Weekly Homework 14

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Introduction to Abstract Math

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Theorem 6.38. If $\sim$ is an equivalence relation on a nonempty set A then the quotient $A/\sim$ is a partition of A.

Proof. Let $\sim$ be an equivalence relation on a nonempty set A.
Let $A/\sim$ be as defined in Def. 6.30.
<u>Claim 1.</u> Every element in $A/\sim$ is nonempty.
P_1: Because $\sim$ is reflexive, $x \in [x]$ for $\forall x$.
So every set in $A/\sim$ is nonempty. //
<u>Claim 2.</u> If [x] and [y] are distinct elements of $A/\sim$, then $[x] \cap [y] = \emptyset$.
P_2: Let S be the statement: "[x] and [y] are distinct elements of $A/\sim$, then $[x] \cap [y] \neq \emptyset$."
Assume S is true. Let $z \in [x]$ and $z \in [y]$.
Since $\sim$ is reflexive, $x \in [x]$ and $y \in [y]$. So $x \sim z$ and $y \sim z$.
Since $\sim$ is transitive, this implies $x \sim y$, or $x \in [y]$.
So by Thm. 6.33, $[x] = [y]$. [x] and [y] cannot be equal and distinct, so S is false. //
<u>Claim 3.</u> $A = \cup_{x \in A/\sim}$. P_2: For every $x \in A$, $x \in [x]$ for some $[x] \in A/\sim$. //
Therefore, by Def. 6.35, $A/\sim$ is a partition on A. $\square$

Theorem 6.41. Suppose that Ω is a partition of a nonempty set A. Then there is an equivalence relation $\sim$ on A so that $A/\sim = \Omega$.

Proof. Let Ω be a partition of a nonempty set A.
Let $\sim$ be a relation on A, s.t. $\sim \subseteq A \times A$, defined $\sim = \{(x, y) | \exists X \in \Omega \text{ s.t. } x, y \in X\}$.
<u>Claim 1.</u> $\sim$ is an E.R.
P_1. Suppose $x, y, z \in A$.
<u>Claim 1a.</u>: $\sim$ is Reflexive.
Suppose x is in the subset containing x, so $x \sim x$. //
<u>Claim 1b.</u>: $\sim$ is Symmetric.
Suppose $x \sim y$. Then $x \in [y]$ and $x \in [x]$, so $x \in [x] \cap [y]$.
Then by Thm. 6.35.2, $[x] = [y]$. Therefore, $y \in [x]$, or $y \sim x$. //
<u>Claim 1c.</u>: $\sim$ is Transitive.
Suppose $z \in [x]$ and $z \in [y]$. Then $z \in [x] \cap [y]$.
Then by Thm. 6.35.2, $[x] = [y]$. So $x \in [y]$ and $x \sim y$. //
<u>Claim 2.</u> $A/\sim = \Omega$.

Claim 2a.: $A/\sim \subseteq \Omega$.
Let $[x] \in A/\sim$. Then $x \in [x]$.
By Def. 6.35.3, $\exists Y \in \Omega$ s.t. $x \in Y$.
Claim 2a-a: $Y \subseteq [x]$.
Take $y \in Y$, so $x, y \in Y$.
By our Def. of $\sim$, $x \sim y$.
So then, $y \in [x]$.
Claim 2a-b: $[x] \subseteq Y$.
Take $y \in [x]$, so $x \sim y$.
We know $x \in Y$, therefore, by our Def. or $\sim$, $y \in [x]$.
Claim 2b.: $\Omega \subseteq A/\sim$.
Suppose $\Omega \neq A/\sim$ and $Z \in \Omega/(A/\sim)$.
Take $x \in Z$. Then $x \in [x]$, so $x \in [x] \cap Z$, which by Def. 6.35.2, cannot be true.
So Z is empty. But Z cannot be empty because Ω is a partition.
Therefore, $\Omega \subseteq A/\sim$ and thus $\Omega = A/\sim$. $\square$