Collision Detection in Math

Note 1. Before we start, lets make some simplifying assumptions:

i. We are only dealing with circles

ii. Circles have unit mass

iii. Collisions are elastic

iv. We don't handle n-body collisions

Assumptions ii., iii. and iv. simplify the collision equations so that it reduces to swapping velocities:

$$\begin{aligned} v_{a_f} &= \frac{v_{a_i}(m_a - m_b) + 2m_b v_{b_i}}{m_a + m_b} = \frac{v_{a_i}(1-1) + 2\cdot 1\cdot v_{b_i}}{1+1} = \frac{v_{a_i}\cdot 0 + 2v_{b_i}}{2} = v_{b_i} \\ v_{b_f} &= \frac{v_{b_i}(m_b - m_a) + 2m_a v_{a_i}}{m_a + m_b} = \frac{v_{b_i}(1-1) + 2\cdot 1\cdot v_{a_i}}{1+1} = \frac{v_{b_i}\cdot 0 + 2v_{a_i}}{2} = v_{a_i} \end{aligned}$$

First we define our data model.

Let $\boldsymbol{C = \{(p, v, r) \,|\, p, v \in \mathbb{R}^2 \text{ and } r \in \mathbb{R}^+\}}$

The set C is all circles on a 2 dimensional plane with position p, velocity v and radius r.

We now recursively define the sequence $\boldsymbol{D = \{w_n\}_0^\infty}$ where

$$w_n = \begin{cases} w_0 \in \mathcal{P}(C) & \text{if } n = 0 \\ \{c_n | c_n = f(c_{n-1}, w_{n-1}) \text{ for } c_{n-1} \in w_{n-1}\} & \text{if } n > 0 \end{cases} \tag{1}$$

We will define $f: C \times \mathcal{P}(C) \to C$ below, f checks if the circle its given intersects with any other circles in the world, handles those collisions, then updates the positions.

First let's define the function $i: C \times \mathcal{P}(C) \to C$ that gives us the set of circles in a world that intersect the given circle

$$i(c, W) = \{x \in W | x \neq c \,\text{and}\, c \text{ intersect } x\} \tag{2}$$

Definition 2. *We define the* intersect *relation over* C

Let $a, b \in C$ *where* $a = (p_a, v_a, r_a)$ *and* $b = (p_b, v_b, r_b)$, *then,*

$$a \,\text{intersect}\, b \iff |p_b - p_a| \leqslant r_a + r_b \tag{3}$$

Great, now lets define f

$$f(c, W) = \begin{cases} u(h(c, e(c, i(c, W)))) & \text{if } i(c, W) \neq \varnothing \\ u(c) & \text{if } i(c, W) = \varnothing \end{cases} \tag{4}$$

Ok, that's a lot of symbols, lets start with the function $u: C \to C$

let $\Delta t = 1/60$

$$u((p, v, r)) = (p + \Delta t\, v, v, r) \tag{5}$$

This updates the circle by adding the velocity to the position.

Ok, lets look at $e: C \times \mathcal{P}(C) \to C$. The function e is a bit of a hack, its purpose is to keep the simulation deterministic when multiple collisions happen at the same time. What it does is it finds and returns the 'minimum' circle out of the set of circles that intersect with the given circle.

Ok lets do it. First we a definition

Definition 3. *We define the relation* $(\leqslant)$ *on* $\mathbb{R}^2$

let $a, b, c, d \in \mathbb{R}$

$$(a, b) \leqslant (c, d) \Longleftrightarrow (a \leqslant c) \wedge (b \leqslant d) \tag{6}$$

Next we define the min function $m: C \times C \to C$, it always returns the 'minimum' of the two circles passed in.

$$m(((p_a, v_a, r_b), (p_b, v_b, r_b))) = \begin{cases} (p_a, v_a, r_a) & \text{if } (p_a \leqslant p_b) \wedge (v_a \leqslant v_b) \wedge (r_a \leqslant r_b) \\ (p_b, v_b, r_b) & \text{if} \neg((p_a \leqslant p_b) \wedge (v_a \leqslant v_b) \wedge (r_a \leqslant r_b)) \end{cases} \tag{7}$$

$$e(c, S) = \begin{cases} x \in S \;\; s.t. \forall a \in S, d(x, a) = x & \text{if } S \neq \varnothing \\ \text{we already proved } S \neq \varnothing & \text{if } S = \varnothing \end{cases} \tag{8}$$

Lastly we define the collision handling function $h: C \times C \to C$ as follows

$$h(((p_a, v_a, r_a), (p_b, v_b, r_b))) = (p_a, v_b, r_a) \tag{9}$$

Ok, so we have the sequence D of worlds, this is our simulation, we just run through this sequence and call some render function $r: \mathcal{P}(C) \to M_{1280 \times 720}$

Theorem 4. *The relation* intersect *is reflexive and symetric*

will probably drop this theorem as it's not important

Proof. *Let's quickly do this proof so we can move on to the interesting bits*

Take $a, b \in C$ *where* $a = (p_a, r_a)$ *and* $b = (p_b, r_b)$

Reflexive:

$$|p_a - p_a| = |(0, 0)| = 0 \leqslant r_a \leqslant r_a + r_a \Longrightarrow |p_a - p_a| \leqslant r_a + r_b \Longrightarrow a \,\text{intersect}\, a$$

This is true since $r_a \geqslant 0$

Symetric: *(Assume* $a \,\text{intersect}\, b$*)*

$$a \,\text{intersect}\, b \Longleftrightarrow |p_b - p_a| \leqslant r_a + r_b \Longleftrightarrow |p_a - p_b| \leqslant r_b + r_a \Longleftrightarrow b \,\text{intersect}\, a$$

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