Analysis of Hurricanes

AMATH463 - Fluid Dynamics

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Contents

Abstract

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Hurricane formation is an impactful topic that is not yet fully understood but we identified many relevant fluid dynamics and thermodynamics mechanisms that play important roles in their creation. Cold fronts above warm water cause evaporation, which upon receiving low vertical wind shear and happening in moist atmosphere areas cause tropical depressions. Given large enough latitude for significant Coriolis force but not too large to have high enough surface temperature, tropical depressions can accumulate energy and gain vorticity through geostrophic flow, at which point they become tropical storms. Tropical storms intensify through drift and conservation of angular momentum, until they have winds strong enough to be called a hurricane. Once a hurricane reaches land, their energy supply (latent heat dissipation from evaporation of sea water) is cut off and they ”rapidly” die. This report discusses hurricanes from their genesis to maturity, relating their dynamics to fluid mechanical concepts and applying a numerical approach to finding temperature profiles through the storm.

Key terms: Geostrophic flow, evaporation and convection, conservation of angular momentum, rotating frame of reference

Introduction

With the growing threat of climate change becoming more of an emphasis in research and policy, the study of weather phenomena has become a larger focus of physical sciences. Ocean flows and air flows in our atmosphere have life-changing impacts on people's lives around the world, and being able to predict such phenomena is part of the role science must play on society. As mentioned by [1], with an average of 17.2 average annual weather and climate disasters in the U.S. alone, a total of above $2 trillion in damages has been caused by hurricanes since 1980, with $20 billion in 2021 alone. There is more and more merit to understanding these phenomena in order to diminish their impact and potentially even avert them.

Meteorology is a science that dates back thousands of years: civilizations as early as Babylonians recorded weather observations, attempting to explain the actions of the atmosphere as well predicting winds and rains. However, by nature of the problem, making accurate predictions was nearly impossible as there was no way to take in account the nearly infinite parameters while retaining sufficient precision in order to make any non-negligible time predictions. [2]

Before the first World War, scientist Lewis Fry Richardson had derived an arithmetic method to solving partial differential equations and had in mind to apply it in Vilhelm Bjerknes' calculational approach to weather forecasting. Although Richardson's computational approach took 6 weeks to calculate a 6 hour advance in weather, thus discouraging others from the computational approach, major developments were made between the World Wars (cold/warm fronts, air masses and polar fronts). After the second war, John von Neumann wanted to demonstrate the potential of computers. By 1956, he had succeeded by computationally forecasting weather as accurately and reliably as human meteorologists but much faster than them. [2]

Since then, computers have become the norm in meteorology and atmospheric sciences. With all the advances in technology, the problem has gotten back to a limitation in theory, especially in the genesis of hurricanes: how hurricanes come to life is still not fully understood. We have a great understanding of the conditions required for hurricanes to form but these conditions do not guarantee a hurricane to form: what decides if a hurricane is born or not from met conditions is a mystery [3].

This, however, does not mean that we cannot glean important information about hurricanes from examining specific aspects. This report will explore a number of concepts relating to hurricanes. First we discuss the necessary fluid mechanical background, including the Coriolis force and geostrophic flow. Then we briefly discuss the conditions for genesis of a hurricane, before deriving the governing equations of mature hurricanes. We then take a numerical approach to finding the temperature profile for a hurricane from the tangential wind speed.

