

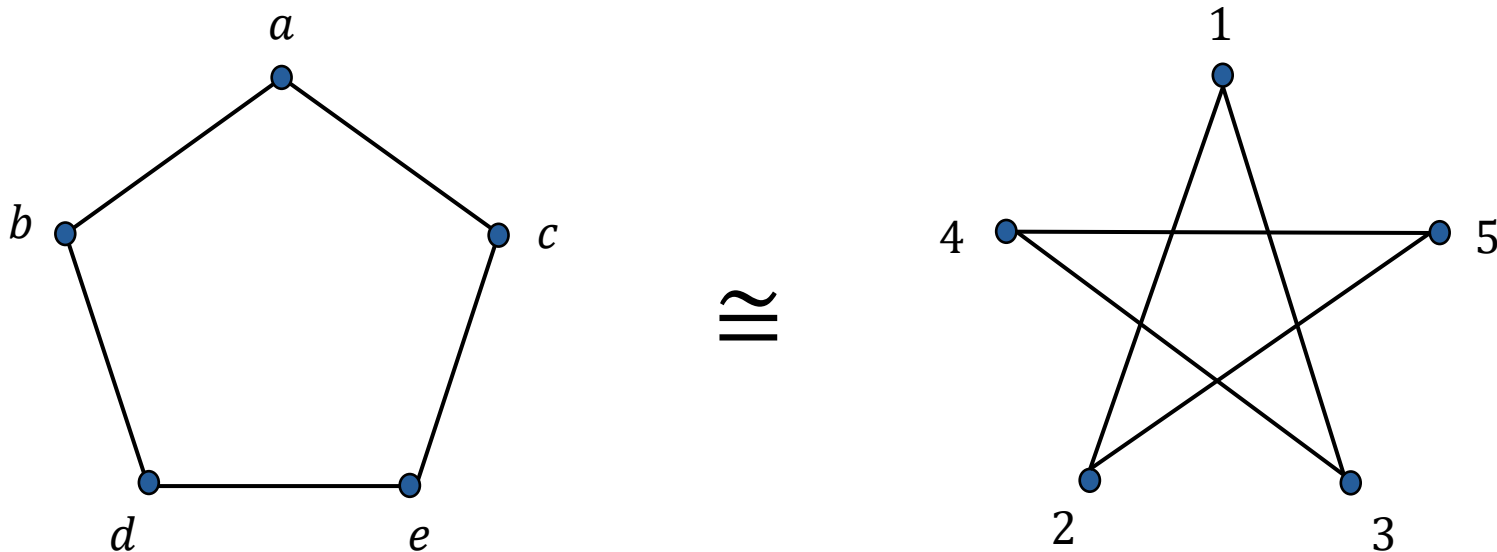
Graph: Isomorphism and Score

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图同构

- **图同构(*Graph isomorphism*)**: 若对图 $G = (V, E)$ 以及图 $G' = (V', E')$ 存在双射函数 $f: V \rightarrow V'$, 满足对任意 $x, y \in V$ 都有 $\{x, y\} \in E$ 当且仅当 $\{f(x), f(y)\} \in E'$ 那么我们称图 G 和图 G' 是同构的。
- 用符号图 $\mathbf{G} \cong \mathbf{G'}$ 表示图同构。
- 直观: 同构的图之间, 仅仅是顶点的名字不同。

