Set Theory
Paradox & Cardinality

Huan Long
Shanghai Jiao Tong University

longhuan@sjtu.edu.cn

Key points one should know of

- Set operations
  - $A \cup B$, $A \cap B$, $A - B$, $\bar{A}$, $A \oplus B$, $P(A)$
- Set identity laws
- Set applications
  - Relation
    - Ordered pairs, $A \times B$, Relation, Equivalence relation, Partition
  - Function
    - Onto function/Surjective function
    - Injective function/One-to-one function/Single-rooted
    - Bijective function

Part II.

Paradox	• **Paradox and ZFC**
Equinumerosity	• Equinumerosity
Cardinal Numbers	• Ordering
Infinite Cardinals	• Countable sets

Barber Paradox[1918]

- Suppose there is a town with just one male barber. The barber shaves *all* and *only* those men in town who do not shave themselves.
- Question: Does the barber shave himself?
  - If the barber does NOT shave himself, then he MUST abide by the rule and shave himself.
  - If he DOES shave himself, according to the rule he will NOT shave himself.

Formal Proof

- **Theorem** There is no set to which every set belongs. [Russell, 1902]

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Proof:

Let A be a set; we will construct a set not belonging to A. Let

$$B=\{x\in A \mid x\notin x\}$$

We claim that $B\notin A$. we have, by the construction of B.

$$B\in B \text{ iff } B\in A \text{ and } B\notin B$$

If $B\in A$, then this reduces to

$B\in B$ iff $B\notin B$, Which is impossible, since one side must be true and the other false. Hence $B\notin A$

Natural Numbers in Set Theory

- Constructing the natural numbers in terms of sets is part of the process of

"Embedding mathematics in set theory"

John von Neumann

- December 28, 1903 – February 8, 1957. Hungarian American mathematician who made major contributions to a vast range of fields:
  - Logic and set theory
  - Quantum mechanics
  - Economics and game theory
  - Mathematical statistics and econometrics
  - Nuclear weapons
  - Computer science

Natural numbers

- By *von Neumann*:

Each natural number is the set of all smaller natural numbers.

