Homework 8

Problem 1. *Which of the following statements about graph G and H are true?*

1. *G and H are isomorphic if and only if for every map $f : V(G) \to V(H)$ and for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.*

2. *G and H are isomorphic if and only if there exists a bijection $f : E(G) \to E(H)$.*

3. *If there exists a bijection $f : V(G) \to V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$, then G and H are isomorphic.*

4. *If G and H are isomorphic, then there exists a bijection $f : V(G) \to V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$.*

5. *If G and H are isomorphic, then there exists a bijection $f : E(G) \to E(H)$.*

6. *G and H are isomorphic if and only if there exists a map $f : V(G) \to V(H)$ such that for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.*

7. *Every graph on n vertices is isomorphic to some graph on the vertex set $\{1, 2, \ldots, n\}$.*

8. *Every graph on $n \geq 1$ vertices is isomorphic to infinitely many graphs.*

Solution. $\underline{4,5,7,8}$. □

Problem 2. *Two simple graphs $G = (V, E)$ and $G' = (V', E')$. A map $f : V \to V'$. Now if f satisfies:*

i) It is a bijective function;

ii) $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$;

Then we say that graph G and G' are isomorphic *to each other. We use $G \cong G'$ to stand for the isomorphism relation.*

Consider the following questions:

1. *$G = K_n$ (Recall: K_n is a clique with n vertices), $g : V \to V'$ is a function which only satisfies requirement ii). Prove that G' must contain a subgraph which is a clique with n-vertices.*

2. *$G = K_{n,m}$ (Recall: $K_{n,m}$ is the so-called* complete bipartite graphs*), g is the same as in question 1. What will be the simplest G' that is related to G under the new relation.*

Solution.

1. Two different vertices in G must be mapped to different vertices in G'. For if $u, v \in V$ are mapped to the same vertex $\omega \in V'$, the edge $\{u, v\} \in E$ cannot be reflected in E', for G' is a simple graph.

2. G' can be just one edge with two incident vertices.

□

Problem 3. *How many graphs on the vertex set $\{1, 2, \ldots, 2n\}$ are isomorphic to the graph consisting of n vertex-disjoint edges (i.e. with edge set {{1,2},{3,4},. . . , {2n-1,2n}}?*

Solution. $\frac{(2n\cdot(2n-1))((2n-2)\cdot(2n-3))\cdots(2\cdot 1)}{2^n \cdot n!} = (2n-1)(2n-3)\cdot\cdots 5\cdot 3.$

□

Problem 4. *Construct an example of a sequence of length n in which each term is some of the numbers* $1, 2, \ldots, n-1$ *and which has an even number of odd terms, and yet the sequence is not a graph score. Show why it is not a graph score.*

Solution. E.g. $(1, 1, 3, 3, 4)$. Use the *Score theorem* to prove that it cannot be a graph score. □

Problem 5. *Let G be a graph with 9 vertices, each of degree 5 or 6. Prove that it has at least 5 vertices of degree 6 or at least 6 vertices of degree 5.*

Solution. x be the number of vertex in G with $deg_G(x) = 6$. Obviously $x \geq 5$ or $x \leq 4$.

1. If $x \geq 5$ then the first part of the argument is true.

2. Otherwise ($x \leq 4$). As the other vertices in graph G are of degree 5, there are at least $9 - x \geq 5$ such vertices. According to the hand-shake lemma, there must be even number of odd-degree vertices. Thus there should be at least 6 vertices with degree 5.

$\square$

Problem 6. *Given a sequence* $(d_1, d_2, \ldots, d_n)$ *of positive integers (where* $n \geq 1$*):*

(i) There exists a tree with score $(d_1, d_2, \ldots, d_n)$*.*

(ii) $\sum_{i=1}^{n} d_i = 2n - 2$*.*

Prove that (i) and (ii) are equivalent.

Solution.

1. $(i) \Rightarrow (ii)$ is obvious.

2. To prove $(ii) \Rightarrow (i)$:

   By induction on the number n.

