Homework 9

Problem 1. *Find an example to verify the claim that '(pairwise) independence does not imply mutual independence'. Pls give a detailed proof.*

Problem 2. *Show that, if $E_1, E_2, \ldots, E_n$ are mutually independent, then so are $\overline{E_1}, \overline{E_2}, \ldots, \overline{E_n}$.*

Problem 3. *A monkey types on a 26 -letter keyboard that has lowercase letters only. Each letter is chosen independently and uniformly at random from the alphabet. If the monkey types 1,000,000 letters. what is the expected number of times the sequence "proof" appears?*

Problem 4. *We have 27 fair coins and one counterfeit coin (28 coins in all), which looks like a fair coin but is a bit heavier. Show that one needs at least 4 weighings to determine the counterfeit coin. We have no calibrated weights, and in one weighing we can only find out which of two groups of some k coins each is heavier, assuming that if both groups consist of fair coins only the result is an equilibrium.*

Problem 5.

1. *Prove that, for every integer n, there exists a coloring of the edges of the complete graph K_n by two colors so that the total number of monochromatic copies of K_4 is at most $\binom{n}{4}2^{-5}$.*
2. *Give a randomized algorithm for finding a coloring with at most $\binom{n}{4}2^{-5}$ monochromatic (i.e. single-color) copies of K_4 that runs in expected time polynomial in n.*

Problem 6. *Use the Lovasz local lemma to show that if*

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1$$

then it is possible to color the edges of K_n with two colors so that it has no monochromatic (i.e. single color) K_k subgraph.

Problem 7. *What is the expected number of trees with k vertices in $G \in \mathcal{G}(n, p)$?*

Problem 8. *Show that if almost all $G \in \mathcal{G}(n, p)$ have a graph property $\mathcal{P}_1$ and almost all $G \in \mathcal{G}(n, p)$ have a graph property $\mathcal{P}_2$, then almost all $G \in \mathcal{G}(n, p)$ have both properties.*