$0 = \emptyset$

$1=\{0\}=\{\emptyset\}$

$2=\{0,1\}=\{\emptyset, \{\emptyset\}\}$

$3=\{0,1,2\}=\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

……

Some properties from the first four natural numbers

$0 = \emptyset$

$1=\{0\}=\{\emptyset\}$

$2=\{0,1\}=\{\emptyset, \{\emptyset\}\}$

$3=\{0,1,2\}=\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

$0 \in 1 \in 2 \in 3 \in \cdots$

$0 \subseteq 1 \subseteq 2 \subseteq 3 \subseteq \cdots$

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Equinumerosity	• **Equinumerosity**
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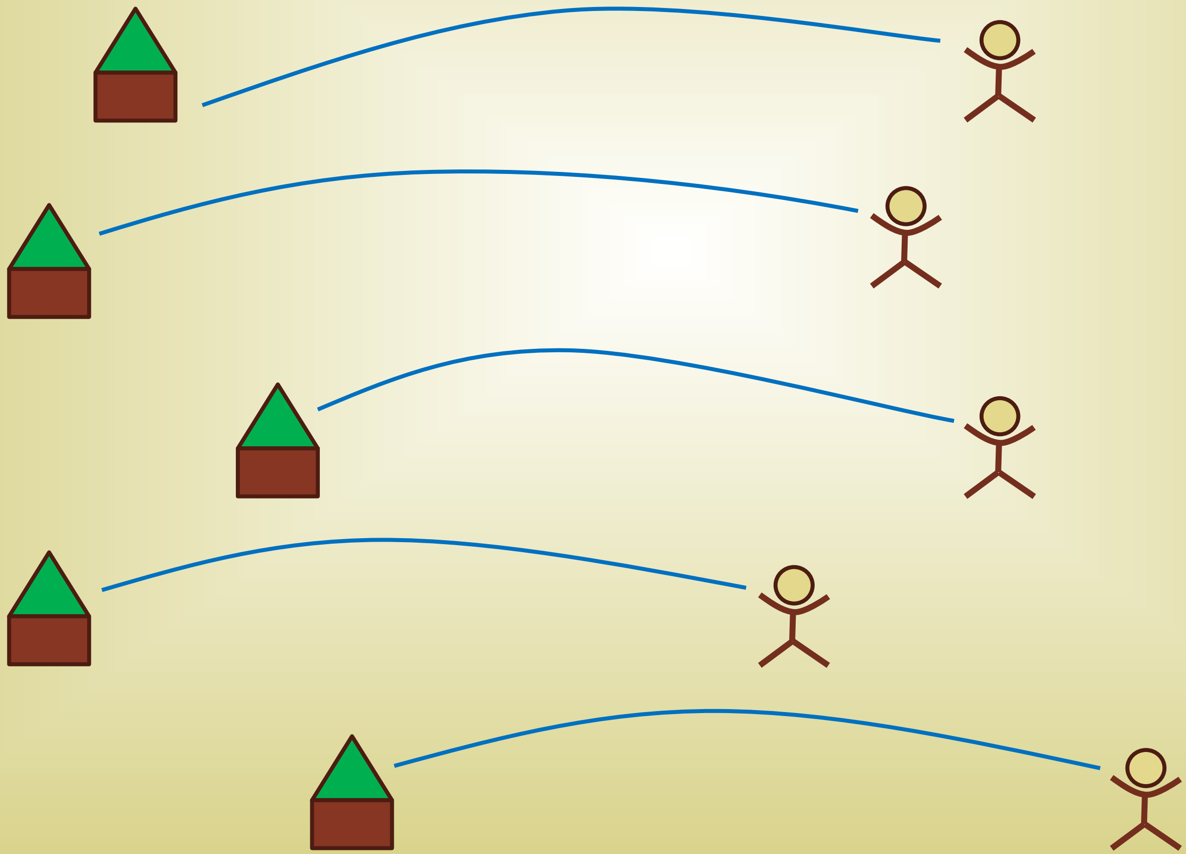
Motivation

- To discuss the **size** of sets. Given two sets A and B, we want to consider such questions as:
  - Do A and B have the same size?
  - Does A have more elements than B?

Example



Equinumerosity

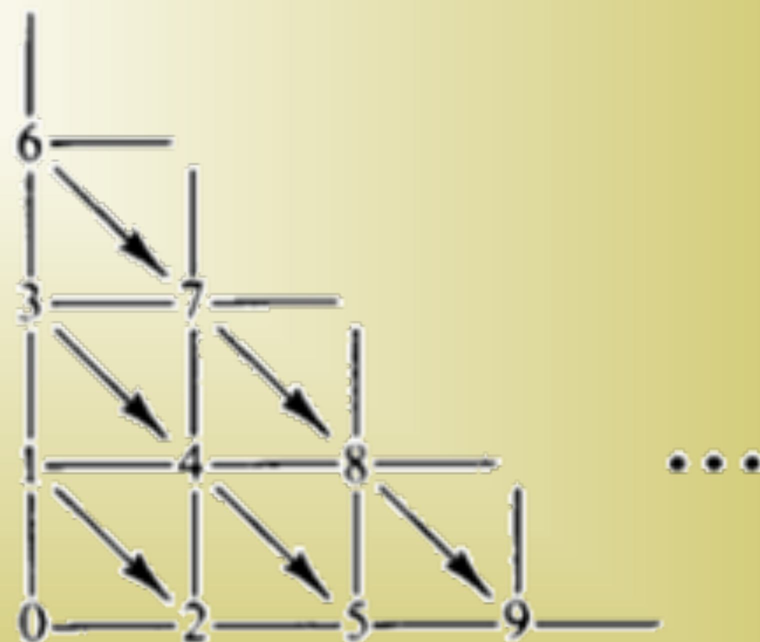
- **Definition** A set $\mathcal{A}$ is ***equinumerous*** to a set $\mathcal{B}$ (written $\mathcal{A} \approx \mathcal{B}$) iff there is a one-to-one function from $\mathcal{A}$ onto $\mathcal{B}$.

