PARTIALLY ORDERED SETS

A **partially ordered set** or **poset** is a set P and a binary relation $\preceq$ such that for all $a, b, c \in P$

1. $a \preceq a$ (reflexivity).
2. $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ (transitivity).
3. $a \preceq b$ and $b \preceq a$ implies $a = b$. (anti-symmetry).

Examples

1. $P = \{1, 2, \ldots, \}$ and $a \leq b$ has the usual meaning.
2. $P = \{1, 2, \ldots, \}$ and $a \preceq b$ if a divides b.
3. $P = \{A_1, A_2, \ldots, A_m\}$ where the A_i are sets and $\preceq = \subseteq$.

A pair of elements a, b are **comparable** if $a \preceq b$ or $b \preceq a$. Otherwise they are **incomparable**.

A poset without incomparable elements (Example 1) is a linear or total order.

We write $a < b$ if $a \preceq b$ and $a \neq b$.

A **chain** is a sequence $a_1 < a_2 < \cdots < a_s$.

A set A is an **anti-chain** if every pair of elements in A are incomparable.

Thus a Sperner family is an anti-chain in our third example.

Theorem

Let P be a finite poset, then $\min\{m: \exists$ *anti-chains* $A_1, A_2, \ldots, A_\mu$ *with* $P = \bigcup_{i=1}^{\mu} A_i\}$ = $\max\{|C|:$ A *is a chain*$\}$.

The minimum number of anti-chains needed to cover P is at least the size of any chain, since a chain can contain at most one element from each anti-chain.

We prove the converse by induction on the maximum length μ of a chain. We have to show that P can be partitioned into μ anti-chains.

If $\mu = 1$ then P itself is an anti-chain and this provides the basis of the induction.

So now suppose that $C = x_1 < x_2 < \cdots < x_\mu$ is a maximum length chain and let A be the set of maximal elements of P.

(An element is x *maximal* if $\nexists y$ such that $y > x$.)

A is an anti-chain.

Now consider $P' = P \setminus A$. P' contains no chain of length μ. If it contained $y_1 < y_2 < \cdots < y_\mu$ then since $y_\mu \notin A$, there exists $a \in A$ such that P contains the chain $y_1 < y_2 < \cdots < y_\mu < a$, contradiction.

Thus the maximum length of a chain in P' is $\mu - 1$ and so it can be partitioned into anti-chains $A_1 \cup A_2 \cup \cdots A_{\mu-1}$. Putting $A_\mu = A$ completes the proof. □

Suppose that $C_1, C_2, \ldots, C_m$ are a collection of chains such that $P = \bigcup_{i=1}^{m} C_i$.

Suppose that A is an anti-chain. Then $m \geq |A|$ because if $m < |A|$ then by the pigeon-hole principle there will be two elements of A in some chain.

Theorem

(Dilworth) Let P be a finite poset, then $\min\{m : \exists \text{ chains } C_1, C_2, \ldots, C_m \text{ with } P = \bigcup_{i=1}^{m} C_i\} = \max\{|A| : A \text{ is an anti-chain}\}$.

We have already argued that $\max\{|A|\} \leq \min\{m\}$.

We will prove there is equality here by induction on $|P|$.

The result is trivial if $|P| = 0$.

Now assume that $|P| > 0$ and that μ is the maximum size of an anti-chain in P. We show that P can be partitioned into μ chains.

Let $C = x_1 < x_2 < \cdots < x_p$ be a *maximal* chain in P i.e. we cannot add elements to it and keep it a chain.

Case 1 Every anti-chain in $P \setminus C$ has $\leq \mu - 1$ elements. Then by induction $P \setminus C = \bigcup_{i=1}^{\mu-1} C_i$ and then $P = C \cup \bigcup_{i=1}^{\mu-1} C_i$ and we are done.

Case 2

There exists an anti-chain $A = \{a_1, a_2, \ldots, a_\mu\}$ in $P \setminus C$. Let

- $P^- = \{x \in P : \; x \preceq a_i$ for some $i\}$.
- $P^+ = \{x \in P : \; x \succeq a_i$ for some $i\}$.

Note that

1. $P = P^- \cup P^+$. Otherwise there is an element x of P which is incomparable with every element of A and so μ is not the maximum size of an anti-chain.
2. $P^- \cap P^+ = A$. Otherwise there exists x, i, j such that $a_i < x < a_j$ and so A is not an anti-chain.
3. $x_p \notin P^-$. Otherwise $x_p < a_i$ for some i and the chain C is not maximal.

