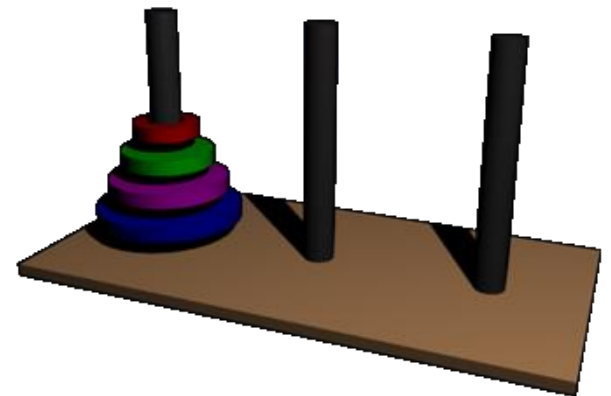


将序列转化成函数

Generating Function

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- Problem: How many ways are there to pay the amount of 21 doublezons if we have
 - 6 one-doublezon coins;
 - 5 two-doublezon coins;
 - 4 five-doublezon coins.

• Solution:

i 对应着取的硬币的个数

$$i_1 + i_2 + i_3 = 21 \quad (*)$$

$$i_1 \in \{0, 1, 2, 3, 4, 5, 6\}; \quad i_2 \in \{0, 2, 4, 6, 8, 10\}; \quad i_3 \in \{0, 5, 10, 15, 20\}.$$

序列, 理解成函数

$$(1 + x + x^2 + x^3 + \dots + x^6)(1 + x^2 + x^4 + x^6 + x^8 + x^{10})$$

$$\cdot (1 + x^5 + x^{10} + x^{15} + x^{20})$$

指数刚好凑出 21

系数

The coefficient of x^{21}

= the number of solutions of (*).

序列组合问题转化为求方程的系数



Recall

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 \cdots \binom{n}{n}x^n$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \sum_{k=0}^n \binom{n}{k} = 2^n$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \cdots = \sum_{k=0}^n \binom{n}{k} (-1)^k = 0$$

$$\sum_{k=0}^n k \binom{n}{k} = n 2^{n-1}$$

两边求导,
代 $x=1$

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}$$

$$(1+x)^n (1+x)^n = (1+x)^{2n}$$

$$\sum_{i=0}^n \binom{n}{i} x^i \binom{n}{n-i} x^{n-i} = \binom{2n}{n} x^n$$

x^n 項的係數

Generating function

生成函数的定义

- (a_0, a_1, a_2, \dots) be a sequence of real numbers, then the **generating function** of this sequence is defined as

$$a(x) = a_0 + a_1x + a_2x^2 + \dots$$

(无限)

Example:

$(1, 1, 1, \dots)$

$$a(x) = 1 + x + x^2 + \dots = \frac{1}{1-x}$$

也可有限

等比求和

Generalized binomial theorem

$$\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}$$

If r is a negative integer

$$\begin{aligned}\binom{r}{k} &= \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!} \\ &= (-1)^k \frac{(-r)(-r+1)(-r+2)\cdots(-r+k-1)}{k!} \\ &= (-1)^k \binom{-r+k-1}{k} = (-1)^k \binom{-r+k-1}{-r-1}\end{aligned}$$

$$\frac{1}{(1-x)^n} = \binom{n-1}{n-1} + \binom{n}{n-1}x + \binom{n+1}{n-1}x^2 + \cdots \binom{n+k-1}{n-1}x^k + \cdots$$

$$(1-x)^{-n} = \binom{-n}{0}(-x)^0 + \binom{-n}{1}(-x)^1 + \binom{-n}{2}(-x)^2 + \cdots$$

函数的操作得到序列的信息

More examples

$$(a_0, a_1, a_2, \dots) \quad a(x) = a_0 + a_1x + a_2x^2 + \dots \quad (\text{定义})$$

$$(0, a_0, a_1, a_2, \dots) \quad a(x) = 0 + a_0x + a_1x^2 + \dots = x \cdot a(x) \quad \text{乘了一个 } x$$

$$(0, 1, \frac{1}{2}, \frac{1}{3}, \dots) \quad a(x) = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \dots = -\ln(1-x) \quad \begin{matrix} \text{泰勒展开} \\ \text{右移 } j-1 \text{ 位} \end{matrix} \quad -1 < x < 1$$

$$(1, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \dots) \quad a(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x$$