Theory

3.1 Fluid mechanics in a rotating frame

We would like to determine the mechanisms behind the generation of vorticity observed in hurricanes. Our first step will be to determine the effect of a rotating frame of reference on flow. To do so, we define an inertial frame of reference S_I as well as a rotating frame of reference S_R which rotates with angular velocity $\vec{\Omega}$ with respect to the inertial frame of reference. Suppose $\vec{A}$ is constant in the rotating frame of reference while $\vec{B}(t)$ changes in the rotating frame (3.1). When looking at how both of these change in the inertial frame, we obtain:

$$\left(\frac{d\vec{A}}{dt}\right)_I = \vec{\Omega} \times \vec{A}$$

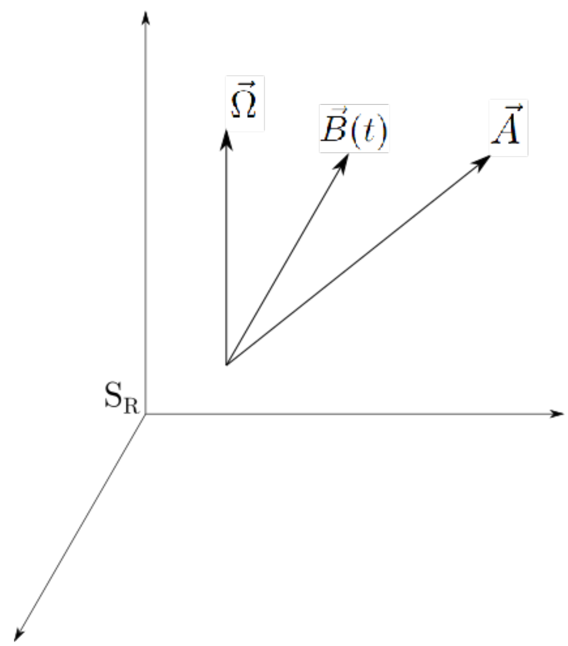

Figure 3.1: $\vec{A}$ and $\vec{\Omega}$ are constant in the rotating frame whereas $\vec{B}$ changes.

As well as:

$$\left(\frac{d\vec{B}}{dt}\right)_I = \left(\frac{d\vec{B}}{dt}\right)_R + \vec{\Omega} \times \vec{B} \tag{3.1}$$

Let us define $\vec{B} = \vec{r}$, a position vector. Then, $\frac{d\vec{B}}{dt} = \vec{u}$ such that our equation above gives:

$$\vec{u}_I = \vec{u}_R + \vec{\Omega} \times \vec{r} \tag{3.2}$$

Applying 3.1 to $\vec{u}_I$ gives:

$$\left(\frac{d\vec{u}_I}{dt}\right)_I = \left(\frac{d\vec{u}_I}{dt}\right)_R + \vec{\Omega} \times \vec{u}_I$$

Then, using 3.2 gives:

$$\left(\frac{d\vec{u}_I}{dt}\right)_I = \frac{d}{dt}\left(\vec{u}_R + \vec{\Omega} \times \vec{r}\right)_R + \vec{\Omega} \times \left(\vec{u}_R + \vec{\Omega} \times \vec{r}\right)_R$$
$$\left(\frac{d\vec{u}_I}{dt}\right)_I = \left(\frac{d\vec{u}_R}{dt}\right)_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)$$

As per Kundu [4] *Ch. 4, section 11*, we can define:

$$\vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) = -\vec{\nabla}\phi_c$$

where:

$$\phi_c = \frac{|\vec{\Omega} \times \vec{r}|^2}{2}$$

Our equation therefore becomes:

$$\vec{a}_I = \vec{a}_R + 2\vec{\Omega} \times \vec{u}_R - \vec{\nabla}\phi_c$$

Now, we input this in the Navier-Stokes equation:

$$\rho\left(\frac{D\vec{u}_R}{Dt} + 2\vec{\Omega} \times \vec{u}_R - \vec{\nabla}\phi_c\right) = -\vec{\nabla}p + \rho\vec{\nabla}\Phi + \mu\nabla^2\vec{u}_R$$

If we redefine the gravitational potential as being:

$$\Phi = \phi_g + \phi_c$$

where ϕ_g is the gravitational potential, we get the following modified Navier-Stokes equation for a rotating frame:

$$\rho\left(\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u}\right) = -\vec{\nabla}p + \rho\vec{\nabla}\Phi + \mu\nabla^2\vec{u}$$

where $\vec{u} = \vec{u}_R$. Let us then show the first of two mechanisms that generates vorticity during hurricane formation: *geostrophic balance*.

