Cyclic Causality: D-separation and Nonlinearity

David Reber

3/24 - 3/30 update

Past To-Do's

- 3/24 - 3/25
  - ✓ Relate problem to the literature
  - ✓ Writeup, submit mid-semester report
- 3/27 - 3/30
  - Come up with an example for each of the following:
    - ✓ Cyclic SCMs
    - ✓ Loss of d-separation
    - ✓ Intrinsic SCMs
      - Validity of d-separation via numerics

(It'll be a short week for me as I'll be out of town 3/31 - 4/3)

[Spirtes 2013]

Example: Cyclic SCM

$$M \begin{cases} \mathbf{u}, \mathbf{v} \in \mathbb{R}^4 \\ U_i \sim \mathcal{N}(0,1) \quad iid. \\ f_1 : V_1 \leftarrow U_1 \\ f_2 : V_2 \leftarrow U_2 \\ f_3 : V_3 \leftarrow V_1 V_4 + U_3 \\ f_4 : V_4 \leftarrow V_2 V_3 + U_4 \end{cases}$$

[Spirtes 2013]

Example: Cyclic SCM

$$
M \left\{
\begin{array}{l}
\mathbf{u}, \mathbf{v} \in \mathbb{R}^4 \\
U_i \sim \mathcal{N}(0,1) \quad iid. \\
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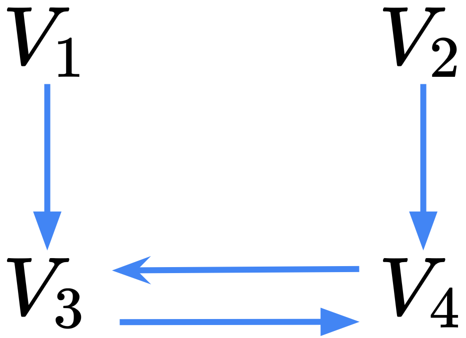
\right.
$$



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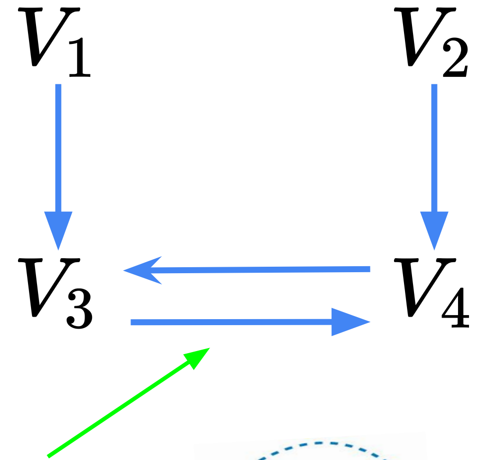

Not latent confounder:

[Spirtes 2013]

Example: Observational Distribution

$$
\begin{aligned}
V_1 &\leftarrow U_1 \\
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[Spirtes 2013]

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Potential Responses

[Spirtes 2013]

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$$P(U_i = u_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u_i^2}$$

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[Spirtes 2013
Bongers et al. 2021
Forr’e et al. 2017]

Example: d-separation

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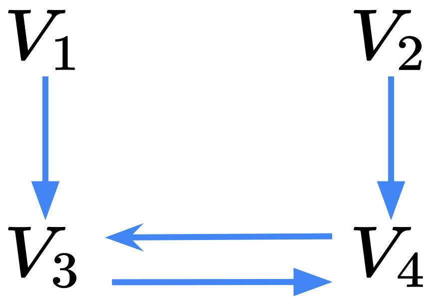

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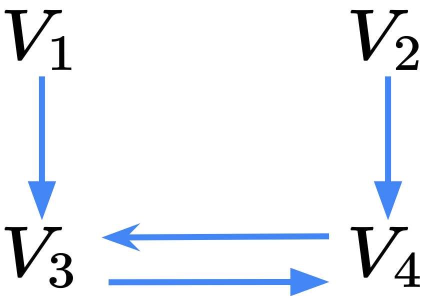

$$(V_1 \perp\!\!\!\perp V_2 | V_3, V_4)_G \text{?}$$

Example: d-separation

[Spirtes 2013
Bongers et al. 2021
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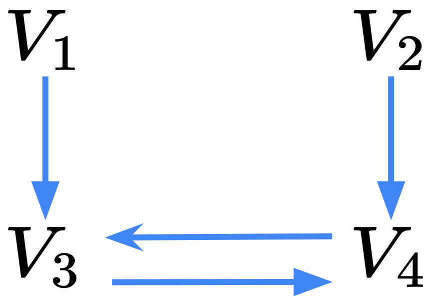