泰勒展开

$$f'''(x) = (x-1)^{-2} = (-2) \cdot (x-1)^{-3} \quad f'(x) = \frac{1}{1-x} \quad \frac{1}{(1-x)^2} = f''(x)$$

$$f^{(4)}(x) = 6 \cdot (x-1)^{-4} \quad f^{(5)}(x) = -24(x-1)^{-5}$$

Operations with Sequences - Addition

$$(a_0, a_1, a_2, \dots) \quad a(x) = a_0 + a_1x + a_2x^2 + \dots$$

$$(b_0, b_1, b_2, \dots) \quad b(x) = b_0 + b_1x + b_2x^2 + \dots$$

$$(a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots)$$

$$a(x) + b(x)$$

可加性

常数线性扩张

Constant linear expansion

$$(a_0, a_1, a_2, \dots) \quad a(x) = a_0 + a_1x + a_2x^2 + \dots$$

$$(\alpha a_0, \alpha a_1, \alpha a_2, \dots)$$

$$\alpha \cdot a(x)$$

移动 (右移)

Shifting-right

$$(a_0, a_1, a_2, \dots) \quad a(x) = a_0 + a_1x + a_2x^2 + \dots$$

$$(\underbrace{0, 0, \dots, 0}_n, a_0, a_1, a_2, \dots)$$

填 n 个 0

$$x^n \cdot a(x)$$

提出 x^n

Shifting-left 左移

$$(a_0, a_1, a_2, \dots) \quad a(x) = a_0 + a_1x + a_2x^2 + \dots$$

$$(a_3, a_4, a_5, \dots)$$

$$\frac{a(x) - a_0 - a_1x - a_2x^2}{x^3}$$

去掉前三项

大考按

$$(a_0, a_1, a_2, \dots) \quad a(x) = a_0 + a_1x + a_2x^2 + \dots$$

$$a(\alpha x)$$

Example:

$$(1,1,1, \dots) \qquad a(x) = \frac{1}{1-x}$$

$$1 + 2x + 4x^2$$

$(1, 2, 4, 8, \dots)$	$a(2x) = \frac{1}{1 - 2x}$
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$$(a_0, 0, a_2, 0, a_4, 0 \dots) \quad \frac{1}{2}(a(x) + a(-x))$$

$$(2 = -1)$$

Substituting $-x^n$

$$(a_0, a_1, a_2, \dots) \quad a(x) = a_0 + a_1x + a_2x^2 + \dots$$

插入常数个0

$$(a_0, 0, 0, a_1, 0, 0, a_2, 0, 0, \dots)$$

$$a_0 + a_1x^3 + a_2x^6 + \dots$$

$$= a(x^3)$$

Example:

$$(1, 1, 2, 2, 4, 4, 8, 8, \dots) \text{ i.e., } a_n = 2^{\lfloor n/2 \rfloor}$$

$$(1, 2, 4, 8, \dots) \quad \frac{1}{1 - 2x}$$

$$(1, 0, 2, 0, 4, 0, 8, \dots) \quad \frac{1}{1 - 2x^2}$$

$$(0, 1, 0, 2, 0, 4, 0, 8, \dots) \quad \frac{x}{1 - 2x^2}$$

$$\frac{1 + x}{1 - 2x^2}$$

向下取整

插入一个0, 变为 x^2 } 相加

Differentiation

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$$

$$2a_2 + 6a_3x$$

$$(a_0, a_1, a_2, \dots) \quad a(x) = a_0 + a_1x + a_2x^2 + \dots + 12a_4x^2$$

$$(a_1, 2a_2, 3a_3, \dots)$$

$$a'(x)$$

$$\frac{d}{dx}$$

Example:

$$(1^2, 2^2, 3^2, 4^2, \dots) \text{ i.e., } a_k = (k+1)^2$$

$$(1, 1, 1, 1, \dots) \quad \frac{1}{1-x}$$

$$(1, 2, 3, 4, \dots, k+1, \dots) \quad \left(\frac{1}{1-x} \right)' = \frac{1}{(1-x)^2}$$