   For $n = 1, 2$ the implication holds trivially, so let $n > 2$. Suppose the implication holds for any $n-1$ long positive sequence $(d_1, d_2, \ldots, d_{n-1})$ with $\sum_{i=1}^{n-1} d_i = 2(n-1) - 2$.

   For the induction step, consider an length n positive sequence $\ell = (d_1, d_2, \ldots, d_n)$ with $\sum_{i=1}^{n} d_i = 2n - 2$:

   Since the sum of the d_i is smaller than $2n$, there exists an i with $d_i = 1$. w.l.o.g. we assume $d_1 = 1$. With a similar argument we can also conclude that there must exist some index j such that $d_j \geq 2$. We take $k = \min\{j \mid d_j \geq 2\}$.
   Now the sequence $\ell = (d_1, d_2, \ldots, d_k, \ldots, d_n)$= $(1, d_2, \ldots, d_k - 1 + 1, \ldots, d_n)$, we can derive a new sequence $\ell' = (d_2, \ldots, d_k - 1, \ldots, d_n)$. Obviously ℓ' is a $n-1$ length sequence (all positive) with the summation to be $2n - 2 - 1 - 1 = 2(n-1) - 2$. Then according to the induction hypothesis, there exists a tree $\mathcal{T}'$ which corresponds to ℓ'.
   Then $\mathcal{T} = (V(\mathcal{T}') \cup \{v_1\}, E(\mathcal{T}') \cup \{v_1, v_k\})$ is the tree which witnesses the validity of the sequence ℓ.

BE CAREFUL: Why is the following 'proof' of the implication (ii)⇒(i) insufficient (or, rather, makes no sense)? We proceed by induction on n. The base case $n = 1$ is easy to check, so let us assume that the implication holds for some $n \geq 1$. We want to prove it for $n + 1$.If $D = (d_1, d_2, ..., d_n)$ is a sequence of positive integers with $\sum_{i=1}^{n} d_i = 2n-2$, then we already know that there exists a tree T on n vertices with D as a score. Add another vertex v to T and connect it to any vertex of T by an edge, obtaining a tree T' on $n+1$ vertices. Let D' be the score of T'. We know that the number of vertices increased by 1, and the sum of degrees of vertices increased by 2 (the new vertex has degree 1 and the degree of one old vertex increased by 1). Hence the sequence D' satisfies condition (ii) and it is a score of a tree, namely of T'. This finishes the inductive step. □

Problem 7. *Let N_k denote the number of spanning trees of K_n in which the vertex n has degree k, $k = 1, 2, \ldots, n-1$ (recall that we assume $V(K_n) = \{1, 2, \ldots, n\}$).*

i) Prove that $(n-1-k)N_k = k(n-1)N_{k+1}$.

ii) Using i), derive $N_k = \binom{n-2}{k-1}(n-1)^{n-1-k}$.

iii) Prove Cayley's formula from ii).

Solution.

i) Both sides of the equality count the number of pairs spanning trees (T, T^*), where $deg_T(n) = k$, $deg_{T^*}(n) = k+1$, and T^* arises from T by the following operation: pick an edge $\{i, j\} \in E(T)$ with $i \neq n \neq j$, delete it, and add either the edge $\{i, n\}$ or the edge $\{j, n\}$, depending on which of these edges connects the two components of $T - \{i, j\}$.

- From one T we can get $n-1-k$ different T^*: the number of different edges in T which are not connected to n at the beginning;
- And one T^* can be obtained from $k(n-1)$ different T: pick any vertex $v \in \{1, 2, \ldots, n-1\}$. If one deletes all edges incident to n in a spanning tree from N_{k+1}, neighbours of n (denoted by $\ell_1, \ell_2, \ldots, \ell_{k+1}$) will lie in exactly $k+1$ different components. Suppose v lies in the last component, namely C_{k+1}. Add an edge between v and some ith leaf ℓ_i ($i \in \{1, 2, \ldots, k\}$) of n and remove the original edge (n, ℓ_i) simultaneously, one will get a different T. In all, there are $n-1$ ways to pick v and k ways to pick ℓ_i.

ii)

iii) $\sum_k N_k$ happens to be the expansion of $((n-1)+1)^{n-2}$ according to the binomial theorem.

$\square$