Homework 8

Problem 1. *Which of the following statements about graph G and H are true?*

1. *G and H are isomorphic if and only if for every map $f : V(G) \to V(H)$ and for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.*

2. *G and H are isomorphic if and only if there exists a bijection $f : E(G) \to E(H)$.*

3. *If there exists a bijection $f : V(G) \to V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$, then G and H are isomorphic.*

4. *If G and H are isomorphic, then there exists a bijection $f : V(G) \to V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$.*

5. *If G and H are isomorphic, then there exists a bijection $f : E(G) \to E(H)$.*

6. *G and H are isomorphic if and only if there exists a map $f : V(G) \to V(H)$ such that for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.*

7. *Every graph on n vertices is isomorphic to some graph on the vertex set $\{1, 2, \ldots, n\}$.*

8. *Every graph on $n \geq 1$ vertices is isomorphic to infinitely many graphs.*

Problem 2. *Two simple graphs $G = (V, E)$ and $G' = (V', E')$. A map $f : V \to V'$. Now if f satisfies:*

i) *It is a bijective function;*

ii) *$\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$;*

Then we say that graph G and G' are isomorphic *to each other. We use $G \cong G'$ to stand for the isomorphism relation.*

Consider the following questions:

1. *$G = K_n$ (Recall: K_n is a clique with n vertices), $g : V \to V'$ is a function which only satisfies requirement ii). Prove that G' must contain a subgraph which is a clique with n-vertices.*

2. *$G = K_{n,m}$ (Recall: $K_{n,m}$ is the so-called* complete bipartite graphs*), g is the same as in question 1. What will be the simplest G' that is related to G under the new relation.*

Problem 3. *How many graphs on the vertex set $\{1, 2, \ldots, 2n\}$ are isomorphic to the graph consisting of n vertex-disjoint edges (i.e. with edge set {{1,2},{3,4},. . . , {2n-1,2n}}?*

Problem 4. *Construct an example of a sequence of length n in which each term is some of the numbers $1, 2, \ldots, n-1$ and which has an even number of odd terms, and yet the sequence is not a graph score. Show why it is not a graph score.*

Problem 5. *Let G be a graph with 9 vertices, each of degree 5 or 6. Prove that it has at least 5 vertices of degree 6 or at least 6 vertices of degree 5.*

Problem 6. *Given a sequence $(d_1, d_2, \ldots, d_n)$ of positive integers (where $n \geq 1$):*

(i) There exists a tree with score $(d_1, d_2, \ldots, d_n)$.

(ii) $\sum_{i=1}^{n} d_i = 2n - 2$.

Prove that (i) and (ii) are equivalent.

Problem 7. *Let N_k denote the number of spanning trees of K_n in which the vertex n has degree k, $k = 1, 2, \ldots, n-1$ (recall that we assume $V(K_n) = \{1, 2, \ldots, n\}$).*

i) Prove that $(n - 1 - k)N_k = k(n - 1)N_{k+1}$.

ii) Using i), derive $N_k = \binom{n-2}{k-1}(n-1)^{n-1-k}$.

iii) Prove Cayley's formula from ii).