$$(2 \cdot 1, 3 \cdot 2, 4 \cdot 3, \dots, (k+2)(k+1), \dots) \quad \left(\frac{1}{1-x} \right)'' = \frac{2}{(1-x)^3}$$

$$(1-x)^{-2}$$

$$2(1-x)^{-3}$$

Differentiation

$$(a_0, a_1, a_2, \dots) \quad a(x) = a_0 + a_1x + a_2x^2 + \dots$$

$$(a_1, 2a_2, 3a_3, \dots)$$

$$a'(x)$$

Example:

$$(1^2, 2^2, 3^2, 4^2, \dots) \text{ i.e., } a_k = (k+1)^2$$

$$(1, 1, 1, 1, \dots) \quad \frac{1}{1-x}$$

$$(1, 2, 3, 4, \dots, \underline{k+1}, \dots)$$

$$\left(\frac{1}{1-x} \right)' = \frac{1}{(1-x)^2}$$

$$(2 \cdot 1, 3 \cdot 2, 4 \cdot 3, \dots, \underline{(k+1)^2 + k + 1}, \dots)$$

$$(k+1)(k+2)$$

$$\frac{2}{(1-x)^3} - \frac{1}{(1-x)^2}$$

$$\left(\frac{1}{1-x} \right)'' = \frac{2}{(1-x)^3}$$

Integration

$$(a_0, a_1, a_2, \dots) \quad a(x) = a_0 + a_1x + a_2x^2 + \dots$$

$$(0, a_0, \frac{1}{2}a_1, \frac{1}{3}a_2, \frac{1}{4}a_3, \dots)$$

$$\int_0^x a(t)dt$$

卷积

Multiplication/Convolution

$$(a_0, a_1, a_2, \dots) \quad a(x) = a_0 + a_1x + a_2x^2 + \dots$$

$$(b_0, b_1, b_2, \dots) \quad b(x) = b_0 + b_1x + b_2x^2 + \dots$$

$$(c_0, c_1, c_2, \dots)$$

$$a(x) \cdot b(x)$$

$$c_0 = a_0b_0$$

$$c_1 = a_0b_1 + a_1b_0$$

$$c_2 = a_0b_2 + a_1b_1 + a_2b_0$$

\vdots

$$c_k = \sum_{i,j \geq 0; i+j=k} a_i b_j$$

卷积操作

解决递归问题

Solving Recurrence

- Recurrence relation

Define g_i recursively. 递归地

- Manipulation: 操作

New equivalence concerning $G(x)$.

- Solving

Closed form for $G(x)$.

- Expanding

New form for g_i .

Application

- A box contains 30 **red**, 40 **blue**, and 50 **green** balls; balls of the same color are indistinguishable. How many ways are there of selecting a collection of 70 balls from the box?

- Solution:

$$\begin{aligned} & (1 + x + x^2 + \dots + x^{30}) \cdot (1 + x + x^2 + \dots + x^{40}) \\ & \cdot (1 + x + x^2 + \dots + x^{50}) \\ &= \left(\frac{1 - x^{31}}{1 - x} \right) \cdot \left(\frac{1 - x^{41}}{1 - x} \right) \cdot \left(\frac{1 - x^{51}}{1 - x} \right) \\ &= \frac{1}{(1 - x)^3} (1 - x^{31})(1 - x^{41})(1 - x^{51}) \end{aligned}$$

x⁷⁰ 的系数

$$\begin{aligned}
 & (1 + x + x^2 + \cdots + x^{30}) \cdot (1 + x + x^2 + \cdots + x^{40}) \\
 & \cdot (1 + x + x^2 + \cdots + x^{50}) \\
 & = \frac{1}{(1-x)^3} (1 - x^{31})(1 - x^{41})(1 - x^{51})
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{(1-x)^n} &= \binom{n-1}{n-1} + \binom{n}{n-1}x + \binom{n+1}{n-1}x^2 + \cdots \binom{n+k-1}{n-1}x^k + \cdots \\
 &= \left(\binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 + \cdots \right) (1 - x^{31} - x^{41} - x^{51} + \cdots)
 \end{aligned}$$