图同构的例子



$$f: a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 5, e \mapsto 4$$

顶点映射关系

History

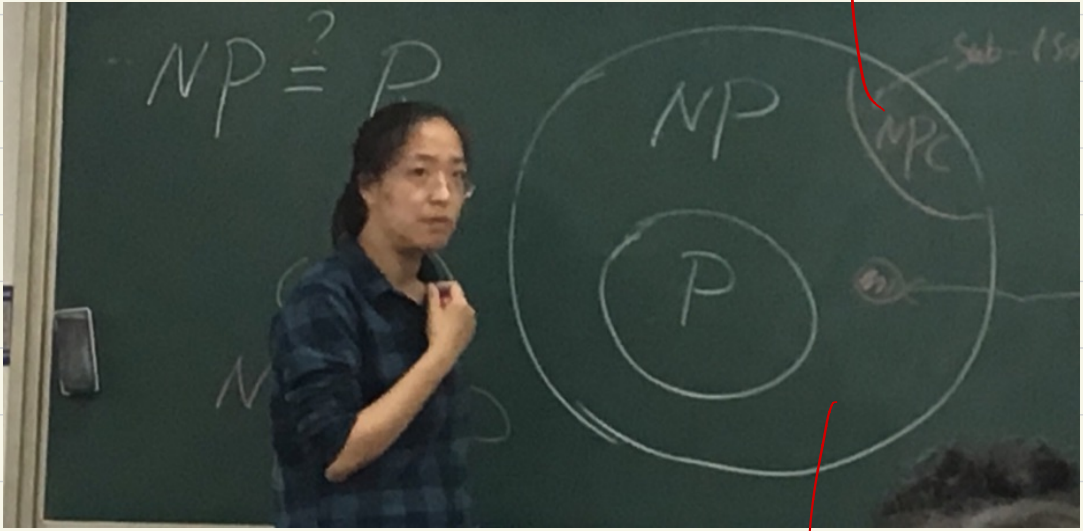
- In November 2015, [László Babai](#), a mathematician and computer scientist at the University of Chicago, claimed to have proven that the graph isomorphism problem is solvable in [quasi-polynomial time](#). This work has not yet been vetted. In January 2017, Babai shortly retracted the quasi-polynomiality claim and stated a [sub-exponential time](#) time complexity bound instead. He restored the original claim five days later. $2^{O(\log^c n)}$
并非多项式
- Interestingly, in July 2016, Wenxue Du, a Chinese mathematician at the Anhui University, devised an algorithm outputting a generating set and a block family of the automorphism group of a graph within time $n^{C \log n}$ for some constant C . patent ! tricky



- ✓ In 1988, Babai won the Hungarian State Prize, in 1990 he was elected as a corresponding member of the Hungarian Academy of Sciences, and in 1994 he became a full member. In 1999 the [Budapest University of Technology and Economics](#) awarded him an honorary doctorate.
- ✓ In 1993, Babai was awarded the [Gödel Prize](#) together with [Shafi Goldwasser](#), [Silvio Micali](#), [Shlomo Moran](#), and [Charles Rackoff](#), for their papers on interactive proof systems.^[17]
- ✓ In 2015, he was elected^[18] a fellow of the [American Academy of Arts and Sciences](#), and won the [Knuth Prize](#).

子图同构问题

$NP \supset P$ (LGI) 相信, NP complex



同构

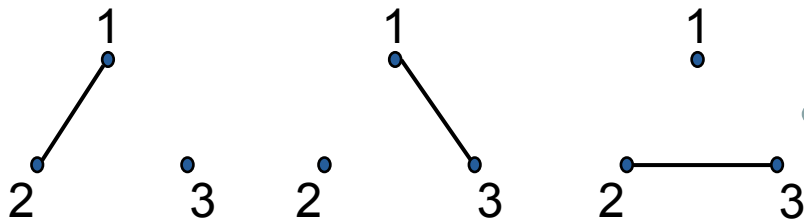
(人们相信)

图的计数

- **问题：**以集合 $V = \{1, 2, \dots, n\}$ 中的元素为顶点构造图， $G = (V, E)$ 其中 $E \subseteq \binom{V}{2}$ ，求问能构成多少个图？