Applying the inductive hypothesis to P^- ($|P^-| < |P|$ follows from 3) we see that P^- can be partitioned into μ chains $C_1^-, C_2^-, \ldots, C_\mu^-$.

Now the elements of A must be distributed one to a chain and so we can assume that $a_i \in C_i^-$ for $i = 1, 2, \ldots, \mu$.

a_i must be the maximum element of chain C_i^-, else the maximum of C_i^- is in $(P^- \cap P^+) \setminus A$, which contradicts 2.

Applying the same argument to P^+ we get chains $C_1^+, C_2^+, \ldots, C_\mu^+$ with a_i as the minimum element of C_i^+ for $i = 1, 2, \ldots, \mu$.

Then from 2 we see that $P = C_1 \cup C_2 \cup \cdots \cup C_\mu$ where $C_i = C_i^- \cup C_i^+$ is a chain for $i = 1, 2, \ldots, \mu$. $\square$

Three applications of Dilworth's Theorem

(i) Another proof of

Theorem

Erdős and Szekerés

$a_1, a_2, \ldots, a_{n^2+1}$ *contains a monotone subsequence of length* $n+1$.

Let $P = \{(i, a_i) : \ 1 \le i \le n^2 + 1\}$ and let say $(i, a_i) \preceq (j, a_j)$ if $i < j$ and $a_i \le a_j$.

A chain in P corresponds to a monotone increasing subsequence. So, suppose that there are no monotone increasing sequences of length $n+1$. Then any cover of P by chains requires at least $n+1$ chains and so, by Dilworths theorem, there exists an anti-chain A of size $n+1$.

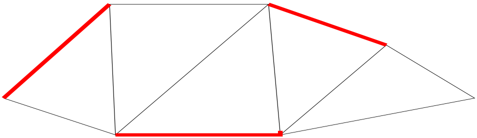
Let $A = \{(i_t, a_{i_t}) : 1 \le t \le n+1\}$ where $i_1 < i_2 \le \cdots < i_{n+1}$.

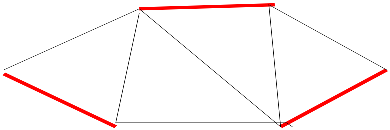
Observe that $a_{i_t} > a_{i_{t+1}}$ for $1 \le t \le n$, for otherwise $(i_t, a_{i_t}) \preceq (i_{t+1}, a_{i_{t+1}})$ and A is not an anti-chain.

Thus A defines a monotone decreasing sequence of length $n+1$. $\square$

Matchings in bipartite graphs

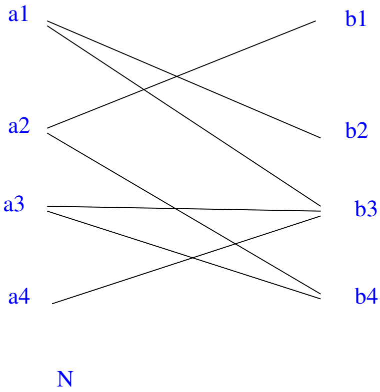
Re-call that a matching is a set of vertex disjoint edges.





P

Let $G = (A \cup B, E)$ be a bipartite graph with bipartition A, B.
For $S \subseteq A$ let $N(S) = \{b \in B : \exists a \in S, (a, b) \in E\}$.

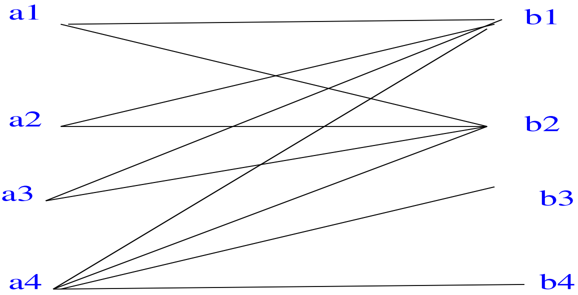


Clearly, $|M| \leq |A|, |B|$ for any matching M of G.

Theorem

(Hall) G contains a matching of size $|A|$ iff

$$|N(S)| \geq |S| \qquad \forall S \subseteq A.$$



$N(\{a_1, a_2, a_3\}) = \{b_1, b_2\}$ and so at most 2 of a_1, a_2, a_3 can be saturated by a matching.