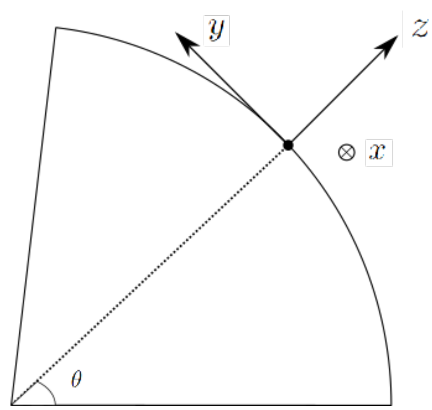

Figure 3.2: Traditional f-plane approximation in the Northern Hemisphere

Consider a small region on the surface of the Earth at latitude θ as shown by 3.2. If we have this region be in a rotating frame of reference with

$$2\vec{\Omega} = (0, f_h, f)$$

where f is the Coriolis parameter such that

$$f = 2\Omega\sin\theta \qquad f_h = 2\Omega\cos\theta$$

defining $\vec{u} = (u, v, w)$ gives us:

$$\begin{aligned} 2\vec{\Omega} \times \vec{u} &= (0, f_h, f) \times (u, v, w) \\ &= (f_h w - fv, fu, f_h u) \end{aligned}$$

Let us assume that viscosity has negligible effects. We can then use each dimensions of our modified Navier-Stokes equation:

$$\begin{aligned} \rho\left(\frac{Du}{Dt} + f_h w - fv\right) &= -p_x \\ \rho\left(\frac{Dv}{Dt} + fu\right) &= -p_y \\ \rho\left(\frac{Dw}{Dt} + f_h u\right) &= -p_z - \rho g \end{aligned}$$

This approximation for large scale flows allows us to neglect f_h terms, giving

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}p_x \tag{3.3}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}p_y \tag{3.4}$$

$$\frac{Dw}{Dt} = -p_z + \rho g \tag{3.5}$$

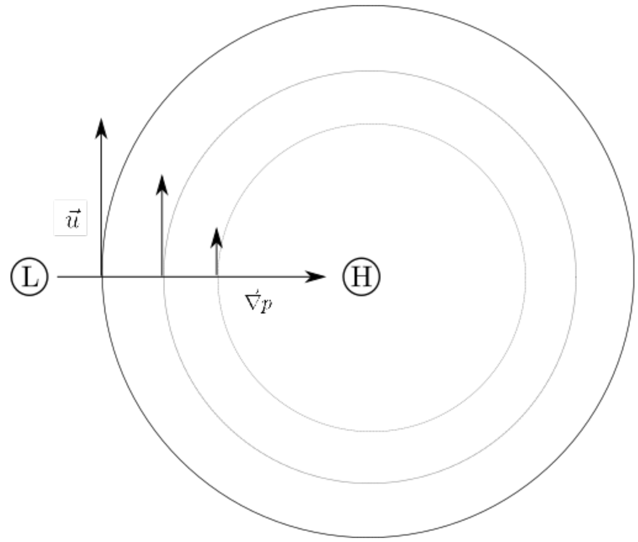

Figure 3.3: Geostrophic flow is created along isobars.

3.2 Geostrophic flow

We can then assume a geostrophic balance in this system, or balance between the Coriolis and pressure gradient forces:

$$-fv = -\frac{p_x}{\rho} \tag{3.6}$$

$$fu = -\frac{p_y}{\rho} \tag{3.7}$$

$$p_z = -\rho g \tag{3.8}$$

From these, we can identify a planar flow caused by geostrophic balance:

$$\vec{u} = (u, v) = \frac{1}{f\rho}\left(-p_y, p_x\right)$$

We see that this planar flow follows lines perpendicular to a given pressure gradient. That is, they follow "isobars", or lines of constant pressure, as shown by Fig. 3.3.

