Recurrence relation

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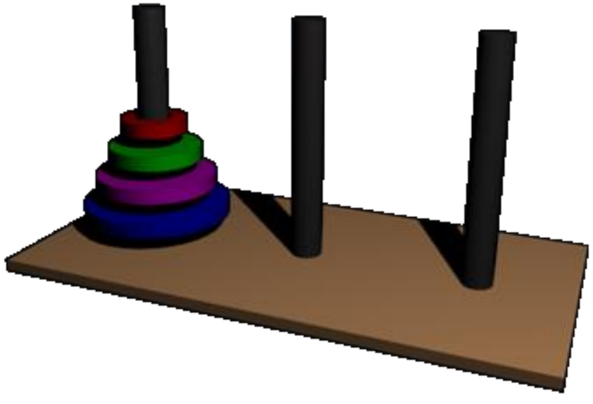
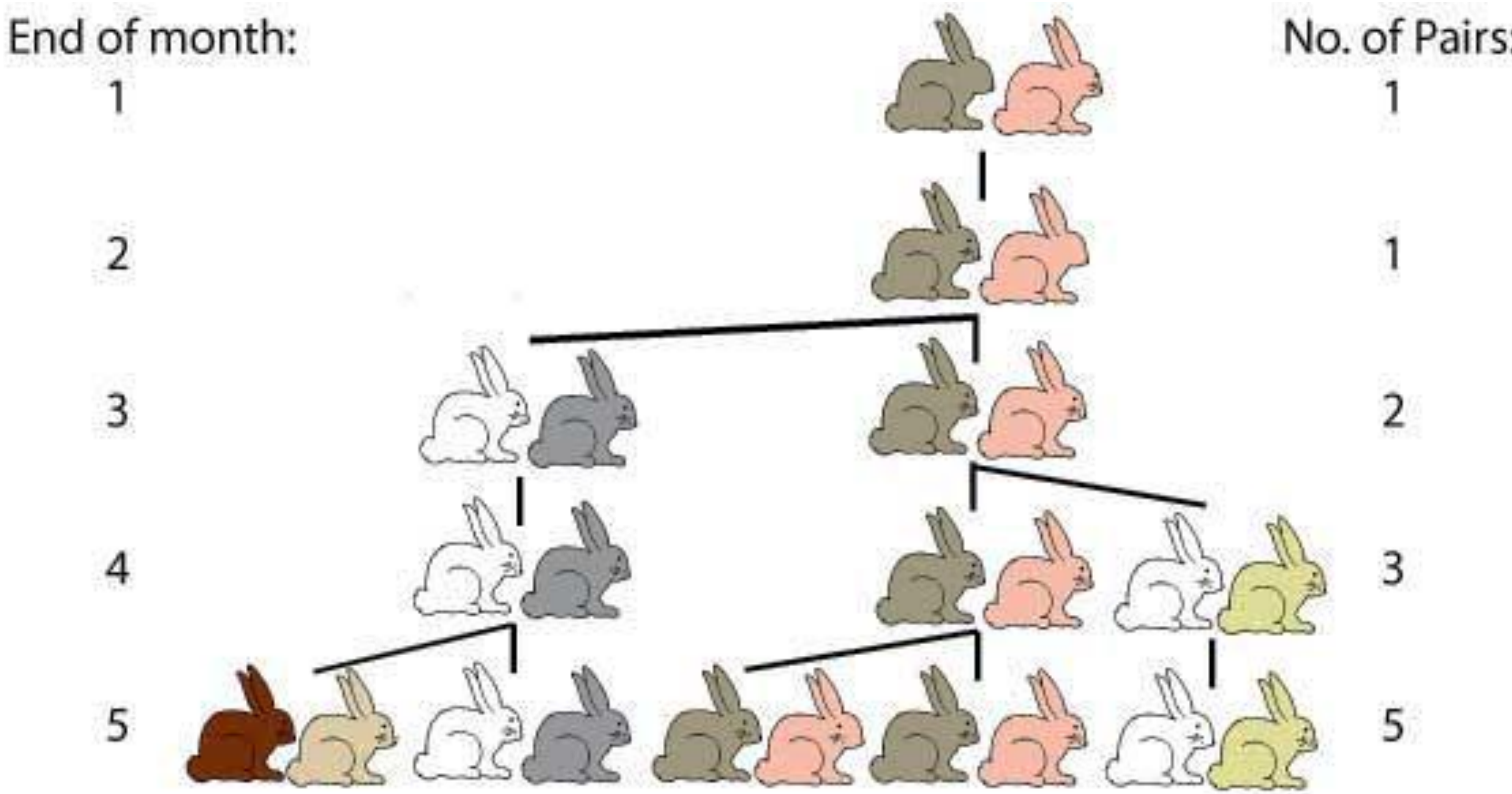
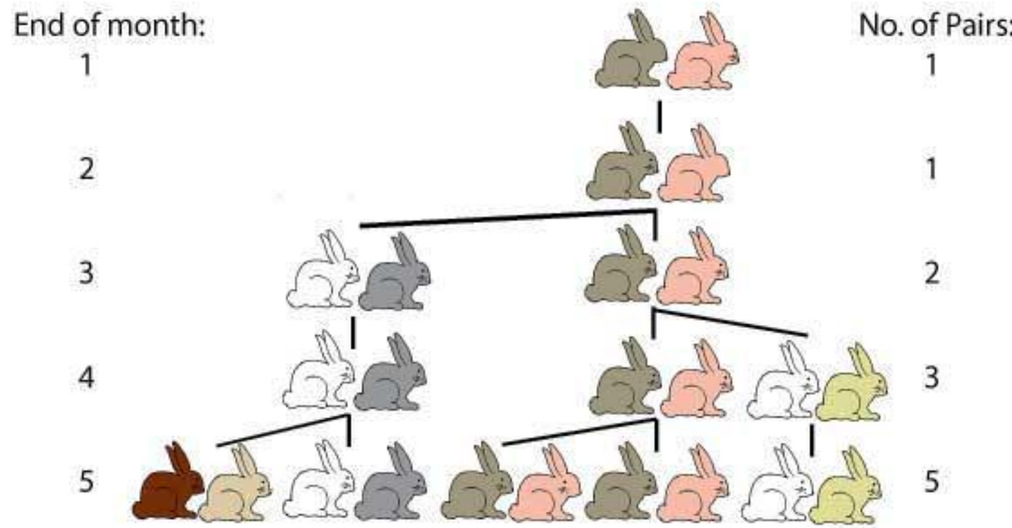