- A one-to-one function from $\mathcal{A}$ onto $\mathcal{B}$ is called a *one-to-one correspondence* between $\mathcal{A}$ and $\mathcal{B}$.

Example: $\omega \times \omega \approx \omega$

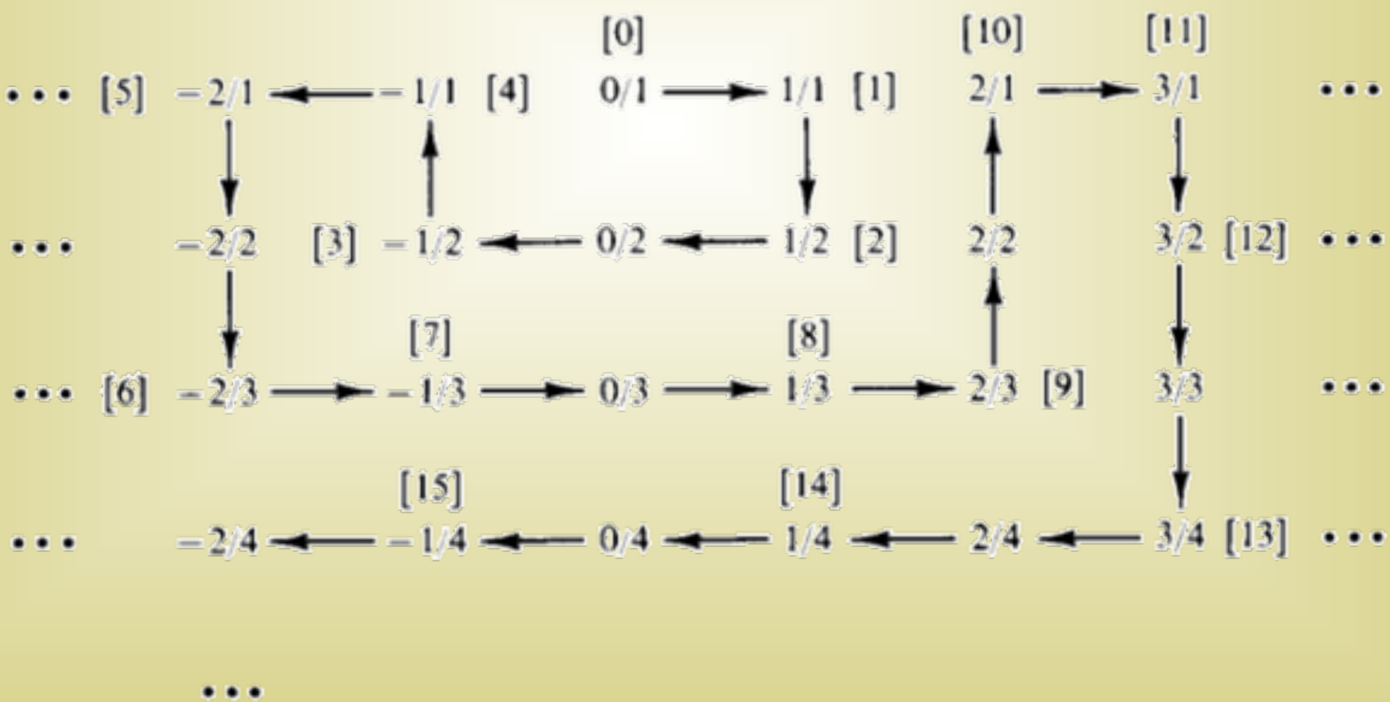
- The set $\omega \times \omega$ is equinumerous to ω. There is a function **J** mapping $\omega \times \omega$ one-to-one onto ω.