2 edges means 2 paths

$$(V_1 \perp\!\!\!\perp V_2 | V_3, V_4)_G \textbf{?}$$

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\begin{tikzcd}
V_1 \arrow[d] & V_2 \arrow[d] \\
V_3 \arrow[r, shift right] & V_4 \arrow[l, shift right]
\end{tikzcd}
$$

2 edges means 2 paths

$$(V_1 \perp\!\!\!\perp V_2 \,|\, V_3, V_4)_G\,?$$

Path 1: $V_1 \rightarrow \boxed{V_2} \rightarrow \boxed{V_3} \leftarrow V_4$

Example: d-separation

[Spirtes 2013
Bongers et al. 2021
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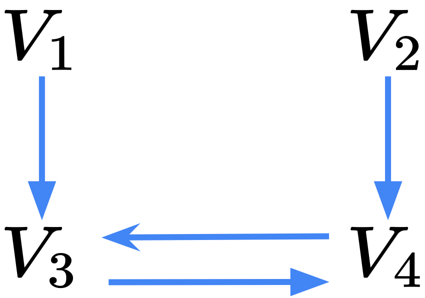

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[Spirtes 2013
Bongers et al. 2021
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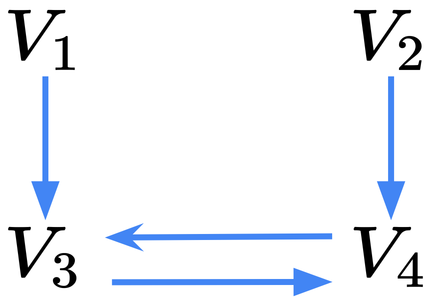

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Path 2: $V_1 \rightarrow \boxed{V_2} \leftarrow \boxed{V_3} \leftarrow V_4$

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$(V_1 \perp\!\!\!\perp V_2 | V_3, V_4)_G$ ✓

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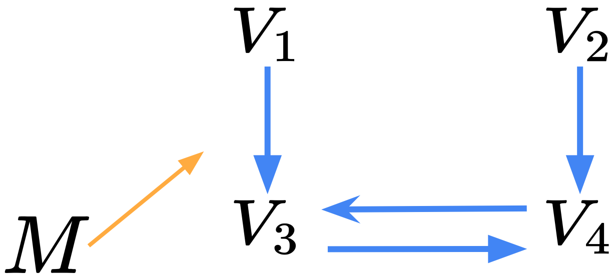
Path 2: $V_1 \rightarrow \boxed{V_2} \leftarrow \boxed{V_3} \leftarrow V_4$ *Blocked*

[Spirtes 2013]

Example: Directed Global Markov Property



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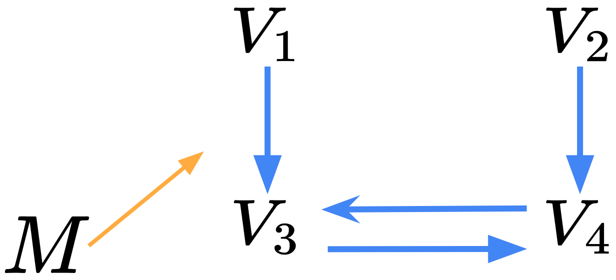

$$\xrightarrow{\text{d-separation}} (V_1 \perp\!\!\!\perp V_2 \mid V_3, V_4)_G$$

[Spirtes 2013]

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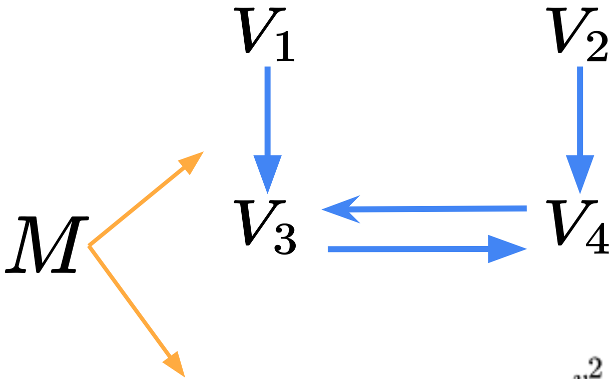