If G contains a matching M of size $|A|$ then $M = \{(a, f(a)) : \ a \in A\}$, where $f : A \to B$ is a 1-1 function.

But then,

$$|N(S)| \geq |f(S)| = S$$

for all $S \subseteq A$.

Let $G = (A \cup B, E)$ be a bipartite graph which satisfies Hall's condition. Define a poset $P = A \cup B$ and define $<$ by $a < b$ only if $a \in A, b \in B$ and $(a, b) \in E$.

Suppose that the largest anti-chain in P is $A = \{a_1, a_2, \ldots, a_h, b_1, b_2, \ldots, b_k\}$ and let $s = h + k$.

Now

$$N(\{a_1, a_2, \ldots, a_h\}) \subseteq B \setminus \{b_1, b_2, \ldots, b_k\}$$

for otherwise A will not be an anti-chain.

From Hall's condition we see that

$$|B| - k \geq h \text{ or equivalently} |B| \geq s.$$

Now by Dilworth's theorem, P is the union of s chains:

A matching M of size m, $|A| - m$ members of A and $|B| - m$ members of B.

But then

$$m + (|A| - m) + (|B| - m) = s \leq |B|$$

and so $m \geq |A|$. $\square$

Marriage Theorem

Theorem

Suppose $G = (A \cup B, E)$ is k-regular. ($k \geq 1$) i.e. $d_G(v) = k$ for all $v \in A \cup B$. Then G has a perfect matching.

Proof

$$k|A| = |E| = k|B|$$

and so $|A| = |B|$.
Suppose $S \subseteq A$. Let m be the number of edges incident with S. Then

$$k|S| = m \leq k|N(S)|.$$

So Hall's condition holds and there is a matching of size $|A|$ i.e. a perfect matching.

König's Theorem

We will use Hall's Theorem to prove König's Theorem. Given a bipartite graph $G = (A \cup B), E)$ we say that $S \subseteq V$ is a cover if $e \cap S \neq \emptyset$ for all $e \in E$.

Theorem

$$\min\{|S| : S \textit{ is a cover}\} = \max\{|M| : M \textit{ is a matching}\}.$$

Proof One part is easy. If S is a cover and M is a matching then $|S| \geq |M|$. This is because no vertex in S can belong to more than one edge in M.

To begin the main proof, we first prove a lemma that is a small generalisation of Hall's Theorem.

Lemma

Assume that $|A| \leq |B|$. *Let* $d = \max\{(|X| - |N(X)|)^+ : X \subseteq A\}$ *where* $\xi^+ = \max\{0, \xi\}$. *Then*

$$\mu = \max\{|M| : M \textit{ is a matching } \} = |A| - d.$$

Proof That $\mu \leq |A| - d$ is easy. For the lower bound, add d dummy vertices D to B and add an edge between each vertex in D and each vertex in A to create the graph Γ. We now find that Γ satisfies the conditions of Hall's Theorem.

If M_1 is a matching of size $|A|$ in Γ then removing the edges of M_1 incident with D gives us a matching of size $|A| - d$ in G. ☐

Continuing the proof of König's Theorem let $S \subseteq A$ be such that $|N(S)| = |S| - d$.

Let $T = A \setminus S$. Then $T \cup N(S)$ is a cover, since there are no edges joining S to $B \setminus N(S)$.

Finally observe that

$$|T \cup N(S)| = |A| - |S| + |S| - d = |A| - d = \mu.$$

$\square$

Intervals Problem

$I_1, I_2, \ldots, I_{mn+1}$ are closed intervals on the real line i.e. $I_j = [a_j, b_j]$ where $a_j \leq b_j$ for $1 \leq j \leq mn + 1$.

Theorem

Either (i) there are $m + 1$ intervals that are pair-wise disjoint or (ii) there are $n + 1$ intervals with a non-empty intersection

Define a partial ordering $\preceq$ on the intervals by $I_r \preceq I_s$ iff $b_r \leq a_s$. Suppose that $I_{i_1}, I_{i_2}, \ldots, I_{i_t}$ is a collection of pair-wise disjoint intervals. Assume that $a_{i_1} < a_{i_2} \cdots < a_{i_t}$. Then $I_{i_1} < I_{i_2} \cdots < I_{i_t}$ form a chain and conversely a chain of length t comes from a set of t pair-wise disjoint intervals.
So if (i) does not hold, then the maximum length of a chain is m.

