} = 3$$

- Dot product in R^n:

  The dot product of $\mathbf{u}=(u_1\,,u_2\,,\cdots,u_n)$ and $\mathbf{v}=(v_1\,,v_2\,,\cdots,v_n)$ returns a scalar quantity

  $$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+\cdots+u_nv_n \quad (\mathbf{u}\cdot\mathbf{v} \text{ is a real number})$$

  (The dot product is defined as the sum of component-by-component multiplications)

- Ex 4: Finding the dot product of two vectors

  The dot product of **u** = (1, 2, 0, –3) and **v** = (3, –2, 4, 2) is

  $$\mathbf{u}\cdot\mathbf{v}=(1)(3)+(2)(-2)+(0)(4)+(-3)(2)=-7$$

- Theorem 1.3: Properties of the dot product

  If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and c is a scalar,

  then the following properties are true

  (1) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$ (commutative property of the dot product)

  (2) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$ (distributive property of the dot product over vector addition)

  (3) $c(\mathbf{u}\cdot\mathbf{v})=(c\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(c\mathbf{v})$ (associative property of the scalar multiplication and the dot product)

  (4) $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2 \Rightarrow \mathbf{v}\cdot\mathbf{v}\geq 0$

  (5) $\mathbf{v}\cdot\mathbf{v}=0$ if and only if $\mathbf{v}=\mathbf{0}$ (straightforwardly true according to (4))

※ The proofs of the above properties simply follow the definition of the dot product in R^n

- Euclidean *n*-space:

  When R^n is combined with the standard operations of **vector addition**, **scalar multiplication**, **vector length**, and **dot product**, the resulting vector space is called **Euclidean *n*-space**

- Ex 5: Find dot products

$$\mathbf{u} = (2\,,-2)\,,\ \mathbf{v} = (5\,,8),\ \mathbf{w} = (-4\,,3)$$

(a) $\mathbf{u}\cdot\mathbf{v}$ (b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ (c) $\mathbf{u}\cdot(2\mathbf{v})$ (d) $\|\mathbf{w}\|^2$ (e) $\mathbf{u}\cdot(\mathbf{v}-2\mathbf{w})$

Sol:

(a) $\mathbf{u}\cdot\mathbf{v} = (2)(5)+(-2)(8) = -6$

(b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w} = -6\mathbf{w} = -6(-4\,,3) = (24\,,-18)$

(c) $\mathbf{u}\cdot(2\mathbf{v}) = 2(\mathbf{u}\cdot\mathbf{v}) = 2(-6) = -12$

(d) $\|\mathbf{w}\|^2 = \mathbf{w}\cdot\mathbf{w} = (-4)(-4)+(3)(3) = 25$

(e) $\mathbf{v}-2\mathbf{w} = (5-(-8)\,,8-6) = (13\,,2)$

$\mathbf{u}\cdot(\mathbf{v}-2\mathbf{w}) = (2)(13)+(-2)(2) = 26-4 = 22$

- Ex 6: Using the properties of the dot product

Given $\mathbf{u}\cdot\mathbf{u}=39,\ \mathbf{u}\cdot\mathbf{v}=-3,\ \mathbf{v}\cdot\mathbf{v}=79,$

find $(\mathbf{u}+2\mathbf{v})\cdot(3\mathbf{u}+\mathbf{v})$

Sol:

$$\begin{aligned}(\mathbf{u}+2\mathbf{v})\cdot(3\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot(3\mathbf{u}+\mathbf{v})+2\mathbf{v}\cdot(3\mathbf{u}+\mathbf{v})\\ &= \mathbf{u}\cdot(3\mathbf{u})+\mathbf{u}\cdot\mathbf{v}+(2\mathbf{v})\cdot(3\mathbf{u})+(2\mathbf{v})\cdot\mathbf{v}\\ &= 3(\mathbf{u}\cdot\mathbf{u})+\mathbf{u}\cdot\mathbf{v}+6(\mathbf{v}\cdot\mathbf{u})+2(\mathbf{v}\cdot\mathbf{v})\\ &= 3(\mathbf{u}\cdot\mathbf{u})+7(\mathbf{u}\cdot\mathbf{v})+2(\mathbf{v}\cdot\mathbf{v})\\ &= 3(39)+7(-3)+2(79)=254\end{aligned}$$

- Theorem 1.4: The Cauchy-Schwarz inequality

  If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then

  $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$ ($|\mathbf{u}\cdot\mathbf{v}|$ denotes the absolute value of $\mathbf{u}\cdot\mathbf{v}$)

  (The geometric interpretation for this inequality is shown on the next slide)

- Ex 7: An example of the Cauchy-Schwarz inequality

  Verify the Cauchy-Schwarz inequality for $\mathbf{u} = (1, -1, 3)$ and $\mathbf{v} = (2, 0, -1)$

Sol:

$$\mathbf{u}\cdot\mathbf{v} = -1,\ \ \mathbf{u}\cdot\mathbf{u} = 11,\ \ \mathbf{v}\cdot\mathbf{v} = 5$$

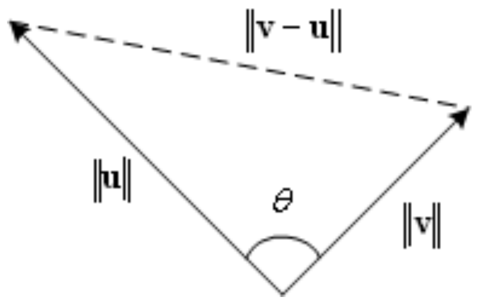
$$\Rightarrow |\mathbf{u}\cdot\mathbf{v}| = |-1| = 1$$

$$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}}\cdot\sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{11}\cdot\sqrt{5} = \sqrt{55}$$

$$\therefore |\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$$

- Dot product and the angle between two vectors

To find the angle $\theta\,(0 \le \theta \le \pi)$ between two nonzero vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in R^2, the Law of Cosines can be applied to the following triangle to obtain



$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{v}\|\|\mathbf{u}\|\cos\theta$$