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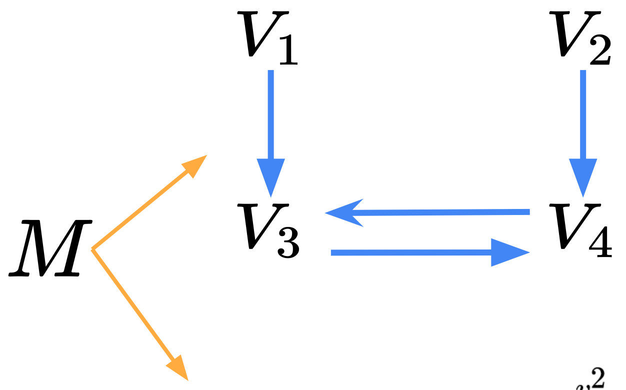

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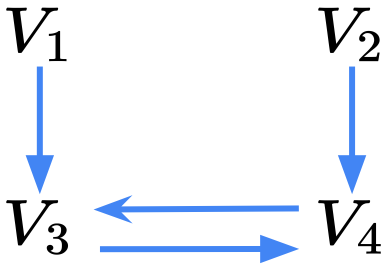
Not Possible to Factor!

[Spirtes 2013]

Example: Directed Global Markov Property

M



d-separation

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Not Possible to Factor!

So $(V_1 \not\perp\!\!\!\perp V_2 | V_3, V_4)_P$

[Spirtes 2013]

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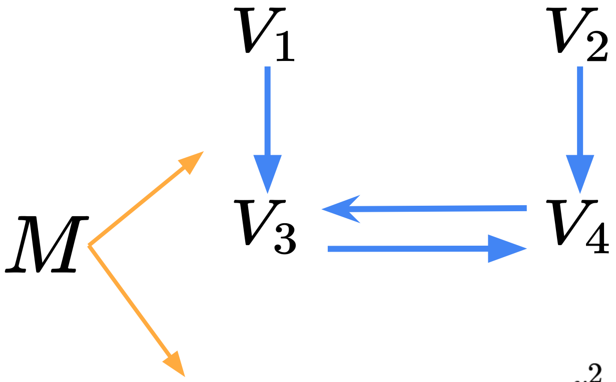

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Not Possible to Factor!

So $(V_1 \not\perp\!\!\!\perp V_2 | V_3, V_4)_P$

Let’s double check with numerics…

Numerics: Directed Global Markov Property

$$V_1(\mathbf{U}) = U_1$$
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$$P(\mathbf{U}) \xrightarrow{V_i(\mathbf{U})} P(\mathbf{V})$$

- Monte Carlo: sample from $P(\mathbf{U})$ and apply $V_i(\mathbf{U})$

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💡 $P(\mathbf{U}) \xrightarrow{V_i(\mathbf{U})} P(\mathbf{V})$

- Monte Carlo: sample from $P(\mathbf{U})$ and apply $V_i(\mathbf{U})$

- Test $(V_1 \overset{?}{\perp\!\!\!\perp} V_2 | V_3, V_4)_P$ numerically…
  - 🤔 Conditional independence + continuous domain? That's hard…

[FCIT: Chalupka et al. 2018]

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- …using FCIT

(a fast independence test which tries to predict V_1 from V_2 after V_3 and V_4 are provided)

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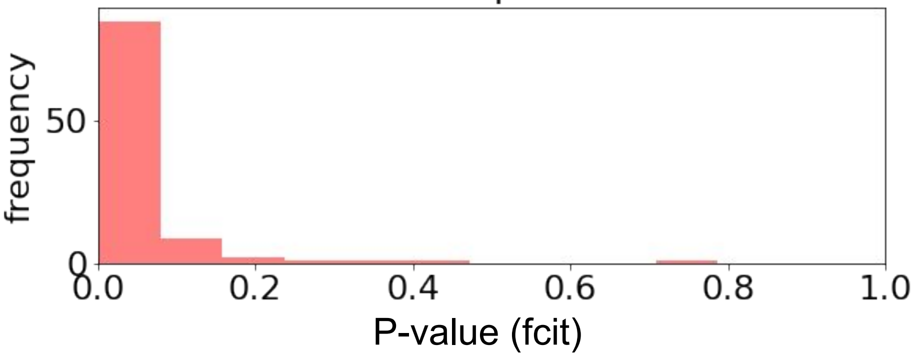
$(V_1 \not\perp\!\!\!\perp V_2 | V_3, V_4)$



