The most important questions of life are, indeed, for the most part, really only problems of probability

Pierre - Simon Laplace

A random experiment (or non-deterministic) is an experiment

(i) whose all possible outcomes are known in advance,

(ii) whose each outcome is not possible to predict in advance, and

(iii) can be repeated under identical conditions.

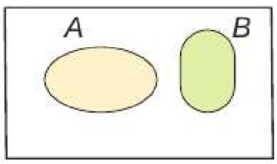
Three coins are tossed simultaneously, what is the probability of getting (i) exactly one head (ii) at least one head (iii) at most one head?

A sample space is the set of all possible outcomes of a random experiment. Each point in sample space is an elementary event.

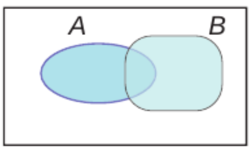
A **simple event** (or elementary event or sample point) is the most basic possible outcome of a random experiment and it cannot be decomposed further.

If a die is rolled, then the sample space $S = \{1, 2, 3, 4, 5, 6\}$.

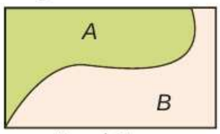
If the experiment consists of choosing a number randomly between 0 and 1, then the sample space is $S = \{\, x : 0 < x < 1\}$.



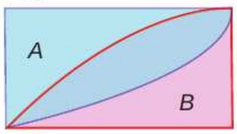
A and *B* are Mutually exclusive



A and *B* are Mutually inclusive



A and *B* are Mutually exclusive and exhaustive



A and *B* are Mutually inclusive and exhaustive

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Random Experiment	Total Number of Outcomes	Sample space
Tossing a fair coin	$2^1 = 2$	$\{H, T\}$
Tossing two coins	$2^2 = 4$	$\{HH, HT, TH, TT\}$
Tossing three coins	$2^3 = 8$	$\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
Rolling fair die	$6^1 = 6$	$\{1, 2,3, 4, 5,6\}$
Rolling Two dice or single die two times.	$6^2 = 36$	$\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$ $(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$ $(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$ $(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$ $(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$ $(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$
Drawing a card from a pack of 52 playing cards	$52^1 = 52$	Heart ♥ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Diamond ♦ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Spade ♠ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour Club ♣ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour

Let S be the sample space associated with a random experiment and A be an event. Let $n(S)$ and $n(A)$ be the number of elements of S and A respectively. Then the probability of the event A is defined as

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to } A}{\text{Exhaustive number of cases in } S}$$

Axioms of probability

Let S be a finite sample space, let $\mathscr{P}(S)$ be the class of events, and let P be a real valued function defined on $\mathscr{P}(S)$. Then $P(A)$ is called probability function of the event A, when the following axioms are hold:

[P_1] For any event A, $\quad P(A) \geq 0 \quad$ (Non-negativity axiom)

[P_2] For any two mutually exclusive events

$P(A \cup B) = P(A) + P(B) \quad$ (Additivity axiom)

[P_3] For the certain event $\quad P(S) = 1 \quad$ (Normalization axiom)

An integer is chosen at random from the first ten positive integers. Find the probability that it is (i) an even number (ii) multiple of three.

$\underbrace{\text{an even number}}_{A}$ $\underbrace{\text{multiple of three}}_{B}$

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{2, 4, 6, 8, 10\}$ $\quad P(A) = \frac{n(A)}{n(S)} = \frac{5}{10} = 1/2$

$B = \{3, 6, 9\} \Rightarrow P(B) = \frac{3}{10}$

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Suppose ten coins are tossed. Find the probability to get (i) exactly two heads (ii) at most two heads (iii) at least two heads

$n(S) = 2^{10} = 1024$

$S = \{(HHH\cdots H), (HH\cdots T) \cdots\}$

$n(A) = {}^{10}C_2 = 45$

$P(A) = 45/1024$

$n(B) = {}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2$

$= 56$

$P(B) = 56/1024$

$n(C) = {}^{10}C_2 + {}^{10}C_3 + \cdots {}^{10}C_{10}$

$= 1024 - [{}^{10}C_0 + {}^{10}C_1]$

$= 1013$

$P(C) = \frac{1013}{1024}$.