(The length of the substense of θ can be expressed in terms of the lengths of the adjacent sides and cos θ)

$$\because \|\mathbf{v}-\mathbf{u}\|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2$$

$$\|\mathbf{v}\|^2 = v_1^2 + v_2^2$$

$$\|\mathbf{u}\|^2 = u_1^2 + u_2^2$$

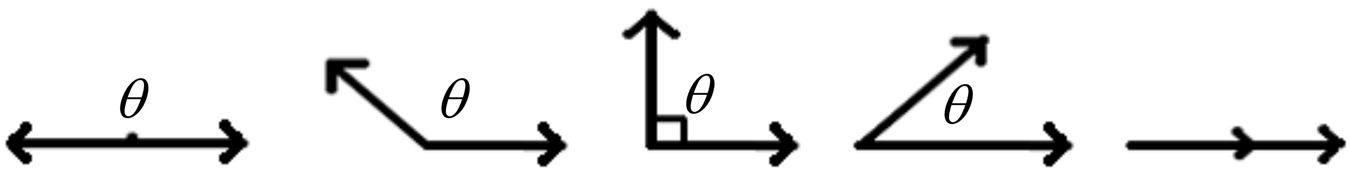
$$\therefore \cos\theta = \frac{u_1 v_1 + u_2 v_2}{\|\mathbf{v}\|\|\mathbf{u}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|\|\mathbf{u}\|}$$

※ You can employ the fact that $|\cos\theta| \le 1$ to prove the Cauchy-Schwarz inequality in R^2

- The angle between two nonzero vectors in R^n:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|},\ 0 \le \theta \le \pi$$

Opposite direction	$\mathbf{u}\cdot\mathbf{v}<\mathbf{0}$	$\mathbf{u}\cdot\mathbf{v}=\mathbf{0}$	$\mathbf{u}\cdot\mathbf{v}>\mathbf{0}$	Same direction
$\theta=\pi$	$\frac{\pi}{2}<\theta<\pi$	$\theta=\frac{\pi}{2}$	$0<\theta<\frac{\pi}{2}$	$\theta=0$
$\cos\theta=-1$	$\cos\theta<0$	$\cos\theta=0$	$\cos\theta>0$	$\cos\theta=1$



- Note:

The angle between the zero vector and another vector is not defined (since the denominator cannot be zero)

- Ex 8: Finding the angle between two vectors

$$\mathbf{u}=(-4\,,0\,,2\,,-2)\quad \mathbf{v}=(2\,,0\,,-1\,,1)$$

Sol:

$$\|\mathbf{u}\|=\sqrt{\mathbf{u}\cdot\mathbf{u}}=\sqrt{(-4)^2+0^2+2^2+(-2)^2}=\sqrt{24}$$

$$\|\mathbf{v}\|=\sqrt{\mathbf{v}\cdot\mathbf{v}}=\sqrt{2^2+(0)^2+(-1)^2+1^2}=\sqrt{6}$$

$$\mathbf{u}\cdot\mathbf{v}=(-4)(2)+(0)(0)+(2)(-1)+(-2)(1)=-12$$

$$\Rightarrow \cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\frac{-12}{\sqrt{24}\sqrt{6}}=-\frac{12}{\sqrt{144}}=-1$$

u and **v** have opposite directions

(In fact, **u** = –2**v** implies that **u** and **v** are parallel and with different directions)

- Orthogonal vectors:

  Two vectors **u** and **v** in R^n are orthogonal (perpendicular) if

$$\mathbf{u} \cdot \mathbf{v} = 0$$

- Note:

  The vector **0** is said to be orthogonal to every vector

- Ex 10: Finding orthogonal vectors

Determine all vectors in R^n that are orthogonal to **u** = (4, 2)

Sol:

$$\mathbf{u}=(4\,,2) \quad \text{Let} \quad \mathbf{v}=(v_1\,,v_2)$$

$$\Rightarrow \quad \mathbf{u}\cdot\mathbf{v}=(4\,,2)\cdot(v_1\,,v_2)$$

$$=4v_1+2v_2$$

$$=0$$

$$\Rightarrow \quad v_1=\frac{-t}{2} \quad , \quad v_2=t$$

$$\therefore \quad \mathbf{v}=\left(\frac{-t}{2},t\right), \quad t\in R$$

- Theorem 1.5: The triangle inequality

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

Pf:

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$$

$$= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}$$

$$= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \leq \|\mathbf{u}\|^2 + 2\,|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 \quad (c \leq |c|)$$

$$\leq \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \quad \text{(Cauchy-Schwarz inequality)}$$

$$= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$$

$$\therefore \|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

(The geometric representation of the triangle inequality: for any triangle, the sum of the lengths of any two sides is larger than the length of the third side (see the next slide))

- Note:

Equality occurs in the triangle inequality if and only if the vectors $\mathbf{u}$ and $\mathbf{v}$ have the same direction (in this situation, $\cos\theta = 1$ and thus $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\ \|\mathbf{v}\| \geq 0$)