Thus the coefficient of x^{70} is:

$$\begin{aligned}
 &= \binom{70+2}{2} - \binom{70+2-31}{2} - \binom{70+2-41}{2} - \binom{70+2-51}{2} \\
 &= 1061 \rightarrow \text{右边都减1}
 \end{aligned}$$

Application

- Fibonacci Number

$$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2}$$

$$F(x) = f_0 + f_1x + f_2x^2 + \cdots + f_{n-2}x^{n-2} + f_{n-1}x^{n-1} + f_nx^n + \cdots$$

原数列 \rightarrow 生成函数
(递推式)

Fibonacci Sequence

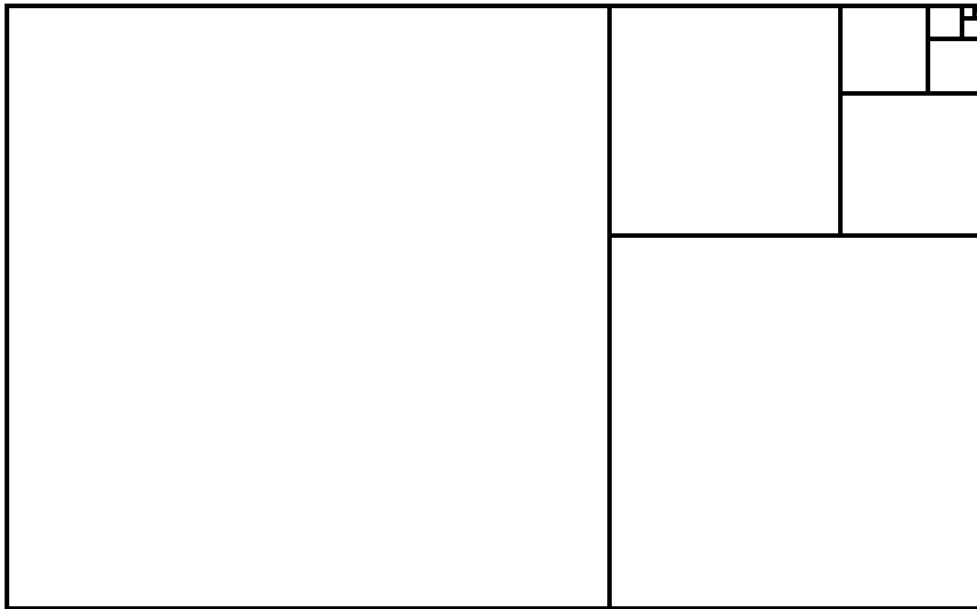
- $f_n = f_{n-1} + f_{n-2}$
- $F(x) = f_0 + f_1x + f_2x^2 + \dots + f_{n-2}x^{n-2} + f_{n-1}x^{n-1} + f_nx^n + \dots$
- $xF(x) = f_0x + f_1x^2 + f_2x^3 + \dots + f_{n-2}x^{n-1} + f_{n-1}x^n + \dots$
- $x^2F(x) = f_0x^2 + f_1x^3 + f_2x^4 + \dots + f_{n-2}x^n + f_{n-1}x^{n+1} + \dots$