- Fibonacci and his rabbits [1202]





$f_0 = 0,\ \ f_1 = 1,\ \ f_2 = 1,\ \ f_3 = 2,\ \ f_4 = 3$

$f_{13} = ?$

$f_n = f_{n-1} + f_{n-2}$

Fibonacci Sequence

- $f_n = f_{n-1} + f_{n-2}$
- $F(x) = f_0 + f_1 x + \boxed{f_2 x^2} + \cdots + f_{n-2}x^{n-2} + f_{n-1}x^{n-1} + \boxed{f_n x^n} + \cdots$
- $xF(x) = \quad f_0 x + \boxed{f_1 x^2} + f_2 x^3 + \cdots\cdots\cdots + f_{n-2}x^{n-1} + \boxed{f_{n-1}x^n} + \cdots$
- $x^2F(x) = \quad \boxed{f_0\, x^2} + f_1 x^3 + f_2 x^4 + \cdots\cdots\cdots\cdots\cdots + \boxed{f_{n-2}x^n} + f_{n-1}x^{n+1} + \cdots$

生成函数，等式

$$F(x) - xF(x) - x^2F(x) = f_0 + (f_1 - f_0)x$$

会被消掉

$$F(x) = \frac{x}{1 - x - x^2}$$

简化

分解

$$= \frac{a}{1-\lambda_1 x} + \frac{b}{1-\lambda_2 x}$$

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

k阶齐次线性递归表达式 会考 重要

Homogeneous linear recurrence of k^{th} degree with constant coefficients

常数 为系数

系数

$$h_n = a_{k-1}h_{n-1} + a_{k-2}h_{n-2} + \cdots + a_1 h_{n-k-1} + a_0 h_{n-k}$$

Characteristic polynomial of the above recurrence

特征式

找h(n)解析解

一元k次

$$p(x) = x^k - a_{k-1}x^{k-1} - a_{k-2}x^{k-2} - \cdots - a_1 x - a_0 = 0$$

$$x^k - a_{k-1}x^{k-1} - a_{k-2}x^{k-2} - \cdots - a_1 x - a_0 = 0 \quad (☆)$$

特征方程

令

$$\Rightarrow (x - \lambda_1)(x - \lambda_2) \ldots (x - \lambda_k) = 0$$

k个解

$$x^k - a_{k-1}x^{k-1} - a_{k-2}x^{k-2} - \cdots - a_1 x - a_0 = 0 \quad (☆)$$

$$(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_k) = 0$$

- If $\lambda_i \neq \lambda_j$ whenever $i \neq j$

  c：常数

  Then $h_n = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_k\lambda_k^n$ ？

- If $p(x) = (x-\lambda_1)^{s_1}(x-\lambda_2)^{s_2}\cdots\left(x-\lambda_q\right)^{s_q}$