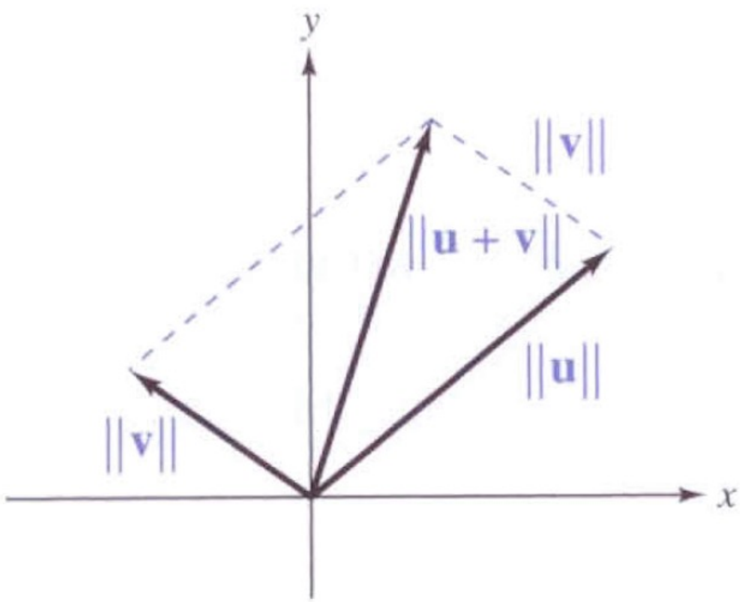
- Theorem 1.6: The Pythagorean theorem

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n,

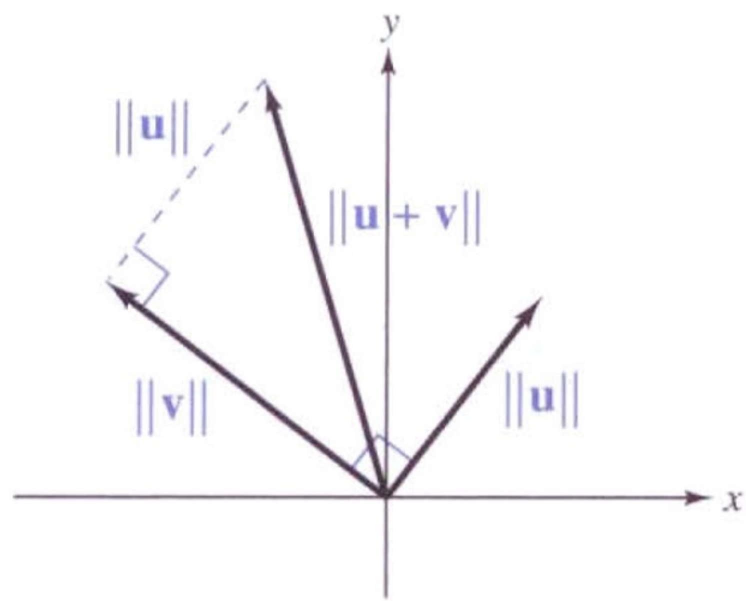
$\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$

(This is because $\mathbf{u}\cdot\mathbf{v} = 0$ in the proof for Theorem 1.5)

※ The geometric meaning: for any right triangle the sum of the squares of the lengths of two legs equals the square of the length of the hypotenuse.



$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$



$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$

- Similarity between dot product and matrix multiplication:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

(A vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in R^n can be represented as an $n \times 1$ column matrix)

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [u_1 v_1 + u_2 v_2 + \cdots + u_n v_n]$$

(The result of the dot product of **u** and **v** is the same as the result of the matrix multiplication of $\mathbf{u}^T$ and **v**)

- length:
- norm:
- unit vector:
- standard unit vector:
- Distance:
- dot product:
- Euclidean *n*-space:
- Cauchy-Schwarz inequality:
- angle:
- triangle inequality:
- Pythagorean theorem:

1.2 Inner Product Spaces

- Inner product: represented by angle brackets $\langle \mathbf{u}\,,\,\mathbf{v}\rangle$

  Let **u**, **v**, and **w** be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $\langle \mathbf{u}\,,\,\mathbf{v}\rangle$ with each pair of vectors **u** and **v** and satisfies the following axioms

(1) $\langle \mathbf{u}\,,\mathbf{v}\rangle = \langle \mathbf{v}\,,\mathbf{u}\rangle$ (commutative property of the inner product)

(2) $\langle \mathbf{u}\,,\mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u}\,,\mathbf{v}\rangle + \langle \mathbf{u}\,,\mathbf{w}\rangle$ (distributive property of the inner product over vector addition)

(3) $c\,\langle \mathbf{u}\,,\mathbf{v}\rangle = \langle c\mathbf{u}\,,\mathbf{v}\rangle$ (associative property of the scalar multiplication and the inner product)

(4) $\langle \mathbf{v}\,,\mathbf{v}\rangle \geq 0$

(5) $\langle \mathbf{v}\,,\mathbf{v}\rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (straightforwardly true according to (4))

- Note:

$$\mathbf{u} \cdot \mathbf{v} = \text{dot product (Euclidean inner product for } R^n)$$

$$<\mathbf{u}\,,\,\mathbf{v}> = \text{general inner product for a vector space } V$$

- Note:

A vector space V with an inner product is called an inner product space

$$\text{Vector space: } \quad (V,\ +,\ \cdot)$$

$$\text{Inner product space: } \quad (V,\ +,\ \cdot\ ,\ <,\ >)$$

- Ex 1: The Euclidean inner product for R^n

Show that the dot product in R^n satisfies the four axioms of an inner product

Sol:

$$\mathbf{u}=(u_1\,,u_2\,,\cdots,u_n)\quad,\quad\mathbf{v}=(v_1\,,v_2\,,\cdots,v_n)$$

$$\langle\mathbf{u}\,,\,\mathbf{v}\rangle=\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+\cdots+u_nv_n$$

By Theorem 1.3, this dot product satisfies the required four axioms. Thus, the dot product can be a sort of inner product in R^n

- Ex 2: A different inner product for R^n

Show that the following function defines an inner product on R^2. Given $\mathbf{u}=(u_1\ ,u_2)$ and $\mathbf{v}=(v_1\ ,v_2)$,

$$\langle \mathbf{u}\ ,\mathbf{v}\rangle = u_1v_1 + 2u_2v_2$$

Sol:

(1) $\langle \mathbf{u}\ ,\mathbf{v}\rangle = u_1v_1 + 2u_2v_2 = v_1u_1 + 2v_2u_2 = \langle \mathbf{v}\ ,\mathbf{u}\rangle$

(2) $\mathbf{w}=(w_1\ ,w_2)$

$$\begin{aligned}\Rightarrow\ \langle \mathbf{u}\ ,\mathbf{v}+\mathbf{w}\rangle &= u_1(v_1+w_1) + 2u_2(v_2+w_2)\\ &= u_1v_1 + u_1w_1 + 2u_2v_2 + 2u_2w_2\\ &= (u_1v_1 + 2u_2v_2) + (u_1w_1 + 2u_2w_2)\\ &= \langle \mathbf{u}\ ,\mathbf{v}\rangle + \langle \mathbf{u}\ ,\mathbf{w}\rangle\end{aligned}$$

$$(3)\quad c\ \langle \mathbf{u}\,,\,\mathbf{v}\rangle = c(u_1v_1 + 2u_2v_2) = (cu_1)v_1 + 2(cu_2)v_2 = \langle c\mathbf{u}\,,\,\mathbf{v}\rangle$$

$$(4)\quad \langle \mathbf{v}\,,\,\mathbf{v}\rangle = {v_1}^2 + 2{v_2}^2 \ge 0$$

$$(5)\quad \langle \mathbf{v}\,,\,\mathbf{v}\rangle = 0 \Rightarrow {v_1}^2 + 2{v_2}^2 = 0 \Rightarrow v_1 = v_2 = 0 \quad (\mathbf{v} = \mathbf{0})$$

- Note: Example 2 can be generalized such that

$$\langle \mathbf{u}\,,\,\mathbf{v}\rangle = c_1u_1v_1 + c_2u_2v_2 + \cdots + c_nu_nv_n,\ \text{for all } c_i > 0$$

can be an inner product on R^n

- Ex 3: A function that is not an inner product

Show that the following function is not an inner product on R^3

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 - 2u_2v_2 + u_3v_3$$

Sol:

Let $\mathbf{v} = (1, 2, 1)$

Then $\langle \mathbf{v}, \mathbf{v} \rangle = (1)(1) - 2(2)(2) + (1)(1) = -6 < 0$

Axiom 4 is not satisfied

Thus this function is not an inner product on R^3

- Theorem 1.7: Properties of inner products

Let **u**, **v**, and **w** be vectors in an inner product space V, and let c be any real number

(1) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

(2) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

(3) $\langle \mathbf{u}, c\mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$

※ To prove these properties, you can use only the four axioms for defining an inner product

Pf:

(1) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle 0\mathbf{u}, \mathbf{v} \rangle \overset{(3)}{=} 0 \langle \mathbf{u}, \mathbf{v} \rangle = 0$

(2) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle \overset{(1)}{=} \langle \mathbf{w}, \mathbf{u} + \mathbf{v} \rangle \overset{(2)}{=} \langle \mathbf{w}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle \overset{(1)}{=} \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

(3) $\langle \mathbf{u}, c\mathbf{v} \rangle \overset{(1)}{=} \langle c\mathbf{v}, \mathbf{u} \rangle \overset{(3)}{=} c \langle \mathbf{u}, \mathbf{v} \rangle$

※ The definition of norm (or length), distance, angle, orthogonal, and normalizing for general inner product spaces closely parallel to those based on the dot product in Euclidean *n*-space

- Norm (length) of **u**:

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

- Distance between **u** and **v**:

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

- Angle between two nonzero vectors **u** and **v**:

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}, \ 0 \le \theta \le \pi$$

- Orthogonal: $(\mathbf{u} \perp \mathbf{v})$

u and **v** are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$

- Normalizing vectors

(1) If $\|\mathbf{v}\| = 1$, then **v** is called a unit vector

(Note that $\|\mathbf{v}\|$ is defined as $\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$)

(2) $\mathbf{v} \neq \mathbf{0} \xrightarrow{\text{Normalizing}} \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ (the unit vector in the direction of **v**)

(if **v** is not a zero vector)

- Ex 6: An inner product in the polynomial space

For $p = a_0 + a_1x + \cdots + a_nx^n$ and $q = b_0 + b_1x + \cdots + b_nx^n$,

and $\langle p\,,\,q\rangle \equiv a_0b_0 + a_1b_1 + \cdots + a_nb_n$ is an inner product

Let $p(x) = 1 - 2x^2$, $q(x) = 4 - 2x + x^2$ be polynomials in P_2

(a) $\langle p\,,\,q\rangle = ?$ (b) $\|q\| = ?$ (c) $d(p\,,\,q) = ?$

Sol:

(a) $\langle p\,,\,q\rangle = (1)(4) + (0)(-2) + (-2)(1) = 2$

(b) $\|q\| = \sqrt{\langle q\,,\,q\rangle} = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$

(c) $\because\ p - q = -3 + 2x - 3x^2$

$$\therefore d(p\,,\,q) = \|p - q\| = \sqrt{\langle p-q, p-q\rangle}$$

$$= \sqrt{(-3)^2 + 2^2 + (-3)^2} = \sqrt{22}$$

- Properties of norm: (the same as the properties for the dot product in R^n)

  (1) $\|\mathbf{u}\| \geq 0$

  (2) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$

  (3) $\|c\mathbf{u}\| = |c|\,\|\mathbf{u}\|$

- Properties of distance: (the same as the properties for the dot product in R^n)

  (1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

  (2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

  (3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

- Theorem 1.8：

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V

(1) Cauchy-Schwarz inequality:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\| \, \|\mathbf{v}\| \qquad \text{Theorem 1.4}$$

(2) Triangle inequality:

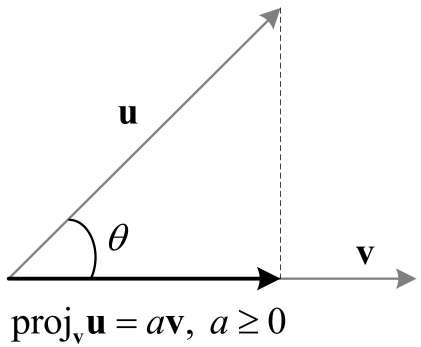
$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\| \qquad \text{Theorem 1.5}$$

(3) Pythagorean theorem:

$\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \qquad \text{Theorem 1.6}$$

- Orthogonal projection: For the dot product function in R^n, we define the orthogonal projection of **u** onto **v** to be $\text{proj}_{\mathbf{v}}\mathbf{u} = a\mathbf{v}$ (a scalar multiple of **v**), and the coefficient a can be derived as follows



Consider $a \geq 0$, $\|a\mathbf{v}\| = |a|\,\|\mathbf{v}\| = a\|\mathbf{v}\| = \|\mathbf{u}\|\cos\theta$

$$= \frac{\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta}{\|\mathbf{v}\|} = \frac{\|\mathbf{u}\|\,\|\mathbf{v}\|}{\|\mathbf{v}\|}\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}$$

$$\Rightarrow a = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} \quad \Rightarrow \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$$

- For inner product spaces:

Let **u** and **v** be two vectors in an inner product space V. If $\mathbf{v} \neq \mathbf{0}$, then the orthogonal projection of **u** onto **v** is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v}$$

