23 May 2023 17:06

Scalars

Vectors $\rightarrow$ one dimensional array

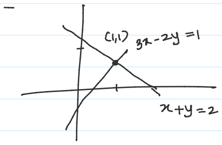
Matrices $\rightarrow$ Two dimensional arrays

Tensors $\rightarrow$ Multi dimensional

Matrix properties, Row-echelon form, Invertibility.

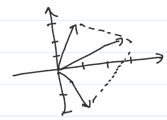
Linear Transformations.

$$x + y = 2$$
$$3x - 2y = 1$$



$$x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$(1)\begin{bmatrix}1\\3\end{bmatrix} + 1\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$



Solution of a linear system:

$$x + y = 2 \rightarrow (1)$$
$$3x - 2y = 1 \rightarrow (2)$$

Interchange two equations.

Multiply any equation by a nonzero constant

We add a multiple any equation to any other equation.

$\downarrow$

New System

$$(2)\times(1) \rightarrow \left.\begin{aligned} 2x + 2y &= 4 \\ 3x - 2y &= 1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x + y &= 2 \\ 5x &= 5 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x + y &= 2 \\ x &= 1 \end{aligned}\right\} \quad y = 1$$

$$5x = 5$$
$$x = 1$$

Elementary row operations:

Consider

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad n\text{-variables}$$

$$x_1\begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{bmatrix} + x_2\begin{bmatrix}a_{12}\\a_{22}\\\vdots\\a_{m2}\end{bmatrix} + \cdots + x_n\begin{bmatrix}a_{1n}\\\vdots\\a_{mn}\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\\vdots\\b_m\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}}_{x} = \underbrace{\begin{bmatrix}b_1\\b_2\\\vdots\\b_m\end{bmatrix}}_{b}$$

$\downarrow$ Coefficient matrix.

$\boxed{Ax = b} \rightarrow b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, A: $m \times n$

$b = 0 \quad \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$

$Ax = 0 \rightarrow$ homogeneous system.

$x = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$

Elementary row operations:

Augmented matrix

$[A|b] \rightarrow$

(i) Multiply any row by a nonzero constant

(ii) Interchange any two rows.

(iii) Add a multiple of any one row to any other row.

$$\left.\begin{aligned} x + y &= 2 \\ 3x - 2y &= 1 \end{aligned}\right\} \rightarrow \underbrace{\begin{bmatrix}1 & 1\\3 & -2\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\y\end{bmatrix}}_{x} = \underbrace{\begin{bmatrix}2\\1\end{bmatrix}}_{b}$$

$$[A|b] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & -2 & 1 \end{array}\right] \sim \quad R_2 \leftarrow R_2 + (-3)R_1$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -5 & -5 \end{array}\right] \sim \quad R_2 \leftarrow \left(-\tfrac{1}{5}\right) R_2$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \quad R_1 \leftarrow R_1 + (-1) R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow x = 1$
$y = 1$

Row-Echelon form:

A rectangular matrix is in echelon form (row echelon form) if it has the following

(i) All nonzero rows are above any rows of all zeros.
(ii) Each leading entry (first nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
(iii) All entries in a column below a leading entry should be zero

} Row echelon form

If in addition,
(iv) The leading entry in each row is 1.
(v) Each leading 1 is the only nonzero entry in its column.

Reduced Row-echelon form

Examples:

$$A = \begin{bmatrix} \underline{\underline{1}} & 0 & 5 \\ 0 & \underline{\underline{1}} & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} \underline{\underline{2}} & 0 & 1 \\ 0 & \underline{\underline{1}} & 0 \\ 0 & 0 & \underline{\underline{1}} \end{bmatrix} \quad D = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

A: Reduced row echelon form (pivot) — B: Non-example — C: Row Echelon form (pivots) — D: Non-example

Column that contains the pivot is called the pivot column.

Consider

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Step 1: Begin with the left most nonzero column. This is the pivot column. The pivot position is at the top.

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 2: Select a nonzero entry in the pivot column as pivot. Use the elementary row operations to create zeros in all position below the pivot.

$R_2 \leftarrow R_2 + (-1) R_1$

$$\begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 4: Apply Step 1 to 2 for the rows below the row that contains the pivot.

$R_3 \leftarrow R_3 + \left(-\tfrac{3}{2}\right) R_2$

Forward phase
$$\begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix} \rightarrow \text{Row-Echelon form}$$

$R_1 \leftarrow \frac{1}{3}R_1 \Rightarrow$

$R_2 \leftarrow \frac{1}{2}R$
$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_1 \leftarrow R_1 + 3R_2$
$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Similarly carry two elementary row operations
$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \text{Reduced row echelon form.}$$

Properties:

$A \pm B = C$ A & B have same sizes

cA,   Each entry should be multiplied

$AB =$

AB is possible   No. of columns of A = No. of rows of B

$\text{Trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}$.