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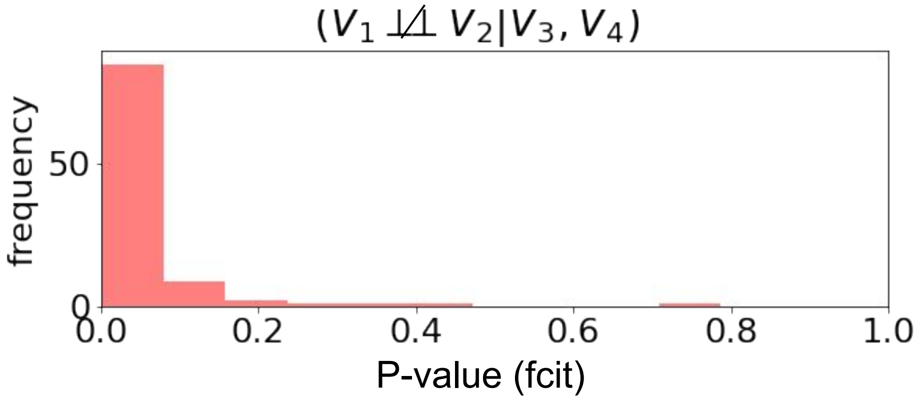

Low p-values $\longrightarrow$ conditional dependence

[FCIT: Chalupka et al. 2018]

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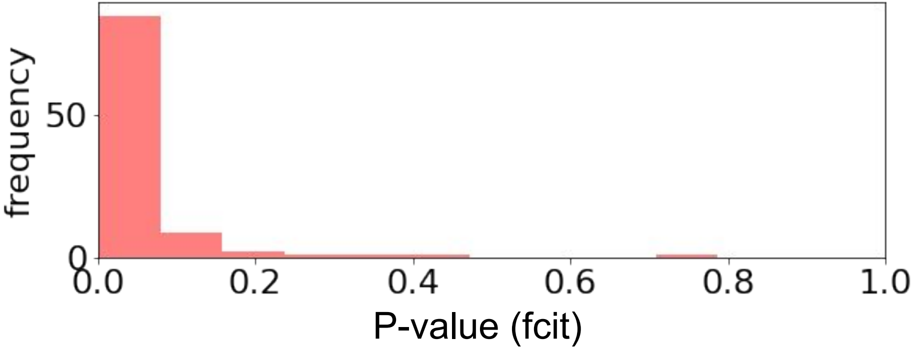
(a fast independence test which tries to predict V_1 from V_2 after V_3 and V_4 are provided)

$(V_1 \not\perp\!\!\!\perp V_2 | V_3, V_4)$



Low p-values ⟶ conditional dependence

$$(V_1 \not\perp\!\!\!\perp V_2 | V_3, V_4)_P$$

$$(V_1 \perp\!\!\!\perp V_2 | V_3, V_4)_G \not\Rightarrow (V_1 \perp\!\!\!\perp V_2 | V_3, V_4)_P$$

Example: Asymptotics

$$V_1(\mathbf{U}) = U_1$$
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$$P(\mathbf{U}) \xrightarrow{V_i(\mathbf{U})} P(\mathbf{V})$$

🤔 $V_i(\mathbf{U})$ is found by fixing $\mathbf{u}$ and solving for the equilibrium of the system

Example: Asymptotics

$$V_1(\mathbf{U}) = U_1$$
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$$P(\mathbf{U}) \xrightarrow{V_i(\mathbf{U})} P(\mathbf{V})$$

- 🤔 $V_i(\mathbf{U})$ is found by fixing $\mathbf{u}$ and solving for the equilibrium of the system
- 🤔 For linear SCMs*, these $V_i(\mathbf{U})$ at equilibrium preserve the conditional independencies of $\boldsymbol{G}$ (hence, d-separation is valid)

*If $P(\mathbf{V}) > 0$ everywhere, and there's no trivial dependencies

Example: Asymptotics

$$V_1(\mathbf{U}) = U_1$$
$$V_2(\mathbf{U}) = U_2$$
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Perhaps nonlinear SCMs whose *asymptomatic dynamics* are bounded by a linear SCM will also inherit these nice properties of $V_i(\mathbf{U})$

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💡 Perhaps nonlinear SCMs whose *asymptomatic dynamics* are bounded by a linear SCM will also inherit these nice properties of $V_i(\mathbf{U})$

- What happens if we try to bound this example??

