# Homework 5

**Problem 1.** *1. Determine the coefficient of* $x^{50}$ *in* $(x^7 + x^8 + x^9 + x^{10} + \cdots)^6$

*2. Determine the coefficient of* $x^3$ *in* $(2+x)^{\frac{3}{2}}/(1-x)$

*3. Determine the coefficient of* $x^4$ *in* $(2+3x)^5\sqrt{1-x}$

*Solution.*

1.
$$\begin{aligned} &= x^{42}(1 + x + x^2 + x^3 + \cdots)^6 \\ &= x^{42} \cdot \tfrac{1}{(1-x)^6} \end{aligned}$$

The coefficient of $x^{50}$ is $\binom{8+6-1}{6-1} = \binom{13}{5}$.

2.
$$\begin{aligned} &= (x+2)^{3/2}(1 + x + x^2 + \cdots) \\ &= \textstyle\sum_{k=0}^{\infty} \binom{3/2}{k} x^k (2)^{3/2-k}(1 + x + x^2 + \cdots) \end{aligned}$$

The coefficient of $x^3$ is $\sum_{k=0}^{3} \binom{3/2}{k}(2)^{3/2-k}$. Then use the Newton formula ....

3.
$$= \textstyle\sum_{k=0}^{5} \binom{5}{k} 2^k (3x)^{5-k} \sum_{j=0}^{\infty} \binom{1/2}{j}(-x)^j$$

The coefficient of $x^4$ is $\sum_{k=1}^{5} \binom{5}{k} 2^k (3)^{5-k} \binom{1/2}{k-1}(-1)^{k-1}$

$\square$

**Problem 2.** *Find generating functions for the following sequences (express them in a closed form, without infinite series!):*

*1.* $0, 0, 0, 0, -6, 6, -6, 6, -6, \cdots$.

*2.* $1, 0, 1, 0, 1, 0, \cdots$ .

*3.* $1, 2, 1, 4, 1, 8 \cdots$

*Solution.*

$\square$

| Sequence | Generating Function |
|---|---|
| $(1, 1, 1, 1, \ldots)$ | $\frac{1}{1-x}$ |
| $(1, -1, 1, -1, \ldots)$ | $\frac{1}{1+x}$ |
| $(-6, 6, -6, 6, \ldots)$ | $\frac{-6}{1+x}$ |
| $(0, 0, 0, 0, -6, 6, -6, 6, \ldots)$ | $\frac{-6x^4}{1+x}$ |
| $(1, 0, 1, 0, \ldots)$ | $\frac{\frac{1}{1-x}+\frac{1}{1+x}}{2} = \frac{1}{1-x^2}$ |
| $(0, 1, 0, 1, \ldots)$ | $\frac{\frac{1}{1-x}-\frac{1}{1+x}}{2} = \frac{x}{1-x^2}$ |
| $(1, 2, 4, 8, \ldots)$ | $\frac{1}{1-2x}$ |
| $(2, 4, 8, \ldots)$ | $\frac{\frac{1}{1-2x}-1}{x} = \frac{2}{1-2x}$ |
| $(1, 0, 2, 0, 4, 0, 8, \ldots)$ | $\frac{1}{1-2x^2}$ |
| $(1, 1, 2, 1, 4, 1, 8, \ldots)$ | $\frac{1}{1-2x^2} + \frac{x}{1-x^2}$ |
| $(1, 2, 1, 4, 1, 8, \ldots)$ | $\frac{\frac{1}{1-2x^2}+\frac{x}{1-x^2}-1}{x} = -\frac{2x^3+2x^2-2x-1}{(1-2x^2)(1-x^2)}$ |

**Problem 3.** *Let $a_n$ be the number of ordered triples $\langle i, j, k\rangle$ of integer numbers such that $i \geq 0, j \geq 1, k \geq 1$, and $i + 3j + 3k = n$. Find the generating function of the sequence $(a_0, a_1, a_2, \ldots)$ and calculate a formula for $a_n$.*

*Solution.*

$(1 + x + x^2 + x^3 + \cdots)(x^3 + x^6 + x^9 + \cdots)(x3 + x^6 + x^9 + \cdots)$
$= \frac{1}{1-x}\frac{x^3}{1-x^3}\frac{x^3}{1-x^3}$
$= \frac{x^6(1+x+x^2)}{(1-x^3)^3} = x^6(1 + x + x^2)(1 - x^3)^{-3}$.
Then use the generalized binomial theorem. □

**Problem 4.** *If $a(x)$ is the generating function of a sequence $(a_0, a_1, a_2, \ldots)$, please find the generating function of the sequence of partial sums $(a_0, a_0 + a_1, a_0 + a_1 +$*

$a_2, \ldots)$.

*Solution*.

$$\begin{aligned}
& a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 \cdots \\
= \;& a_0(1 + x + x^2 + \cdots) \\
& + a_1 x(1 + x + x^2 + \cdots) \\
& + a_2 x^2(1 + x + x^2 + \cdots) \\
= \;& \tfrac{a_0}{1-x} + \tfrac{a_1 x}{1-x} + \tfrac{a_2 x^2}{1-x} + \cdots \\
= \;& \tfrac{1}{1-x} a(x)
\end{aligned}$$

□