Generating Function

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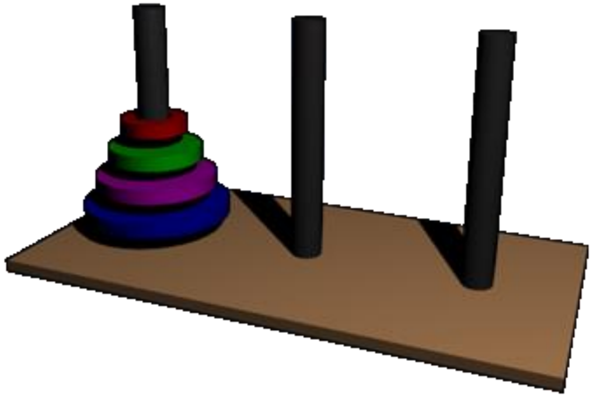


- Problem: How many ways are there to pay the amount of 21 doublezons if we have
  - 6 one-doublezon coins;
  - 5 two-doublezon coins;
  - 4 five-doublezon coins.
- Solution:

$$\boldsymbol{i_1 + i_2 + i_3 = 21} \quad \textbf{(*)}$$

$$\boldsymbol{i_1} \in \{\mathbf{0, 1, 2, 3, 4, 5, 6}\}; \ \boldsymbol{i_2} \in \{\mathbf{0, 2, 4, 6, 8, 10}\}; \ \boldsymbol{i_3} \in \{\mathbf{0, 5, 10, 15, 20}\}.$$

$$\boldsymbol{(1 + x + x^2 + x^3 + \cdots + x^6)(1 + x^2 + x^4 + x^6 + x^8 + x^{10})}$$
$$\cdot \boldsymbol{(1 + x^5 + x^{10} + x^{15} + x^{20})}$$

The coefficient of $\boldsymbol{x^{21}}$
= the number of solutions of (*).

Recall

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 \cdots \binom{n}{n} x^n$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \cdots = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$$

$$\sum_{k=0}^{n} k \binom{n}{k} = n\, 2^{n-1}$$

Generating function

- $(a_0, a_1, a_2, \ldots)$ be a sequence of real numbers, then the generating function of this sequence is defined as

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

Example:

$(1,1,1, \ldots)$ $\quad a(x) = 1 + x + x^2 + \cdots = \dfrac{1}{1 - x}$

Generalized binomial theorem

$$\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}$$

If r is a negative integer

$$\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}$$

$$= (-1)^k \frac{(-r)(-r+1)(-r+2)\cdots(-r+k-1)}{k!}$$

$$= (-1)^k \binom{-r+k-1}{k} = (-1)^k \binom{-r+k-1}{-r-1}$$

$$\frac{1}{(1-x)^n} = \binom{n-1}{n-1} + \binom{n}{n-1}x + \binom{n+1}{n-1}x^2 + \cdots \binom{n+k-1}{n-1}x^k + \cdots$$

More examples

$(a_0, a_1, a_2, \ldots)$ $\quad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(0, a_0, a_1, a_2, \ldots)$ $\quad a(x) = 0 + a_0 x + a_1 x^2 + \cdots = x \cdot a(x)$

$(0, 1, \frac{1}{2}, \frac{1}{3}, \ldots)$ $\quad a(x) = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\ln(1 - x) \quad -1 < x < 1$

$(1, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots)$ $\quad a(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$

Operations with Sequences -Addition

$(a_0, a_1, a_2, \ldots)$ $\quad$ $a(x) = a_0 + a_1x + a_2x^2 + \cdots$

$(b_0, b_1, b_2, \ldots)$ $\quad$ $b(x) = b_0 + b_1x + b_2x^2 + \cdots$

$(a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots)$ $\quad$ $a(x) + b(x)$

Constant linear expansion

$(a_0, a_1, a_2, \ldots)$ $\qquad$ $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(\alpha a_0, \alpha a_1, \alpha a_2, \ldots)$ $\qquad$ $\alpha \cdot a(x)$