*If $P(\mathbf{V}) > 0$ everywhere, and there's no trivial dependencies

Example: Linear Bounding

Bound each f_i by $\quad g_i = (\sum_j a_{ij} V_j) + U_i \quad a_{ij} = \max_{\mathbf{v}} |\frac{\partial f_i}{\partial V_j}(\mathbf{v}, \mathbf{u})|$

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$$
\begin{aligned}
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\end{aligned}
$$

$$
\begin{aligned}
g_1 &: V_1 \leftarrow U_1 \\
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$f_1 : V_1 \leftarrow U_1 \longrightarrow a_{1j} = 0 \longrightarrow g_1 : V_1 \leftarrow U_1$

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$f_3 : V_3 \leftarrow V_1 V_4 + U_3 \longrightarrow g_3 : V_3 \leftarrow \boxed{a_{31}} V_1 + \boxed{a_{34}} V_4 + U_3$

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$a_{31} = \max_{\mathbf{v}} |V_4| \quad a_{34} = \max_{\mathbf{v}} |V_1| \quad a_{42} = \max_{\mathbf{v}} |V_3| \quad a_{43} = \max_{\mathbf{v}} |V_2|$

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- Some terms go to infinity, so we can't bound this example!
  - (these correspond directly to unboundedness of the potential response function)

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- Some terms go to infinity, so we can’t bound this example!
  - (these correspond directly to unboundedness of the potential response function)
- What if we tweak the example?

Example: Intrinsic SCM

$$M \begin{cases} \mathbf{u}, \mathbf{v} \in [-0.5, 0.5] \\ U_i \sim \mathcal{N}(0,1) \cap [-0.5, 0.5] \quad iid. \\ f_1 : V_1 \leftarrow U_1 \\ f_2 : V_2 \leftarrow U_2 \\ f_3 : V_3 \leftarrow V_1 V_4 + U_3 \\ f_4 : V_4 \leftarrow V_2 V_3 + U_4 \end{cases}$$

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$$M\left\{\begin{array}{l}\mathbf{u}, \mathbf{v} \in [-0.5, 0.5] \\ U_i \sim \mathcal{N}(0,1) \cap [-0.5, 0.5] \quad iid. \\ \\ f_1 : V_1 \leftarrow U_1 \\ f_2 : V_2 \leftarrow U_2 \\ f_3 : V_3 \leftarrow V_1 V_4 + U_3 \\ f_4 : V_4 \leftarrow V_2 V_3 + U_4 \end{array}\right.$$

$$\begin{tikzcd} V_1 \arrow[d] & V_2 \arrow[d] \\ V_3 \arrow[r, shift right] & V_4 \arrow[l, shift right] \end{tikzcd}$$

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\mathbf{u}, \mathbf{v} \in [-0.5, 0.5] \\
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f_3 : V_3 \leftarrow V_1 V_4 + U_3 \\
f_4 : V_4 \leftarrow V_2 V_3 + U_4
\end{cases}
\qquad
\mathcal{L}(M) \begin{cases}
\mathbf{u}, \mathbf{v} \in [-0.5, 0.5] \\
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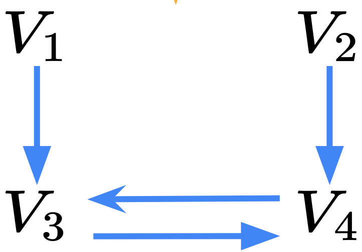
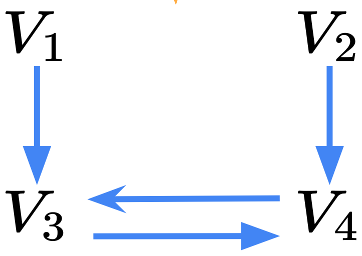
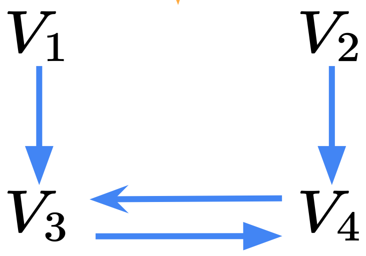
\end{cases}
$$



