Collision Detection in Math

Note 1. Before we start, lets make some simplifying assumptions:

i. We are only dealing with circles

ii. Circles have unit mass

iii. Collisions are elastic

iv. We don't handle n-body collisions

Assumptions ii., iii. and iv. simplify the collision equations so that it reduces to swapping velocities:

$$\begin{aligned} v_{a_f} &= \frac{v_{a_i}(m_a - m_b) + 2m_b v_{b_i}}{m_a + m_b} = \frac{v_{a_i}(1-1) + 2\cdot 1\cdot v_{b_i}}{1+1} = \frac{v_{a_i}\cdot 0 + 2v_{b_i}}{2} = v_{b_i} \\ v_{b_f} &= \frac{v_{b_i}(m_b - m_a) + 2m_a v_{a_i}}{m_a + m_b} = \frac{v_{b_i}(1-1) + 2\cdot 1\cdot v_{a_i}}{1+1} = \frac{v_{b_i}\cdot 0 + 2v_{a_i}}{2} = v_{a_i} \end{aligned}$$

First we define our data model.

Let $\mathbf{Circles} = \{(\boldsymbol{p}, \boldsymbol{v}, \boldsymbol{r}) \,|\, \boldsymbol{p}, \boldsymbol{v} \in \mathbb{R}^2 \, \textbf{and} \, \boldsymbol{r} \in \mathbb{R}^+\}$

The set **Circles** is all circles on a 2 dimensional plane with position $\boldsymbol{p}$, velocity $\boldsymbol{v}$ and radius $\boldsymbol{r}$.

We now recursively construct the sequence of worlds

let $\mathbf{Worlds} = \{\mathbf{world}_{\boldsymbol{n}}\}_{\mathbf{0}}^{\infty}$ where $\mathbf{world}_{\boldsymbol{n}} \in \boldsymbol{\mathcal{P}}(\mathbf{Circles})$

$$\mathrm{world}_n = \begin{cases} \mathrm{world}_0 \,\forall\, \mathrm{world}_0 \in \mathcal{P}(\mathrm{Circles}) & \text{if } n = 0 \\ \mathrm{stepWorld}(\mathrm{world}_{n-1}) & \text{if } n > 0 \end{cases} \tag{1}$$

This sequence $\mathbf{Worlds} = \{\mathbf{world_0}, \mathbf{world_1}, \mathbf{world_2}, \mathbf{...}\}$ can be played back by tranforming it with a render function that maps each world to an image ie. $\boldsymbol{r} \colon \boldsymbol{\mathcal{P}}(\mathbf{Circles}) \rightarrow \boldsymbol{M}_{\mathbf{640 \times 480}}$

As we can see from the definition of **Worlds**, every world $\mathbf{world}_{\boldsymbol{i}}$ is fully dependent on the previous world $\mathbf{world}_{\boldsymbol{i}-\mathbf{1}}$ except for the very first world $\mathbf{world_0}$. Thus, once the first world is chosen, the entire future of that world is predetermined.

Alright, lets look at the **stepWorld** function

let stepWorld: $\boldsymbol{\mathcal{P}}(\mathbf{Circles}) \rightarrow \boldsymbol{\mathcal{P}}(\mathbf{Circles})$

$$\mathrm{stepWorld}(\mathrm{world}_n) = \{c_{n+1} \in \mathrm{Circles} \,|\, c_{n+1} = \mathrm{stepCircle}(\mathrm{circle}, \mathrm{world}_n) \text{ for circle} \in \mathrm{world}_n\} \tag{2}$$

We will define **stepCirlce:** $\mathbf{Circles} \times \boldsymbol{\mathcal{P}}(\mathbf{Circles}) \rightarrow \mathbf{Circle}$ later, we need to create some helper functions first.