  Then $h_n = (c_{11}+c_{12}n + \cdots + c_{1s_1}n^{s_1-1})\lambda_1^n$

  构造k维线性空间

  $+(c_{21}+c_{22}n + \cdots + c_{2s_2}n^{s_2-1})\lambda_2^n$

  求证明与书上

  表达式等价

  $\vdots$

  $+(c_{q1}+c_{q2}n + \cdots + c_{qs_q}n^{s_q-1})\lambda_q^n$

  各不相同

Example

- $f_n = f_{n-1} + f_{n-2} \quad (n \geq 2)$

  $f_0 = 0, f_1 = 1, f_2 = 1$
- $x^2 - x - 1 = 0 \quad (☆)$ 特征方程
- $x = \frac{1 \pm \sqrt{5}}{2}$ 求解 (排重根)
- $f_n = a\left(\underbrace{\frac{1+\sqrt{5}}{2}}_{\lambda_1}\right)^n + b\left(\underbrace{\frac{1-\sqrt{5}}{2}}_{\lambda_2}\right)^n$ $\quad c_1\lambda_1^n + c_2\lambda_2^n$
- $f_0 = 0 \Rightarrow a + b = 0 \Rightarrow a = -b$

$$f_1 = 1 \Rightarrow a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right) = 1 \Rightarrow a = \frac{1}{\sqrt{5}}$$

- $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Example

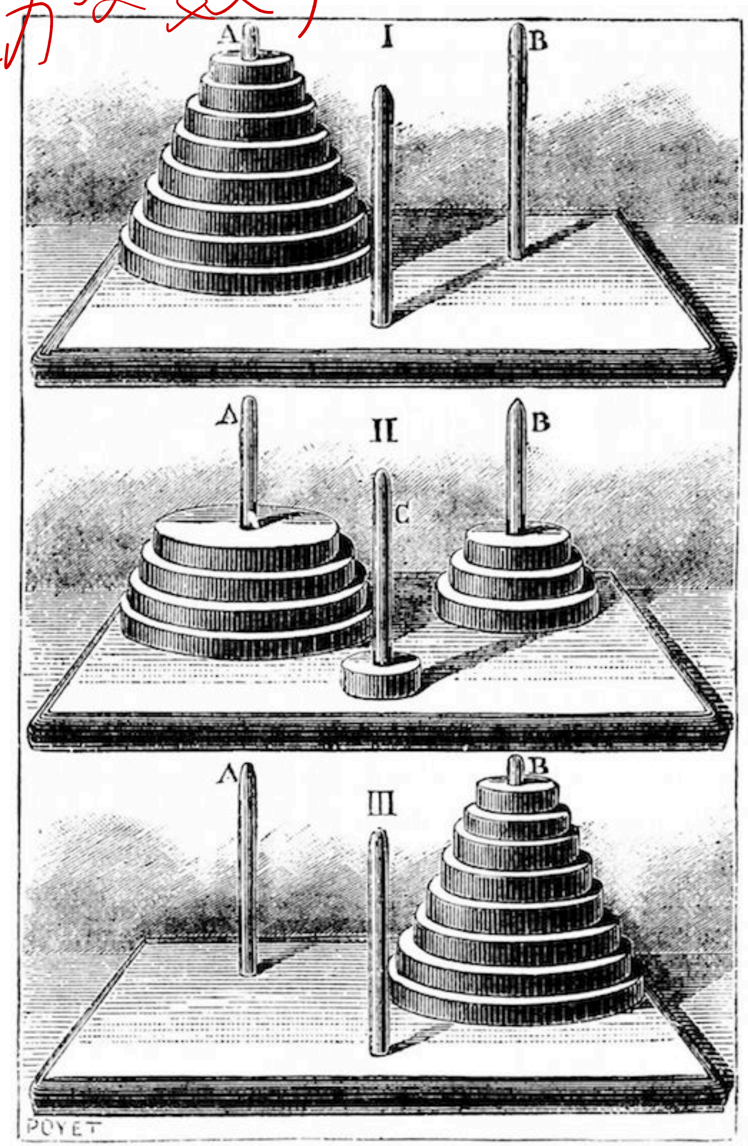
- $h_n = -h_{n-1} + 3h_{n-2} + 5h_{n-3} + 2h_{n-4} \quad (n \geq 4)$
  $h_0 = 1, h_1 = 0, h_2 = 1, h_3 = 2$
- $x^4 + x^3 - 3x^2 - 5x - 2 = 0 \quad (☆)$ 特征方程
- $x = \underbrace{-1, \overset{\text{三重}}{-1}, -1}, 2$ 求解
- $h_n = c_1(-1)^n + c_2 n(-1)^n + c_3 n^2(-1)^n + c_4 2^n$
- $$\begin{aligned} (n=0) \quad & c_1 + c_4 = 1 \\ (n=1) \quad & -c_1 - c_2 - c_3 + 2c_4 = 0 \\ (n=2) \quad & c_1 + 2c_2 + 4c_3 + 4c_4 = 1 \\ (n=3) \quad & -c_1 - 3c_2 - 9c_3 + 8c_4 = 2 \end{aligned}$$ 代 $h_0 \sim h_3$
- $c_1 = \frac{7}{9},\ c_2 = -\frac{3}{9},\ c_3 = 0,\ c_4 = \frac{2}{9}$
- $h_n = \frac{7}{9}(-1)^n - \frac{3}{9}n(-1)^n + \frac{2}{9}2^n$