- Ex 10: Finding an orthogonal projection in R^3

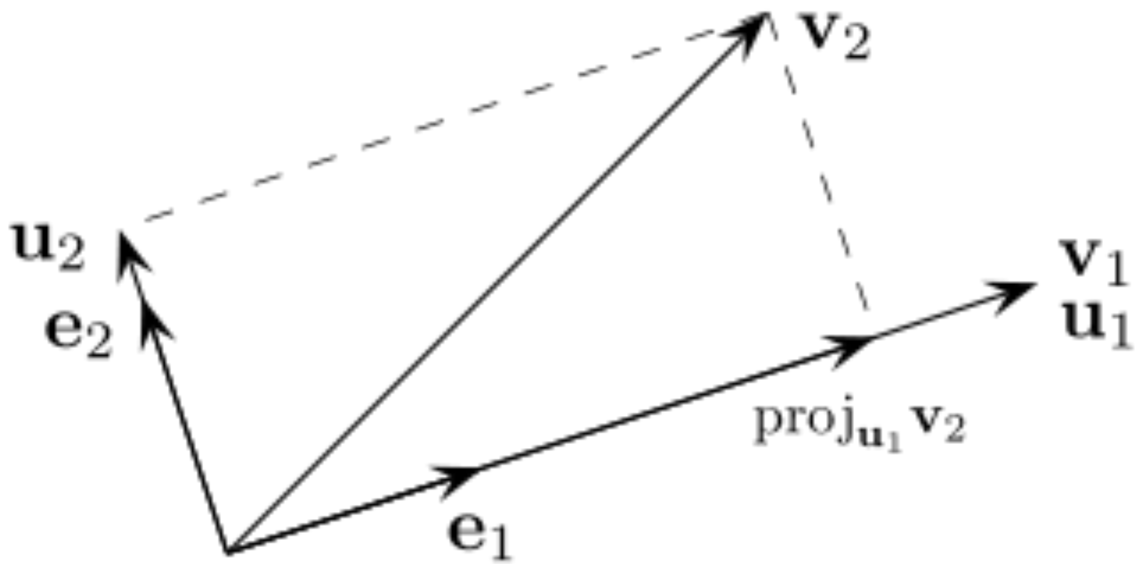
  Use the Euclidean inner product in R^3 to find the orthogonal projection of $\mathbf{u}$ = (6, 2, 4) onto $\mathbf{v}$ = (1, 2, 0)

Sol:

$$\because \langle \mathbf{u}, \mathbf{v} \rangle = (6)(1) + (2)(2) + (4)(0) = 10$$

$$\langle \mathbf{v}, \mathbf{v} \rangle = 1^2 + 2^2 + 0^2 = 5$$

$$\therefore \operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{10}{5}(1, 2, 0) = (2, 4, 0)$$

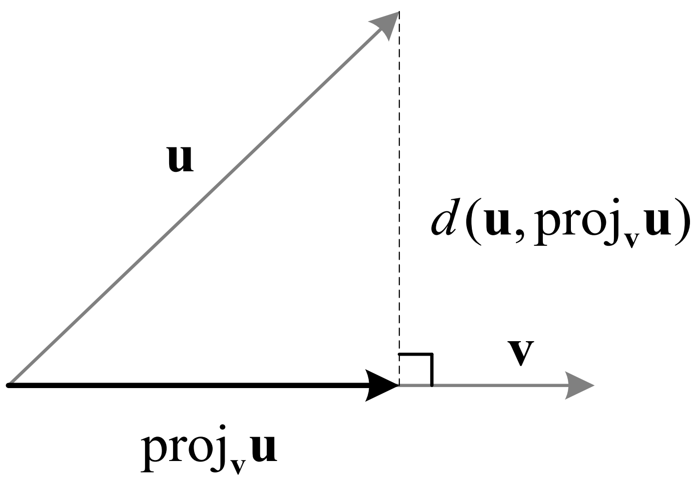


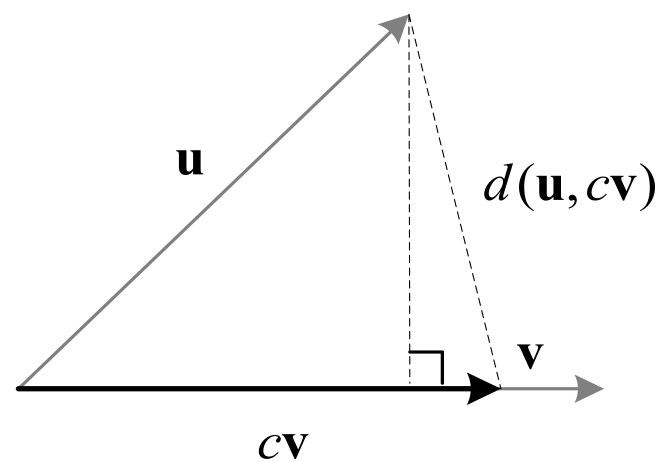
$\mathbf{v}_2$
$\mathbf{u}_2$
$\mathbf{e}_2$
$\mathbf{v}_1$
$\mathbf{u}_1$
$\text{proj}_{\mathbf{u}_1}\mathbf{v}_2$
$\mathbf{e}_1$

- Theorem 1.9: Orthogonal projection and distance

  Let **u** and **v** be two vectors in an inner product space V, and if $\mathbf{v} \neq \mathbf{0}$, then

$$d(\mathbf{u}, \mathrm{proj}_{\mathbf{v}}\mathbf{u}) < d(\mathbf{u}, c\mathbf{v}), \quad c \neq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$$





※ Theorem 1.9 can be inferred straightforwardly by the Pythagorean Theorem, i.e., in a right triangle, the hypotenuse is longer than both legs.

- inner product:
- inner product space:
- norm:
- distance:
- angle:
- orthogonal:
- unit vector:
- normalizing:
- Cauchy-Schwarz inequality:
- triangle inequality:
- Pythagorean theorem:
- orthogonal projection:

1.3 Orthonormal Bases: Gram-Schmidt Process

- Orthogonal set :

A set S of vectors in an inner product space V is called an orthogonal set if every pair of vectors in the set is orthogonal

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$
$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0, \text{ for } i \neq j$$

- Orthonormal set :

An orthogonal set in which each vector is a unit vector is called orthonormal set

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$
$$\begin{cases} \text{For } i = j,\ \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \langle \mathbf{v}_i, \mathbf{v}_i \rangle = \|\mathbf{v}_i\|^2 = 1 \\ \text{For } i \neq j,\ \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \end{cases}$$

- Note:
  - If S is also a basis, then it is called an orthogonal basis or an orthonormal basis
  - The standard basis for R^n is orthonormal. For example,

$$S = \{(1,0,0),(0,1,0),(0,0,1)\}$$

    is an orthonormal basis for R^3

- This section identifies some advantages of orthonormal bases, and develops a procedure for constructing such bases, known as Gram-Schmidt orthonormalization process

- Ex 1: A nonstandard orthonormal basis for R^3

Show that the following set is an orthonormal basis

$$S = \Big\{ \underbrace{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}^{\mathbf{v}_1}, \quad \underbrace{\left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)}^{\mathbf{v}_2}, \quad \underbrace{\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)}^{\mathbf{v}_3} \Big\}$$

Sol:

First, show that the three vectors are mutually orthogonal

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = -\tfrac{1}{6} + \tfrac{1}{6} + 0 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$$