- **解：** $|\binom{V}{2}| = \binom{n}{2}$ ，为 K_n 的边数目。

每条边有两种可能，故以 V 为顶点的图共有 $2^{\binom{n}{2}}$ 种。

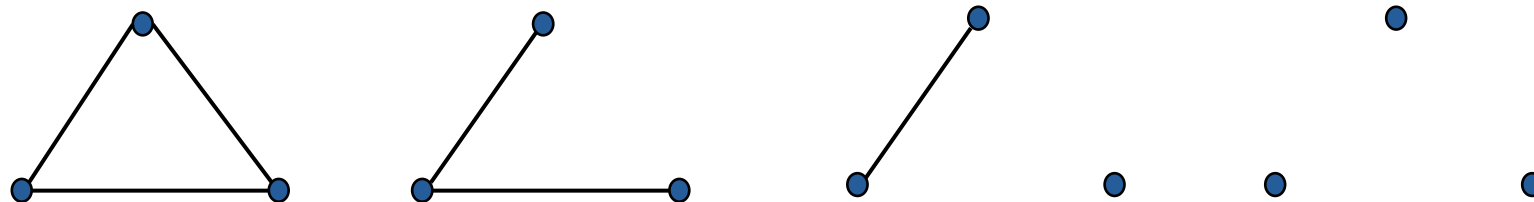


同构

考虑结构上的相似性，有多少图是彼此不同的？

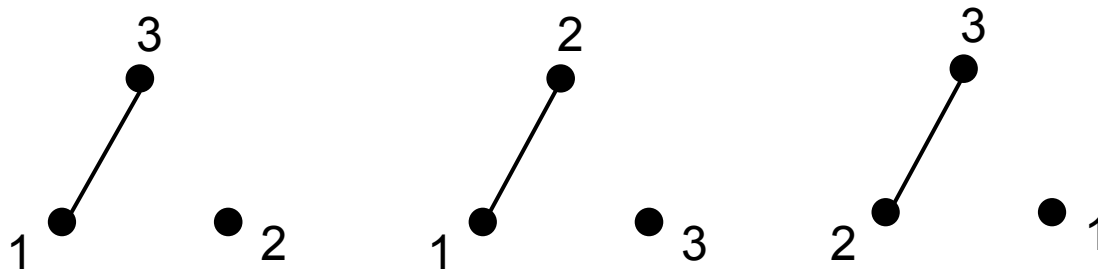
非同构图计数

- **问题：**以集合 $V = \{1, 2, \dots, n\}$ 中的元素为顶点构造图， $G = (V, E)$ 其中 $E \subseteq \binom{V}{2}$ ，求问**彼此不同构**的图有多少个？
- 例：含三个顶点的彼此不同构的图只有以下4种：



$$4 < 2^{\binom{3}{2}} = 8$$

- 显然，（同构）图的个数不会超过所有图的个数（是 $2^{\binom{n}{2}}$ ）。
- 与此同时，任一 $G = (V, E)$ 至多与 $n!$ 个 V 上不同的图同构。
- 例： $3! = 6$ ， 但与第一张图同构且互不相同的图只有三种。



- 解： 设 n 个顶点且不同构的图有 x 个， 则：

$$\frac{2^{\binom{n}{2}}}{n!} \leq x \leq 2^{\binom{n}{2}}$$

- 我们可以对上下界估值：

$$-\log_2 2^{\binom{n}{2}} = \binom{n}{2} = \frac{n^2}{2} \left(1 - \frac{1}{n}\right)$$

$$-\log_2 \frac{2^{\binom{n}{2}}}{n!} = \binom{n}{2} - \log_2 n!$$

$$\geq \binom{n}{2} - \log_2 n^n$$

$$= \frac{n^2}{2} \left(1 - \frac{1}{n} - \frac{2 \log_2 n}{n}\right)$$

$$x = \Theta\left(2^{\frac{n^2}{2}}\right)$$

同一数量级

Graph Score

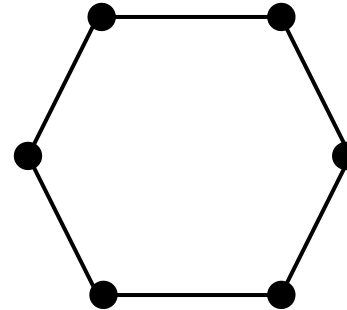
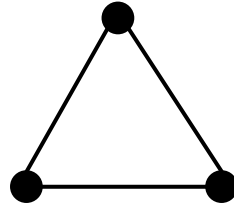
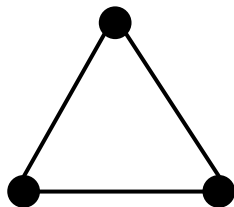
- Let G be a graph. The vertices of G be v_1, v_2, \dots, v_n . The the **degree sequence** of G , or a **score** of G is:

$$(\deg_G(v_1), \deg_G(v_2), \dots, \deg_G(v_n))$$

- Two scores are **equal** to each other if one can be obtained from the other by rearranging the order of the numbers.

- Isomorphic graphs \Rightarrow The same scores.
- The same scores $\neq \Rightarrow$ Isomorphic graphs.

反之不一定对



$(2,2,2,2,2,2)$

$(2,2,2,2,2,2)$

Not every finite sequence is a graph Score.

Score Theorem

Let $D = (d_1, d_2, \dots, d_n)$ be a sequence of natural numbers, $n > 1$. Suppose that $d_1 \leq d_2 \leq \dots \leq d_n$, and let the symbol D' denote the sequence $(d_1', d_2', \dots, d_{n-1}')$, where

$$d_i' = \begin{cases} d_i & \text{if } i < n - d_n \\ d_i - 1 & \text{if } i \geq n - d_n \end{cases}$$

Then D is a graph score iff D' is a graph score.

