Homework 4

Problem 1. *Count the number of linear extensions for the following partial ordering:*

X is a disjoint union of sets $X_1, X_2, \ldots, X_k$ of sizes $r_1, r_2, \ldots, r_k$, respectively. Each X_i is linearly ordered by $\preceq$, and no two elements from the different X are comparable.

Solution. $\binom{r_1+r_2+\cdots+r_k}{r_1,r_2,\cdots,r_k}$. □

Problem 2. *Given a set X with $|X| = n$, determine the number of ordered set pairs $\langle A, B\rangle$ where $A \subseteq B \subseteq X$.*

Solution. As a set B of size b has exactly 2^b subsets A, the number of such ordered pairs is

$$\sum_{0\le b\le n} 2^b \binom{n}{b} = (1+2)^n = 3^n.$$

□

Problem 3. *There are n married couples attending a dance. How many ways are there to form n pairs for dancing if no wife should dance with their husband.*

Solution. It is $D(n)$. □

Problem 4. *Count the permutations with exactly k fixed points. (Remark: π is a permutation of the set $\{1,2,\ldots,n\}$. Call an index i with $\pi(i) = i$ a* fixed point *of the permutation π.)*

Solution. First choose the points that are fixed. It will have $\binom{n}{k}$ possible choices.

The rest is counting the number of permutation without a fixed point, which is $D(n-k)$.

In all, the answer is $\binom{n}{k} \cdot D(n-k)$. □

Problem 5. *What is wrong with the following inductive proof that $D(n) = (n-1)!$ for all $n \ge 2$? Can you find a false step in it? For $n = 2$, the formula holds, so assume $n \ge 3$. Let π be a permutation of $\{1, 2, ..., n-1\}$ with no fixed point. We want to extend it to a permutation π' of $\{1, 2, ..., n\}$ with no fixed point. We choose* *a number* $i \in \{1, 2, ..., n-1\}$*, and we define* $\pi'(n) = \pi(i), \pi'(i) = n$*, and* $\pi'(j) = \pi(j)$ *for* $j \neq i, n$*. This defines a permutation of* $\{1, 2, ..., n\}$*, and it is easy to check that it has no fixed point. For each of the* $D(n-1) = (n-2)!$ *possible choices of* π*, the index* i *can be chosen in* $n-1$ *ways. Therefore,* $D(n) = (n-2)! \cdot (n-1) = (n-1)!$.

Solution. Basically it says that the extended function π' is totally decided by the choice of i. However this is not the case: after letting $\pi'(n) = \pi(i)$, $\pi'(i) = n$ could not be the only choice for keeping π' bijective. In another word, the construction is an undercount. □

Problem 6. *How many ways are there to seat n married couples at a round table with* $2n$ *chairs in such a way that the couples never sit next to each other?*

Solution.(hint)

A_i = {the ways of seating in which the ith couple is adjacent.}

- $|A_i| = (2n-1)! \cdot 2^1/(2n-1)$;
- $|A_i \cap A_j| = (2n-2)! \cdot 2^2/(2n-2)$ for $i \neq j$;
- $\cdots$;
- $|A_{i1} \cap \cdots \cap A_{ik}| = (2n-k)! \cdot 2^k/(2n-k)$ for different A_{ij}s.

The final result should be $(2n)!/(2n) - |A_1 \cup A_2 \cup \cdots \cup A_k|$. Then by PIE....

□