- Tower of Hanoi [Édouard Lucas,1883]

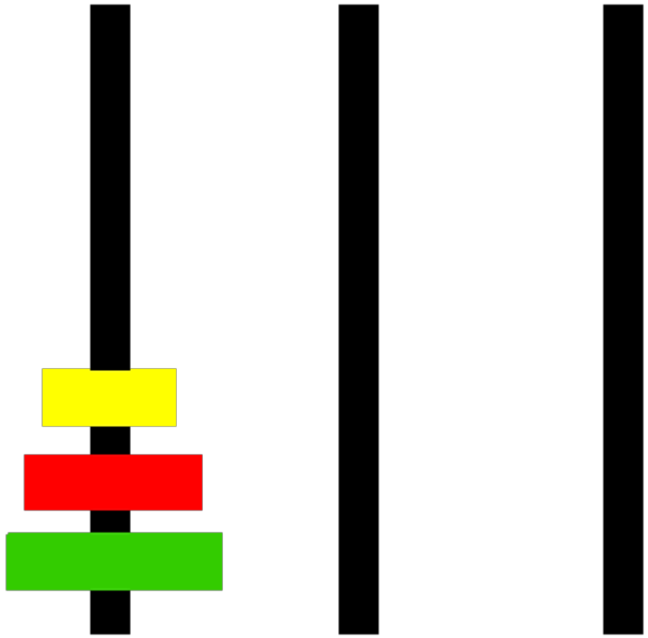
(最少移动次数)

Game rules:

1. Only one disk can be moved at a time.
2. Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack i.e. a disk can only be moved if it is the uppermost disk on a stack.
3. No disk may be placed on top of a smaller disk.

