Graph: Isomorphism and Score

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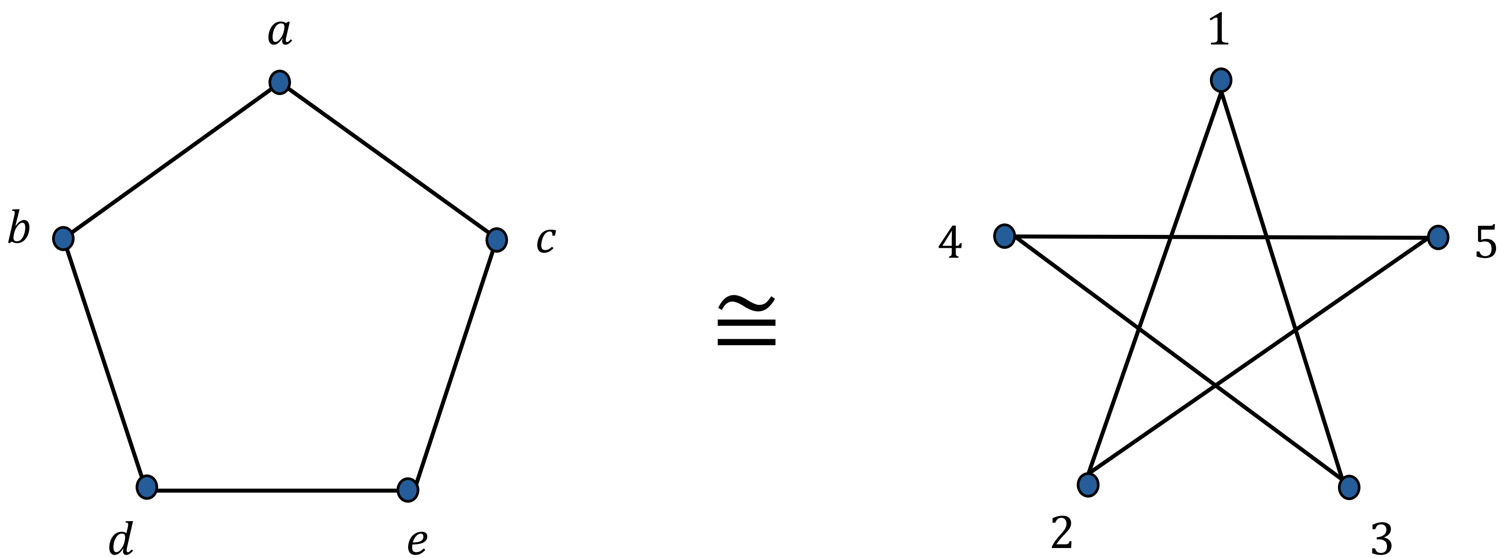
图同构

- **图同构(*Graph isomorphism*)：** 若对图$G=(V,E)$ 以及图$G'=(V',E')$ 存在双射函数$f\colon V\to V'$，满足对任意$x,y\in V$ 都有 $\{x,y\}\in E$ 当且仅当 $\{f(x),f(y)\}\in E'$ 那么我们称图G和图G'是同构的。
- 用符号图$\boldsymbol{G\cong G'}$表示图同构。
- 直观：同构的图之间，仅仅是顶点的名字不同。

图同构的例子



$$f\colon\ a \mapsto 1, b \mapsto 2, \mathrm{c} \mapsto 3, d \mapsto 5, e \mapsto 4$$

History

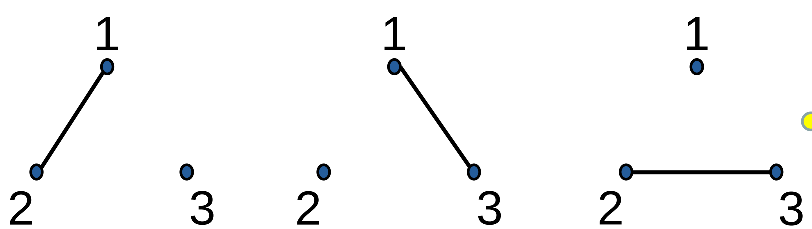
- In November 2015, László Babai, a mathematician and computer scientist at the University of Chicago, claimed to have proven that the graph isomorphism problem is solvable in quasi-polynomial time. This work has not yet been vetted. In January 2017, Babai shortly retracted the quasi-polynomiality claim and stated a sub-exponential time complexity bound instead. He restored the original claim five days later.

