1. 解：

$f(n)$	$g(n)$	$f = O(g)$	$f = \Omega(g)$	$f = \Theta(g)$
$2n^3 + 3n$	$100n^2 + 2n + 100$	×	✓	×
$50n + \log n$	$10n + \log\log n$	✓	✓	✓
$50n \log n$	$10n \log\log n$	×	✓	×
$\log n$	$\log^2 n$	✓	×	×
$n!$	5^n	×	✓	×

(1) $\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = \infty$

(2) $\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = \frac{50}{10} = 5$

(3) $\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = \frac{50 \log n}{10 \log(\log n)} = \infty$

(4) $\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = \frac{1}{\log n} = 0$

(5) $\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = \frac{n!}{5^n} = \infty$

2 解: (1): $f(x) = \sin x$

$g(x) = \cos x$

(2) $f(x) = x^{x+\sin x}$

$g(x) = x^{x+\cos x}$

3 解: (a) $f(x) = \left(1+\frac{1}{x}\right)^x = e^{x\ln\left(1+\frac{1}{x}\right)}$

$$f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ln\left(1+\frac{1}{x}\right) + x \cdot \frac{-\frac{1}{x^2}}{1+\frac{1}{x}}\right]$$

$$= \left(1+\frac{1}{x}\right)^x \left[\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]$$

$$g(x) = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}$$

$$g'(x) = \frac{-\frac{1}{x^2}}{1+\frac{1}{x}} + \frac{1}{(x+1)^2}$$

$$= \frac{-(x+1)+x}{(x+1)^2 \cdot x} < 0$$

又∵ $\lim_{x\to\infty} g(x) = \frac{1}{x} - \frac{1}{x+1} = 0$

又∵ $g'(x) < 0$ ∴ $g(x) > 0$ 恒成立

∴ $f'(x) > 0$ 而 $\lim_{n\to\infty} (1+\frac{1}{n})^n$

$$= e^{n\ln(1+\frac{1}{n})} = e^{n\cdot\frac{1}{n}} = e$$

又∵ $f'(x) > 0$ ∴ $f(x) < e$

∴ $f(x) \le e$

(2) $f(x) = (1+\frac{1}{x})^{x+1}$ $f'(x)$

$$= (1+\frac{1}{x})^{x+1}\left[\ln(1+\frac{1}{x}) + (x+1)\cdot\left(-\frac{1}{x^2}\right)\right]$$

$$g(x) = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x} - \frac{1}{x^2}$$

$$\lim_{x\to\infty} g(x) = 0$$

$$g'(x) = \frac{-\frac{1}{x^2}}{1+\frac{1}{x}} + \frac{1}{x^2} + \frac{2x}{x^4}$$

$$= -\frac{1}{x^2+x} + \frac{1}{x^2} + \frac{2}{x^3}$$

$$= \frac{-x^2 + x(x+1) + 2(x+1)}{x^3(x+1)}$$

$$= \frac{3x+2}{x^3(x+1)} > 0$$

$\therefore g(x) < 0$ 即 $f'(x) < 0$

又 $\because \lim_{x\to\infty} \left(1+\frac{1}{n}\right)^{n+1} = e \cdot 1 = e$ $\therefore f(x) > e$

$\therefore f(x) \geq e$

(c) $\because \left(1+\frac{1}{n}\right)^n \le e$

而 $\left(1+\frac{1}{n}\right)^{n+1} \ge e$

$\therefore \left(1+\frac{1}{n}\right)^n \ge \frac{e}{1+\frac{1}{n}}$

而 $\lim_{n\to\infty} \frac{e}{1+\frac{1}{n}} = e$

由夹逼定理，$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

4: 解：$n=0$ 时，$1 \ge 1$，成立，

假设 $n=k$ 时成立，

当 $n=k+1$ 时，求证

$(1+x)^{k+1} \ge 1+x(k+1) = 1+kx+x$

已知 $n=k$ 时 $(1+x)^k \ge 1+kx$

故 $(1+x)^{k+1} \geq (1+kx)(1+x)$

$= kx^2 + (k+1)x + 1$

$= kx^2 + x + kx + 1$

$\geq 1 + kx + x$

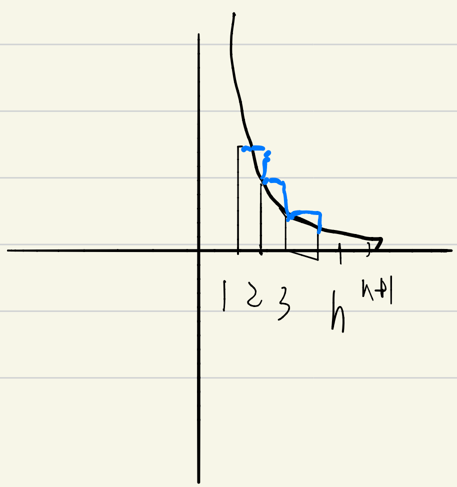
得证

5、解：$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$

$= \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$

考虑 $f(x) = \frac{1}{\sqrt{x}}$

$f(x)$ 单减



$$\int_0^n \frac{1}{\sqrt{x}}\,dx \geq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \int_1^{n+1} \frac{1}{\sqrt{x}}\,dx$$

即 $$2\sqrt{x}\Big|_1^{n+1} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{x}\Big|_0^{n}$$

$$\therefore 2\sqrt{n+1}-2 \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n}$$

又 $$\because \int_2^{n+1} \cdots \leq \sum_{k=2}^{n} \frac{1}{\sqrt{k}} \leq \int_1^{n} \cdots$$

$$\therefore 2\sqrt{x}\Big|_2^{n+1} \leq \sum_{k=2}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{x}\Big|_1^{n}$$

$$\therefore 2\sqrt{n+1}-2\sqrt{2} \leq \sum_{k=2}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n}-2$$

$$\therefore 1+\sum_{k=2}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n}-1$$

$$\therefore \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n}-1$$

6. 解：(a) $\binom{2m}{m} = \frac{(2m)(2m-1)\cdots(m+1)}{m(m-1)\cdots 2\cdot 1}$

$= 2\cdot\left(2+\frac{1}{m-1}\right)\cdot\left(2+\frac{2}{m-2}\right)\cdot\cdots\cdot\left(2+\frac{m-1}{m-m+1}\right)$

$= \prod_{k=1}^{m}\left[2+\frac{k-1}{m-k+1}\right]$ ① 若 m 为偶数。

原式 $= \prod_{k=1}^{\frac{m}{2}}\left[2+\frac{k-1}{m-k+1}\right]\left[2+\frac{m-k}{k}\right]$

$\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$

$\therefore \binom{2m}{m} \geq \left(\frac{2m}{m}\right)^m = 2^m$

若m偶数则原式$=$

$$(2m-1)(2m-3)\cdots(m+3)(m+1)\cdot\frac{2m\cdot(2m-2)\cdots(m+4)(m+2)}{m\cdot(m-1)\cdots2\cdot1}$$

$$=(2m-1)(2m-3)\cdots(m+1)\cdot\frac{2^{\frac{m}{2}}}{\frac{m}{2}\left(\frac{m}{2}-1\right)\cdots2\cdot1}$$

$$\binom{2m}{m}=\sum_{k=0}^{m}\binom{m}{k}\binom{m}{m-k}$$