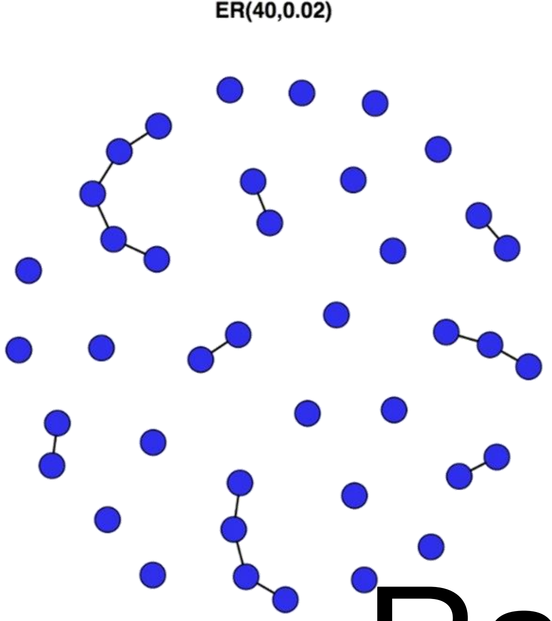

Random Graphs

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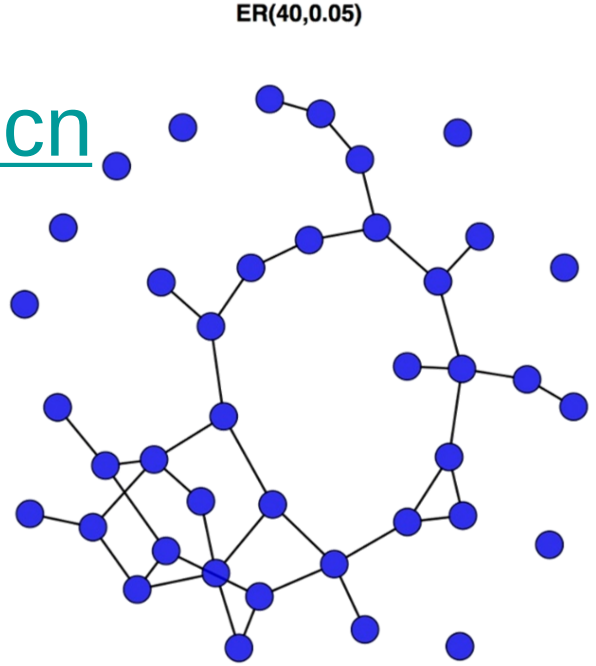
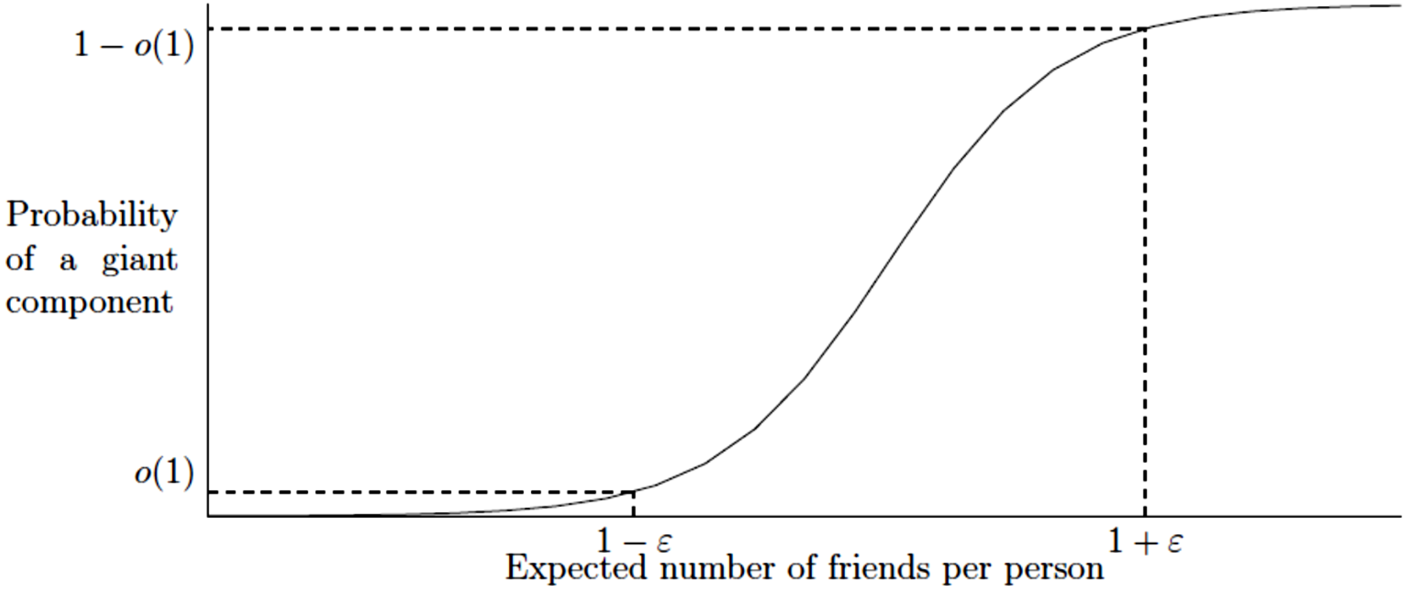


