Definition 1 (Unique Solvability)**.** *Let* $M = \langle V, U, F, P(U) \rangle$ *be an SCM and* $\mathbf{Z} \subseteq \mathbf{V}$ *a subset of endogenous variables. We say that* M *is uniquely solvable if for almost every* $\mathbf{u} \in dom(\mathbf{U})$ *the equations*

$$\mathbf{V} = F(\mathbf{V}, \mathbf{u})$$

have a unique solution.

Throughout this paper we only need consider uniquely solvable SCMs, which simplifies our definition of potential response.

Definition 2 (Potential Response)**.** *Let* $M = \langle V, U, F, P(U) \rangle$ *be a uniquely solvable SCM. The potential response function is the mapping* $\overline{\mathbf{V}}(\mathbf{U}) : dom(\mathbf{U}) \to dom(\mathbf{V})$ *which associates each* $\mathbf{u} \in dom(\mathbf{U})$ *with the unique solution* $\mathbf{v}^*$ *of* $\mathbf{V} = F(\mathbf{V}, \mathbf{u})$*.*

Definition 3 (Simple SCM)**.** *Let* $M = \langle V, U, F, P(U) \rangle$ *be an SCM. We call* M *simple if* M *is uniquely solvable w.r.t. every subset* $\mathbf{Z} \subseteq \mathbf{V}$*.*

The following defintion is adopted from [**?**].

Definition 4 (Lipschitz Matrix)**.** *Let* $M = \langle V, U, F, P(U) \rangle$ *be an SCM, with* $dom(\mathbf{U}) = \mathbb{R}^m$*,* $dom(\mathbf{V}) = \mathbb{R}^n$*, and* $F : dom(\mathbf{U}) \times dom(\mathbf{V}) \to dom(\mathbf{V})$ *differentiable and Lipschitz.*

Let $\mathbf{Z} \subseteq \mathbf{V}$ *be a subset of endogenous variables. For each pair of vertex indices* i*,* $j \in \mathbf{Z}$*, define*

$$a_{ij} = \sup_{\mathbf{u},\mathbf{v}} \left| \frac{\partial f_i}{\partial v_j}(\mathbf{u}, \mathbf{v}) \right|$$

We call the matrix $A_{\mathbf{z}} = [a_{ij}]_{i,j \in \mathbf{Z}}$ *the Lipschitz matrix of* $F_{\mathbf{Z}}$*.*

When $\mathbf{Z} = \mathbf{V}$*, we simply call* $A = [a_{ij}]$ *the Lipschitz matrix of* F*.*

The matrix A can be viewed as a collection of Lipschitz constants along the dirction of each partial derivative.

Definition 5 (stable-Lipschitz SCM)**.** *Let* $M = \langle V, U, F, P(U) \rangle$ *be an SCM with causal diagram* G*. We say* M *is stable-Lipschitz if, for every strongly connected component* $\mathbf{Z}$ *of* G*, the following conditions hold:*

- $F_{\mathbf{Z}}$ *is differentiable and Lipschitz.*
- $\rho(A_{\mathbf{Z}}) < 1$*.*

No constraints are placed on components of F which are not part of strongly connected components of G. This immediately implies the following:

Theorem 1 (acylic $\subset$ stable-Lipschitz)**.** *Let* M *be an acyclic SCM. Then* M *is stable-Lipschitz.*

Theorem 2 (Closed under Interventions)**.** *Let M be a stable-Lipshitz SCM, $\mathbf{X} \subseteq \mathbf{V}$, and $\mathbf{x} \in dom(\mathbf{X})$. Then $M_{do(\mathbf{X}=\mathbf{x})}$ is stable-Lipschitz.*

Lemma 3 (Unique Solvability)**.** *Let M be a stable-Lipschitz SCM. Then M is uniquely solvable.*

Theorem 4 (stable-Lipschitz $\subset$ simple)**.** *Let M be a stable-Lipschitz SCM. Then M is simple.*

New Results

Lemma 5 (Injectivity)**.** *Let M have structural functions of the form $F(\mathbf{V}, \mathbf{U}) = H(\mathbf{V}) + \mathbf{U}$ (additive noise). Furthermore, assume that for every ancestral $\mathbf{W} \subseteq \mathbf{V}$,*

- *$F_{\mathbf{W}}$ is Lipschitz*
- *$det(I_{|\mathbf{W}|} - A_{\mathbf{W}}) \neq 0$ (uniquely solvable)*

Then $h_{\mathbf{V}} := \mathbf{V} - H(Pa(\mathbf{V}))$ is injective for ancestral $\mathbf{W} \subseteq \mathbf{V}$.

Lemma 6 (Intrinsic Stability)**.** *Let A be a Lipschitz matrix for differentiable $H(\mathbf{V})$. Then $\rho(A) < 1$ implies $\rho(\frac{dK}{d\mathbf{V}}(\mathbf{v})) < 1$ for all $\mathbf{v} \in \mathbf{V}$.*

Theorem 7 (Observational dGMP)**.** *Let M be stable Lipschitz with 1. structural equations of the form $F(\mathbf{V}, \mathbf{U}) = H(\mathbf{V}) + \mathbf{U}$ (additive noise), 2. each $U_i \cap U_j = \emptyset$ for $i \neq j$ (independent noise), and 3. $P_M(\mathbf{V})$ has density according the the Legesgue measure on $\mathbb{R}^{|\mathbf{V}|}$ (positivity).*

Then M satisfies the directed global Markov property.

Remark 1. *I am confident that conditions 1 and 2 in the hypothesis can be substantially weakened with further research.*

This is a very new result, so while I believe the proof to be accurate and comprehensive, I'm still vetting it for errors: I'd place 5:1 odds against finding an irrecoverable error in the proof.

Corollary 1 (Adjustment Formula)**.** *Let M be as (before) and $Q = P(y|do(x))$ a causal query. If the BDC is satisfied, then Q can be found via backdoor adjustment.*

Remark 2. *One of the motivations for weakening the condition of independent noise is to be able to similarly prove that the front-door criteria is also valid.*

Theorem 8 (Interventional dGMP)**.** *Let M be as in Theorem 7. For any $\mathbf{X} \subseteq \mathbf{Z}$, $M_{do(\mathbf{X}=\mathbf{x})}$ satisfies the directed global Markov property.*

Theorem 9 (Closure under Twin Operation)**.** *Let M be stable-Lipschitz. Then M^{twin} is also stable-Lipschitz.*

Conjecture 1 (Counterfactual dGMP)**.** *Let M be stable-Lipschtiz. Then the counterfactual distributions of M satisfy the directed global Markov property relative to the corresponding twin network.*

Remark 3. *I believe the counterfactual dGMP holds if the condition of independent noise can weakened, as an immediate consequence of the other results so far. However, I would place 2:1 odds that I'm missing some additional aspect of the proof.*