Shifting-right

$(a_0, a_1, a_2, \dots)$ $\quad$ $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(\underbrace{0,0,\cdots,0}_{n}, a_0, a_1, a_2, \dots)$ $\quad$ $x^n \cdot a(x)$

Shifting-left

$(a_0, a_1, a_2, \ldots)$ $\qquad$ $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(a_3, a_4, a_5, \ldots)$ $\qquad$ $\boxed{\dfrac{a(x) - a_0 - a_1 x - a_2 x^2}{x^3}}$

Substituting $-\alpha x$

$(a_0, a_1, a_2, \ldots)$ $\qquad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(a_0, \alpha a_1, \alpha^2 a_2, \ldots)$ $\qquad \boxed{a(\alpha x)}$

Example:

$(1,1,1,\ldots)$ $\qquad a(x) = \dfrac{1}{1-x}$

$(1,2,4,8,\ldots)$ $\qquad \boxed{a(2x) = \dfrac{1}{1-2x}}$

$(a_0, 0, a_2, 0, a_4, 0 \ldots)$ $\qquad \boxed{\dfrac{1}{2}(a(x) + a(-x))}$

Substituting $-x^n$

$(a_0, a_1, a_2, \dots)$ $\qquad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(a_0, 0,0, a_1, 0,0, a_2, 0,0, \dots)$ $\qquad a_0 + a_1 x^3 + a_2 x^6 + \cdots = a(x^3)$

Example:

$(1,1,2,2,4,4,8,8, \dots)$ i.e., $a_n = 2^{\lfloor n/2 \rfloor}$ $\qquad \dfrac{1+x}{1-2x^2}$

$(1,2,4,8, \dots)$ $\qquad \dfrac{1}{1-2x}$

$(1,0,2,0,4,0,8, \dots)$ $\qquad \dfrac{1}{1-2x^2}$

$(0,1,0,2,0,4,0,8, \dots)$ $\qquad \dfrac{x}{1-2x^2}$

Differentiation

$(a_0, a_1, a_2, \dots)$ $\qquad$ $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(a_1, 2a_2, 3a_3, \dots)$ $\qquad$ $a'(x)$

Example:

$(1^2, 2^2, 3^2, 4^2, \dots)$ i.e., $a_k = (k+1)^2$

$(1,1,1,1,\dots)$ $\quad \dfrac{1}{1-x}$

$(1,2,3,4,\dots, k+1, \dots)$ $\qquad \left(\dfrac{1}{1-x}\right)' = \dfrac{1}{(1-x)^2}$

$(2\cdot 1, 3\cdot 2, 4\cdot 3, \dots, (k+2)(k+1), \dots$ $\qquad \left(\dfrac{1}{1-x}\right)'' = \dfrac{2}{(1-x)^3}$

Differentiation

$(a_0, a_1, a_2, \dots)$ $\qquad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(a_1, 2a_2, 3a_3, \dots)$ $\qquad \boxed{a'(x)}$

Example:

$(1^2, 2^2, 3^2, 4^2, \dots)$ i.e., $a_k = (k+1)^2$ $\qquad \boxed{\dfrac{2}{(1-x)^3} - \dfrac{1}{(1-x)^2}}$

$(1,1,1,1,\dots)$ $\qquad \dfrac{1}{1-x}$

$(1,2,3,4,\dots,\underline{k+1},\dots)$ $\qquad \left(\dfrac{1}{1-x}\right)' = \dfrac{1}{(1-x)^2}$

$(2\cdot 1, 3\cdot 2, 4\cdot 3, \dots, \underline{(k+1)^2 + k + 1}, \dots$ $\qquad \left(\dfrac{1}{1-x}\right)'' = \dfrac{2}{(1-x)^3}$