- ✓ In 1988, Babai won the Hungarian State Prize, in 1990 he was elected as a corresponding member of the Hungarian Academy of Sciences, and in 1994 he became a full member. In 1999 the Budapest University of Technology and Economics awarded him an honorary doctorate.
- ✓ In 1993, Babai was awarded the Gödel Prize together with Shafi Goldwasser, Silvio Micali, Shlomo Moran, and Charles Rackoff, for their papers on interactive proof systems.[17]
- ✓ In 2015, he was elected[18] a fellow of the American Academy of Arts and Sciences, and won the Knuth Prize.

- Interestingly, in July 2016, Wenxue Du, a Chinese mathematician at the Anhui University, devised an algorithm outputting a generating set and a block family of the automorphism group of a graph within time $n^{C\log n}$ for some constant C.

图的计数

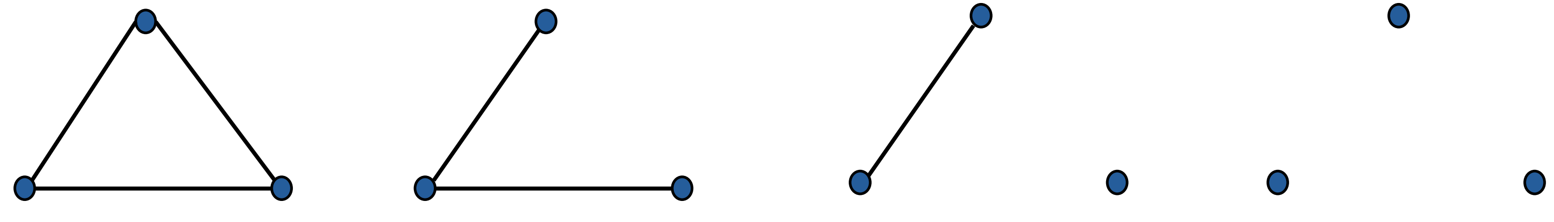
- **问题：** 以集合$V=\{1,2,\dots,n\}$中的元素为顶点构造图，$G=(V,E)$其中$E\subseteq\binom{V}{2}$，求问能构成多少个图？
- **解：** $\left|\binom{V}{2}\right|=\binom{n}{2}$，为$K_n$的边数目。
  每条边有两种可能，故以V为顶点的图共有$2^{\binom{n}{2}}$种。



考虑结构上的相似性，有多少图是彼此不同的？

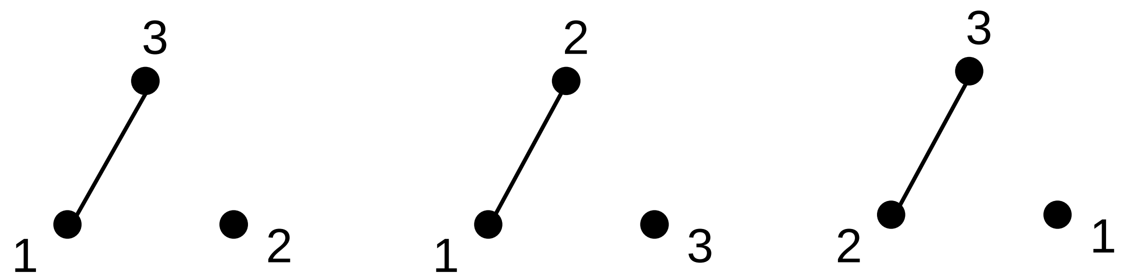
非同构图计数

- **问题：** 以集合$V=\{1,2,\ldots,n\}$中的元素为顶点构造图，$G=(V,E)$其中$E\subseteq\binom{V}{2}$，求问**彼此不同构**的图有多少个？
- 例：含三个顶点的彼此不同构的图只有以下4种：



$$4<2^{\binom{3}{2}}=8$$

- 显然，（同构）图的个数不会超过所有图的个数（是$2^{\binom{n}{2}}$）。
- 与此同时，任一$G=(V,E)$至多与$n!$个V上不同的图同构。
- 例： $3!=6$， 但与第一张图同构且互不相同的图只有三种。