Second, show that each vector is of length 1

$$\|\mathbf{v}_1\| = \sqrt{\mathbf{v}_1 \cdot \mathbf{v}_1} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2} + 0} = 1$$

$$\|\mathbf{v}_2\| = \sqrt{\mathbf{v}_2 \cdot \mathbf{v}_2} = \sqrt{\tfrac{2}{36} + \tfrac{2}{36} + \tfrac{8}{9}} = 1$$

$$\|\mathbf{v}_3\| = \sqrt{\mathbf{v}_3 \cdot \mathbf{v}_3} = \sqrt{\tfrac{4}{9} + \tfrac{4}{9} + \tfrac{1}{9}} = 1$$

Thus S is an orthonormal set

Because these three vectors are linearly independent (you can check by solving $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$) in R^3 (of dimension 3), by Theorem (given a vector space with dimension n, then n linearly independent vectors can form a basis for this vector space), these three linearly independent vectors form a basis for R^3.

$\Rightarrow$ S is a (nonstandard) orthonormal basis for R^3

- Ex : An orthonormal basis for $P_2(x)$

In $P_2(x)$, with the inner product $\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2$,
the standard basis $B = \{1, x, x^2\}$ is orthonormal

Sol:

$$\mathbf{v}_1 = 1 + 0x + 0x^2, \quad \mathbf{v}_2 = 0 + x + 0x^2, \quad \mathbf{v}_3 = 0 + 0x + x^2,$$

Then

$$\langle \mathbf{v}_1, \mathbf{v}_2\rangle = (1)(0) + (0)(1) + (0)(0) = 0$$
$$\langle \mathbf{v}_1, \mathbf{v}_3\rangle = (1)(0) + (0)(0) + (0)(1) = 0$$
$$\langle \mathbf{v}_2, \mathbf{v}_3\rangle = (0)(0) + (1)(0) + (0)(1) = 0$$

$$\|\mathbf{v}_1\| = \sqrt{\langle \mathbf{v}_1, \mathbf{v}_1\rangle} = \sqrt{(1)(1) + (0)(0) + (0)(0)} = 1$$
$$\|\mathbf{v}_2\| = \sqrt{\langle \mathbf{v}_2, \mathbf{v}_2\rangle} = \sqrt{(0)(0) + (1)(1) + (0)(0)} = 1$$
$$\|\mathbf{v}_3\| = \sqrt{\langle \mathbf{v}_3, \mathbf{v}_3\rangle} = \sqrt{(0)(0) + (0)(0) + (1)(1)} = 1$$

- Theorem 1.10: Orthogonal sets are linearly independent

  If $S=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space V, then S is linearly independent

  Pf:

  S is an orthogonal set of nonzero vectors,

  i.e., $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$, and $\langle \mathbf{v}_i, \mathbf{v}_i \rangle > 0$

  For $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ (If there is only the trivial solution for c_i's, i.e., all c_i's are 0, S is linearly independent)

  $$\Rightarrow \langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n, \mathbf{v}_i \rangle = \langle \mathbf{0}, \mathbf{v}_i \rangle = 0 \quad \forall i$$

  $$\Rightarrow c_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \cdots + c_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle + \cdots + c_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle$$

  $= c_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 0$ (because S is an orthogonal set of nonzero vectors)

  $\because \langle \mathbf{v}_i, \mathbf{v}_i \rangle \neq 0 \quad \Rightarrow c_i = 0 \quad \forall i \quad \therefore S$ is linearly independent

- Corollary to Theorem 1.10:

  If V is an inner product space with dimension n, then any orthogonal set of n nonzero vectors is a basis for V

1. By Theorem 1.10, if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of n vectors, then S is linearly independent
2. According to Theorem, if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of n vectors in V (with dimension n), then S is a basis for V

※ Based on the above two arguments, it is straightforward to derive the above corollary to Theorem 1.10

- Ex 4: Using orthogonality to test for a basis

Show that the following set is a basis for R^4

$$S = \{\overset{\mathbf{v}_1}{(2,3,2,-2)}, \overset{\mathbf{v}_2}{(1,0,0,1)}, \overset{\mathbf{v}_3}{(-1,0,2,1)}, \overset{\mathbf{v}_4}{(-1,2,-1,1)}\}$$

Sol:

$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$: nonzero vectors

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 2+0+0-2=0 \qquad \mathbf{v}_2 \cdot \mathbf{v}_3 = -1+0+0+1=0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = -2+0+4-2=0 \qquad \mathbf{v}_2 \cdot \mathbf{v}_4 = -1+0+0+1=0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_4 = -2+6-2-2=0 \qquad \mathbf{v}_3 \cdot \mathbf{v}_4 = 1+0-2+1=0$$

$\Rightarrow S$ is orthogonal

$\Rightarrow S$ is a basis for R^4 (by Corollary to Theorem 1.10)

※ The corollary to Thm. 1.10 shows an advantage of introducing the concept of orthogonal vectors, i.e., it is not necessary to solve linear systems to test whether S is a basis (e.g., Ex 1) if S is a set of orthogonal vectors

- Theorem 1.11: Coordinates relative to an orthonormal basis

If $B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, then the unique coordinate representation of a vector $\mathbf{w}$ with respect to B is

$$\mathbf{w}=\langle \mathbf{w}, \mathbf{v}_1\rangle \mathbf{v}_1+\langle \mathbf{w}, \mathbf{v}_2\rangle \mathbf{v}_2+\cdots+\langle \mathbf{w}, \mathbf{v}_n\rangle \mathbf{v}_n$$

※ The above theorem tells us that it is easy to derive the coordinate representation of a vector relative to an orthonormal basis, which is another advantage of using orthonormal bases

Pf:

$B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthonormal basis for V

$\mathbf{w}=k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_n\mathbf{v}_n \in V$ (unique representation)

Since $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$, then

$$\Rightarrow \mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{w}, \mathbf{v}_n \rangle \mathbf{v}_n$$

- Note:

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthonormal basis for V and $\mathbf{w} \in V$,

Then the corresponding coordinate matrix of $\mathbf{w}$ relative to B is

$$[\mathbf{w}]_B = \begin{bmatrix} \langle \mathbf{w}, \mathbf{v}_1 \rangle \\ \langle \mathbf{w}, \mathbf{v}_2 \rangle \\ \vdots \\ \langle \mathbf{w}, \mathbf{v}_n \rangle \end{bmatrix}$$

