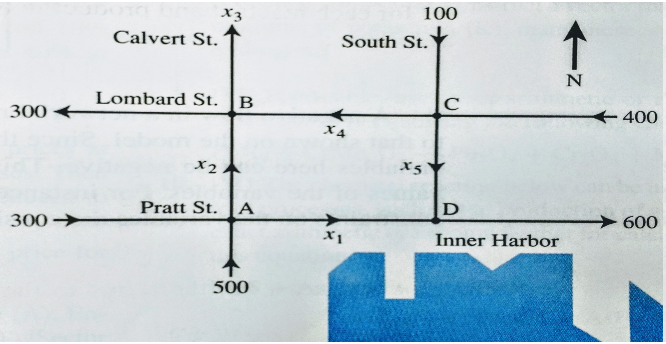
Dynamical Stability despite Time-Varying Network Structure

David Reber and Benjamin Webb

Brigham Young University



Vehicular Traffic Network[1]

[1] Lay, David C., et al. *Linear Algebra and Its Applications*. 5th ed., Pearson, 2016.

Outline

What is a network?

Basic Definition: A network is a collection of elements that interact in some way.

Structure of a Network

The structure of a network is represented by a graph $G = (V, E)$ with vertices V and edges E where

(i) V represent the network elements; and

(ii) E represent the interactions between network elements.

Types of Network Dynamics

Most real networks are dynamic in two distinct ways:

- **Dynamics between the Network Elements**:
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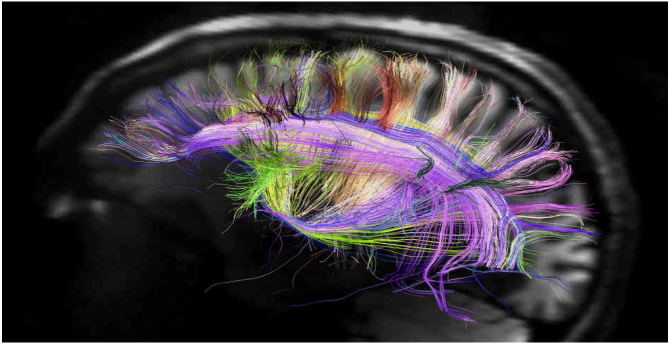
- **Dynamics of the Network Structure**: The network's graph structure evolves as new elements and interactions are added or removed over time.

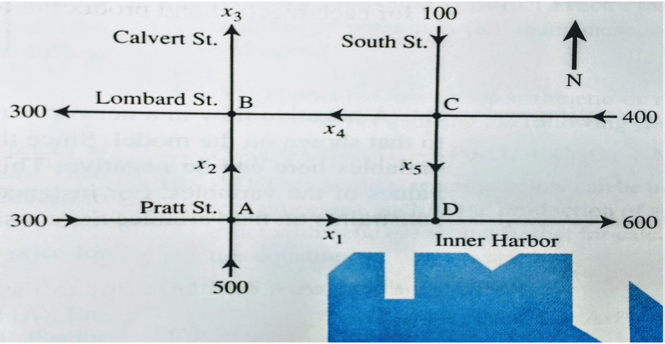
Example: Biological Networks



Neural Network[2]

[2] http://www.livescience.com/40855-brain-connections-no-neuron-is-an-island.html

Traffic Network



What type of questions can we ask about a network's dynamics?

When is a dynamical network (F, X) stable?

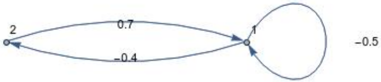
Definition: Stability

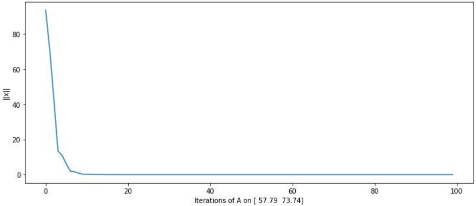
The dynamical system $F : X \to X$ has a globally attracting fixed point $\mathbf{y} \in X$ if for all $\mathbf{x} \in X$,

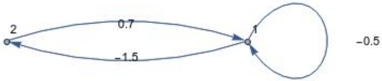
$$\lim_{k \to \infty} F^k(\mathbf{x}) = \mathbf{y}.$$

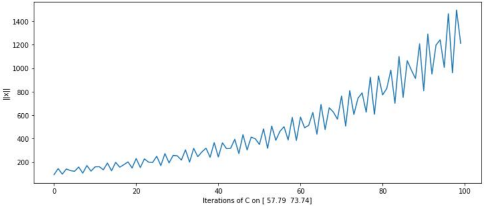
If this is the case we call (F, X) stable.

Dynamical Stability









Stability of Dynamical Networks

Definition: Stability Matrix

If the function $F : X \to X$ is differentiable with $X = \mathbb{R}^n$, then let $S_F \in \mathbb{R}^{n\times n}$ be the matrix given by

$$(S_F)_{ij} = \sup_{\mathbf{x}\in X} \left| \frac{\partial F_i}{\partial x_j}(\mathbf{x}) \right| .$$

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Remark: S_F is a constant square matrix representing the worst-case linear approximation to F.

What is the underlying structure of a dynamical network?

Definition: Graph of Interactions

The graph of interactions of the dynamical network (F, X) is the graph $G = (V, E)$ on n vertices, where the edge e_{ij} is given weight $(S_F)_{ij}$.

Intuitively, G is the directed graph for which S_F is the adjacency matrix.