Definition 2. *We define the relation* **intersect** *over Circles*

Let $a, b \in \mathrm{Circles}$ *where* $a = (p_a, v_a, r_a)$ *and* $b = (p_b, v_b, r_b)$

$$a \,\mathrm{intersect}\, b \iff |p_b - p_a| \leqslant r_a + r_b \tag{3}$$

Now we define the function **colliding** which gives us all circles in a world that intersect a circle

let colliding: Circles $\times\, \boldsymbol{\mathcal{P}}$**(Circles)** $\rightarrow \boldsymbol{\mathcal{P}}$**(Circles)**

$$\text{colliding}(\text{circle}, \text{world}) = \{x \in \text{world} \,|\, x \neq \text{circle} \,\text{and}\, (\text{circle}, x) \in \text{intersect}\} \tag{4}$$

Great, now lets define **stepCircle**

$$\text{stepCircle}(\text{circle}, \text{world}) = \begin{cases} \text{physics}(\text{collide}(c, \text{world})) & \text{if } \text{colliding}(c, \text{world}) \neq \varnothing \\ \text{physics}(c) & \text{if } \text{colliding}(c, \text{world}) = \varnothing \end{cases} \tag{5}$$

let physics: Circles $\rightarrow$ **Circles**

let $\boldsymbol{\Delta t = 1/60}$

$$\text{physics}((p, v, r)) = (p + \Delta t\, v, v, r) \tag{6}$$

This updates the circle by integrating the position with respect to time.

let collide: Circles $\times\, \boldsymbol{\mathcal{P}(C)} \rightarrow$ **Circles**

$$\text{collide}(\text{circle}, \text{world}) = \text{bounce}(\text{circle}, \text{chooseColliding}(\text{circle}, \text{world})) \tag{7}$$

$$\text{bounce}(((p_a, v_a, r_a), (p_b, v_b, r_b))) = (p_a, v_b, r_a) \tag{8}$$

Lets look at **chooseColliding:** $\boldsymbol{C \times \mathcal{P}(C) \rightarrow C}$. This function is a bit of a hack, its purpose is to keep the simulation deterministic when multiple collisions happen at the same time. The way it does this is it finds and returns the 'minimum' circle out of the set of circles that intersect with the a circle.

Ok lets do it. First a definition

Definition 3. *We define the relation* $(\leqslant)$ *on* $\mathbb{R}^2$

let $a, b, c, d \in \mathbb{R}$

$$(a, b) \leqslant (c, d) \Longleftrightarrow (a \leqslant c) \wedge (b \leqslant d) \tag{9}$$

let $\boldsymbol{m}$**: Circles** $\times$ **Circles** $\rightarrow$ **Circles** be a function that returns the minimum circle of two given circles

$$m(((p_a, v_a, r_b), (p_b, v_b, r_b))) = \begin{cases} (p_a, v_a, r_a) & \text{if}\, (p_a \leqslant p_b) \wedge (v_a \leqslant v_b) \wedge (r_a \leqslant r_b) \\ (p_b, v_b, r_b) & \text{if}\, \neg((p_a \leqslant p_b) \wedge (v_a \leqslant v_b) \wedge (r_a \leqslant r_b)) \end{cases} \tag{10}$$

$$\text{chooseColliding}(c, w) = x \,\text{for}\, x \in \text{colliding}(c, w) \;\; s.t. \forall a \in \text{colliding}(c, w), m(x, a) = x \tag{11}$$

Lastly we define the collision handling function **collide** as follows

Ok, so we have the sequence D of worlds, this is our simulation, we just run through this sequence and call some render function $r\colon \mathcal{P}(C) \rightarrow M_{1280 \times 720}$

Theorem 4. *The relation* intersect *is reflexive and symetric*

will probably drop this theorem as it's not important

Proof. *Let's quickly do this proof so we can move on to the interesting bits*

Take $a, b \in C$ *where* $a = (p_a, r_a)$ *and* $b = (p_b, r_b)$

Reflexive:

$$|p_a - p_a| = |(0,0)| = 0 \leqslant r_a \leqslant r_a + r_a \Longrightarrow |p_a - p_a| \leqslant r_a + r_b \Longrightarrow a \,\text{intersect}\, a$$

This is true since $r_a \geqslant 0$

Symetric: *(Assume* $a \,\text{intersect}\, b$*)*

$$a \,\text{intersect}\, b \Longleftrightarrow |p_b - p_a| \leqslant r_a + r_b \Longleftrightarrow |p_a - p_b| \leqslant r_b + r_a \Longleftrightarrow b \,\text{intersect}\, a$$

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