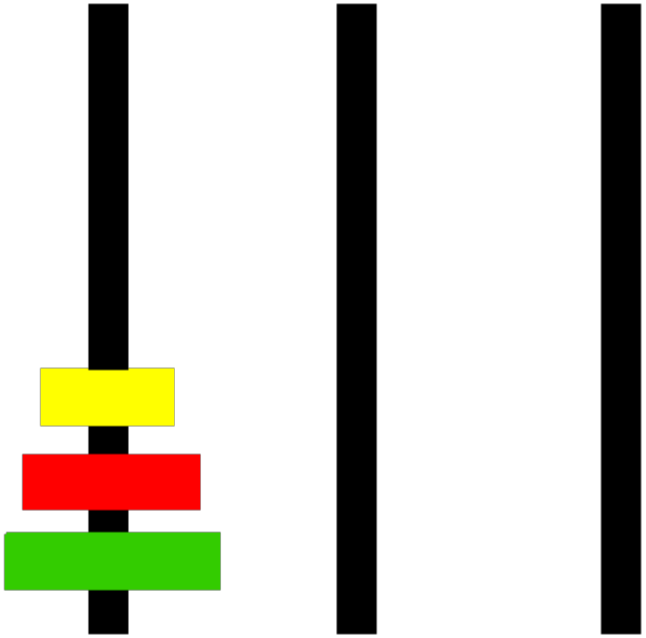


- Tower of Hanoi



$n = 3$

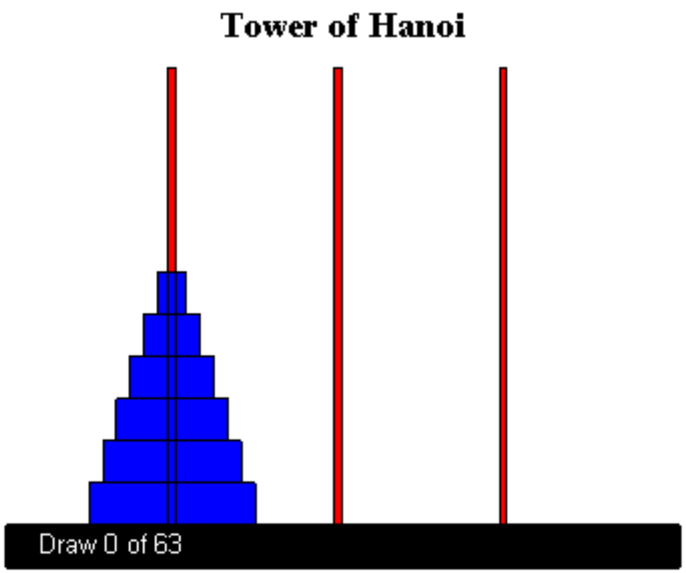
- Tower of Hanoi



$n = 3$

$h_n = 7$

- Tower of Hanoi



$n = 5$

$h_n = 63$

$$h_0 = 0$$
$$h_1 = 1$$
$$h_2 = 3$$
$$\vdots$$
$$h_n = 2h_{n-1} + 1$$

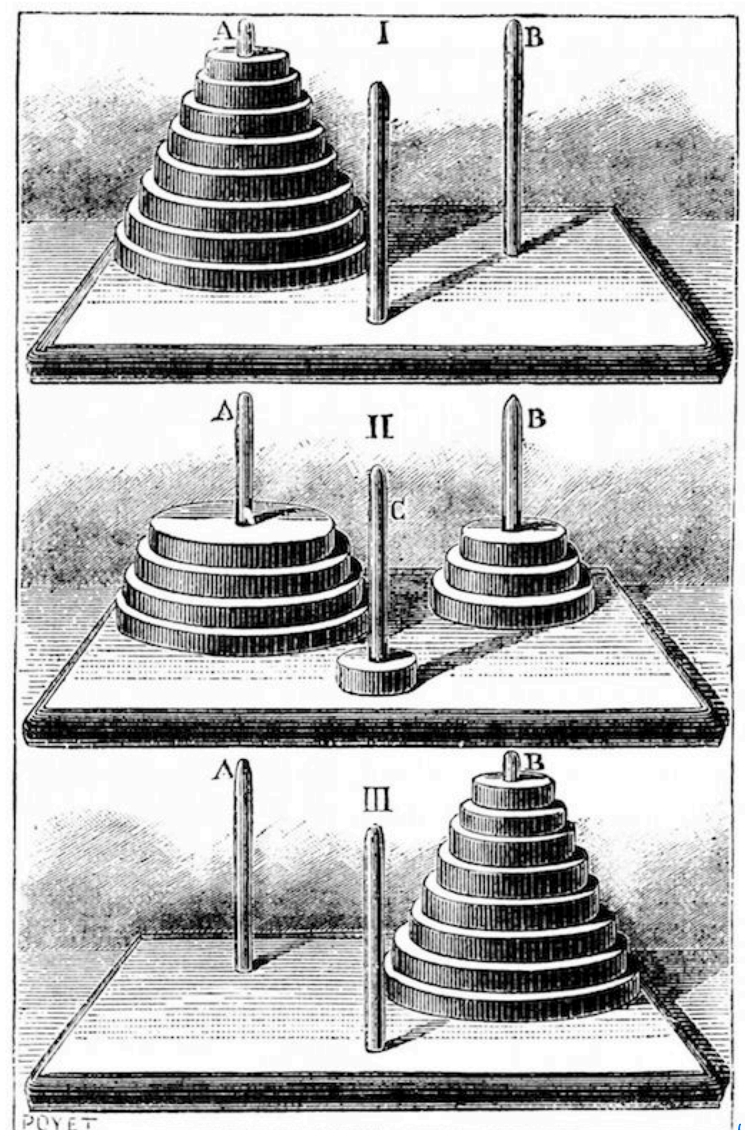
非齐次

$$\begin{cases} h_n \le 2h_{n-1} + 1 \\ h_n \ge 2h_{n-1} + 1 \ ? \end{cases}$$

两个方向都应该证

反过来说

必须先把最大的上面的都移走(并暂存



$$h_0 = 0$$
$$h_1 = 1$$
$$h_2 = 3$$
$$\vdots$$
$$h_n = 2h_{n-1} + 1$$
$$= 2(2h_{n-2}+1) + 1 = 2^2 h_{n-2} + 2 + 1$$
$$= 2^2(2h_{n-3}+1) + 2 + 1 = 2^3 h_{n-3} + 2^2 + 2 + 1$$
$$\vdots$$
$$= 2^{n-1}(h_0 + 1) + 2^{n-2} + \cdots + 2^2 + 2 + 1$$
$$= 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1$$
$$= 2^n - 1$$

非齐次　　　　　　　　　指数为1　　　　仍为线性

Non-homogeneous linear recurrence of k^{th} degree with constant coefficients　齐次找通解，非齐次找特解，加

$$h_n = a_{k-1}h_{n-1} + a_{k-2}h_{n-2} + \cdots + a_0 h_{n-k} + b_n$$ 起来

Every solution to nonhomogeneous equation is of the form:

Some specific solution + Solution to homogeneous.

一些经验建议

Some suggestions

- If b_n is of n's k −degree polynomial, then the specific solution is more likely to be n's k −degree polynomial as well.
  - If $b_n = c$ try $h_n = r$ 尝试常数特解
  - If $b_n = dn + c$ try $h_n = rn + s$ 尝试线性项
  - If $b_n = rn^2 + sn + t$ try $h_n = fn^2 + dn + c$
- If b_n is of n's exponential form, then the specific solution is more likely to be n's exponential form as well.