Integration

$(a_0, a_1, a_2, \dots)$ $\qquad$ $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(0, a_0, \frac{1}{2} a_1, \frac{1}{3} a_2, \frac{1}{4} a_{3,} \dots)$ $\qquad$ $\int_0^x a(t)dt$

Multiplication/Convolution

$(a_0, a_1, a_2, \ldots)$ $\qquad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(b_0, b_1, b_2, \ldots)$ $\qquad b(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

$(c_0, c_1, c_2, \ldots)$ $\qquad a(x) \cdot b(x)$

$$c_0 = a_0 b_0$$

$$c_1 = a_0 b_1 + a_1 b_0$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$$

$$\vdots$$

$$c_k = \sum_{i,j \geq 0; i+j=k} a_i b_j$$

Solving Recurrence

- Recurrence relation
  Define g_i recursively.
- Manipulation:
  New equivalence concerning $G(x)$.
- Solving
  Closed form for $G(x)$.
- Expanding
  New form for g_i.

Application

- A box contains 30 red, 40 blue, and 50 green balls; balls of the same color are indistinguishable. How many ways are there of selecting a collection of 70 balls from the box?
- Solution:

$$(1 + x + x^2 + \cdots + x^{30}) \cdot (1 + x + x^2 + \cdots + x^{40}) \cdot (1 + x + x^2 + \cdots + x^{50})$$

$$= \left(\frac{1 - x^{31}}{1 - x}\right) \cdot \left(\frac{1 - x^{41}}{1 - x}\right) \cdot \left(\frac{1 - x^{51}}{1 - x}\right)$$

$$= \frac{1}{(1 - x)^3}(1 - x^{31})(1 - x^{41})(1 - x^{51})$$

$$(1 + x + x^2 + \cdots + x^{30}) \cdot (1 + x + x^2 + \cdots + x^{40})$$
$$\cdot (1 + x + x^2 + \cdots + x^{50})$$
$$= \frac{1}{(1-x)^3}(1 - x^{31})(1 - x^{41})(1 - x^{51})$$

$$\frac{1}{(1-x)^n} = \binom{n-1}{n-1} + \binom{n}{n-1}x + \binom{n+1}{n-1}x^2 + \cdots \binom{n+k-1}{n-1}x^k + \cdots$$

$$= \left(\binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 + \cdots\right)(1 - x^{31} - x^{41} - x^{51} + \cdots)$$

Thus the coefficient of x^{70} is：

$$= \binom{70+2}{2} - \binom{70+2-31}{2} - \binom{70+2-41}{2} - \binom{70+2-51}{2}$$
$$= 1061$$

Application

- Fibonacci Number

$$f_0 = 0, \quad f_1 = 1, \qquad f_n = f_{n-1} + f_{n-2}$$

$$F(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_{n-2} x^{n-2} + f_{n-1} x^{n-1} + f_n x^n + \cdots$$

Fibonacci Sequence

- $f_n = f_{n-1} + f_{n-2}$
- $F(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_{n-2} x^{n-2} + f_{n-1} x^{n-1} + f_n x^n + \cdots$
- $xF(x) = \quad f_0 x + f_1 x^2 + f_2 x^3 + \cdots\cdots\cdots + f_{n-2} x^{n-1} + f_{n-1} x^n + \cdots$
- $x^2 F(x) = \quad f_0 x^2 + f_1 x^3 + f_2 x^4 + \cdots\cdots\cdots\cdots\cdots + f_{n-2} x^n + f_{n-1} x^{n+1} + \cdots$