If A and B are any two events, then

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

(i) $P(A \cup B \cup C) = \{P(A) + P(B) + P(C)\}$
$- \{P(A \cap B) + P(B \cap C) + P(C \cap A)\} + P(A \cap B \cap C)\}$

Given that $P(A) = 0.52$, $P(B) = 0.43$, and $P(A \cap B) = 0.24$, find

(i) $P(A \cap \overline{B})$ (ii) $P(A \cup B)$ (iii) $P(\overline{A} \cap \overline{B})$ (iv) $P(\overline{A} \cup \overline{B})$.

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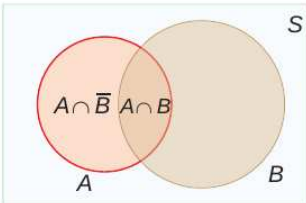
If $\overline{A}$ is the complementary event of A, then

$$\boxed{P(\overline{A}) = 1 - P(A)}$$

$P(A \cap \overline{B}) = P(A) + P(\overline{B}) - P(A \cup \overline{B})$

If A and B are any two events and $\overline{B}$ is the complementary events of B, then

$$\boxed{P(A \cap \overline{B}) = P(A) - P(A \cap B)}$$



$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$
$$= 0.52 - 0.24 = 0.28$$
$$P(A \cap \overline{B}) = 0.28.$$

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) \quad \text{(By de Morgan's law)}$$
$$= 1 - P(A \cup B)$$
$$= 1 - 0.71 = 0.29.$$

$$P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B}) \quad \text{(By de Morgan's law)}$$
$$= 1 - P(A \cap B) = 1 - 0.24$$
$$= 0.76.$$

The probability that a girl, preparing for competitive examination will get a State Government service is 0.12, the probability that she will get a Central Government job is 0.25, and the probability that she will get both is 0.07. Find the probability that (*i*) she will get atleast one of the two jobs (*ii*) she will get only one of the two jobs.

$P(S) = 0.12$

$P(C) = 0.25$

$P(S \cap C) = 0.07$

$P(S \cup C) = P(S) + P(C) - P(S \cap C)$
$= 0.3$

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$P(S \cup C) - P(S \cap C)$

$= 0.3 - 0.07$

$= 0.23$

Conditional Probability

Suppose a fair die is rolled once, then the sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Now we ask two questions

Q_1: What is the probability of getting an odd number which is greater than 2?

Q_2: If the die shows an odd number, then what is the probability that it is greater than 2?

$Q_1 = \{3, 5\}$ $\quad P(Q_1) = \frac{2}{6} = 1/3$

$Q_2 = \{3, 5\}$ $\quad S = \{1, 3, 5\}$

$P(Q_2) = 2/3$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

If $P(A) = 0.6$, $P(B) = 0.5$, and $P(A \cap B) = 0.2$

Find (i) $P(A/B)$ (ii) $P(\overline{A}/B)$ (iii) $P(A/\overline{B})$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.5} = 0.4$$

$$P(\overline{A}/B) = \frac{P(\overline{A} \cap B)}{P(B)}$$

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$$= \frac{P(B) - P(A \cap B)}{P(B)} = 3/5$$

$$P(A|\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

$$= 4/5$$

A die is rolled. If it shows an odd number, then find the probability of getting 5.

$$S = \{1, 3, 5\} \qquad P(A) = \frac{1}{3}$$

$$A = \{5\}$$

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{1, 3, 5\} \qquad B = \{5\}$$

$$B \cap A = \{5\}$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$= \frac{1/6}{3/6}$$

$$= 1/3$$

Two events A and B are said to be independent if and only if

$$\boxed{P(A \cap B) = P(A) \cdot P(B)}$$

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$$P(A/B) = P(A) \quad \text{if } P(B) > 0$$
$$P(B/A) = P(B) \quad \text{if } P(A) > 0$$

If A and B are independent then

(i) $\overline{A}$ and $\overline{B}$ are independent.

(ii) A and $\overline{B}$ are independent.

(iii) $\overline{A}$ and B are also independent.