Example: Intrinsic SCM

$$M \left\{ \begin{array}{l} \color{red}{\mathbf{u}, \mathbf{v} \in [-0.5, 0.5]} \\ \color{red}{U_i \sim \mathcal{N}(0,1) \cap [-0.5, 0.5] \quad iid.} \\ f_1 : V_1 \leftarrow U_1 \\ f_2 : V_2 \leftarrow U_2 \\ f_3 : V_3 \leftarrow V_1 V_4 + U_3 \\ f_4 : V_4 \leftarrow V_2 V_3 + U_4 \end{array} \right. \longrightarrow \quad \mathcal{L}(M) \left\{ \begin{array}{l} \color{red}{\mathbf{u}, \mathbf{v} \in [-0.5, 0.5]} \\ \color{red}{U_i \sim \mathcal{N}(0,1) \cap [-0.5, 0.5] \quad iid.} \\ g_1 : V_1 \leftarrow U_1 \\ g_2 : V_2 \leftarrow U_2 \\ g_3 : V_3 \leftarrow a_{31} V_1 + a_{34} V_4 + U_3 \\ g_4 : V_4 \leftarrow a_{42} V_2 + a_{43} V_3 + U_4 \end{array} \right.$$





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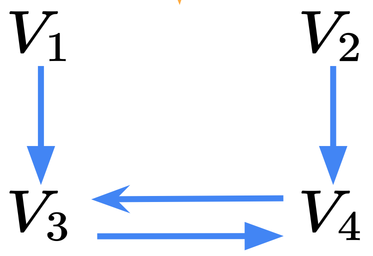
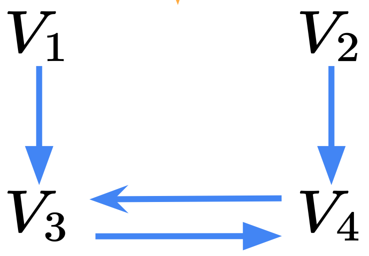


$(V_1 \perp\!\!\!\perp V_2 | V_3, V_4)_G$ ✓



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$(V_1 \overset{?}{\perp\!\!\!\perp} V_2 | V_3, V_4)_P$?

$(V_1 \perp\!\!\!\perp V_2 | V_3, V_4)_P \checkmark$

Numerics: Intrinsic SCM

$U_i \sim \mathcal{N}(0,1) \cap [-0.5, 0.5] \quad iid.$

$$M \quad \boxed{\begin{aligned} V_1(\mathbf{U}) &= U_1 \\ V_2(\mathbf{U}) &= U_2 \\ V_3(\mathbf{U}) &= \tfrac{U_1U_4+U_3}{1-U_1U_2} \\ V_4(\mathbf{U}) &= \tfrac{U_1U_4+U_3}{1-U_1U_2}U_2 + U_4 \end{aligned}}$$

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Test $(V_1 \overset{?}{\perp\!\!\!\perp} V_2 | V_3, V_4)_P$ again, numerically

- Using $P(\mathbf{U}) \xrightarrow{V_i(\mathbf{U})} P(\mathbf{V})$ but with restricted domain

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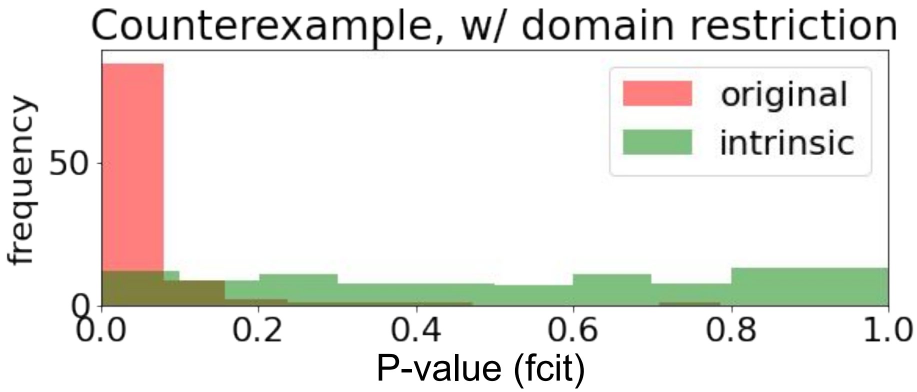