$$F(x) - xF(x) - x^2F(x) = f_0 + (f_1 - f_0)x$$

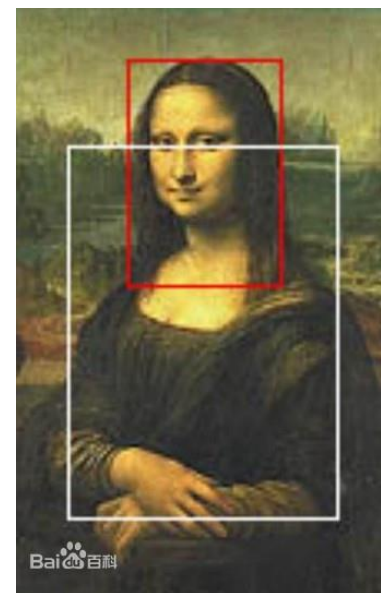
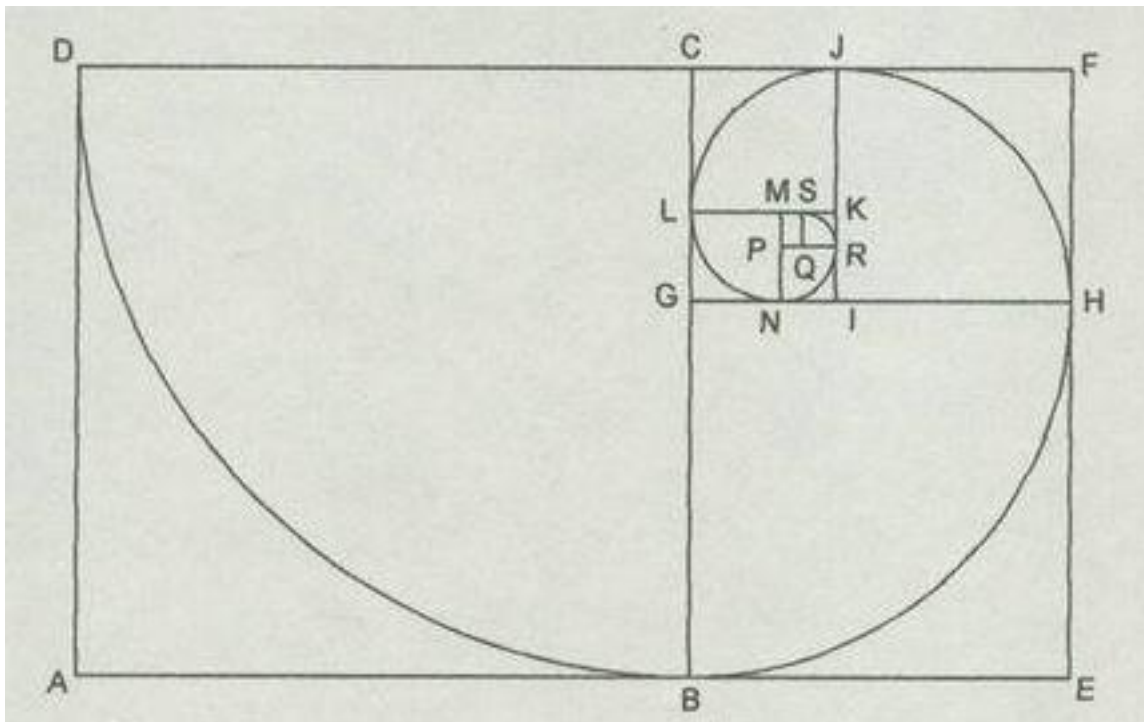
$$\begin{aligned} F(x) &= \frac{x}{1 - x - x^2} \\ &= \frac{a}{1 - \lambda_1 x} + \frac{b}{1 - \lambda_2 x} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \frac{1 + \sqrt{5}}{2}x} - \frac{1}{1 - \frac{1 - \sqrt{5}}{2}x} \right) \end{aligned}$$