Phase transition

The interesting thing about the $\boldsymbol{G}(n, p)$ model is that even though edges are chosen independently, certain global properties of the graph emerge from the independent choice.



Probability	Transition
$p = o(\frac{1}{n})$	Forest of trees, no component of size greater than $O(\log n)$
$p = \frac{d}{n}, d < 1$	All components of size $O(\log n)$
$p = \frac{d}{n}, d = 1$	Components of size $O(n^{\frac{2}{3}})$
$p = \frac{d}{n}, d > 1$	Giant component plus $O(\log n)$ components
$p = \sqrt{\frac{2 \ln n}{n}}$	Diameter two
$p = \frac{1}{2} \frac{\ln n}{n}$	Giant component plus isolated vertices
$p = \frac{\ln n}{n}$	Disappearance of isolated vertices Appearance of Hamilton circuit Diameter $O(\log n)$
$p = \frac{1}{2}$	Clique of size $(2 - \epsilon) \ln n$

Phase transition

Definition. If there exists a function $p(n)$ such that

- when $\lim_{n\to\infty}\left(\frac{p_1(n)}{p(n)}\right) = 0$, $\quad \boldsymbol{G}(n, p_1(n))$ almost surely does not have the property.
- when $\lim_{n\to\infty}\left(\frac{p_2(n)}{p(n)}\right) = \infty$, $\quad \boldsymbol{G}(n, p_2(n))$ almost surely has the property.

Then we say phase transition occurs and $p(n)$ is the threshold.

Every increasing property has a threshold.

Increasing property

- **Definition:** The probability of a graph having the property increases as edges are added to the graph.
- Example:
  - Connectivity
  - Having no isolated vertices
  - Having a cycle
  - ……

Lemma: If Q is an increasing property of graphs and $0 \le p \le q \le 1$, then the probability that $\boldsymbol{G}(n, q)$ has property Q is greater than or equal to the probability that $\boldsymbol{G}(n, p)$ has property Q.

Proof:

Independently generate graph $\boldsymbol{G}(n, p)$ and $\boldsymbol{G}(n, \frac{q-p}{1-p})$.

$H = \boldsymbol{G}(n, p) \cup \boldsymbol{G}(n, \frac{q-p}{1-p})$ (the union of the edge set).

Graph H has the same distribution as $\boldsymbol{G}(n, q)$:

$$\Pr\big(\{u, v\} \in E(H)\big) = p + (1-p)\frac{q-p}{1-p} = q.$$

And edges in H are independent.

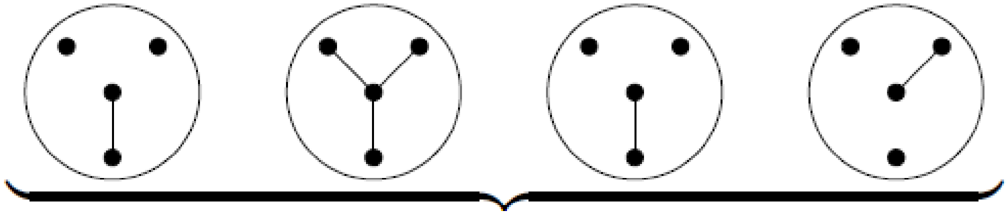
The result follows naturally.

Replication

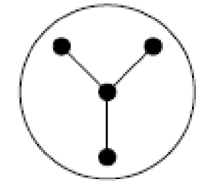
m-fold replication of $\boldsymbol{G}(n, p)$:

- Independently generate m copies of $\boldsymbol{G}(n, p)$ (on the same vertex set);
- Take the union of the m copies;

The result graph H has the same distribution as $G(n, q)$, where $q = 1 - (1 - p)^m$.