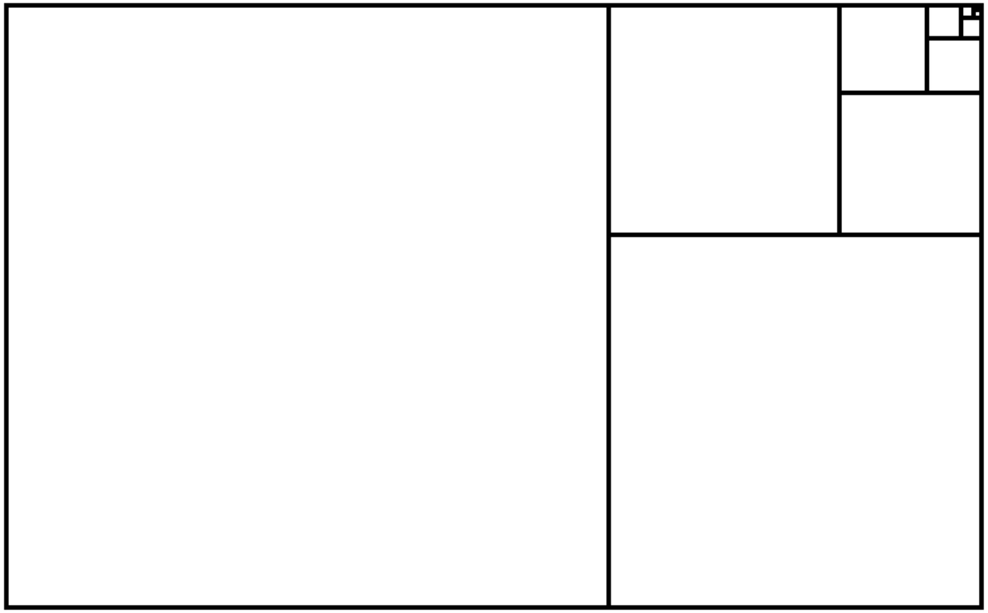
$$F(x) - xF(x) - x^2 F(x) = f_0 + (f_1 - f_0)x$$

$$F(x) = \frac{x}{1 - x - x^2}$$

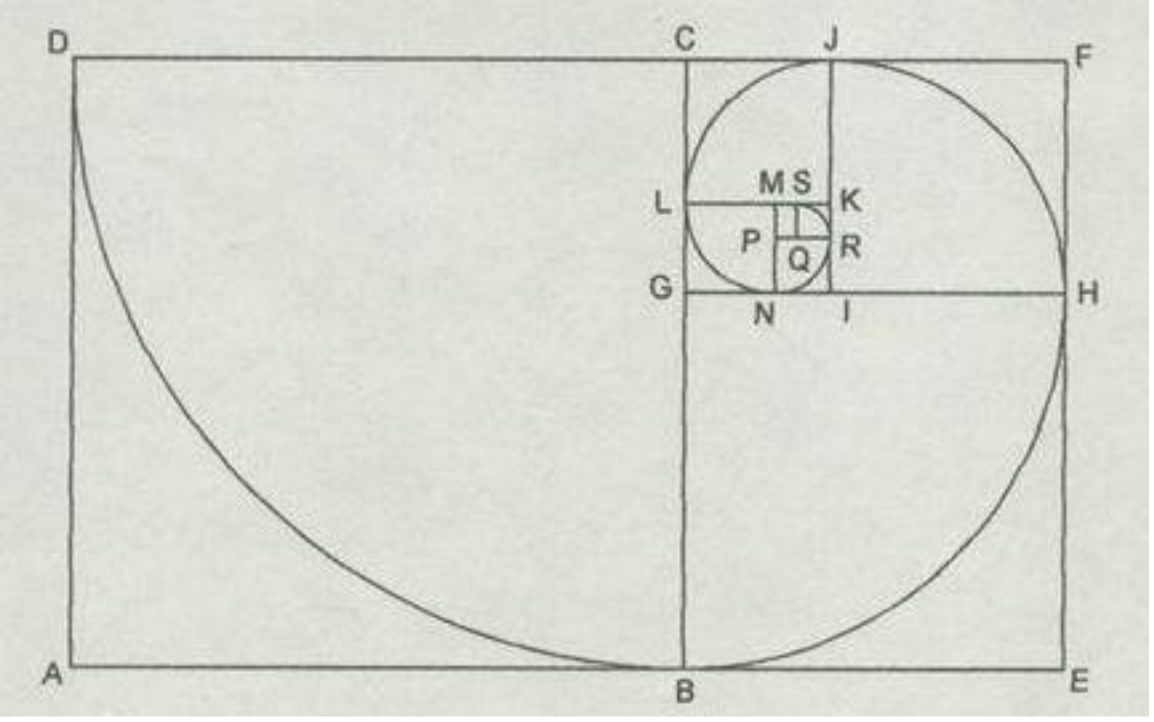
$$= \frac{a}{1 - \lambda_1 x} + \frac{b}{1 - \lambda_2 x} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \frac{1+\sqrt{5}}{2} x} - \frac{1}{1 - \frac{1-\sqrt{5}}{2} x} \right)$$