- Ex

For $\mathbf{w} = (5, -5, 2)$, find its coordinates relative to the standard basis for R^3

$$\langle \mathbf{w}, \mathbf{v}_1 \rangle = \mathbf{w} \cdot \mathbf{v}_1 = (5\,, -5\,, 2) \cdot (1\,, 0\,, 0) = 5$$

$$\langle \mathbf{w}, \mathbf{v}_2 \rangle = \mathbf{w} \cdot \mathbf{v}_2 = (5, -5\,, 2) \cdot (0\,, 1\,, 0) = -5$$

$$\langle \mathbf{w}, \mathbf{v}_3 \rangle = \mathbf{w} \cdot \mathbf{v}_3 = (5\,, -5\,, 2) \cdot (0\,, 0\,, 1) = 2$$

$$\Rightarrow [\mathbf{w}]_B = \begin{bmatrix} 5 \\ -5 \\ 2 \end{bmatrix}$$

※ In fact, it is not necessary to use Thm. 1.11 to find the coordinates relative to the standard basis, because we know that the coordinates of a vector relative to the standard basis are the same as the components of that vector

※ The advantage of the orthonormal basis emerges when we try to find the coordinate matrix of a vector relative to an nonstandard orthonormal basis (see the next slide)

- Ex 5: Representing vectors relative to an orthonormal basis

  Find the coordinates of $\mathbf{w} = (5, -5, 2)$ relative to the following orthonormal basis for R^3

$$B = \{\underbrace{(\tfrac{3}{5}, \tfrac{4}{5}, 0)}^{\mathbf{v}_1}, \underbrace{(-\tfrac{4}{5}, \tfrac{3}{5}, 0)}^{\mathbf{v}_2}, \underbrace{(0, 0, 1)}^{\mathbf{v}_3}\}$$

Sol:

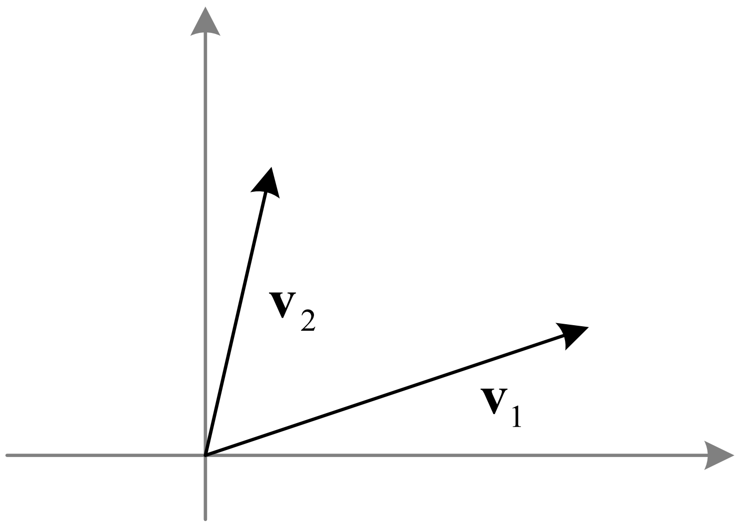
$$\langle \mathbf{w}, \mathbf{v}_1 \rangle = \mathbf{w} \cdot \mathbf{v}_1 = (5, -5, 2) \cdot (\tfrac{3}{5}, \tfrac{4}{5}, 0) = -1$$

$$\langle \mathbf{w}, \mathbf{v}_2 \rangle = \mathbf{w} \cdot \mathbf{v}_2 = (5, -5, 2) \cdot (-\tfrac{4}{5}, \tfrac{3}{5}, 0) = -7$$

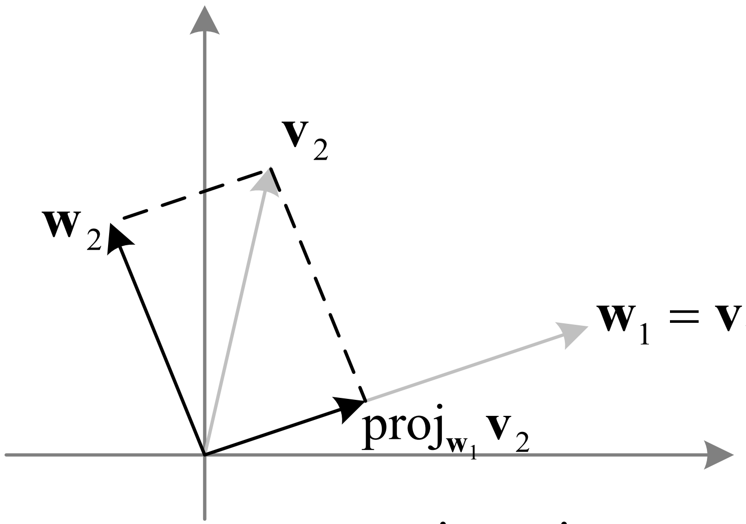
$$\langle \mathbf{w}, \mathbf{v}_3 \rangle = \mathbf{w} \cdot \mathbf{v}_3 = (5, -5, 2) \cdot (0, 0, 1) = 2$$

$$\Rightarrow [\mathbf{w}]_B = \begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}$$

- The geometric intuition of the Gram-Schmidt process to find an orthonormal basis in R^2



$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for R^2



$\mathbf{w}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{w}_1}\mathbf{v}_2$ is orthogonal to $\mathbf{w}_1 = \mathbf{v}_1$

$$\Rightarrow \left\{\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}\right\} \text{ is an orthonormal basis for } R^2$$

- Gram-Schmidt orthonormalization process:

$B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for an inner product space V

Let $\mathbf{w}_1=\mathbf{v}_1$

$S_1=\text{span}(\{\mathbf{w}_1\})$

$$\mathbf{w}_2=\mathbf{v}_2-\text{proj}_{S_1}\mathbf{v}_2=\mathbf{v}_2-\frac{\langle\mathbf{v}_2,\mathbf{w}_1\rangle}{\langle\mathbf{w}_1,\mathbf{w}_1\rangle}\mathbf{w}_1$$

$S_2=\text{span}(\{\mathbf{w}_1,\mathbf{w}_2\})$

$$\mathbf{w}_3=\mathbf{v}_3-\text{proj}_{S_2}\mathbf{v}_3=\mathbf{v}_3-\frac{\langle\mathbf{v}_3,\mathbf{w}_1\rangle}{\langle\mathbf{w}_1,\mathbf{w}_1\rangle}\mathbf{w}_1-\frac{\langle\mathbf{v}_3,\mathbf{w}_2\rangle}{\langle\mathbf{w}_2,\mathbf{w}_2\rangle}\mathbf{w}_2$$