- **解**：设n个顶点且不同构的图有x个，则：

$$\frac{2^{\binom{n}{2}}}{n!} \le x \le 2^{\binom{n}{2}}$$

- 我们可以对上下界估值：
  - $\log_2 2^{\binom{n}{2}} = \binom{n}{2} = \frac{n^2}{2}\left(1 - \frac{1}{n}\right)$
  - $\log_2 \frac{2^{\binom{n}{2}}}{n!} = \binom{n}{2} - \log_2 n!$
    $$\ge \binom{n}{2} - \log_2 n^n$$
    $$= \frac{n^2}{2}\left(1 - \frac{1}{n} - \frac{2\log_2 n}{n}\right)$$

$$x = 2^{\Theta(\frac{n^2}{2})}$$

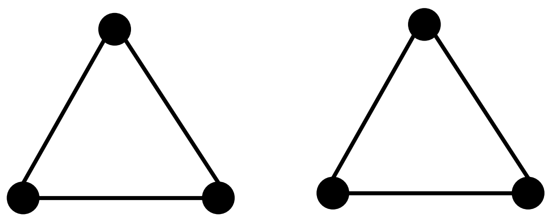
Graph Score

- Let G be a graph. The vertices of G be $v_1, v_2, \ldots, v_n$. The the degree sequence of G, or a score of G is:

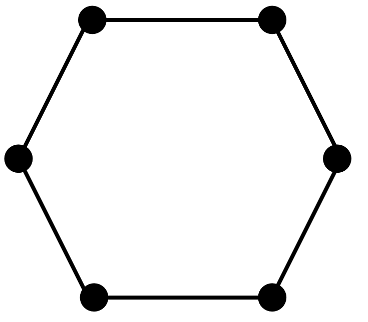
$$(\deg_G(v_1), \deg_G(v_2), \ldots, \deg_G(v_n))$$

- Two scores are equal to each other if one can be obtained form the other by rearranging the order of the numbers.

- Isomorphic graphs $\Longrightarrow$ The same scores.
- The same scores $\not\Longrightarrow$ Isomorphic graphs.



$(2,2,2,2,2,2)$



$(2,2,2,2,2,2)$

Not every finite sequence is a graph Score.

Score Theorem

Let $D = (d_1, d_2, \ldots, d_n)$ be a sequence of natural numbers, $n > 1$. Suppose that $d_1 \leq d_2 \leq \cdots \leq d_n$, and let the symbol D' denote the sequence $(d_1{}', d_2{}', \ldots, d_{n-1}{}')$, where

$$d_i' = \begin{cases} d_i & if \;\; i < n - d_n \\ d_i - 1 & if \; i \geq n - d_n \end{cases}$$

Then D is a graph score iff D' is a graph score.

Application

Thm: Let $D = (d_1, d_2, \dots, d_n)$ be a sequence of natural numbers, $n > 1$. Suppose that $d_1 \le d_2 \le \cdots \le d_n$, and let the symbol D' denote the sequence $({d_1}', {d_2}', \dots, {d_{n-1}}')$, where

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- $(1,1,1,2,2,3,4,5,5)$
- $(1,1,1,1,1,2,3,4)$
- $\cancel{(1,1,1,0,0,1,2)}$
- $(0,0,1,1,1,1,2)$
- $\cancel{(0,0,1,1,0,0)}$
- $(0,0,0,0,1,1)$
- $(0,0,0,0,0)$