When evaporation happens, high pressure appears at points where evaporation is the highest, and causes this geostrophic flow near the surface. This flow, which has other sources, is commonly referred as primary circulation. [3]

In addition, due to conservation of angular momentum, we can show that circulation is generated when a system moves in latitude. We recall the equation for circulation:

$$\Gamma(t) = \iint_A \vec{\omega} \cdot \hat{n} dA = \oint_{\partial A} \vec{u} \cdot d\vec{r} \tag{3.9}$$

where A is an area, ∂A is the boundary of that area and $d\vec{r}$ is an infinitesimal piece of the boundary directed counter-clockwise. We can write our circulations for the inertial and rotating frame as

$$\Gamma_I(t) = \oint_{\partial A} \vec{u}_I \cdot d\vec{r} \tag{3.10}$$

$$\Gamma_R(t) = \oint_{\partial A} \vec{u}_R \cdot d\vec{r} \tag{3.11}$$

$$\tag{3.12}$$

Then, recalling the relation between $\vec{u}_I$ and $\vec{u}_R$:

$$\vec{u}_I = \vec{u}_R + \vec{\Omega} \times \vec{r} \tag{3.13}$$

our inertial circulation becomes:

$$\begin{aligned}
\Gamma_I(t) &= \oint_{\partial A} (\vec{u}_R + \vec{\Omega} \times \vec{r}) \cdot d\vec{r} \\
&= \oint_{\partial A} \vec{u}_R \cdot d\vec{r} + \oint_{\partial A} (\vec{\Omega} \times \vec{r}) \cdot d\vec{r} \\
&= \Gamma_R(t) + \iint_A \vec{\nabla} \times (\vec{\Omega} \times \vec{r}) \cdot \hat{n} dA \\
&= \Gamma_R(t) + \iint_A 2\vec{\Omega} \cdot \hat{n} dA \\
&= \Gamma_R(t) + 2|\vec{\Omega}| A_\perp
\end{aligned}$$

where $A_\perp$ is the area of interest projected onto a plane perpendicular to $\vec{\Omega}$ (in our case, a plane that goes through the equator) as demonstrated by Fig. 3.4. If we take the time derivative:

$$\frac{d\Gamma_I}{dt} = \frac{d\Gamma_R}{dt} + 2|\vec{\Omega}| \frac{dA_\perp}{dt}$$

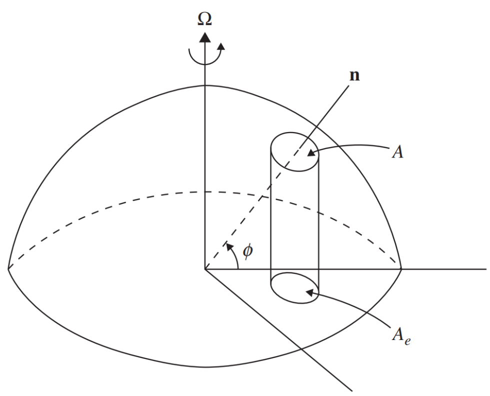

Figure 3.4: Depiction of the area of interest and it's projection onto the plane perpendicular to $\vec{\Omega}$ from [5]

we notice that change in projected area is a generator for inertial circulation. Now let us assume rotating circulation is constant and recall 3.9 and 3.10:

$$\begin{aligned}
\frac{d}{dt} \iint_A \vec{\omega}_I \cdot \hat{n} dA &= 2|\vec{\Omega}| \frac{dA_\perp}{dt} \\
\iint_A \frac{d\vec{\omega}_I}{dt} \cdot \hat{n} dA &= 2|\vec{\Omega}| \frac{dA_\perp}{dt}
\end{aligned}$$

By taking the divergence of 3.13 we get that $\vec{\omega}_I = \vec{\omega}_R + 2\vec{\Omega}$, so

$$\begin{aligned}
\iint_A \frac{d}{dt} \left(\vec{\omega}_R + 2\vec{\Omega}\right) \cdot \hat{n} dA &= 2|\vec{\Omega}| \frac{dA_\perp}{dt} \\
\iint_A \frac{d\vec{\omega}_R}{dt} \cdot \hat{n} dA &= 2|\vec{\Omega}| \frac{dA_\perp}{dt}
\end{aligned}$$

We see from this previous equation that whenever the projected area changes, the total rotational vorticity in the area of interest must also change. As hurricanes move in latitude, their total vorticity will change.