$\vdots$

$$\mathbf{w}_n=\mathbf{v}_n-\text{proj}_{S_{n-1}}\mathbf{v}_n=\mathbf{v}_n-\sum_{i=1}^{n-1}\frac{\langle\mathbf{v}_n,\mathbf{w}_i\rangle}{\langle\mathbf{w}_i,\mathbf{w}_i\rangle}\mathbf{w}_i$$

The orthogonal projection onto a subspace is actually the sum of orthogonal projections onto the vectors in an orthogonal basis for that subspace

$\Rightarrow B'=\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\}$ is an orthogonal basis

$\Rightarrow B''=\{\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \cdots, \frac{\mathbf{w}_n}{\|\mathbf{w}_n\|}\}$ is an orthonormal basis

- Ex 7: Applying the Gram-Schmidt orthonormalization process

Apply the Gram-Schmidt process to the following basis for R^3

$$B = \{\overset{\mathbf{v}_1}{(1,1,0)},\ \overset{\mathbf{v}_2}{(1,2,0)},\ \overset{\mathbf{v}_3}{(0,1,2)}\}$$

Sol:

$$\mathbf{w}_1 = \mathbf{v}_1 = (1,1,0)$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = (1,2,0) - \frac{3}{2}(1,1,0) = (-\frac{1}{2},\frac{1}{2},0)$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2$$

$$= (0,1,2) - \frac{1}{2}(1,1,0) - \frac{1/2}{1/2}(-\frac{1}{2},\frac{1}{2},0) = (0,0,2)$$

Orthogonal basis

$$\Rightarrow B' = \{\mathbf{w}_1\ , \mathbf{w}_2\ , \mathbf{w}_3\} = \{(1,1,0), (\frac{-1}{2}, \frac{1}{2}, 0), (0,0,2)\}$$

Orthonormal basis

$$\Rightarrow B'' = \{\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|}\} = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (0,0,1)\}$$

- Ex 10: Alternative form of Gram-Schmidt orthonormalization process

  Find an orthonormal basis for the solution space of the homogeneous system of linear equations

$$\begin{aligned} x_1 + x_2 \phantom{{}+2x_3} + 7x_4 &= 0 \\ 2x_1 + x_2 + 2x_3 + 6x_4 &= 0 \end{aligned}$$

- Ex 10: Alternative form of Gram-Schmidt orthonormalization process

  Find an orthonormal basis for the solution space of the homogeneous system of linear equations

$$\begin{aligned} x_1 + x_2 + 7x_4 &= 0 \\ 2x_1 + x_2 + 2x_3 + 6x_4 &= 0 \end{aligned}$$

Sol:

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s+t \\ 2s-8t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -8 \\ 0 \\ 1 \end{bmatrix}$$

Thus one basis for the solution space is

$$B=\{\mathbf{v}_1,\mathbf{v}_2\}=\{(-2,2,1,0),(1,-8,0,1)\}$$

$$\mathbf{w}_1=\mathbf{v}_1 \text{ and } \mathbf{u}_1=\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}=\frac{1}{3}(-2,\ 2,\ 1,\ 0)=\left(\frac{-2}{3},\frac{2}{3},\frac{1}{3},0\right)$$

$$\mathbf{w}_2=\mathbf{v}_2-\langle\mathbf{v}_2,\mathbf{u}_1\rangle\mathbf{u}_1 \text{ (due to } \mathbf{w}_2=\mathbf{v}_2-\frac{\langle\mathbf{v}_2,\mathbf{u}_1\rangle}{\langle\mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1 \text{ and } \langle\mathbf{u}_1,\mathbf{u}_1\rangle=1)$$

$$=(1,\ -8,\ 0,\ 1)-\left[(1,\ -8,\ 0,\ 1)\cdot\left(\frac{-2}{3},\frac{2}{3},\frac{1}{3},0\right)\right]\left(\frac{-2}{3},\frac{2}{3},\frac{1}{3},0\right)$$

$$=(-3,\ -4,\ 2,\ 1)$$

$$\mathbf{u}_2=\frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}=\frac{1}{\sqrt{30}}(-3,\ -4,\ 2,\ 1)$$

$$\Rightarrow B''=\left\{\left(\frac{-2}{3},\frac{2}{3},\frac{1}{3},0\right),\left(\frac{-3}{\sqrt{30}},\frac{-4}{\sqrt{30}},\frac{2}{\sqrt{30}},\frac{1}{\sqrt{30}}\right)\right\}$$

※ In this alternative form, we always normalize $\mathbf{w}_i$ to be $\mathbf{u}_i$ before processing $\mathbf{w}_{i+1}$

※ The advantage of this method is that it is easier to calculate the orthogonal projection of $\mathbf{w}_{i+1}$ on $\mathbf{u}_1$, $\mathbf{u}_2$,…, $\mathbf{u}_i$

- Alternative form of the Gram-Schmidt orthonormalization process:

$B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for an inner product space V

$$\mathbf{u}_1=\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}=\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$

$$\mathbf{u}_2=\frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \text{ where } \mathbf{w}_2=\mathbf{v}_2-\langle\mathbf{v}_2, \mathbf{u}_1\rangle\mathbf{u}_1$$

$$\mathbf{u}_3=\frac{\mathbf{w}_3}{\|\mathbf{w}_3\|}, \text{ where } \mathbf{w}_3=\mathbf{v}_3-\langle\mathbf{v}_3, \mathbf{u}_1\rangle\mathbf{u}_1-\langle\mathbf{v}_3, \mathbf{u}_2\rangle\mathbf{u}_2$$

$$\vdots$$

$$\mathbf{u}_n=\frac{\mathbf{w}_n}{\|\mathbf{w}_n\|}, \text{ where } \mathbf{w}_n=\mathbf{v}_n-\sum_{i=1}^{n-1}\langle\mathbf{v}_n, \mathbf{u}_i\rangle\mathbf{u}_i$$

$\Rightarrow \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is an orthonormal basis for V

Keywords in Section 1.3:

- orthogonal set:
- orthonormal set:
- orthogonal basis:
- orthonormal basis:
- linear independent:
- Gram-Schmidt Process: Gram-Schmidt