AMATH 361 Final Project

Surface Waves & Earthquakes

Group 21

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Contents

Abstract

Abstract

A surface elastic wave is a wave travelling on the surface of a material without causing permanent changes. There are several factors affecting the propagation of these waves and the majority of them stem from physical material properties. A prime example of surface elastic waves are earthquakes, with their propagation being majorly dictated by ground composition. These waves can be broadly classified as p-waves or s-waves according to whether the wave is displaced in the direction of wave propagation, or in a direction orthogonal to the wave propagation, respectively. This report investigates the behaviour of p-waves through a simple material.
In doing so, the wave equation was derived from the equation of motion in order to obtain a value for wave speed in terms of density and Lamé coefficients. The 3D spherical solution to this equation was found in addition to a planar wave approximate solution. Finally, the governance of Snell's law in the propagation of p-waves through physical interfaces was explored. We attempted to provide computational analysis of p-wave propagation through layered media to no avail.

Key terms: p-wave, seismic fault, wave propagation, wavefunction

Introduction

A real world application of continuum mechanics is the study of how different elastic waves propagate in the Earth's crust. This field, called Seismology, commonly thought of as a very much applied science, stands on a large theoretical base. With applications ranging from detection of ground composition and structure[1] to providing Earthquake early warnings[2], our group found interest in delving deeper into this theoretical base.

As part of venturing into this field, we identified a problem that seemed solvable and attainable within the scope of this project. The first step when taking a set of equations and making a physical model out of it is to evaluate the different effects caused by the real world onto our equations. In our case, we put the focus on what the waves propagate in during seismic events: the soil. Typically, due to the large scale of seismic events, surface elastic waves will encounter different types of soils and these soils will have considerable effects on the propagation and amplitude of waves.

Earthquakes are composed of two different types of waves: p-waves (compressional) and s-waves (shear). These two form the complex phenomenon that are seismic waves and although quite interesting, the whole of this phenomenon is larger than the scope of this project. For this reason, we have decided to focus on the compressional part of the waves created by earthquakes: the p-waves.

P-waves are named as such from the fact they are the *primary* waves that are detected, due to their faster propagation than s-waves. In addition to being faster than s-waves, p-waves are far less destructive than s-waves, if destructive at all. These two properties of p-waves make them a natural warning for upcoming earthquakes.

The ability to accurately and automatically detect p-waves, as well as localizing their origin offers the chance to prevent civilian casualties by alerting nearby people at risk before the arrival of s-waves. With modelling of p-wave propagation in soil being an integral part of localizing their origin, we found the study of how these waves interface between several mediums to be relevent and of real-world use.

Specifically, the goal of this project is to model p-wave propagation between 2 physical media as a naive first step to a full model of p-wave propagation.

Theory

3.1 Background information

Seismic faults are planar fractures/discontinuities typically caused by significant displacements in solid ground.

Two particular types of waves caused by seismic faults are p-waves and s-waves. The distinction between the types draws from the order in which these wave arrive to a given location: p-waves are primary waves and s-waves are secondary waves.[3, 57]. Stein provides an excellent visual of the difference between the two, shown in Figure 6.2.

When talking about seismic waves, it is common to refer the direction of propagation of the wave as the z-axis, with the x-axis pointing outside the ground and the y-axis pointing parallel to the ground.

S-waves displace the field in a direction orthogonal to the propagation of the wave. This is also known as a *transverse wave*. As is typical for transverse waves, such as light [3, 57], s-waves have two independent polarizations. Therefore, we refer to the two polarization as SH waves, where displacements happen in the $y - z$ plane and SV planes, where displacements happen in the $x - z$ plane. On the other hand, a p-wave displaces the field in the same direction as the propagation of the wave, in other words, on the z-axis. This is also known as a *longitudinal wave*[3, 57].

The focus of this project will be on the longitudinal waves caused by seismic faults: the p-waves.