Hurricanes typically follow the path of major wind flows and will often experience beta drift, a phenomenon which causes hurricanes to move in latitude. This will indirectly increase their vorticity, as shown above.

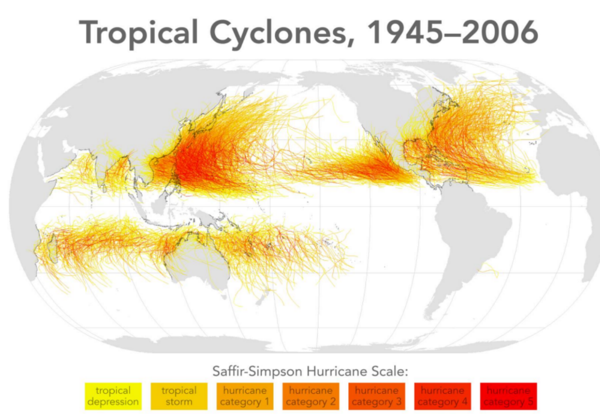

Figure 3.5: Path taken by hurricanes and their intensity, 1945-2006, from [6]

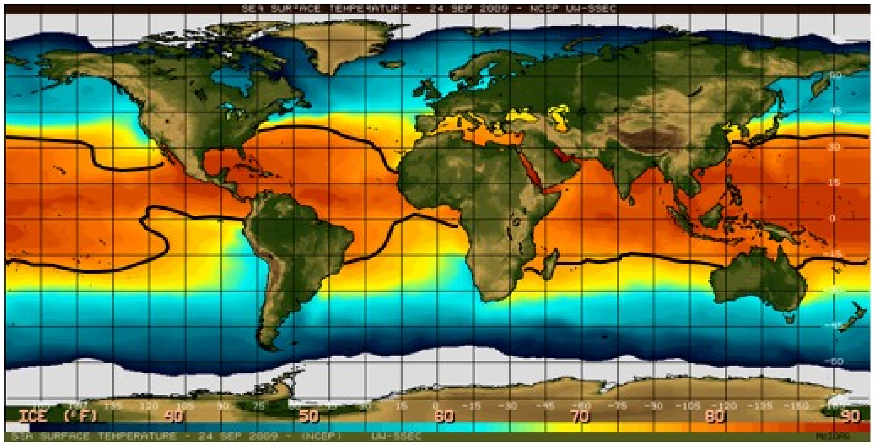

Figure 3.6: Sea surface temperature around the world. Black lines represent locations where the temperature is sufficiently high enough for hurricane formation (26.5°C)

3.3 Conditions for genesis

As mentioned in the introduction, there are certain conditions that are necessary (but not sufficient) for hurricanes to form. A high-level description of these required conditions are as follow:

1. *A sea surface temperature of at least 26.5°C*. Hurricanes take their energy from the heat energy of the ocean through a mix evaporation and latent heat and as such this condition is crucial. For this reason, nearly all hurricanes grow from depressions born within the tropics. Specifically, most hurricanes that hit the east coast of the U.S. are born near the west coast of northern Africa, and grow as they are pushed westward by streams, as seen in figure 3.6. [3]

2. *The proper temperature gradient in height* in order to support thunderstorm activity. [3]

3. *Sufficient water vapor* in the middle of the troposphere. Often, the middle of the troposphere contains dry air which inhibits the formation of tropical depressions, which are what spawn hurricanes. [3]

4. *Sufficient distance from the equator*. As mentioned in the above section on fluid mechanics in a rotated frame: the effects of conservation of angular momentum on hurricane vorticity depend on the Coriolis parameter, which increases as latitude goes away from the equator. Typically, the Coriolis Force becomes significant at about 483km above and below the equator. Outside of these bounds, cyclonic rotation becomes difficult to generate. [3]

5. *Little vertical wind shear* from the surface towards the atmosphere. Wind shears inhibit the formation of hurricane and while it is not fully understood why, some research shows that this my be tied to point 3, as it would drag dry air in the middle of the troposphere.

