

Homework 7

Problem 1. Fill in the blanks with either true (✓) or false (×)

$f(n)$	$g(n)$	$f = O(g)$	$f = \Omega(g)$	$f = \Theta(g)$
$2n^3 + 3n$	$100n^2 + 2n + 100$	×	✓	×
$50n + \log n$	$10n + \log \log n$	✓	✓	✓
$50n \log n$	$10n \log \log n$	×	✓	×
$\log n$	$\log^2 n$	✓	×	×
$n!$	5^n	×	✓	×

Problem 2. 1. Find two functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.

2. Furthermore, we say a function $h : \mathbb{R} \rightarrow \mathbb{R}$ is monotonically increasing if it satisfies the property ' $x \leq y \Rightarrow h(x) \leq h(y)$ '.

Find two monotonically increasing functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.

(Please give the detailed proof that your functions satisfy the requirements.)

Solution.

$$1. \begin{cases} f(x) = \sin(x); \\ g(x) = \cos(x). \end{cases}$$

$$2. \begin{cases} f(x) = x^{\sin(x)+x}; \\ g(x) = x^{\cos(x)+x}. \end{cases}$$

The detailed proof are omitted. Just stick to the definition of $O(-)$.

□

Problem 3. Prove that

(a) $\left(1 + \frac{1}{n}\right)^n \leq e$ for all $n \geq 1$.

(b) $\left(1 + \frac{1}{n}\right)^{n+1} \geq e$ for all $n \geq 1$.

(c) Using (a) and (b), conclude that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$.

Solution.

$$(a) \left(1 + \frac{1}{n}\right)^n \leq \left(e^{\frac{1}{n}}\right)^n = e.$$

$$(b) \left(1 + \frac{1}{n}\right)^{n+1} = \left(\frac{n+1}{n}\right)^{n+1} = \left(\frac{1}{\frac{n}{n+1}}\right)^{n+1} = \left(\frac{1}{1 - \frac{1}{n+1}}\right)^{n+1} \geq \left(e^{\frac{1}{n+1}}\right)^{n+1} = e.$$

$$(c) \because \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)^n}{\left(1 + \frac{1}{n}\right)^{n+1}} = 1 \therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+1}.$$

$$\text{While } e \stackrel{(b)}{\leq} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+1} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right) \stackrel{(a)}{\leq} e \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = e.$$

□

Problem 4. Prove Bernoulli's inequality: for each natural number n and for every real $x \geq -1$, we have $(1 + x)^n \geq 1 + nx$.

Solution. Apply binomial theorem to the left.

□

Problem 5. Prove that for $n = 1, 2, \dots$, we have

$$2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1.$$

Solution. Proof by induction.

□

Problem 6.

a) Show that the product of all primes p with $m < p \leq 2m$ is at most $\binom{2m}{m}$.

b) Using a), prove the estimate $\pi(x) = O\left(\frac{x}{\ln x}\right)$, where $\pi(x)$ denote the number of primes not exceeding the number x .

Solution.

1. $B = \binom{2m}{m} = \frac{(m+1) \times (m+2) \times \dots \times (2m)}{1 \times 2 \times \dots \times m}$. It is easy to find that if p is a prime number and $p \in (m, 2m]$ then $p|B$. Thus $\prod_{m < p \leq 2m} p | B$. It follows that the upper bound of the products of prime numbers between m and $2m$ is B .

2. There are several ways to prove the second problem.

First proof: Combing $a)$, w.l.o.g. assume n is even and $n = 2m$. It is obvious that

$$B \leq \sum_{i=0}^{2m} \binom{2m}{i} = 4^m$$

With $a)$ we have $\prod_{m < p \leq 2m} p \leq B \leq 4^m$ (p is prime, as above). It follows that

$$\sum_{m < p \leq 2m} \log p \leq m \log 4 = 2m \quad (\star)$$

Then count the number of primes between m and $2m$, i.e. the number of $p \in (m, 2m]$,

$$\pi(2m) - \pi(m) = \sum_{m < p \leq 2m} 1 \leq \sum_{m < p \leq 2m} \frac{\log p}{\log m} = \frac{1}{\log m} \left(\sum_{m < p \leq 2m} \log p \right) \stackrel{(\star)}{\leq} \frac{2m}{\log m}.$$

For any given x , there exists $k \geq 1$ such that $x \in (2^{k-1}, 2^k]$.

Finally with the above analysis

$$\pi(x) \leq \pi(2^k) = \sum_{i=1}^k (\pi(2^i) - \pi(2^{i-1})) = O\left(\sum_{j=1}^k \frac{2^j}{j}\right) = O\left(\frac{2^k}{k}\right) = O\left(\frac{x}{\ln x}\right).$$

Second proof: Proof by contradiction

□