# Homework 6

**Problem 1.** *Prove that any natural number $n \in \mathbb{N}$ can be written as a sum of mutually distinct Fibonacci numbers.*

*Solution.*[Hint (from the textbook *Invitation to Discrete Mathematics*)] Induction on $n$. Call a representation of $n$ as a sum of distinct Fibonacci numbers *reduced* if it uses no two consecutive Fibonacci numbers. Any representation can be converted to a reduced one by repeatedly replacing the largest two consecutive Fibonacci numbers by their sum. To go from $n$ to $n+1$ consider a reduced representation of $n$, add a 1 to it, and make it reduced again. Let us remark that a reduced representation is unique. □

**Problem 2.** *Express the $n^{th}$ term of the sequences given by the following recurrence relations*

1. $a_0 = 2, a_1 = 3, a_{n+2} = 3a_n - 2a_{n+1}$ $(n = 0, 1, 2, \ldots)$.
2. $a_0 = 1, a_{n+1} = 2a_n + 3$ $(n = 0, 1, 2, \ldots)$.

*Solution.*

1. Characteristic function is $x^2 + 2x - 3 = (x+3)(x-1) = 0$.

   Let $f_n = a(-3)^n + b \cdot 1^n$. Then $\left\{\begin{array}{lcl} 2 & = & a+b \\ 3 & = & -3a+b \end{array}\right. \Rightarrow a = -1/4, b = 9/4$.

   $\therefore$ the $n-$th term is $f_n$.

2. Characteristic function for the homogeneous part is $x = 2$. Take $a_n = p2^n + \lambda$

   $a_0 = 1, a_1 = 5$. Now $\left\{\begin{array}{lcl} 1 & = & p+\lambda \\ 5 & = & 2p+\lambda \end{array}\right. \Rightarrow p = 4, \lambda = -3$.

□

**Problem 3.** *Solve the recurrence relation $a_{n+2} = \sqrt{a_{n+1}a_n}$ with initial conditions $a_0 = 2, a_1 = 8$ and find $\lim_{n\to\infty} a_n$.*

*Solution.* Consider the sequence $b_n = \log_2 a_n$. Then

$$2\log_2 a_{n+2} = \log_2 a_{n+1} + \log_2 a_n$$

i.e. $2b_{n+2} = b_{n+1} + b_n$. $b_0 = 1, b_1 = 3$. One can find $b_n = (-\frac{4}{3})(-\frac{1}{2})^n + \frac{7}{3}$. $\therefore a_n = 2^{(-\frac{4}{3})(-\frac{1}{2})^n + \frac{7}{3}}$. $\lim_{n\to\infty} a_n = 2^{\frac{7}{3}}$. □

**Problem 4.** *Show that for any $n \geq 1$, the number $\frac{1}{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n]$ is an integer.*

*Solution*. Consider $\lambda_1 = (1+\sqrt{2})^n$, $\lambda_2 = (1-\sqrt{2})^n$. They are solutions to the characteristic function $(x-1-\sqrt{2}) \cdot (x-1+\sqrt{2}) = x^2 - 2x - 1$.

Thus the original sequence satisfies the recurrence $a_{n+2} = 2a_{n+1} + a_n$, with $a_0 = a_1 = 1$. $\square$