3.4 Governing equations of mature hurricanes

A common way of approaching hurricane modeling is to assume axial symmetry. To take advantage of this, as described in [5], let us put Eqns. 3.3 and 3.4 in vector form:

$$\frac{D\vec{V}}{Dt} + f\vec{k} \times \vec{V} = -\frac{1}{\rho}\vec{\nabla}p \tag{3.14}$$

for the horizontal velocity $\vec{V} = u\hat{\imath} + v\hat{\jmath}$, or alternately we can write the right hand side in terms of the geopotential $\Phi = gh$ using the hydrostatic equation 3.8:

$$\frac{D\vec{V}}{Dt} + f\vec{k} \times \vec{V} = -\vec{\nabla}\Phi|_{p=const} \tag{3.15}$$

where the gradient is taken along isobars under our assumption of geostrophic flow, so p is constant. From 3.15, we can find that in cylindrical co-ordinates (radial position $\vec{r}$, angular position $\vec{\lambda}$, vertical position $\vec{z}$)

$$\frac{v_\lambda^2}{r} + fv_\lambda = \frac{\partial\Phi}{\partial r} \tag{3.16}$$

for tangential velocity v_λ. Under the axisymmetric assumption, we can say that the angular momentum $M = rv_\lambda + f\frac{r^2}{2}$ is conserved above the frictional boundary layer [7]. [5] states that Eqn. 3.16 can therefore be written as

$$\frac{M^2}{r^3} - \frac{f^2 r}{4} = \frac{\partial\Phi}{\partial r} \tag{3.17}$$

To eliminate $\frac{\partial\Phi}{\partial r}$ we can continue to follow [5] in taking a quick step over from isobaric coordinates to log-pressure coordinates, i.e. where the vertical coordinate is

$$z^* = -H\ln\left(p/p_s\right)$$

for reference pressure p_s and scale height $H = RT_s/g$, where T_s is a global average temperature. We can put the hydrostatic balance equation 3.8 in terms of $\Phi = gz$ using the ideal gas law $p = \rho RT$

$$gdz = d\Phi = -\frac{RT}{p}dp = -RTd\ln p$$

so

$$\frac{\partial\Phi}{\partial\ln p} = -RT$$

Then finally putting into log-pressure coordinates:

$$\frac{\partial\Phi}{\partial z^*} = \frac{RT}{H}$$

We can now differentiate our relation in Eqn. 3.17 with respect to z^* to get a relationship between the radial temperature gradient in the hurricane and the vertical shear of the angular momentum:

$$\frac{1}{r^3}\frac{\partial M^2}{\partial z^*} = \frac{R}{H}\frac{\partial T}{\partial r} \tag{3.18}$$

3.4.1 The effects of vertical wind shear

The relationship between the radial temperature gradient and the vertical shear of the angular momentum in Eqn. 3.18 is interesting for numerous reasons, but principally because the vertical wind shear has an impact on the structure of the hurricane vortex [7].

For instance, vertical wind shear can cause wave-like motion [8] that couples with the convection of the hurricane [9], causing eddy terms in the equations of motion. It can also tilt the vortex axis from the vertical [10] and introduce new pathways for dryer air to enter the vortex and reduce instability [9].