Proof

Thm: Let $D = (d_1, d_2, \dots, d_n)$ be a sequence of natural numbers, $n > 1$. Suppose that $d_1 \leq d_2 \leq \cdots \leq d_n$, and let the symbol D' denote the sequence $(d_1{}', d_2{}', \dots, d_{n-1}{}')$, where

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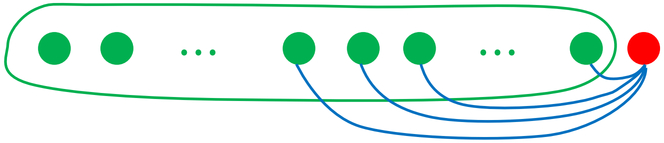
- (if)

$G' = (V', E')$, where

$V' = \{v_1, v_2, \dots, v_{n-1}\}$

New vertex v_n



$G = (V, E)$

$V = V' \cup \{v_n\}$

$E = E' \cup \{\{v_i, v_n\}: i = n - d_n, n - d_n + 1, \dots, n - 1\}.$

Thm: Let $D = (d_1, d_2, \dots, d_n)$ be a sequence of natural numbers, $n > 1$. Suppose that $d_1 \leq d_2 \leq \cdots \leq d_n$, and let the symbol D' denote the sequence $(d_1{}', d_2{}', \dots, d_{n-1}{}')$, where

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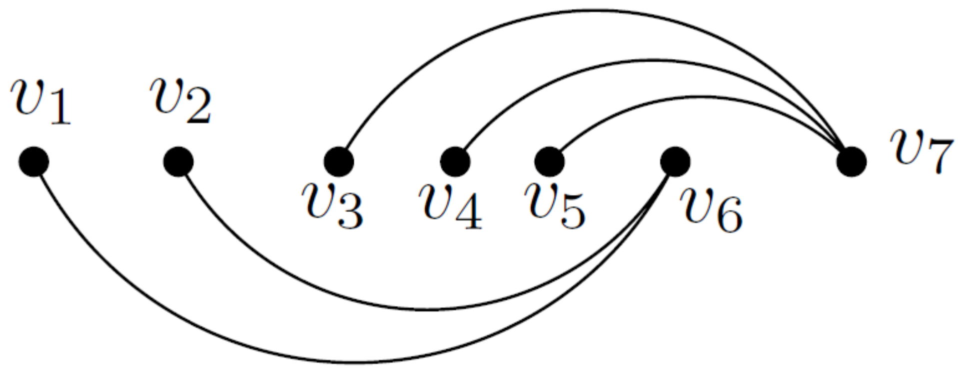

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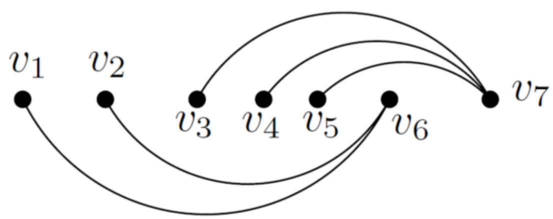

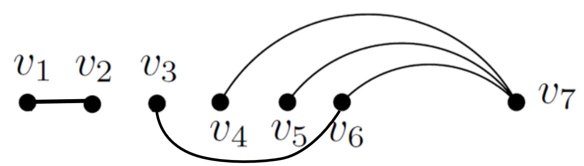

The set $\hat{G}$ of all graphs on the vertex set $\{v_1, \dots, v_n\}$ in which the degree of each vertex v_i equals d_i. $i = 1,2, \dots, n$. It will be *sufficient* to prove the following claim

Claim. The set $\hat{G}$ contains a graph G_0 in which the vertex v_n is adjacent *exactly* to the *last d_n vertices*, i.e. to vertices $v_{n-d_n}, v_{n-d_n+1}, \dots, v_{n-1}$.