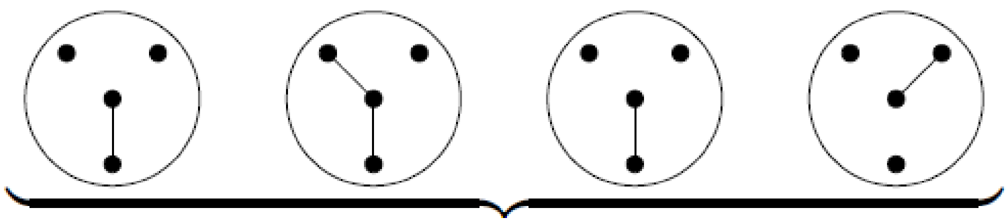


copies of G

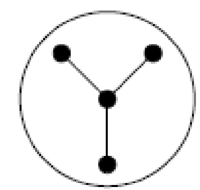


The m-fold replication H

If any graph has three or more edges, then the m-fold replication has three or more edges.



copies of G



The m-fold replication H

Even if no graph has three or more edges, the m-fold replication might have three or more edges.

John’s book: Figure 8.10

Replication

m-fold replication of $\boldsymbol{G}(n, p)$:

- Independently generate m copies of $\boldsymbol{G}(n, p)$ (on the same vertex set);
- Take the union of the m copies;

The result graph H has the same distribution as $G(n, q)$, where $q = 1 - (1 - p)^m$.

One of the copies of $\boldsymbol{G}(n, p)$ has the increasing property

$\downarrow$

$G(n, q)$ has the increasing property.

As $q \leq 1 - (1 - mp) = \boldsymbol{mp}$

$$\therefore \Pr(\boldsymbol{G}(n, \boldsymbol{mp}) \text{has } Q) \geq \Pr(\boldsymbol{G}(n, q) \text{has } Q)$$

Theorem: Every increasing property Q of $\boldsymbol{G}(n,p)$ has a phase transition at $p(n)$, where for each n, $p(n)$ is the minimum real number a_n for which the probability that $\boldsymbol{G}(n,a_n)$ has property Q is $\frac{1}{2}$.

Proof:

First prove that for any function $p_0(n)$ with $\lim_{n\to\infty} \frac{p_0(n)}{p(n)} = 0$, almost surely $\boldsymbol{G}(n,p_0)$ does not have the property Q.

Suppose otherwise: the probability that $\boldsymbol{G}(n,p_0)$ has the property Q *does not converge to 0*.

Then there exists $\epsilon > 0$ for which the probability that $\boldsymbol{G}(n,p_0)$ has the property Q is $\geq \epsilon$ on an infinite set I of n. Let $m = \lceil (1/\epsilon) \rceil$

First prove that for any function $p_0(n)$ with $\lim_{n\to\infty} \frac{p_0(n)}{p(n)} = 0$, almost surely $\boldsymbol{G}(n, p_0)$ does not have the property Q.

Let $\boldsymbol{G}(n, q)$ be the m-fold replication of $\boldsymbol{G}(n, p_0)$.

For all $n \in I$, the probability that $\boldsymbol{G}(n, q)$ does not have Q : $\leq (1-\epsilon)^m \leq e^{-1} \leq 1/2$

$$\Pr(\boldsymbol{G}(n, \boldsymbol{mp_0})\text{has } Q) \geq \Pr(\boldsymbol{G}(n, q)\text{has } Q) \geq 1/2$$

As $p(n)$ is the minimum real number a_n for which $\Pr(\boldsymbol{G}(n, a_n)\text{has } Q) = \frac{1}{2}$, it follows that $\boldsymbol{mp_0(n)} \geq p(n)$.

$\therefore\ \frac{\boldsymbol{p_0(n)}}{p(n)} \geq \frac{1}{m}$ infinitely often.

Contradict to the fact that $\lim_{n\to\infty} \frac{p_0(n)}{p(n)} = 0$.

Theorem: Every increasing property Q of $\boldsymbol{G}(n,p)$ has a phase transition at $p(n)$, where for each n, $p(n)$ is the minimum real number a_n for which the probability that $\boldsymbol{G}(n,a_n)$ has property Q is $\frac{1}{2}$.

Proof:

Secondly prove that for any function $p_1(n)$ with $\lim_{n\to\infty}\frac{p(n)}{p_1(n)}=0$, almost surely $\boldsymbol{G}(n,p_1)$ almost surely has the property Q.