3.2 Continuum mechanics

3.2.1 Stress tensor and wave equation

We would like to solve the equation of motion to obtain solutions representing the wave functions of our compressional waves. We assume the soil is an isotropic elastic material. We have our equation of motion [3, 47]:

$$\frac{\partial \sigma_{ij}(\mathbf{x}, t)}{\partial x_j} + f_i(\mathbf{x}, t) = \rho \frac{\partial^2 u_i(\mathbf{x}, t)}{\partial t^2} \tag{3.1}$$

Where σ is our strain tensor, f_i are body forces and u is our displacement vector. In our case, we assume the fault has already passed and there is therefore no body force, such that:

$$\frac{\partial \sigma_{ij}(\mathbf{x}, t)}{\partial x_j} = \rho \frac{\partial^2 u_i(\mathbf{x}, t)}{\partial t^2} \tag{3.2}$$

We observe the specific case where $i = x$ and let j be the free index:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \rho \frac{\partial^2 u_x}{\partial t^2} \tag{3.3}$$

Now, we have that the constitutive equation for an isotropic material is [3, 50]:

$$\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} \tag{3.4}$$

Let us define:

$$\gamma = \frac{\partial u_i}{\partial x_i} = e_{ii}$$

Where γ represents dilatation, giving us:

$$\sigma_{ij} = \lambda\gamma\delta_{ij} + 2\mu e_{ij} \tag{3.5}$$

Where e is our strain tensor and λ and μ are Lamé coefficients.

$$\begin{aligned}
\sigma_{xx} &= \lambda\gamma + 2\mu e_{xx} = \lambda\gamma + 2\mu\frac{\partial u_x}{\partial x} \\
\sigma_{xy} &= 2\mu e_{xy} = 2\mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\
\sigma_{xz} &= 2\mu e_{xz} = 2\mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)
\end{aligned}$$

We then find the relevent spacial partials:

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} &= \lambda\frac{\partial \gamma}{\partial x} + 2\mu\frac{\partial e_{xx}}{\partial x} \\
\frac{\partial \sigma_{xy}}{\partial y} &= 2\mu\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial y \partial x}\right) \\
\frac{\partial \sigma_{xz}}{\partial z} &= 2\mu\left(\frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial z \partial x}\right)
\end{aligned}$$

We evaluate $\frac{\partial \gamma}{\partial x}$:

$$\begin{aligned}
\frac{\partial \gamma}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\
&= \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z}
\end{aligned}$$

Finally, we input everything back into 3.3:

$$\lambda\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z}\right) + 2\mu\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial y \partial x}\right) + 2\mu\left(\frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial z \partial x}\right) = \rho\frac{\partial^2 u_x}{\partial t^2} \tag{3.6}$$

We rearrange slightly:

$$(\lambda + \mu)\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z}\right) + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) = \rho\frac{\partial^2 u_x}{\partial t^2}$$

We recognize the x partial of the dilatation in the first term as well as the laplacian in the second term:

$$(\lambda + \mu)\frac{\partial \gamma}{\partial x} + \mu\nabla^2 u_x = \rho\frac{\partial^2 u_x}{\partial t^2} \tag{3.7}$$

We can repeat this process for $i = y$ and $i = z$ and see that we can generalize this previous equation as such:

$$(\lambda + \mu)\nabla\gamma + \mu\nabla^{\mathbf{2}}\mathbf{u} = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{3.8}$$

In terms of displacement alone:

$$(\lambda + \mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^{\mathbf{2}}\mathbf{u} = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{3.9}$$

Using the following identity [3, 54]:

$$\nabla^{\mathbf{2}}\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\times(\nabla\times\mathbf{u}) \tag{3.10}$$

Equation 3.9 becomes:

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times (\nabla \times \mathbf{u}) = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{3.11}$$

If we define our displacement field as a sum of scalar potentials and vector potentials as such:

$$\mathbf{u}(\mathbf{x}, t) = \nabla\phi(\mathbf{x}, t) + \nabla \times \mathbf{\Psi}(\mathbf{x}, t)$$

And use the following vector identities[3, 54]:

$$\nabla \times (\nabla\phi) = 0$$
$$\nabla \cdot (\nabla \times \mathbf{\Psi}) = 0$$

We get:

$$(\lambda + 2\mu)\nabla(\nabla^2\phi) - \mu\nabla \times \nabla \times (\nabla \times \mathbf{\Psi}) = \rho\frac{\partial^2}{\partial t^2}(\nabla\phi + \nabla \times \mathbf{\Psi})$$

If we reuse 3.10, we can calculate the second term on the left hand side to be:

$$\nabla \times \nabla \times (\nabla \times \mathbf{\Psi}) = -\nabla^2(\nabla \times \mathbf{\Psi}) + \nabla(\nabla \cdot (\nabla \times \mathbf{\Psi})) \tag{3.12}$$
$$= -\nabla^2(\nabla \times \mathbf{\Psi}) \tag{3.13}$$

Yielding:

$$(\lambda + 2\mu)\nabla(\nabla^2\phi) + \mu\nabla^2(\nabla \times \mathbf{\Psi}) = \rho\frac{\partial^2}{\partial t^2}(\nabla\phi + \nabla \times \mathbf{\Psi})$$

