Homework 7

Problem 1. *Fill in the blanks with either true (✓) or false (×)*

$f(n)$	$g(n)$	$f = O(g)$	$f = \Omega(g)$	$f = \Theta(g)$
$2n^3 + 3n$	$100n^2 + 2n + 100$	×	✓	×
$50n + \log n$	$10n + \log\log n$	✓	✓	✓
$50n \log n$	$10n \log\log n$	×	✓	×
$\log n$	$\log^2 n$	✓	×	×
$n!$	5^n	×	✓	×

Problem 2. 1. *Find two functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.*

2. *Furthermore, we say a function $h : \mathbb{R} \to \mathbb{R}$ is* monotonically increasing *if it satisfies the property '$x \leq y \;\Rightarrow\; h(x) \leq h(y)$'.*
Find two monotonically increasing functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.

(Please give the detailed proof that your functions satisfy the requirements.)

Problem 3. *Prove that*

(a) $\left(1 + \frac{1}{n}\right)^n \leq e$ *for all* $n \geq 1$.

(b) $\left(1 + \frac{1}{n}\right)^{n+1} \geq e$ *for all* $n \geq 1$.

(c) Using (a) and (b), conclude that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

Problem 4. *Prove* Bernoulli's inequality*: for each natural number n and for every real $x \geq -1$, we have $(1 + x)^n \geq 1 + nx$.*

Problem 5. *Prove that for $n = 1, 2, \ldots,$ we have*

$$2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1.$$

Problem 6.

a) Show that the product of all primes p with $m < p \leq 2m$ is at most $\binom{2m}{m}$.

b) Using a), prove the estimate $\pi(x) = O(\frac{x}{\ln x})$, where $\pi(x)$ denote the number of primes not exceeding the number x.