As mentioned above, the dynamics of hurricanes are complex, and efforts to model and predict them are just as complex and varied. However, for smaller aspects, we can apply simpler models. Let us consider a model of uniform vertical shear flow in pressure-based coordinates

$$\bar{U} = U_0 + U_z z \tag{3.19}$$

Profile	v_0	a	b
Standard	71.521	0.3398	5.377×10^{-4}
30 m/s	53.641	0.3398	5.377×10^{-4}
20 m/s	35.761	0.3398	5.377×10^{-4}

Table 3.1

where U_0 and U_z are constants, with an axisymmetric barotropic vortex superimposed as described in [10]. The tangential wind for this system is given by [10] to be

$$v_\lambda = v_0 \frac{s(1 + \frac{5b}{2a} s^4)}{(1 + as^2 + bs^6)^2} \tag{3.20}$$

where s is the ratio of the radius r to the radius of maximum winds $r_{max} = 100$ km and v_0, a, and b are constants. We have the constants for three different runs from [10] in Table 3.1.

Recalling from above that $M = rv_\lambda + f\frac{r^2}{2}$, we can investigate the relationship between the vertical wind shear and temperature, and model temperature profiles for hurricanes of this model. Substituting the definition for angular momentum into Eqn. 3.18 and applying chain rule gives us

$$\frac{1}{r^3}\frac{\partial r}{\partial z^*}\frac{\partial}{\partial r}(rv_\lambda(r) + f\frac{r^2}{2})^2 = \frac{R}{H}\frac{\partial T}{\partial r} \tag{3.21}$$

or

$$2\frac{1}{r^3}\frac{\partial r}{\partial z^*}(rv_\lambda(r) + f\frac{r^2}{2})(v_\lambda(r) + r\frac{\partial v_\lambda}{\partial r} + fr) = \frac{R}{H}\frac{\partial T}{\partial r} \tag{3.22}$$

We clearly need a relation between r and z^* to know anything about T. Let's choose the relation between r and p found by [11] for the Mexican Pacific Ocean $p = 3.8256r + 828.7300$. Given that

$$p = p_a e^{-z^*/H}$$

we can choose $p_a = 101.325$ kPa and $T_s = 13.9$ °C to get

$$r = 26486.04e^{-z^*/243.28} - 216.63$$

or

$$\frac{dr}{dz^*} = -108.87e^{-z^*/243.28}$$

Let's choose $z^* = 0$ for this exercise. So we have

$$\frac{\partial T}{\partial r} = 2\frac{36.32}{r^3}(rv_\lambda(r) + f\frac{r^2}{2})(v_\lambda(r) + r\frac{\partial v_\lambda}{\partial r} + fr)$$

We can integrate this numerically. Let us assume that given we're using a relation from the Mexican Pacific Ocean that our latitude is 20 °N, so $f = 2\Omega \sin\theta = 4.99 \times 10^{-5}$ s^{-1}. We can then get a temperature profile for radii from 0.1 km to $r_{max} = 100$ km, as shown in Fig. 3.7.

We see that, for an isobar of pressure $p = p_a$ (given that we took a constant $z^* = 0$) the temperature is non-physical for small radii and then approaches the average global temperature as it increases. This is expected, as we know that in the center of a hurricane is the eye, and the dynamics of convection work much differently. Therefore this simple model is not expected to produce physical results for smaller radii. Then, moving outward in the hurricane, the storm decreases in intensity, so it makes sense for the temperatures to approach that of the outside.

If we chose a larger pressure than the atmospheric pressure, we can see from our equations that we would expect the temperature to be scaled down, and vice versa for smaller pressures.