Graph of Interactions

Example: Cohen-Grossberg Neural Network

$$F(\mathbf{x}) = \begin{bmatrix} (1-\epsilon)x_1 + a\tanh(x_2) + c_1 \\ (1-\epsilon)x_2 + a\tanh(x_1) + c_2 \end{bmatrix}, \quad \mathbf{x} \in \mathbb{R}^2.$$

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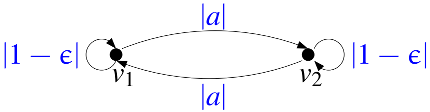

G_F

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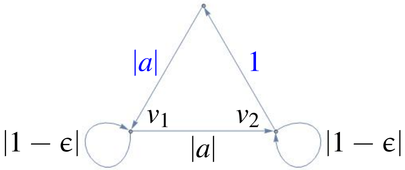

G_F

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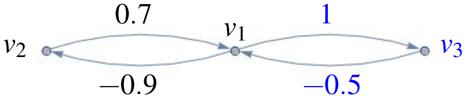
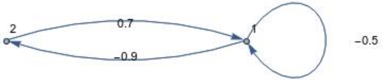
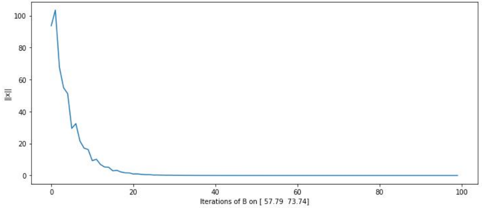
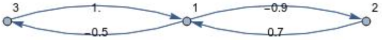
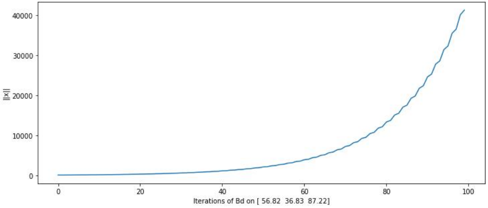


Time-Delayed Dynamical Networks









Outline

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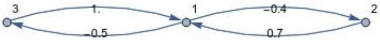
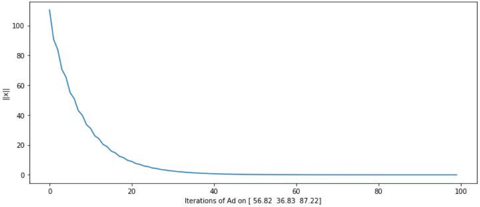
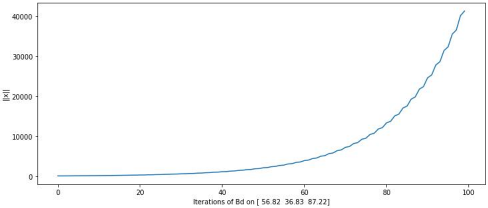
Theorem: B. Webb, et al.

- If (F, X) is intrinsically stable, then (F, X) is also stable.
- If (F, X) is intrinsically stable, then any constantly-time-delayed version of (F, X) is also stable.

Intrinsic Stability

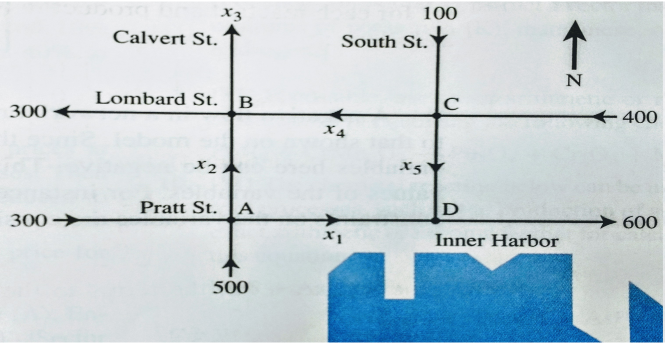






Outline

Time-Varying Time Delays



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Hence, *intrinsic stability* is a much stronger condition than previously believed, and provides insights into the dynamical behavior of certain adaptive networks.

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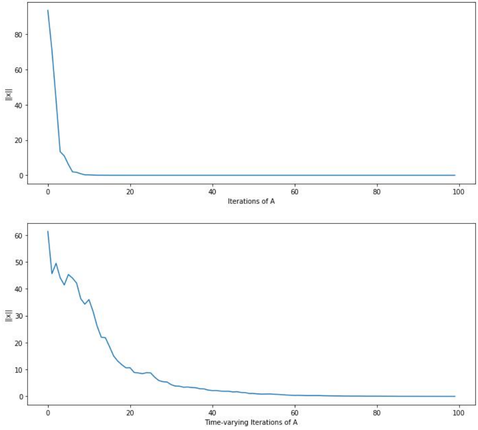

Outline

Future Questions

- How does adding new cycles to a network affect the state dynamics? (World Wide Web)

Dynamics both on and of Networks

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- How does adding new cycles to a network affect the state dynamics? (World Wide Web)
- How does changing the weights of a network affect the state dynamics? (Machine Learning, Deep Learning)

Thank You

References

[1] L. Bunimovich and B. Webb, Restrictions and Stability of Time-Delayed Dynamical Networks. *Nonlinearity* **26**, 2013, 2131-2156.
[2] L. Bunimovich and B. Webb, Isospectral graph transformations, spectral equivalence, and global stability of dynamical networks. *Nonlinearity* **25** (2012) 211-254.
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