Rearranging to isolate the scalar potential and the vector potential gives us:

$$\nabla\left[(\lambda + 2\mu)\nabla^2\phi - \rho\frac{\partial^2\phi}{\partial t^2}\right] = -\nabla \times \left[\mu\nabla^2\mathbf{\Psi} - \rho\frac{\partial^2\mathbf{\Psi}}{\partial t^2}\right] \tag{3.14}$$

The left hand side of this equation represents the scalar wave function and the right hand side represents the vector wave function. Setting either side equal to 0 gives us differential equations allowing us to solve independently for either scalar or vector wave functions:

$$(\lambda + 2\mu)\nabla^2\phi = \rho\frac{\partial^2\phi}{\partial t^2} \tag{3.15}$$
$$\mu\nabla^2\mathbf{\Psi} = \rho\frac{\partial^2\mathbf{\Psi}}{\partial t^2} \tag{3.16}$$

In our case, p-waves are strictly scalar waves and as such, working with 3.15 will be sufficient. If we rearrange it, we obtain:

$$\nabla^2\phi = \frac{\rho}{(\lambda + 2\mu)}\frac{\partial^2\phi}{\partial t^2}$$

And define:

$$\nu = \sqrt{\frac{(\lambda + 2\mu)}{\rho}}$$

We get:

$$\nabla^2\phi = \frac{1}{\nu^2}\frac{\partial^2\phi}{\partial t^2} \tag{3.17}$$

We recognize this to be the 3D scalar wave equation with wave speed ν.

3.2.2 Plane wave approximation

While seismic waves caused by faults are spherical, as is shown in figure 6.1, far from their origin, these waves can be well approximated as planar waves. We recall the 3D scalar wave equation:

$$\nabla^2\phi(\mathbf{x},t) = \frac{1}{\nu^2}\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2} \tag{3.18}$$

Here, ν is a constant typically associated with wave speed, and ϕ corresponds to our wavefunction. The family of solutions to the 1D equivalent of the scalar wave equation is:

$$\phi(x,t) = Ae^{i(\omega t \pm kx)} \tag{3.19}$$

With ω representing the frequency of the wave, k representing the *wavenumber* and A being a constant of integration. This 1D solution can easily be generalized to a 3D solution:

$$\phi(\mathbf{x},t) = Ae^{i(\omega t \pm \mathbf{k}\cdot\mathbf{x})} \tag{3.20}$$

Where $\mathbf{k}$ and $\mathbf{x}$ are now vectors representing the same physical values. In our case, we are working with p-waves only, which act longitudinally. Therefore, we can clearly define:

$$\mathbf{k} = (0,0,k)$$

This gives us the following solution for the approximation of a p-wave as a planar wave:

$$\phi(\mathbf{x},t) = Ae^{i(\omega t \pm kz)} \tag{3.21}$$

We can then find the displacement caused by p-waves by taking the gradient of the wavefunction. Note that we arbitrarily choose the sign in the exponent to be negative:

$$\mathbf{u}(z,t) = \nabla\phi = (0,0,-ik)Ae^{i(\omega t - kz)}, \tag{3.22}$$

Noting that this vector exists purely as a z-component, meaning that the displacement changes *only* in the direction of the propagation of the wave. We take the divergence of this $\mathbf{u}$ to obtain the expression for the dilatation of the material:

$$\nabla\cdot\mathbf{u}(z,t) = -k^2Ae^{i(\omega t - kz)} \tag{3.23}$$

Note that this divergence is nonzero, so a volume change occurs. In particular, the material compresses and expands periodically with respect to the z and t coordinates. The curl here is identically zero, since $\nabla\times(\nabla\phi) = 0$.

3.2.3 Spherical wave

We recall the Laplacian operator in spherical coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \tag{3.24}$$

Such that:

$$\nabla^2\phi(\mathbf{r},t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi(\mathbf{r},t)}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi(\mathbf{r},t)}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi(\mathbf{r},t)}{\partial\varphi^2} \tag{3.25}$$

Where $\mathbf{r}$ is our spherical position vector:

$$\mathbf{r} = (r,\theta,\varphi)$$

For spherical waves, we know our wavefunction depends only on the radial component r:

$$\phi(\mathbf{r},t) = \phi(r,t)$$

Such that 3.25 has only non-zero terms for partials of radius:

$$\nabla^2\phi(\mathbf{r},t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi(\mathbf{r},t)}{\partial r}\right) \tag{3.26}$$