Two cards are drawn from a pack of 52 cards in succession. Find the probability that both are Jack when the first drawn card is (i) replaced (ii) not replaced

case i

$n(A) = 4$

$n(B) = 4$

$n(S) = 52$

$P(A \cap B) = P(A) \cdot P(B)$

$= \frac{4}{52} \times \frac{4}{52}$

$=$

case ii

$P(A \cap B) =$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(A \cap B) = P(A|B)\, P(B)$

$= \frac{3}{51} \frac{4}{52}$

$n(A|B) = 3$

$n(S) = 51$

$= \frac{1}{221}$

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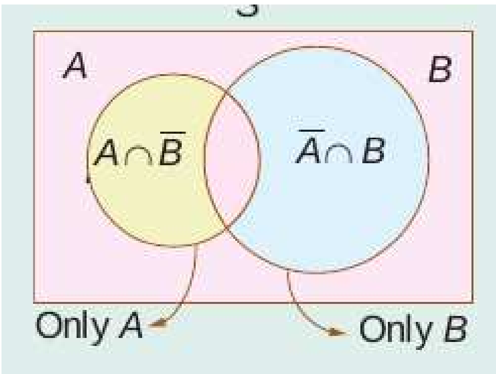
Suppose A and B are two events, such that $P(A) \neq 0$, $P(B) \neq 0$.

(1) If A and B are mutually exclusive, they cannot be independent.

(2) If A and B are independent they cannot be mutually exclusive.

X speaks truth in 70 percent of cases, and Y in 90 percent of cases. What is the probability that they likely to contradict each other in stating the same fact?



$P(A) = 0.7$

$P(B) = 0.9$

$P(\bar{A}) = 0.3$

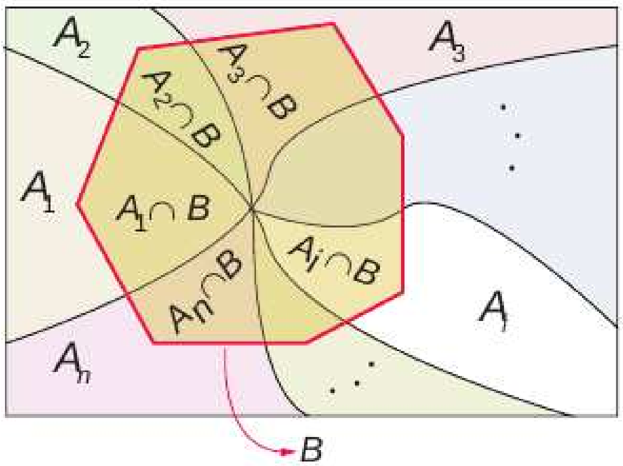
$P(\bar{B}) = 0.1$

$$P[(A \cap \bar{B}) \cup (\bar{A} \cap B)] = P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

$$= P(A)P(\bar{B}) + P(\bar{A})P(B)$$

$$= (0.7)(0.1) + (0.3)(0.9)$$

$$= 0.34$$



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If $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive and exhaustive events and B is any event in S then $P(B)$ is called the total probability of event B and

$$\boxed{P(B) = P(A_1).P(B/A_1) + P(A_2).P(B/A_2) + \cdots + P(A_n).P(B/A_n) = \sum_{i=1}^{n} P(A_i).P(B/A_i)}$$

Urn-I contains 8 red and 4 blue balls and urn-II contains 5 red and 10 blue balls. One urn is chosen at random and two balls are drawn from it. Find the probability that both balls are red.

	Red balls	Blue balls	Total
Urn-I	8	4	12
Urn-II	5	10	15
Total	13	14	27

$A_1 \to$ Event of choosing urn-I

$A_2 \to$ " " " urn-II

$P(A_1) = 1/2 \qquad P(A_2) = 1/2$

$B \Rightarrow$ Event of choosing two red balls

$$P(B|A_1) = \frac{^8C_2}{^{12}C_2} = 14/33$$

$$P(B|A_2) = \frac{^5C_2}{^{15}C_2} = 2/21$$

$$P(B) = P(A_1)P(B/A_1) + P(A_2)P(B/A_2)$$
$$= \frac{1}{2}\frac{14}{33} + \frac{1}{2}\frac{2}{21} = \frac{20}{77}$$

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A factory has two machines I and II. Machine-I produces 40% of items of the output and Machine-II produces 60% of the items. Further 4% of items produced by Machine-I are defective and 5% produced by Machine-II are defective. If an item is drawn at random, find the probability that it is a defective item. S