$$\mathbf{J}(m,n)=((m+n)^2+3m+n)/2$$

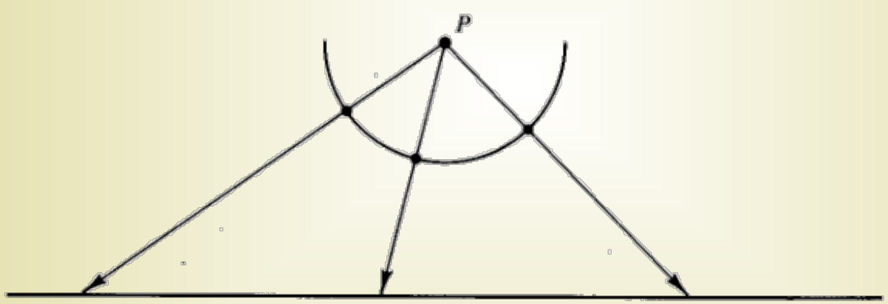


Example: ω≈ℚ

- f: ω→ ℚ



Example: $(0,1) \approx \mathbb{R}$

- $(0,1)=\{x \in \mathbf{R} \mid 0<x<1\}$, then $(0,1) \approx \mathbf{R}$
  - 

  - $f(x)= \tan(\pi(2x-1)/2)$

Examples

- $(0,1) \approx (n,m)$
  - Proof: $f(x) = (n-m)x+m$
- $(0,1) \approx \{x \mid x \in \mathbf{R} \wedge x>0\} = (0,+\infty)$
  - Proof: $f(x)=1/x -1$

Examples

- $[0,1] \approx [0,1)$
  - Proof: $f(x)=x$ if $0 \le x<1$ and $x \neq 1/(2^n)$, $n \in \omega$
    $f(x)=1/(2^{n+1})$ if $x=1/(2^n)$, $n \in \omega$
- $[0,1) \approx (0,1)$
  - Proof: $f(x)=x$ if $0<x<1$ and $x \neq 1/(2^n)$, $n \in \omega$
    $f(0)=1/2$ $x=0$
    $f(x)=1/(2^{n+1})$ if $x=1/(2^n)$, $n \in \omega$
- $[0,1] \approx (0,1)$

Example: $\wp(A) \approx {}^{A}2$

- For any set A, we have $P(A) \approx {}^{A}2$.

Proof: Define a function H from $P(A)$ onto ${}^{A}2$ as:

For any subset B of A, $H(B)$ is the characteristic function of B:

$$f_B(x) = \begin{cases} 1 & \text{if} \quad x \in B \\ 0 & \text{if} \quad x \in A - B \end{cases}$$

H is one-to-one and onto.

Theorem

- For any sets A, B and C:
  - $A \approx A$
  - If $A \approx B$ then $B \approx A$
  - If $A \approx B$ and $B \approx C$ then $A \approx C$.

  Proof:

Theorem(Cantor 1873)

- The set ω is not equinumerous to the set $\mathbf{R}$ of real numbers.
- No set is equinumerous to its power set.

The set ω is not equinumerous to the set $\mathbf{R}$ of real numbers.

Proof: show that for any functon $f: \omega \rightarrow \mathbf{R}$, there is a real number z not belonging to *ran f*

$$f(0) = 32.4345\ldots,$$

$$f(1) = -43.334\ldots,$$

$$f(2) = 0.12418\ldots,$$

$$\ldots\ldots$$

z: the integer part is 0, and the $(n+1)^{st}$ decimal place of z is 7 unless the $(n+1)^{st}$ decimal place of $f(n)$ is 7, in which case the $(n+1)^{st}$ decimal place of z is 6.

Then z is a real number not in *ran f*.

- No set is equinumerous to its power set.

No set is equinumerous to its power set.

Proof: Let $g: A \to \wp(A)$; we will construct a subset B of A that is not in *ran g*. Specifically, let

$$B = \{x \in A \mid x \notin g(x)\}$$

Then $B \subseteq A$, but for each $x \in A$

$$x \in B \text{ iff } x \notin g(x)$$

Hence $B \neq g(x)$.