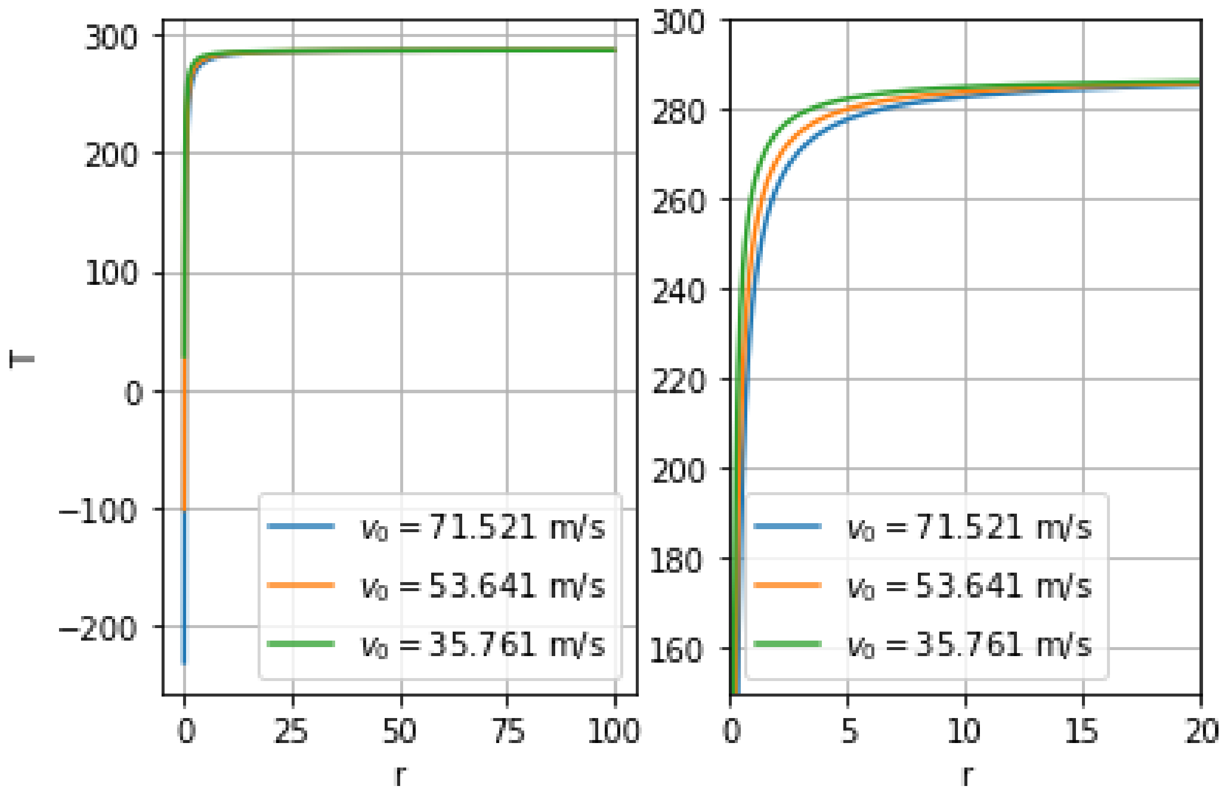

Figure 3.7: The temperature profile with respect to radius. The second plot here is a subset of the first, to see more clearly the differences between different runs.

Conclusion

4.1 Discussion

Hurricane formation is an ever changing topic with many of its mechanism still not fully explained: for example, how the eye of the storm is created and maintained, how vertical wind shears inhibits hurricane formation and what decides whether a tropical depression with the right conditions becomes a hurricane or not are all unclear subjects even today.

Despite the complexity of the subject matter, we were able to identify many of the mechanisms that drive the formation of hurricanes: from fluid mechanics to thermodynamics. Topics of the latter were not deeply investigated as they were not part of the scope of this course and project, however we were able to come up with a simple toy model for the temperature profile in a hurricane with the knowledge of its tangential wind speed.

4.2 Next steps

As mentioned, the dynamics of hurricanes are extremely complex. We saw in our temperature profile that our temperatures were non-physical for radii near the eye of the hurricane. Further investigation into the dynamics of the eyewall and convection at the centre of the storm would be an interesting theoretical and computational exercise. Works such as [7] and [12] would provide a good basis for this exercise.

4.3 Acknowledgement

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List of Figures

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