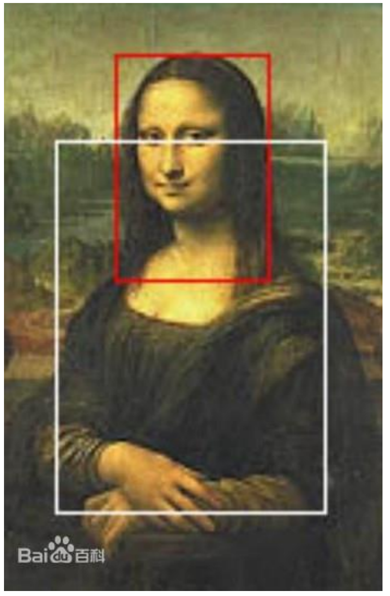
$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

- $\lim_{n \to \infty} \frac{\mathrm{F}_n}{F_{n+1}} = 0.6180339$



- $\lim_{n\to\infty} \frac{\mathrm{F}_n}{F_{n+1}} = 0.6180339$

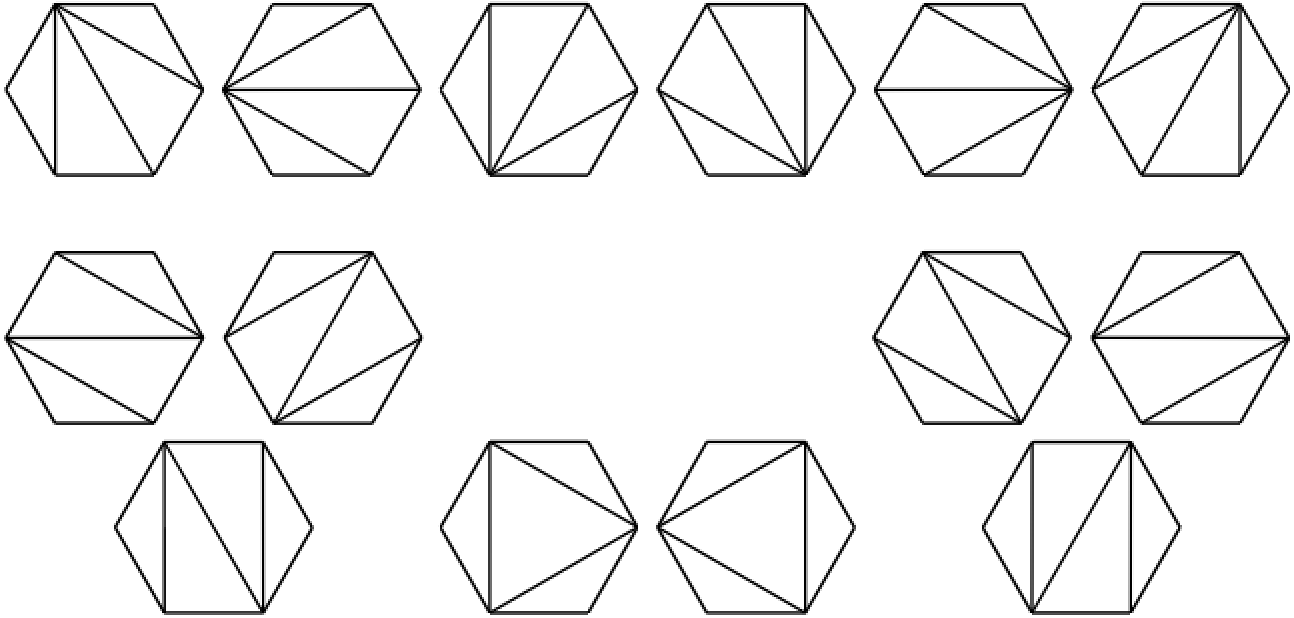




Catalan Number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

- Number of different ways a convex polygon with $n + 2$ sides can be cut into triangles by connecting