Identity
$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & & 0 & \cdots & 1 \end{bmatrix} \qquad AI = IA = A$$

Inverse:

B will be inverse of A

$$AB = BA = I \qquad B = A^{-1}$$
$$(A^{-1})^{-1} = A$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Elementary matrices:

A square matrix is called an elementary matrix if it can be obtained from the $n \times n$ identity matrix I_n by performing single elementary row operation.

Examples

$E_2 =$ $A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ ✓ $B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓ $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Non example

$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $E = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

Non example.

$$\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$AB \neq BA$

$$\begin{bmatrix} 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \end{bmatrix} \quad \begin{bmatrix} 3 \end{bmatrix}$$

$AB \neq BA$

$(A+B)^2 \neq A^2 + 2AB + B^2$

$A^2 + AB + BA + B^2$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2+0 & 4+0 \\ 0+3 & 0+4 \end{bmatrix}$$

Elementary

$$= \begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2+0 & 0+2 \\ 6+0 & 0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 6 & 4 \end{bmatrix}$$

Row operation:	Inverse operation
Multiply a row i by $c \neq 0$	multiply the row i by $1/c$
Interchange row i and j	Interchange i and j
Add c times row i to row j	Add $-c$ times row i to row j

All elementary matrices are invertible.

$$A = \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$R_2 \leftarrow R_2 + \left(\frac{2}{3}\right) R_1$

$$E_1 A = \begin{bmatrix} -3 & 1 \\ 0 & 8/3 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 0 & 8/3 \end{bmatrix} \leftarrow E_1 A \qquad \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix} \rightarrow E_1$$

$R_1 \leftarrow R_1 + \left(\frac{-3}{8}\right) R_2$

$$\begin{bmatrix} -3 & 0 \\ 0 & 8/3 \end{bmatrix} \leftarrow E_2 (E_1 A) \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3/8 \\ 0 & 1 \end{bmatrix} \rightarrow E_2$$

$R_2 \leftarrow \frac{3}{8} R_2$

$$\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} \leftarrow E_3 (E_2 E_1 A) \qquad \begin{bmatrix} 1 & 0 \\ 0 & 3/8 \end{bmatrix}$$

$R_1 \leftarrow \left(\frac{-1}{3}\right) R_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \frac{-1}{3} & 0 \\ 0 & 1 \end{bmatrix} \rightarrow E_4$$

$I \leftarrow E_4 (E_3 E_2 E_1 A)$

$E_4 E_3 E_2 E_1 A = I$

$E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} I = A.$

$(AB)^{-1} = B^{-1} A^{-1}$

$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$

$A^{-1} = E_k E_{k-1} \cdots E_1 I_n$

Transformation: $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$w_1 = x_1 + x_2$

$w_2 = 3x_1 x_2$

$w_3 = x_1^2 - x_2^2$

$$f(x_1, x_2) = (\underset{w_1}{x_1 + x_2}, \underset{w_2}{3x_1 x_2}, \underset{w_3}{x_1^2 - x_2^2})$$

$(1, 1) \longrightarrow (2, 3, 0)$

$\mathbb{R}^n \rightarrow \mathbb{R}^m \quad [x_1, x_2, \ldots, x_n]$

$w_1 = a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n$

$w_2 = a_{12} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n$

$\vdots \qquad \vdots$

$w_m = a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n$.

$w \in \mathbb{R}^m \quad x \in \mathbb{R}^n \qquad T(x) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$

$A(x+y) = Ax + Ay$

$A(cx) = c(Ax)$

$\mathbb{R}^2 \to \mathbb{R}^2$

$(x_1, x_2) \to (w_1, w_2)$

$w_1 = -x_1 + 0x_2 = -x_1 \qquad (x_1, x_2) \to (-x_1, x_2)$

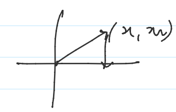
$w_2 = 0x_1 + x_2 \quad x_2$

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\mathbb{R}^2 \to \mathbb{R}$

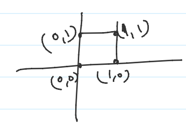
$w_1 = x_1 + 0x_2 = x_1 \qquad (x_1, x_2) \to (x_1, 0)$

$w_2 \quad 0x_1 + 0x_2 \quad 0$



$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \theta = \pi/4 \qquad \pi, \quad \pi/2$

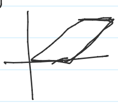
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ +1 \end{bmatrix}$



$w_1 = x + 3y \qquad = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$w_2 = 0 + y$

$\begin{bmatrix} & \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



$\begin{bmatrix} & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\begin{bmatrix} & \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

- $T: \mathbb{R}^n \to \mathbb{R}^m$

$T(u+v) = T(u) + T(v)$

$T(cv) = cT(u)$

height	weight	BMI