Uniform p-values $\longrightarrow$ conditional independence

Numerics: Intrinsic SCM

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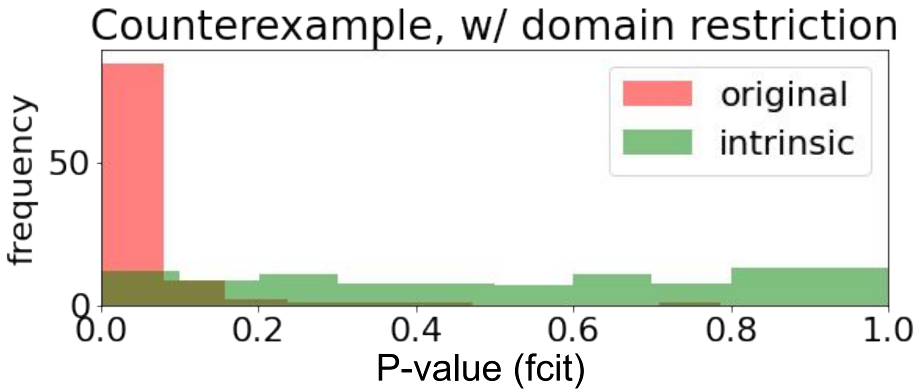

Uniform p-values $\longrightarrow$ conditional independence

$$(V_1 \perp\!\!\!\perp V_2 | V_3, V_4)_P$$

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Test $(V_1 \overset{?}{\perp\!\!\!\perp} V_2 | V_3, V_4)_P$ again, numerically

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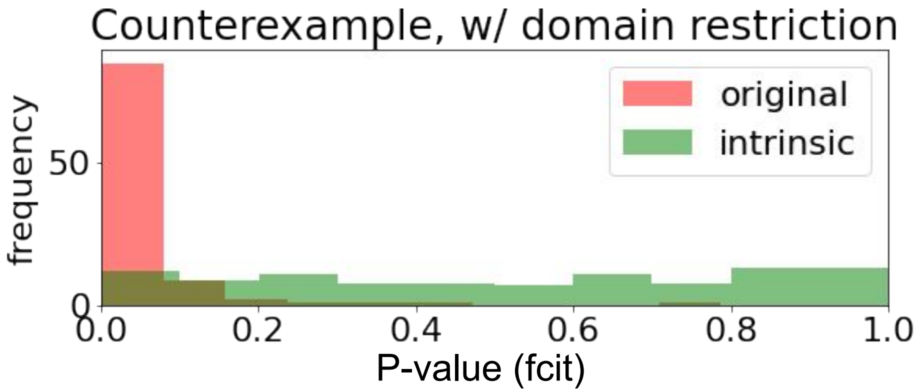


Uniform p-values $\longrightarrow$ conditional independence

$$(V_1 \perp\!\!\!\perp V_2 | V_3, V_4)_P$$

$$(V_1 \perp\!\!\!\perp V_2 | V_3, V_4)_G \implies (V_1 \perp\!\!\!\perp V_2 | V_3, V_4)_P$$

was inherited by the linear bound!

