Homework 1

Problem 1. *Show the Venn-diagram representation for the following sets:*

(a) $(A - B) \cap C$

(b) $\overline{A \oplus (B \cup C)}$

Problem 2. *For any sets A, B and C, prove that*

$$A \cup B = A \cup C, A \cap B = A \cap C \text{ implies } B = C.$$

Solution. (Proof by contradiction)

Suppose $B \neq C$. As B and C are symmetric, then without loss of generality, take $x \in B - C$:

1. if $x \in A$: then $x \in A \cap B$ and $x \notin A \cap C$;
2. if $x \notin A$: then $x \in A \cup B$ and $x \notin A \cup C$.

neither of the above case could be true. Thus the assumption $B \neq C$ is not correct, which leads to $B = C$. □

Problem 3. *Show that a nonempty set has the same number of odd subsets (i.e., subsets with an odd number of elements) as even subsets.*

Solution. As the set S is non-empty, there is some $a \in S$. Now consider all the subsets of $S' = S - \{a\}$. Let X_0 be the set of even subsets of S', X_1 be the set of odd subsets of S'.

We use the symbol X_i^a to stand for the set getting by adding a into every element of X_i, where $i = 0, 1$. Obviously $|X_i^a \models |X_i|$.

To the original set S, $X_0 \cup X_1^a$ is the set of even subsets of S. Similarly $X_1 \cup X_0^a$ is the set of odd subsets of S. It is easy to prove that x_i and x_{1-i}^a are disjoint (i.e., their intersection is empty). $|X_0 \cup X_1^a \models |X_1 \cup X_0^a|$ by the previous problem. □

Problem 4. A, B, C *are three sets. and two functions* $g : A \to B$, $f : B \to C$

a) *If* $f \circ g$ *is an injective function and* g *is surjective, show that* f *is injective.*

b) *If* $f \circ g$ *is an surjective function and* f *is injective, show that* g *is surjective.*

(Note that $f \circ g(x) = f(g(x))$.)

Proof. a) for any $f(x) = f(y)$ with $x, y \in B$, as g is surjective, there exit $u, v \in A$ such that $g(u) = x, g(v) = y$. Now $f(x) = f(y) \Rightarrow f(g(u)) = f(g(v))$. Since $f \circ g$ is injective, we have $u = v$. It follows that $x = g(u) = g(v) = y$. Thus f is injective.

b) suppose g is not surjective and take $b \in B - Ran(g)$. As f is injective we know that $f(x) \neq f(b) \in C$ for any $x \in B \wedge x \neq b$. Then we will have that $f(b) \notin Ran(f \circ g)$ which betrays that $f \circ g$ is surjective.

□

Problem 5. *$\mathcal{R}$ is a binary relation,*

1. *Show that $\mathcal{R}$ is symmetric iff $\mathcal{R}^{-1} \subseteq \mathcal{R}$.*

2. *Show that $\mathcal{R}$ is transitive iff $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$.*

Proof. We prove the the first statement and omit the second one.

$$
\begin{aligned}
& \mathcal{R} \text{ is symmetric} \\
\Longleftrightarrow\ & \forall x \forall y (x\mathcal{R}y \longrightarrow y\mathcal{R}x) \\
\Longleftrightarrow\ & \forall x \forall y (y\mathcal{R}^{-1}x \longrightarrow y\mathcal{R}x) \\
\Longleftrightarrow\ & \mathcal{R}^{-1} \subseteq \mathcal{R}
\end{aligned}
$$

□

Problem 6. *A and B are countable sets. Prove that*

1. *$A \cup B$ is countable*

2. *$A \times B$ is countable*

Solution.(Hint) As $A \preceq \omega$ and $B \preceq \omega$, suppose $f : A \rightarrow \omega$, and $g : B \rightarrow \omega$ are both injective functions.

Then

1. $$h(x) = \begin{cases} 2 \cdot f(x) & x \in A \\ 2 \cdot g(x) + 1 & x \in B - A \end{cases}$$

   and then prove that $h(x)$ is injective.

2. $h(\langle x, y\rangle) = \langle f(x), g(y)\rangle$.

   Then prove that $h(x)$ is injective.

   Function h shows $A \times B \preceq \omega \times \omega$.

   As we know $\omega \times \omega \approx \omega$, we finally get $A \times B \preceq \omega$.

$\square$

Problem 7. *Draw the Hasse diagram of the set of all subsets of* $\{1, 2, 3\}$ *ordered by inclusion.*

Solution. $\square$