We substitute this in our wave equation (3.18):

$$\nabla^2\phi(\mathbf{r},t) = \frac{1}{\nu^2}\frac{\partial^2\phi(\mathbf{r},t)}{\partial t} \tag{3.27}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi(\mathbf{r},t)}{\partial r}\right) = \frac{1}{\nu^2}\frac{\partial^2\phi(\mathbf{r},t)}{\partial t^2} \tag{3.28}$$

Let us define:

$$\phi(r,t) = \frac{\xi(r,t)}{r}$$

Such that 3.27 becomes:

$$\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{\xi(r,t)}{r}\right) &= \frac{1}{\nu^2}\frac{\partial^2}{\partial t^2}\frac{\xi(r,t)}{r} \\
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\xi(r,t)}{\partial r} - \xi(r,t)\right) &= \frac{1}{\nu^2}\frac{\partial^2\xi(r,t)}{\partial t^2} \\
\frac{1}{r}\left(r\frac{\partial^2\xi(r,t)}{\partial r^2} + \xi(r,t) - \xi(r,t)\right) &= \frac{1}{\nu^2}\frac{\partial^2\xi(r,t)}{\partial t^2} \\
\frac{\partial^2\xi(r,t)}{\partial r^2} &= \frac{1}{\nu^2}\frac{\partial^2\xi(r,t)}{\partial t^2}
\end{aligned}$$

We notice that this is the 1D wave equation, which has the following solution:

$$\xi(r,t) = Be^{i(\omega t \pm \nu r)}$$

Such that our wavefunction is of the family:

$$\phi(r,t) = \frac{B}{r}e^{i(\omega t \pm \nu r)} \tag{3.29}$$

3.2.4 Interface between materials

As expected, the Earth is not an isotropic material. However, at very large scales it may be possible to invoke simplifying assumptions such that a usable model is possible. Since the diameter of the earth is large, concentric spherical layers of the various materials making up earth's crust can be approximated as planes. This depends on the scale of the problem being studied.

In this project, we assume the simplified case of two interfacing homogeneous materials. The familiar Snell's Law can actually be used here to model the propagation of a wave through the boundary between two materials. As the wave propagates beneath the surface of the earth in its particular direction of travel, the wave reaches the surface after some time delay. This travel velocity as it would appear from an observer's point of view on the earth's surface is called the *apparent velocity*, given by c_x [3, 65].

A wave traveling from medium 1 to medium 2 will partly transmit through the boundary layer, and partly reflect back into medium 1. Snell's Law is given by [3, 67] as

$$c_x = \frac{\nu_1}{\sin i_1} = \frac{\nu_2}{\sin i_2}, \tag{3.30}$$

where c_x is the apparent velocity. Figure 6.3 illustrates the reflection and transmission onto the interface between two materials with Snell's Law. This is given by the following potential:

$$\phi(x,z,t) = \text{incident P} + \text{reflected P} \tag{3.31}$$

$$= A_1 e^{i(\omega t - k_x x - k_x r_{\nu_1} z)} + A_2 e^{i(\omega t - k_x x + k_x r_{\nu_1} z)}, \tag{3.32}$$

where we note that the reflected wave is proportional to the incident wave, but reflected in its direction of travel (z). The wavenumbers correlate to material properties and wave types [3, 66]. As for the transmission of the p-wave, the potential is given by,

$$\phi(x,z,t) = \text{transmitted P} = A' e^{i(\omega t - k_x x - k_x r_{\nu_2} z)} \tag{3.33}$$

where A' is the amplitude of the wave.

CHAPTER 4

Computational Model

4.1 Description

We attempted to use FEniCS to numerically solve the wave equation in the case of spherical and planar solutions but were not able. This unfortunately wasted a lot of our time. We therefore wrote code to make animations of waves with the next goal being to incorporate an interface and deal with boundary conditions. We were not able to do so.