Next week’s directions

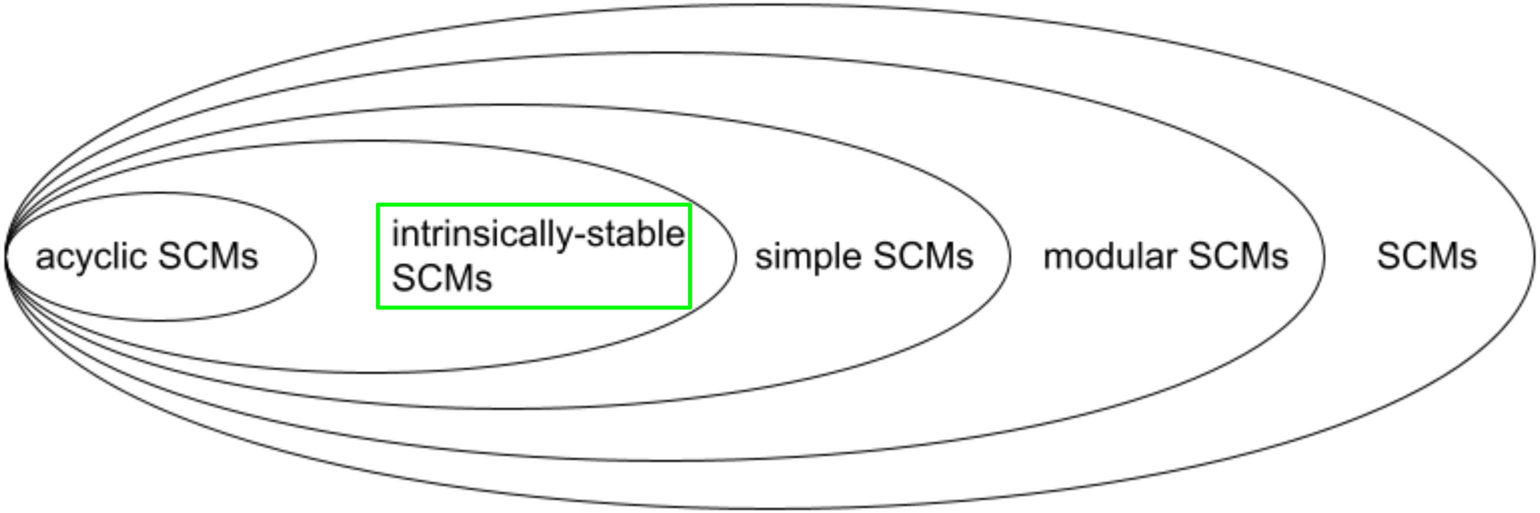
Observational Case: Proof Sketch

Strategy:

- Relate M to a linear SCM L(M) which satisfies the directed global Markov property.
- Show that M and L(M):
  - respect the same graph
  - are observationally equivalent.

$$
\begin{tikzcd}
 & & P_M(V) \arrow[d, equal] & \\
 & & P_{\mathscr{L}(M)}(V) \arrow[r] & (X \perp\!\!\!\perp Y \mid Z)_{P_{\mathscr{L}(M)}} \\
M \arrow[r] \arrow[uurr] \arrow[ddrr] & \mathscr{L}(M) \arrow[ur] \arrow[dr] & & \\
 & & G_{\mathscr{L}}(M) \arrow[r] \arrow[d, equal] & (X \perp\!\!\!\perp Y \mid Z)_{G_{\mathscr{L}(M)}} \arrow[uu, Rightarrow] \\
 & & G_M &
\end{tikzcd}
$$

Conjectured relation to other spaces of cyclic SCMs



Continuing Discussions: Game Theory + Causality

- Lewis Hammond (Oxford, Computer Science: Dphil student)
  - Extending Pearl's causal hierarchy to the game-theoretic domain

- Christian Kroer (Columbia, IOER department: Assistant professor)
  - Research: Game Theory, scalable optimization methods (for finding the potential responses)

- How relates:
  - Game theory is a primary motivation for cyclic causality
  - Local intrinsic stability

Next To-do's

- Prove Observational Equivalence
  - Get examples for each lemma
- Example of finding proper P(U) for linear bound
  - show numerically as well
  - Also an example of when it fails
- Prove (or disprove):
  - acyclic SCMs ⊂ intrinsic SCMs ⊂ simple SCMs
  - Example of each:
    - Linear bound for acyclic
    - Demonstrating unique solvability for each subset of intrinsic example
    - Example of simple SCM that's not intrinsic
- Meet with Christian Kroer, see if worth collaborating

References

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- Peter Spirtes. Directed cyclic graphical representations of feedback models. CoRR, abs/1302.4982, 2013.
- Stephan Bongers, Patrick Forré, Jonas Peters, and Joris M. Mooij. Foundations of structural causal models with cycles and latent variables. The Annals of Statistics, 49(5):2885 – 2915, 2021.