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

- $\lim_{n \rightarrow \infty} \frac{F_n}{F_{n+1}} = 0.6180339$



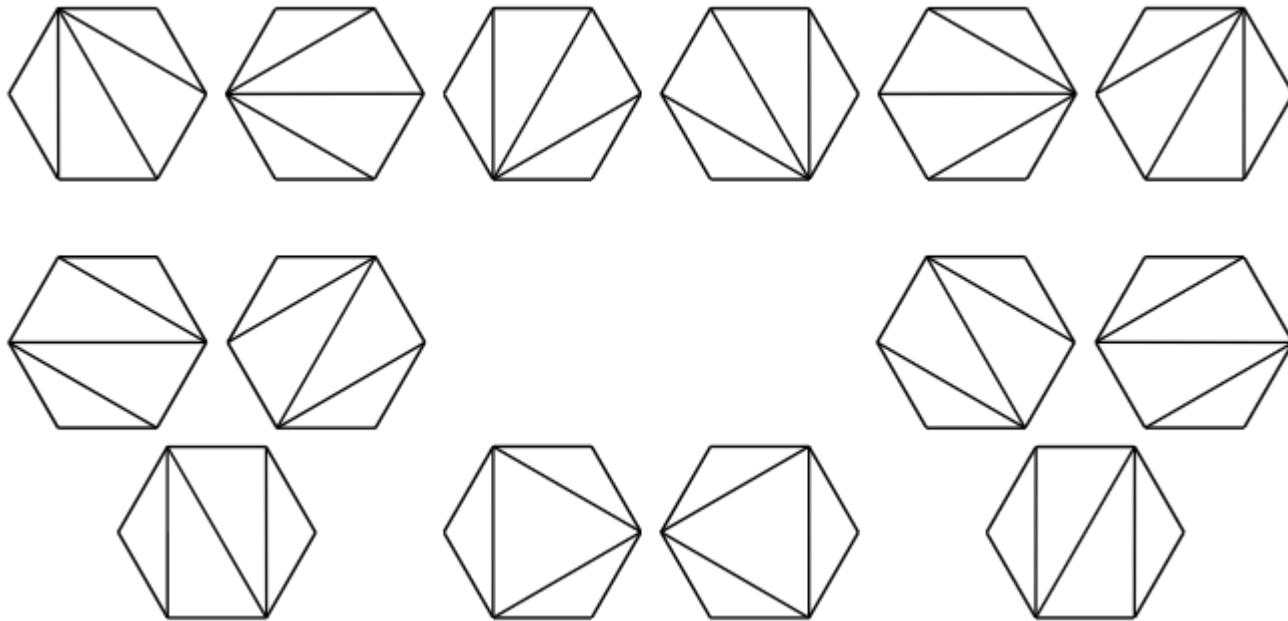
- $$\lim_{n \rightarrow \infty} \frac{F_n}{F_{n+1}} = 0.6180339$$



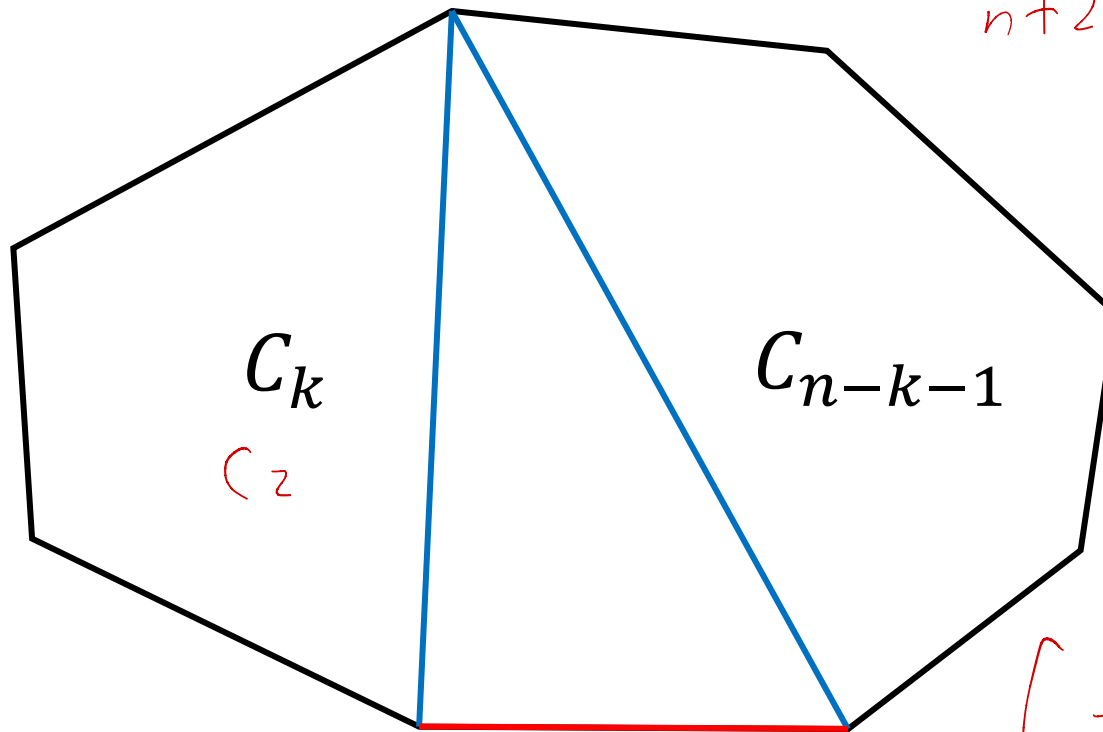
Catalan Number

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

- Number of different ways a convex polygon with $n + 2$ sides can be cut into triangles by connecting vertices with straight lines (a form of **Polygon triangulation**). 凸多边形
1つに
分割



- Number of different ways a convex polygon with $n + 2$ sides can be cut into triangles by connecting vertices with straight lines (a form of **Polygon triangulation**).