vertices with straight lines (a form of Polygon triangulation).



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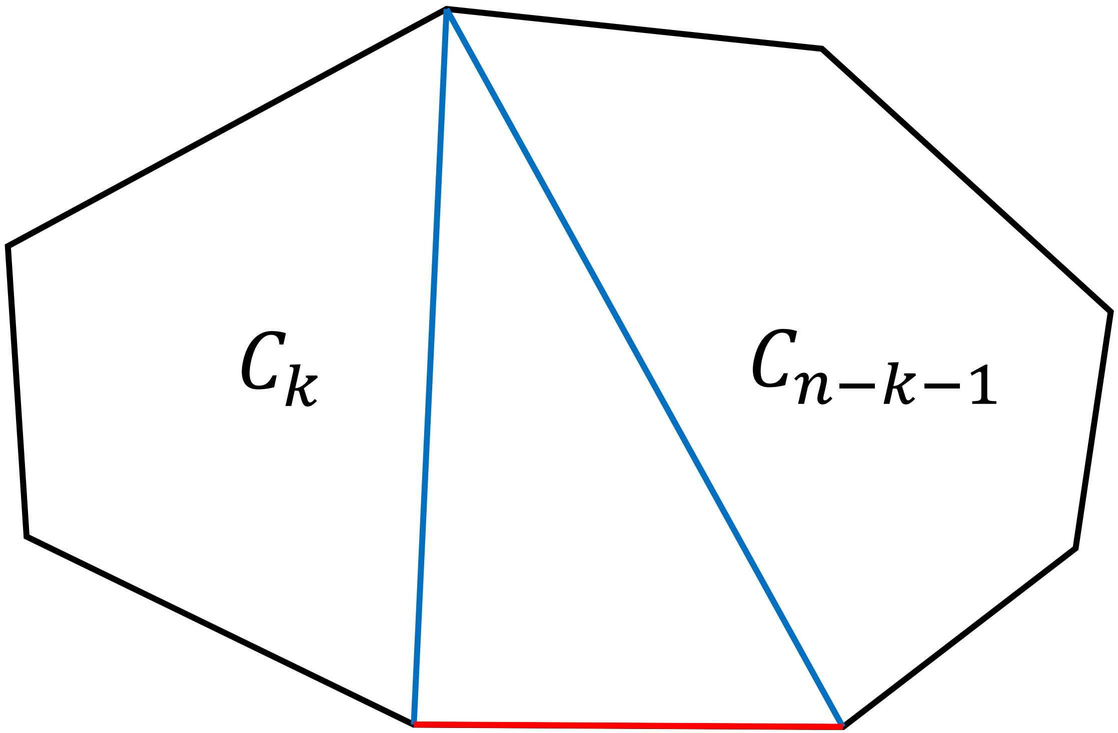


$$C_0 = 1,\ C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$C_0 = 1,$$
$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = \sum_{n \geq 0} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$\left(G(x)\right)^2 = \sum_{n \geq 0} \sum_{k=0}^{n} C_k C_{n-k} x^n$$