This means that the minimum number of chains needed to cover the poset is at least $\lceil \frac{mn+1}{m} \rceil = n+1$.

Dilworth's theorem implies that there must exist an anti-chain $\{I_{j_1}, I_{j_2}, \ldots, I_{j_{n+1}}\}$.

Let $a = \max\{a_{j_1}, a_{j_2}, \ldots, a_{j_{n+1}}\}$ and $b = \min\{b_{j_1}, b_{j_2}, \ldots, b_{j_{n+1}}\}$.

We must have $a \leq b$ else the two intervals giving a, b are disjoint.

But then every interval of the anti-chain contains $[a, b]$.

Möbius Inversion

Suppose that $|P| = n$. We argue next that we can label the elements of $P = \{p_1, p_2, \ldots, p_n\}$ so that

$$p_i \preceq p_j \text{ implies } i \leq j. \tag{1}$$

We prove this by induction on n. The base case $n = 1$ is trivial.

Choose a maximal element of P and label it p_n. Assume that (1) can be achieved for posets with fewer than n elements. Let $P' = P \setminus \{p_n\}$.

We can, by induction, re-label $P' = \{p_1, p_2, \ldots, p_{n-1}\}$ so that (1) holds. Because p_n is maximal, we now have a labelling for all of P.

We now define $\zeta : P^2 \to \{0, 1\}$ by

$$\zeta(x, y) = \begin{cases} 1 & x \preceq y. \\ 0 & \textit{Otherwise}. \end{cases}$$

Given (1) the $n \times n$ matrix $A_\zeta = [\zeta(x, y)]$ is an upper triangular matrix with an all 1's diagonal.

A_ζ is invertible and its inverse is called $A_\mu = [\mu(x, y)]$. The function μ is called the Möbius function of P. The equation $A_\mu A_\zeta = I$ implies the following:

$$\sum_{z \in P} \mu(x, z)\zeta(z, y) = \sum_{x \preceq z \preceq y} \mu(x, z) = \begin{cases} 1 & x = y. \\ 0 & \textit{Otherwise}. \end{cases} \qquad (2)$$

Theorem

(a) *For P equal to the subsets of some finite set X and $\preceq = \subseteq$ we have*

$$\mu(A,B) = \begin{cases} (-1)^{|A|-|B|} & A \subseteq B \\ 0 & \textit{Otherwise.} \end{cases}$$

(b) *For $P = [n]$ and $a \preceq b$ if a divides b we have*

$$\mu(a,b) = \begin{cases} (-1)^{r} & b/a \textit{ is the product of } r \textit{ distinct primes} \\ 0 & \textit{Otherwise.} \end{cases}$$

Proof

We just have to verify (2):
(a) We have

$$\sum_{A \subseteq C \subseteq B} x^{|C|-|A|} = (1+x)^{|B|-|A|}.$$

Putting $x = -1$ we get a RHS of zero, unless $A = B$, in which case we get $0^0 = 1$.

(b) Suppose that $b/a = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ where $p_1, p_2, \ldots, p_r$ are primes and $k_1, k_2, \ldots, k_r \geq 1$.

$$\sum_{a|c|b} \mu(c, b) = \sum_{S \subseteq [r]} (-1)^{|S|} = \begin{cases} 1 & r = 0. \\ 0 & r \geq 1. \end{cases}$$

$\square$

Möbius Inversion

Theorem

Suppose that f, g, h are functions from P to $\mathbf{R}$ such that

$$g(x) = \sum_{a \preceq x} f(a) \quad \textit{and} \quad h(x) = \sum_{b \succeq x} f(b). \tag{3}$$

Then,

$$f(x) = \sum_{a \preceq x} \mu(a, x) g(a) \quad \textit{and} \quad f(x) = \sum_{b \succeq x} \mu(x, b) h(b). \tag{4}$$

Proof Treating f, g, h as column vectors $\mathbf{f}, \mathbf{g}, \mathbf{h}$ we see that (3) is equivalent to $\mathbf{g} = A_\zeta^T \mathbf{f}$ and $\mathbf{h} = A_\zeta \mathbf{f}$. Thus

$$\mathbf{f} = A_\zeta^{-T} \mathbf{g} = A_\mu^T \mathbf{g} \quad \text{and} \quad \mathbf{f} = A_\zeta^{-1} \mathbf{h} = A_\mu \mathbf{h}.$$

Inclusion-Exclusion

Let $A_i, i \in I$ be a family of subsets of a finite set X.