$n+2$ 'edges' $\rightarrow C_n$

$n = 5$

$$C_0 = 1, \quad C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$C_2 = C_0 C_1 + C_1 C_0 = 2$$

$$C_0 = 1,$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = \sum_{n \geq 0} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

$$(G(x))^2 = \sum_{n \geq 0} \sum_{k=0}^n C_k C_{n-k} x^n$$

$$x(G(x))^2 = \sum_{n \geq 0} \sum_{k=0}^n C_k C_{n-k} x^{n+1}$$

$$= \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-k-1} x^n$$

$$C_0 = 1,$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = \sum_{n \geq 0} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

$$x(G(x))^2 = \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-k-1} x^n$$

$$\begin{aligned} G(x) &= \sum_{n \geq 0} C_n x^n = C_0 + \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-k-1} x^n \\ &= 1 + x(G(x))^2 \end{aligned}$$

$$C_0 = 1,$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$xy^2 - y + 1 = 0$$

$$y = \frac{1 \pm \sqrt{1-4x}}{2x}$$

$$G(x) = 1 + x(G(x))^2 \Rightarrow G(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

$$\lim_{x \rightarrow 0} G(x) = C_0 = 1 \Rightarrow G(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

$$\sqrt{1-4x} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n = 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} (-4x)^n$$

$$= 1 + \sum_{n=0}^{\infty} \binom{1/2}{n+1} (-4x)^{n+1} = 1 - 4x \sum_{n=0}^{\infty} \binom{1/2}{n+1} (-4x)^n$$

$$C_0 = 1,$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \qquad \sqrt{1 - 4x} = 1 - 4x \sum_{n \geq 0} \binom{1/2}{n+1} (-4x)^n$$

$$G(x) = \frac{4x \sum_{n \geq 0} \binom{1/2}{n+1} (-4x)^n}{2x} = 2 \sum_{n \geq 0} \binom{1/2}{n+1} (-4x)^n$$

$$C_0 = 1,$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = 2 \sum_{n \geq 0} \binom{1/2}{n+1} (-4x)^n$$

$$C_n = 2 \binom{1/2}{n+1} (-4)^n = 2 \frac{(\frac{1}{2})(\frac{1}{2}-1) \cdots (\frac{1}{2}-n)}{(n+1)!} (-4)^n$$

$$\checkmark \quad ? \quad = \frac{2^n}{(n+1)!} \prod_{k=1}^n (2k-1) = \frac{2^n}{(n+1)!} \prod_{k=1}^n \frac{(2k-1)2k}{2k}$$

$$\checkmark \quad ? \quad = \frac{1}{n! (n+1)!} \prod_{k=1}^n (2k-1)2k$$

$$= \frac{(2n)!}{n! (n+1)!} = \frac{\mathbf{1}}{\mathbf{n+1}} \binom{\mathbf{2n}}{\mathbf{n}}$$

- Number of Dyck words of length $2n$:

A string consisting of n X 's and n Y 's such that no initial segment of the string has more Y 's than X 's.