$$x\left(G(x)\right)^2 = \sum_{n \geq 0} \sum_{k=0}^{n} C_k C_{n-k} x^{n+1}$$

$$= \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-k-1} x^n$$

$$C_0 = 1,$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = \sum_{n\geq 0} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$x\big(G(x)\big)^2 = \sum_{n\geq 1} \sum_{k=0}^{n-1} C_k C_{n-k-1} x^n$$

$$G(x) = \sum_{n\geq 0} C_n x^n = C_0 + \sum_{n\geq 1} \sum_{k=0}^{n-1} C_k C_{n-k-1} x^n$$

$$= 1 + x\big(G(x)\big)^2$$

$$C_0 = 1,$$
$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = 1 + x\big(G(x)\big)^2 \qquad \Longrightarrow \qquad G(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

$$\lim_{x\to 0} G(0) = C_0 = 1 \qquad \Longrightarrow \qquad G(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

$$\sqrt{1-4x} = \sum_{n\geq 0}^{\infty} \binom{1/2}{n} (-4x)^n = 1 + \sum_{n\geq 1}^{\infty} \binom{1/2}{n} (-4x)^n$$

$$= 1 + \sum_{n\geq 0}^{\infty} \binom{1/2}{n+1} (-4x)^{n+1} = 1 - 4x \sum_{n\geq 0}^{\infty} \binom{1/2}{n+1} (-4x)^n$$

$$C_0 = 1,$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

$$\sqrt{1-4x} = 1 - 4x \sum_{n\geq 0}^{\infty} \binom{1/2}{n+1} (-4x)^n$$

$$G(x) = \frac{4x \sum_{n\geq 0}^{\infty} \binom{1/2}{n+1} (-4x)^n}{2x} = 2 \sum_{n\geq 0}^{\infty} \binom{1/2}{n+1} (-4x)^n$$

$$C_0 = 1,$$
$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = 2\sum_{n\geq 0}^{\infty} \binom{1/2}{n+1} (-4x)^n$$

$$C_n = 2\binom{1/2}{n+1}(-4)^n = 2\,\frac{(\frac{1}{2})(\frac{1}{2}-1)\cdots(\frac{1}{2}-n)}{(n+1)!}(-4)^n$$
$$= \frac{2^n}{(n+1)!}\prod_{k=1}^{n}(2k-1) \quad = \frac{2^n}{(n+1)!}\prod_{k=1}^{n}\frac{(2k-1)2k}{2k}$$
$$= \frac{1}{n!\,(n+1)!}\prod_{k=1}^{n}(2k-1)2k$$
$$= \frac{(2n)!}{n!\,(n+1)!} \quad = \boldsymbol{\frac{1}{n+1}\binom{2n}{n}}$$

- Number of Dyck words of length $2n$:

A string consisting of n X's and n Y's such that no initial segment of the string has more Y's than X's.

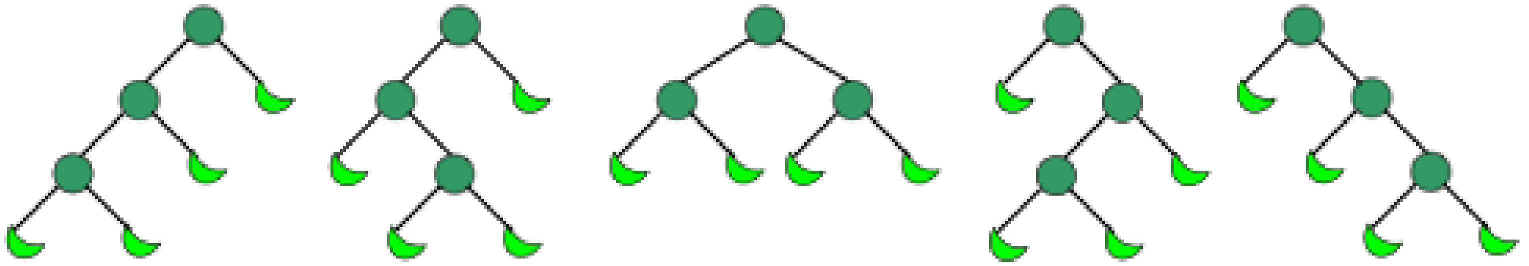
- For example, the following are the Dyck words of length 6:

$$XXXYYY \quad XYXXYY \quad XYXYXY$$
$$XXYYXY \quad XXYXYY$$

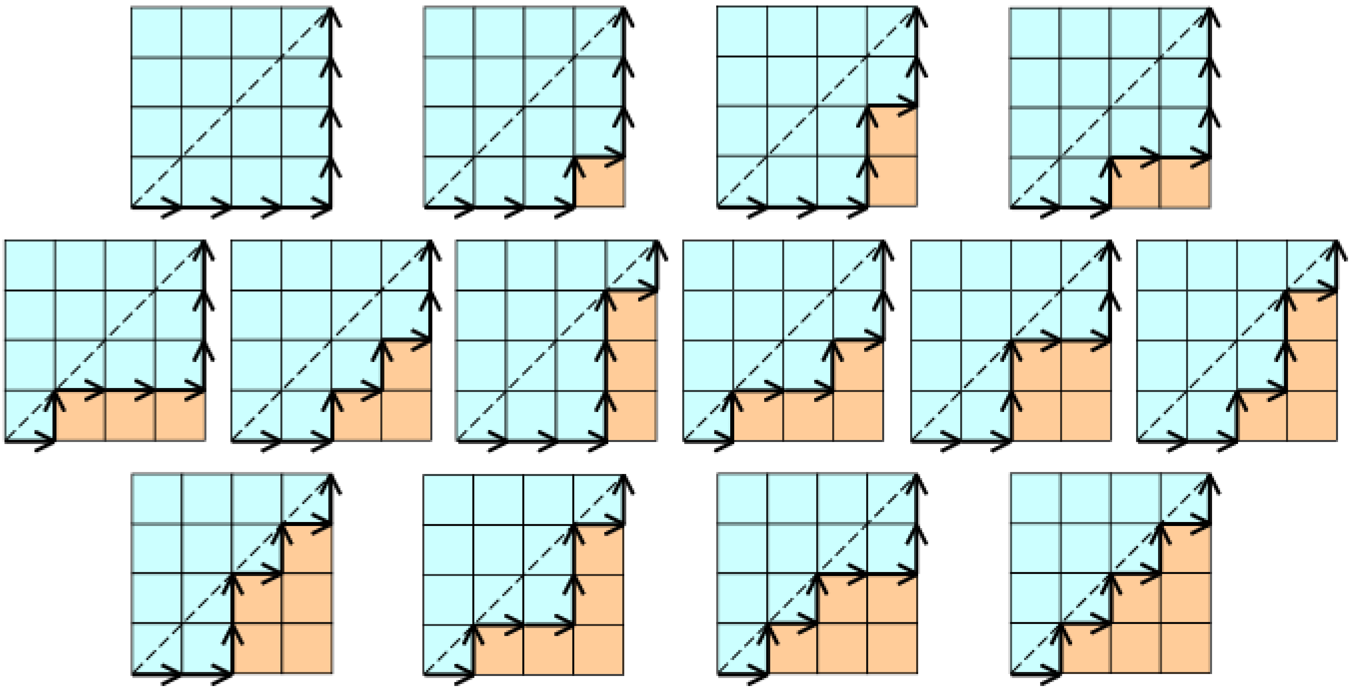
- Number of expressions containing n pairs of parentheses which are correctly matched.

((()))   ()(())   ()()()   (())()   (()())

- Number of full binary trees with $n + 1$ leaves



- Number of monotonic lattice paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal.



Catalan Number

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