Homework 3

Problem 1. *Prove the formula*

1. $\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$
2. $\sum_{k=0}^{n} \binom{m+k-1}{k} = \binom{n+m}{n}$

Solution.

1. Use the equivalence $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$ iteratively.
2. Note that $\binom{m-1}{0} = \binom{m}{0} = 1$. The rest is just like above.

□

Problem 2. *For natural numbers $m \le n$ calculate (i.e. express by a simple formula not containing a sum) $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k}$.*

Solution. $\binom{k}{m}\binom{n}{k} = \frac{k!}{m!(k-m)!} \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{m!(n-m)!} \cdot \frac{(n-m)!}{(k-m)!(n-k)!} = \binom{n}{m}\binom{n-m}{n-k}$.

Thus $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k} = \sum_{k=m}^{n} \binom{n}{m}\binom{n-m}{n-k} = \binom{n}{m} \sum_{k=m}^{n} \binom{n-m}{n-k} = \binom{n}{m}2^{n-m}$. □

Problem 3. *Calculate (i.e. express by a simple formula not containing a sum)*

1. $\sum_{k=1}^{n} \binom{k}{m}\frac{1}{k}$
2. $\sum_{k=0}^{n} \binom{k}{m}k$

Solution.

1. It can be verified that $\frac{1}{k}\binom{k}{m} = \frac{1}{m}\binom{k-1}{m-1}$.

   Thus $\sum_{k=1}^{n} \binom{k}{m}\frac{1}{k} = \frac{1}{m} \sum_{k=1}^{n} \binom{k-1}{m-1} = \frac{1}{m}\binom{n}{m}$.

2. It can be verified that $k\binom{k}{m} = (k+1)\binom{k}{m} - \binom{k}{m} = (m+1)\binom{k+1}{m+1} - \binom{k}{m}$.

   Thus $\sum_{k=0}^{n} \binom{k}{m}k = \sum_{k=0}^{n} \left((m+1)\binom{k+1}{m+1} - \binom{k}{m}\right) = (m+1) \sum_{k=0}^{n} \binom{k+1}{m+1} - \sum_{k=0}^{n} \binom{k}{m}$

   $= (m+1)\binom{n+2}{m+2} - \binom{n+1}{m+1} = \cdots$

$\square$

Problem 4. *(a) Using* **Problem 1.** *for $r = 2$, calculate the sum $\sum_{i=2}^{n} i(i-1)$ and $\sum_{i=1}^{n} i^2$.*

(b) Use (a) and **Problem 1.** *for $r = 3$, calculate $\sum_{i=1}^{n} i^3$.*

Solution.

1. $$r = 2: \qquad \binom{2}{2} + \binom{3}{2} + \cdots + \binom{i}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}$$

   Thus $\frac{\sum_{i=2}^{n} i(i-1)}{2!} = \binom{n+1}{3} \therefore \sum_{i=2}^{n} i(i-1) = 2\binom{n+1}{3}$

   $$r = 1: \qquad \binom{1}{1} + \binom{2}{1} + \cdots + \binom{i}{1} + \cdots + \binom{n}{1} = \binom{n+1}{2}$$

   Thus $\therefore \sum_{i=1}^{n} i = \binom{n+1}{2}$.

   Finally, $\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} (i(i-1) + i) = \sum_{i=1}^{n} i(i-1) + \sum_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6}$.

2. $$r = 3: \qquad \binom{3}{3} + \binom{4}{3} + \cdots + \binom{i}{3} + \cdots + \binom{n}{3} = \binom{n+1}{4}$$

   Thus $\frac{\sum_{i=3}^{n} i(i-1)(i-2)}{3!} = \binom{n+1}{4}$. $\therefore \sum_{i=3}^{n} i^3 - 3i^2 + 2i = 6\binom{n+1}{4}$,

   ...

   The final result is $\binom{n+1}{2}^2$.

$\square$

Problem 5. *How many functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ are there that are* monotone*; that is, for $i < j$ we have $f(i) \leq f(j)$?*

Solution.

Set $k_i = f(i+1) - f(i)$, $i = 0, 1, \ldots, n$, where we add $f(0) = 1$ and $f(n+1) = n$. Then the desired number is the number of nonnegative integer solutions to the equation $k_0 + k_1 + \cdots k_n = n - 1$.

Thus the final solution will be $\binom{2n-1}{n}$.

$\square$