Application

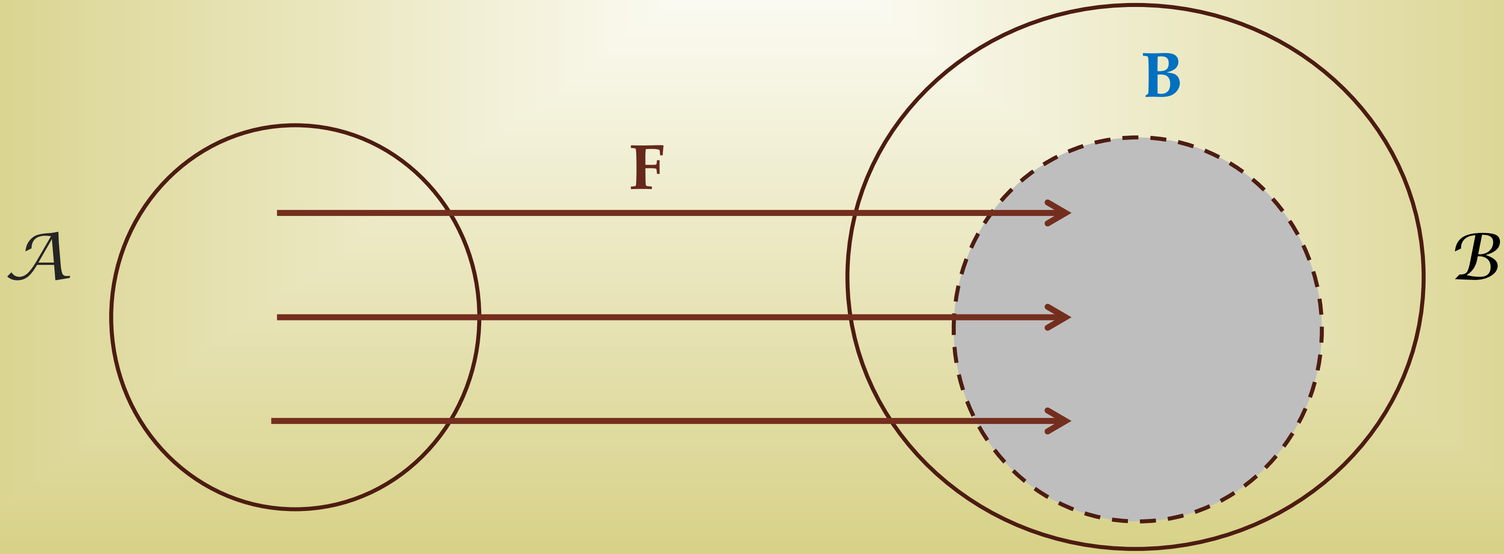
Paradox	• Paradox and ZFC
Equinumerosity	• Equinumerosity
Cardinal Numbers	• **Ordering**
Infinite Cardinals	• Countable sets

Ordering Cardinal Numbers

- **Definition** A set $\mathcal{A}$ is **dominated** by a set $\mathcal{B}$ (written $\mathcal{A} \preccurlyeq \mathcal{B}$) iff there is a ***one-to-one*** function from $\mathcal{A}$ into $\mathcal{B}$.

Examples

- Any set dominates itself.
- If $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A}$ is dominated by $\mathcal{B}$.
- $\mathcal{A} \preccurlyeq \mathcal{B}$ iff $\mathcal{A}$ is equinumerous to some subset of $\mathcal{B}$.