4.2 Implementation

The code is implemented in Python3 using the Matplotlib library. Some functions are defined and then animated in GIF format. The codebase can be accessed here: `https://github.com/dreneuw/AMATH361-Project`.

```python
def alpha(x, z):
    lamb, mu, rho = params(x,z)
    return np.sqrt((lamb+2*mu)/rho)

def params(x, z):
    if z<L/2:
        return [lamb1, mu1, rho1]
    else:
        return [lamb2, mu2, rho2]

def phi_space_plane(x, z, vert = True):
    return np.exp(complex(0,1)*(-alpha(x,z)*z)) if vert else np.exp(complex(0,1)*(-alpha(x,z)*x))

def phi_space_sphere(x,z):
    r = np.sqrt(x**2+z**2)
    return (1/r)*np.exp(complex(0,1)*(-k*r))

def phi_time(x,z,t,verti, sphere=False):
    return phi_space_plane(x, z, vert = verti)*np.exp(complex(0,1)*alpha(x,z)*t) if not sphere else phi_space_sphere(x, z)*np.exp(complex(0,1)*alpha(x,z)*t)
```

Listing 4.1: Function Definitions for Wavefunctions

```python
from matplotlib.animation import FuncAnimation

fig = plt.figure()

cont = plt.contourf(f(xlst,zlst,t[0]))

# animationof the f function
def animate(i):
    global cont
    for c in cont.collections:
        c.remove()  # removes only the contours, leaves the rest intact
    cont = plt.contourf(f(xlst,zlst, i))
    plt.title('t = {}'.format(i))
    return cont

anim = animation.FuncAnimation(fig, animate, frames=t, repeat=False)
```

```
anim.save('wave_animation_horizontal.gif', writer=animation.FFMpegWriter())
```

Listing 4.2: Sample animation code

4.3 Results

The plots obtained are linked below. Since the plots are animations, they are not attached within this document.

- **Vertical wave animation:** https://github.com/dreneuw/AMATH361-Project/blob/main/src/wave_animation.gif
- **Horizontal wave animation:** https://github.com/dreneuw/AMATH361-Project/blob/main/src/wave_animation_horizontal.gif
- **Spherical wave animation:** https://github.com/dreneuw/AMATH361-Project/blob/main/src/wave_animation_spherical.gif

Conclusion

As part of this project, we have derived the wave equation from the equation of motion and evaluated a plane wave approximate solution as well as a spherical wave solution. We also investigated how each of these solutions interact at the interface between two media.

We then attempted to compute these solutions to demonstrate visually the governance of Snell's law in layered media, but we could not get a solution on time. We spent a large sum of time trying to work with FEniCS in order to find spherical and planar numerical solutions to the wave equation, to no avail.

A next step in modelling the propagation of p-wave would require:

- Completing a graphical analysis of spherical and planar waves traveling through layered media
- Providing an in-depth analysis of the effects of density as well as Lamé coefficients in the propagation of these waves
- Studying the effects of non-planar interfaces between layered media and assert their importance based on Earth's composition
- Moving towards an increased number of media as a first step to continuous differences

All in all, we are glad to have chosen this topic but disappointed we weren't able to delve deeper into it especially in the computational section.

List of figures

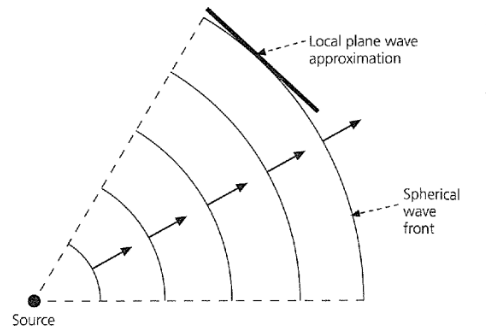

Figure 6.1: At a far distance from the fault, the seismic waves can be validly approximated to plane waves.[3]