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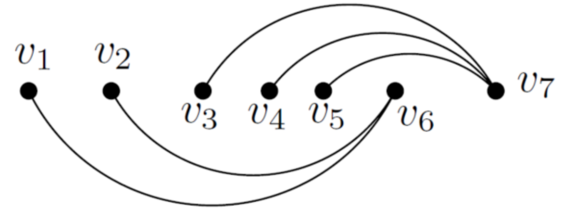
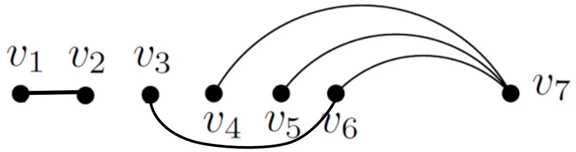
- If $d_n = n - 1$, then any graph from $\hat{G}$ satisfies the claim.
- O.W. $d_n < n - 1$: $\forall G \in \hat{G}$
  - $j(G) =$
    $Max\,\{\, j \in \{1,2,\dots,n-1\} \mid \{v_j, v_n\} \notin E(G)\}$





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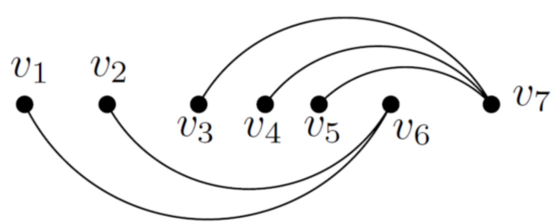

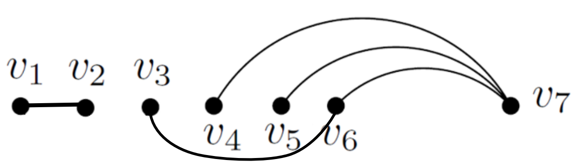

- Let G_0 be a graph in $\hat{G}$ with *smallest* possible value of $j(G)$.
- Prove: $j(G_0) = n - d_n - 1$

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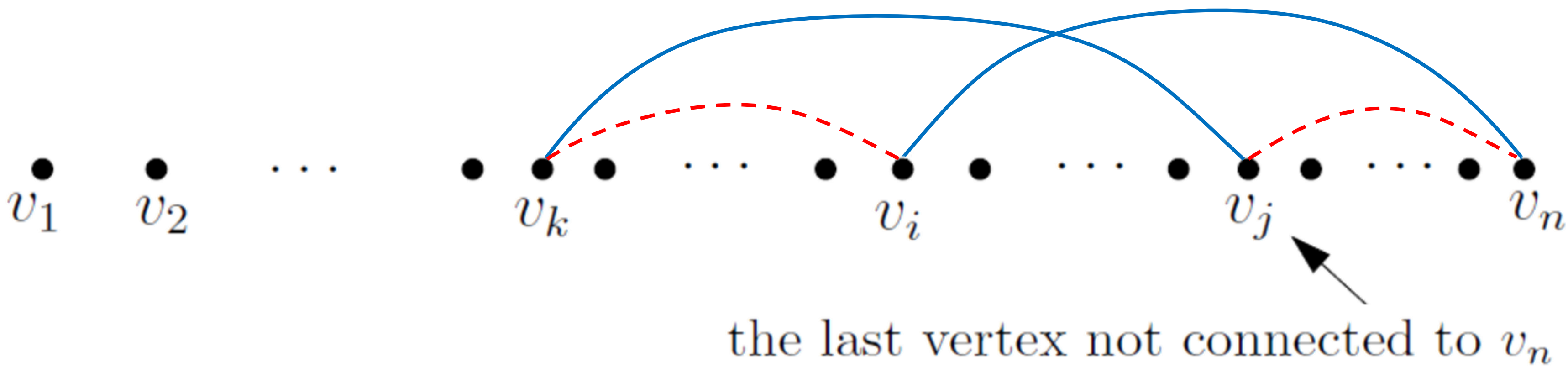
- (Proof by contradiction) Suppose

$$j = j(G_0) > n - d_n - 1$$



$G' = (V, E')$ where

$$E' = \left(E(G_0) \setminus \left\{\{v_i, v_n\}, \{v_j, v_k\}\right\}\right) \cup \left\{\{v_j, v_n\}, \{v_i, v_k\}\right\}$$

The score of G' and G_0 are the same. There is a contradiction as $J(G') \leq J(G_0) - 1$.