Introduction to the Course on Mathematics for Machine Learning

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Outline

1. Introduction
2. Vector Space
3. Subspace

Binary Operator

Any function $f : V \times V \to V$ is called as **binary operator**. For example,

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

So, let us define what is a vector space.

What is a vector?

Definition 2.1 (vector space).

A non-empty set V with a field $\mathbb{F}$ is called as vector space with respect to the operations $+ : V \times V \to V$ (binary operation) and $\cdot : \mathbb{F} \times V \to V$ (scalar multiplication), if the following holds:

(i) for all $u, v \in V$, $u + v = v + u$ (commutativity)

(ii) for all $u, v, w \in V$, $(u + v) + w = u + (v + w)$ (associativity)

(iii) there exists $0 \in V$ such that $u + 0 = u$ for all $u \in V$

(iv) for all $u \in V$ there exists $-u \in V$ such that $u + (-u) = 0$

and

(v) for all $v \in V$, $1\,u = u$

(vi) for all $\alpha, \beta \in \mathbb{F}$ and for all $u \in V$, $(\alpha\beta)u = \alpha(\beta u)$

(vii) for all $\alpha \in \mathbb{F}$ and for all $u, v \in V$, $\alpha(u + v) = \alpha u + \alpha v$

(viii) for all $\alpha, \beta \in \mathbb{F}$ and for all $u \in V$, $(\alpha + \beta)u = \alpha u + \beta u$.

Example 2.2.

Set of all real numbers is a vector space over real numbers with respect to the usual addition and scalar multiplication. That is $(\mathbb{R}, +, \cdot)$ is a vector space over $\mathbb{R}$.

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Example 2.3.

Consider $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. Now, let us define the addition operation as co-ordinate vice addition. Thats is,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and the scalar multiplication is defined as

$$\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1).$$

Then, $(\mathbb{R}^2, +, \cdot)$ is a vector space over $\mathbb{R}$. the set of vectors in $\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

Example 2.4.

More generally, $\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R} \text{ for all } i \in \{1, 2, \ldots, n\}\}$ is a vector space over $\mathbb{R}$.

In fact, one can generalise little more and define set of all $m \times n$ matrices.

Example 2.5.

Let $M_{m\times n} =$ set of all matrices of size $m \times n$ where m and n are positive integers. Then $M_{m\times n}$ over $\mathbb{R}$ is a vector space over $\mathbb{R}$. The addition and scalar multiplication is defined similar to the previous example as

$$(A+B)_{ij} = A_{ij} + B_{ij},$$
$$(\alpha A)_{ij} = \alpha A_{ij}.$$

Example 2.6.

Let V be set of all functions from a set X to the set Y. And define the addition of two functions as

$$(f+g)(x) = f(x) + g(x) \tag{1}$$

and scalar multiplication as

$$(\alpha f)(x) = \alpha f(x).$$

Then, $(V, +, \cdot)$ is a vector space over any field $\mathbb{F}$.

Theorem 2.7 (Cancellation law for vector addition).

If x, y and z are vectors in a vector space V, then

$$x + z = y + z \implies x = y.$$

Proof.

There exists $-z \in V$ such that $z + (-z) = 0$. And so,

$$\begin{aligned} x &= x + 0 \\ &= x + z + (-z) \\ &= y + z + (-z) \\ &= y + 0 \\ &= y. \end{aligned}$$

□

Corollary 2.8.

For any vector space V over $\mathbb{F}$, the following are true.

(i) The zero vector is unique.

(ii) Additive inverse of a vector is unique

Proof.

Let V be a vector space over $\mathbb{F}$.

(i) Suppose there are two zero vectors, say, 0 and $0'$. Take $x = 0$ and $y = 0'$, z being any vector and apply 2.7.

(ii) Suppose there are two inverse, say, v_2 and v_2 for v. Take $x = v_1$ and $y = v_2$, $z = v$ and apply 2.7.

$\square$

Theorem 2.9.

For any vector space V over $\mathbb{F}$, the following are true.

(i) $0v = 0$ *for all* $v \in V$.

(ii) $(-a)v = -(av) = a(-v)$ *for all* $a \in \mathbb{F}$ *and for all* $v \in V$.

(iii) $a0 = 0$ *for all* $a \in \mathbb{F}$.

What is a subspace?

THANK YOU!