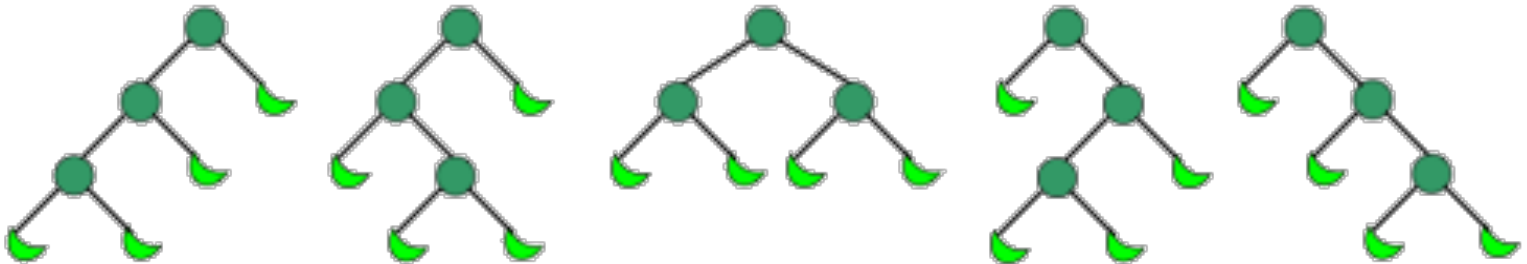
- For example, the following are the Dyck words of length 6:

XXXXYY XYXXYY XYXYXY
XXYYXY XXYYYY

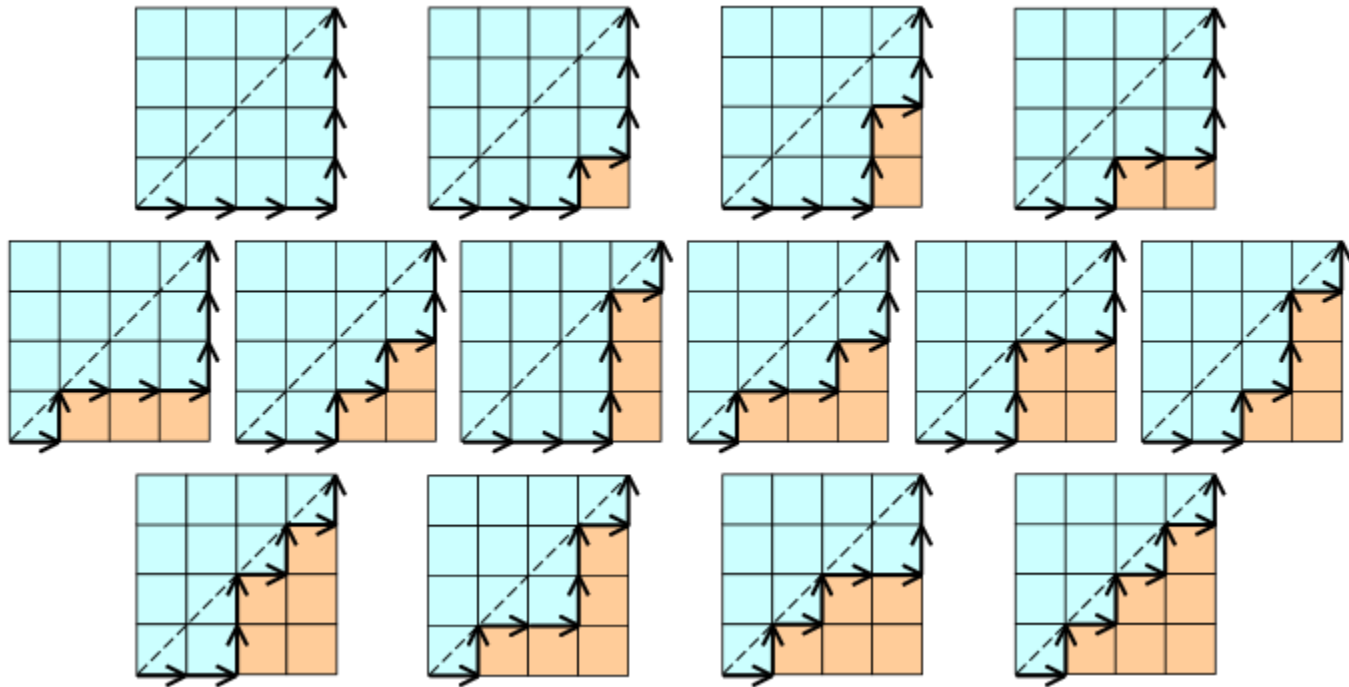
- Number of expressions containing n pairs of parentheses which are correctly matched.

((())) ()(()) ()()() (()()) (()())

- Number of **full binary trees** with $n + 1$ leaves



- Number of ^{单调的}monotonic ^{路径}lattice paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal. 对角线



Catalan Number

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$