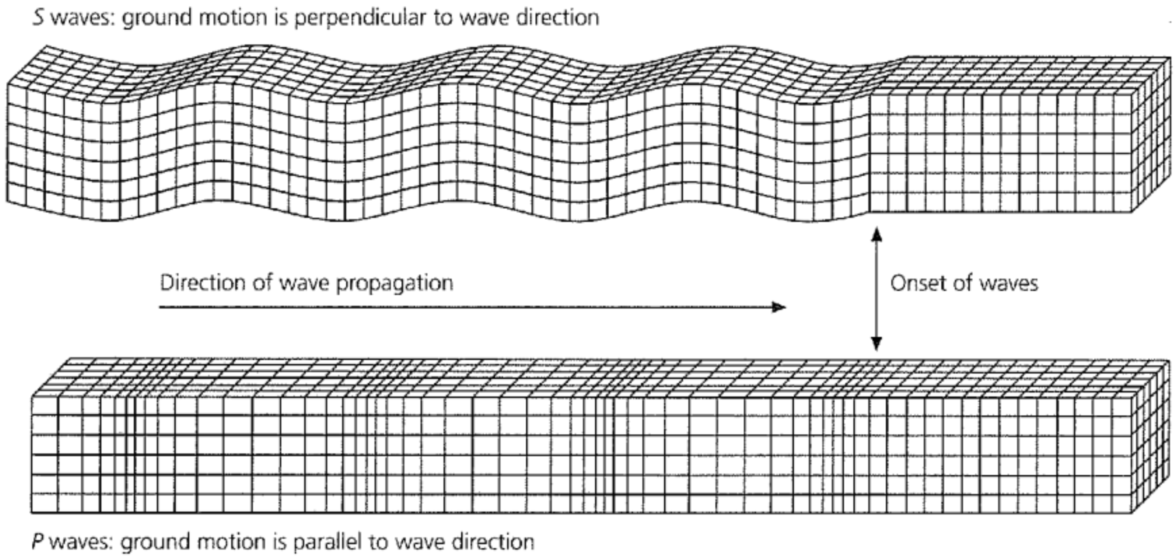

Figure 6.2: From [3, Fig 2.4-3], a visual of the difference between s-waves and p-waves.

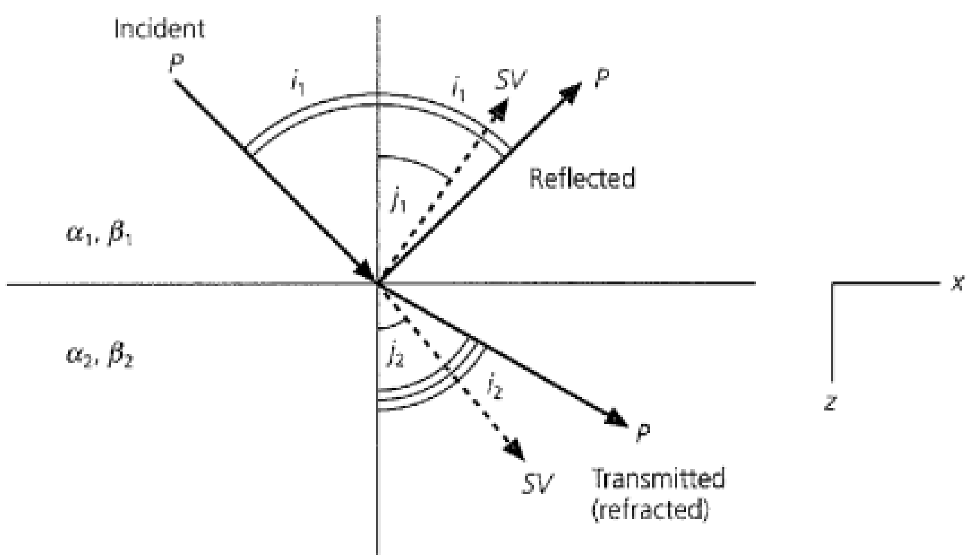

Figure 6.3: A p-wave incident with angle i_1 to a boundary layer. The wave is partially reflected at angle i_1 and transmitted at angle i_2.[3, Figure 2.5-5]

Bibliography

[1] N. I. Christensen and W. D. Mooney, "Seismic velocity structure and composition of the continental crust: A global view," *Journal of Geophysical Research: Solid Earth*, vol. 100, no. B6, pp. 9761–9788, 1995.

[2] N. R. C. Government of Canada, "Government of canada / gouvernement du canada," Mar 2019.

[3] S. Stein and M. Wysession, *An introduction to seismology, earthquakes, and earth structure*. John Wiley & Sons, 2009.

[4] B. Lautrup, *Physics of continuous matter: exotic and everyday phenomena in the macroscopic world*. CRC press, 2011.

[5] W. A. Strauss, *Partial differential equations: An introduction*. John Wiley & Sons, 2007.

[6] H. Aochi, T. Ulrich, A. Ducellier, F. Dupros, and D. Michea, "Finite difference simulations of seismic wave propagation for understanding earthquake physics and predicting ground motions: Advances and challenges," in *Journal of Physics: Conference Series*, vol. 454, p. 012010, IOP Publishing, 2013.

[7] R. W. Graves, "Simulating seismic wave propagation in 3d elastic media using staggered-grid finite differences," *Bulletin of the seismological society of America*, vol. 86, no. 4, pp. 1091–1106, 1996.

[8] D. M. Boore, "Finite difference methods for seismic wave propagation in heterogeneous materials," *Methods in computational physics*, vol. 11, pp. 1–37, 1972.

[9] D. E. Rene, Halbertsma, "AMATH361 Final project," 4 2022.