  - If $b_n = d^n$ try $h_n = pd^n$

根据非齐次项

猜特解形式(不一定对)

对整个关系找特解

Example

非齐次

利用非齐次部分猜

- $h_n = 3h_{n-1} - 4n \quad (n \geq 1)$ with $h_0 = 2$

特解

- Homogeneous part: $h_n = 3h_{n-1}, \quad x - 3 = 0 \ (☆)$
- $h_n = c3^n \quad (n \geq 1)$
- Find one specific solution for $h_n = 3h_{n-1} - 4n \quad (n \geq 1)$
  - Try $h_n = rn + s$ 猜（原始关系特解） 代 $n=0$
  - $rn + s = 3(r(n-1) + s) - 4n$
  - $rn + s = (3r - 4)n + (-3r + 3s)$

    $\Rightarrow r = 2, s = 3 \Rightarrow h_n = 2n + 3$ √
- $h_n = c3^n + 2n + 3$ 齐次通解 + 特解
- $(n = 0) \quad 2 = c \times 3^0 + 2 \times 0 + 3 \ \Rightarrow \ c = -1$ 代 $n=0$ 求 c
- $h_n = -3^n + 2n + 3 \qquad (n \geq 0)$

$h_{n-1} = -3^{n-1} + 2n+$

Example

- $h_n = 3h_{n-1} + 3^n \quad (n \geq 1)$ with $h_0 = 2$
- Homogeneous part: $h_n' = c3^n$ (原方程的一部分，加一撇来区分)
- Find one specific solution for $h_n = 3h_{n-1} + 3^n \quad (n \geq 1)$
  - Try $h_n = p3^n$
  - $p3^n = 3p3^{n-1} + 3^n \Rightarrow p = p + 1 \Rightarrow$ Impossible!
  - Try $h_n = pn3^n$ (经验总)
  - $pn3^n = 3p(n-1)3^{n-1} + 3^n \Rightarrow p = 1 \Rightarrow h_n = n3^n$ √
- $h_n = c3^n + n3^n$
- $(n = 0) \quad c(3^0) + 0(3^0) = 2 \quad \Rightarrow \quad c = 2$

- $h_n = 2 \times 3^n + n3^n = (2+n)3^n \qquad (n \geq 0)$

(一种分析方法)

Recall

- Master theorem (analysis of algorithms)

$h_n = h_{n-1} \cdot h_{n-2}$ 如何变线性?

取对数

$$\ln(h_n) = \ln(h_{n-1}) + \ln(h_{n-2})$$