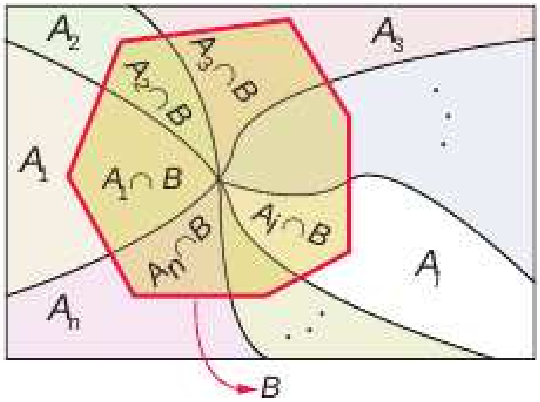
$P(D|M_1) = 0.04 \qquad P(M_1) = 0.4$

$P(D|M_2) = 0.05 \qquad P(M_2) = 0.6$

$P(D) = (0.04)(0.4) + (0.05)(0.6)$

$= P(M_1)P(D|M_1) + P(M_2)P(D|M_2)$

$= 0.046$



If $A_1, A_2, A_3, ..., A_n$ are mutually exclusive and exhaustive events such that $P(A_i) > 0$, $i = 1,2,3,\ldots.n$ and B is any event in which $P(B) > 0$, then

$$P(A_i/B) = \frac{P(A_i)\,P(B/A_i)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + \cdots + P(A_n)\,P(B/A_n)}$$

A factory has two machines I and II. Machine I produces 40% of items of the output and Machine II produces 60% of the items. Further 4% of items produced by Machine I are defective and 5% produced by Machine II are defective. An item is drawn at random. If the drawn item is defective, find the probability that it was produced by Machine II.

$$P(M_2|D) = \frac{P(M_2)\,P(D|M_2)}{P(M_1)P(D|M_1) + P(M_2)P(D|M_2)}$$

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$$= \frac{(0.6)(0.05)}{(0.4)(0.04) + (0.6)(0.05)}$$

$$= 15/23$$

A construction company employs 2 executive engineers. Engineer-1 does the work for 60% of jobs of the company. Engineer-2 does the work for 40% of jobs of the company. It is known from the past experience that the probability of an error when engineer-1 does the work is 0.03, whereas the probability of an error in the work of engineer-2 is 0.04. Suppose a serious error occurs in the work, which engineer would you guess did the work?

$P(Er/E1) = 0.03 \qquad P(E1) = 0.6$

$P(Er/E2) = 0.04 \qquad P(E2) = 0.4$

$$P(E1/Er) = \frac{P(E1)\,P(Er/E1)}{P(E1)\,P(Er/E1) + P(E2)\,P(Er/E2)}$$

$$= 9/17$$

$$P(E2/Er) = \frac{P(E2)\,P(Er/E2)}{P(E1)\,P(Er/E1) + P(E2)\,P(Er/E2)}$$

$$= 8/17$$

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A random variable X *is a function defined on a sample space* S *into the real numbers* $\mathbb{R}$ *such that the inverse image of points or subset or interval of* $\mathbb{R}$ *is an event in* S*, for which probability is assigned.*

$$S = \{\underset{2}{\underline{HH}}, \underset{1}{\underline{HT}}, \underset{1}{\underline{TH}}, \underset{0}{\underline{TT}}\}$$

X : No of Heads

X	0	1	2

2 5 10 17 26 37

5 8 13 20 29 40

$$S = \begin{Bmatrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{Bmatrix}$$

X : sum

X : 2 3 4 ⋯ 12

X : product

1/36 2/36 2/36

X = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36

X : sum of squares

X	2	3	4	. .	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	.	

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Two balls are chosen randomly from an urn containing 6 white and 4 black balls. Suppose that we win ₹ 30 for each black ball selected and we lose ₹ 20 for each white ball selected. If X denotes the winning amount, find the values of X and number of points in its inverse images.

x	-40	10	60
$P(X=x)$	$1/3$	1B 1W	2B
	$\frac{6C_2}{10C_2}$	$\frac{4C_1 \times 6C_1}{10C_2}$	$\frac{4C_2}{10C_2}$
	$(15/45)$	$(24/45)$	$6/45$

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