Schröder-Bernstein Theorem

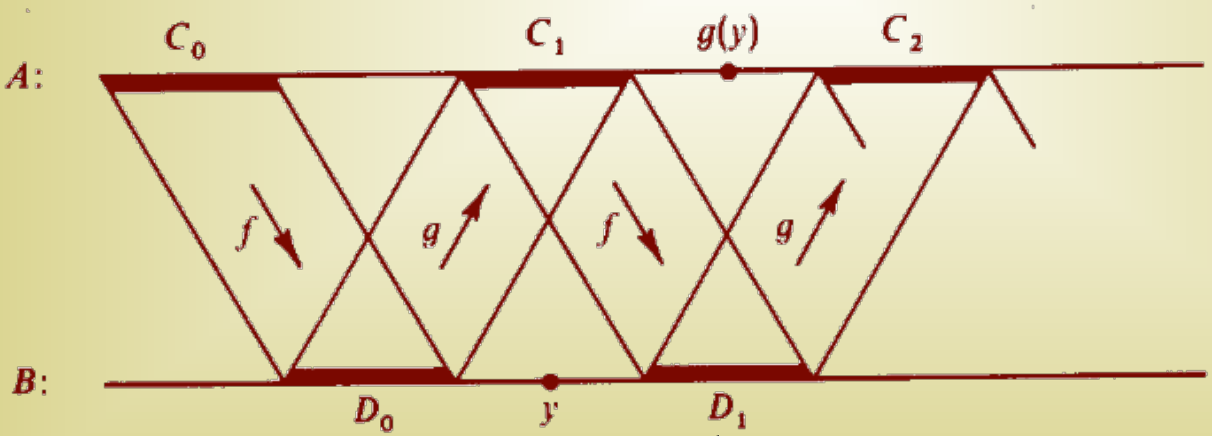
- If $A \preccurlyeq B$ and $B \preccurlyeq A$, then $A \approx B$.

Proof:

$f\colon A \to B,\ g\colon B \to A$. Define C_n by recursion:

$C_0 = A - ran\ g$ and $C_n^+ = g[f[C_n]]$

$$h(x) = \begin{cases} f(x) & \text{if } x \in C_n \text{ for some } n, \\ g^{-1}(x) & \text{otherwise} \end{cases}$$



$h(x)$ is one-to-one and onto.

Application of the Schröder-Bernstein Theorem

- Example
  - If $A \subseteq B \subseteq C$ and $A \approx C$, then all three sets are equinumerous.
  - The set **R** of real numbers is equinumerous to the closed unit interval $[0,1]$.

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Countable Sets

- **Definition** A set $\boldsymbol{A}$ is countable iff $\boldsymbol{A} \preccurlyeq \omega$,
- Intuitively speaking, the elements in a countable set can *be counted by* means of the natural numbers.
- An equivalent definition: A set $\boldsymbol{A}$ is countable iff either $\boldsymbol{A}$ is finite or $\boldsymbol{A} \approx \boldsymbol{\omega}$.

Example

- ω is countable, as is **Z** and **Q**
- **R** is uncountable
- A, B are countable sets
  - $\forall\, C \subseteq A$, C is countable
  - $A \cup B$ is countable
  - $A \times B$ is countable

Example

- ω is countable, as is **Z** and **Q**
- **R** is uncountable
- A, B are countable sets
  - $\forall\, C \subseteq A$, C is countable
  - $A \cup B$ is countable
  - $A \times B$ is countable
- For any infinite set A, $\wp(A)$ is uncountable.

Continuum Hypothesis

- Are there any sets with cardinality between $\aleph_0$ and $2^{\aleph_0}$?
- Continuum hypothesis (Cantor): No.

  i.e., there is no λ with $\aleph_0 < \lambda < 2^{\aleph_0}$.

  Or, equivalently, it says: Every uncountable set of real numbers is equinumerous to the set of all real numbers.

GENERAL VERSION: for any infinite cardinal κ, there is no cardinal number between κ and 2^{κ} .

HISTORY

- Georg Cantor: 1878, proposed the conjecture
- David Hilbert: 1900, the first of Hilbert's 23 problems.
- Kurt Gödel: 1939, $\mathrm{ZF} \nvdash \neg\mathrm{CH}$.
- Paul Cohen: 1963, $\mathrm{ZF} \nvdash \mathrm{CH}$.

Thanks!