For $J \subseteq I$ let $f(J)$ equal the number of elements in $\bigcap_{i \in J} A_i$ that are also in $\bigcap_{i \notin I}(X \setminus A_i)$.

Let $h(J)$ be the number of elements in $\bigcap_{i \in J} A_i$. Then

$$h(J) = \sum_{K \supseteq J} f(K) = \sum_{K \succeq J} f(K).$$

Möbius inversion gives us

$$f(J) = \sum_{K \succeq J} \mu(K, J) h(K) = \sum_{K \supseteq J} (-1)^{|K|-|J|} h(K).$$

Putting $J = \emptyset$ we get

$$\left| \bigcap_{i \in I} (X \setminus A_i) \right| = \sum_{K \subseteq I} (-1)^{|K|-|J|} \left| \bigcap_{j \in K} A_j \right| .$$

Divisibility Poset

Supose now that $f : \boldsymbol{N} \to \boldsymbol{R}$ and that g is given by

$$g(n) = \sum_{d|n} f(d).$$

Then Möbius inversion gives

$$f(n) = \sum_{d|n} \mu(d,n) g(d) = \sum_{\substack{d|n \\ n/d \text{ square free}}} (-1)^{p(n/d)} g(d)$$

where $p(m)$ is the number of distinct prime factors of m.

Totient function

For a natural number n, let $\phi(n)$ denote the number of integers $m \leq n$ such that m, n have n common factors (other than one) – *co-prime*.

Lemma

$$n = \sum_{d|n} \phi(d) = \sum_{d|n} \phi(n/d). \qquad (5)$$

Proof If $(m, n) = d$ then $m = m_1 d, n = n_1 d$ where $(m_1, n_1) = 1$. So the number of choices for m is the number of choices for m_1 i.e. $\phi(n_1) = \phi(n/d)$. □

Möbius inversion with $g(n) = n$ and $f(n) = \phi(n)$ applied to (5) gives

$$\phi(n) = \sum_{d|n} (-1)^{p(n/d)} d = \sum_{d|n} (-1)^{p(d)} \frac{n}{d}. \tag{6}$$

$$\begin{aligned} \phi(n) &= n \sum_{d|n} \frac{(-1)^{p(d)}}{d} \\ &= n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right), \end{aligned} \tag{7}$$

assuming that $n = p_1^{k_1} p_2^{k_2} \ldots, p_r^{k_r}$ where $p_1, p_2, \ldots, p_r$ are primes and $k_1, k_2, \ldots, k_r \geq 1$.

2-colored necklace

A *necklace* is a sequence $x_1x_2\cdots x_n$ of n 0' and 1's arranged in circle.

Two necklaces x, y are said to *equivalent* if there exists $d|n$ such that $y_i = x_{i+d}, i = 1, 2, \ldots, n$ where we interpret $i + d \mod n$. In this case we say that x is *periodic* with period d.

Let N_n denote the number of distinct i.e. non-equivalent necklaces and let $M(d)$ denote the number of aperiodic necklaces of length d.

Thus

$$N_n = \sum_{d|n} M(d) \quad \text{and} \quad \sum_{d|n} dM(d) = 2^n.$$

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For the second equation think about rotating a periodic necklace one step at a time for d steps. If we do this for all periodic necklaces then we get all 2^n sequences.

Applying Möbius inversion to the second equation with $f(d) = dM(d), g(n) = 2^n$, we get

$$M(n) = \frac{1}{n} \sum_{d|n} \mu(d, n) 2^d.$$

So,

$$N_n = \sum_{d|n} M(d) = \sum_{d|n} \sum_{\ell|d} \frac{1}{d} \mu(\ell, d) 2^d = \sum_{d|n} \frac{1}{d} \sum_{\ell|d} \mu(\ell, d) 2^\ell.$$

Now substitute $d = k\ell$ and tidy up to get

$$N_n = \sum_{\ell|n} \frac{2^\ell}{\ell} \sum_{k|\frac{n}{\ell}} \frac{\mu(1,k)}{k} = \frac{1}{n} \sum_{\ell|n} \phi(n/\ell) 2^\ell.$$

For the second equation, we use the expression (7).