Application

Thm: Let $D = (d_1, d_2, \dots, d_n)$ be a sequence of natural numbers, $n > 1$. Suppose that $d_1 \leq d_2 \leq \dots \leq d_n$, and let the symbol D' denote the sequence

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Then D is a graph score iff D' is a graph score.

- $(1, 1, 1, 2, 2, 3, 4, 5, 5)$
- $(1, 1, 1, 1, 1, 2, 3, 4)$
- ~~$(1, 1, 1, 0, 0, 1, 2)$~~
- $(0, 0, 1, 1, 1, 1, 2)$
- ~~$(0, 0, 1, 1, 0, 0)$~~
- $(0, 0, 0, 0, 1, 1)$
- $(0, 0, 0, 0, 0)$

Proof

Thm: Let $D = (d_1, d_2, \dots, d_n)$ be a sequence of natural numbers, $n > 1$. Suppose that $d_1 \leq d_2 \leq \dots \leq d_n$, and let the symbol D' denote the sequence

$(d'_1, d'_2, \dots, d'_{n-1})$, where

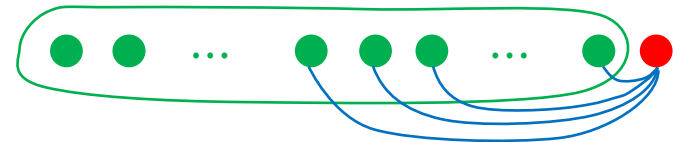
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Then D is a graph score iff D' is a graph score.

• (if)

$G' = (V', E')$, where $V' = \{v_1, v_2, \dots, v_{n-1}\}$

New vertex v_n



$G = (V, E)$

$V = V' \cup \{v_n\}$

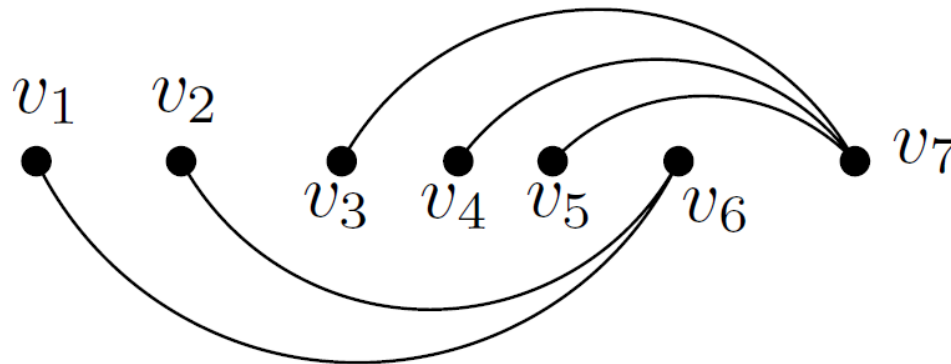
$$E = E' \cup \{\{v_i, v_n\} : i = n - d_n, n - d_n + 1, \dots, n - 1\}.$$

Thm: Let $D = (d_1, d_2, \dots, d_n)$ be a sequence of natural numbers, $n > 1$. Suppose that $d_1 \leq d_2 \leq \dots \leq d_n$, and let the symbol D' denote the sequence $(d'_1, d'_2, \dots, d'_{n-1})$, where

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Then D is a graph score iff D' is a graph score.

- (Only if)

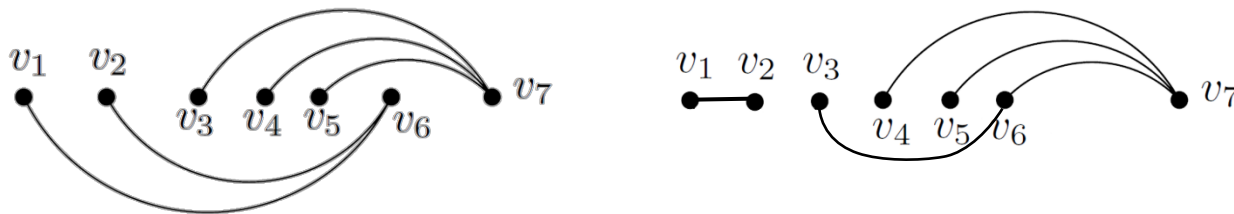


Thm: Let $D = (d_1, d_2, \dots, d_n)$ be a sequence of natural numbers, $n > 1$. Suppose that $d_1 \leq d_2 \leq \dots \leq d_n$, and let the symbol D' denote the sequence $(d'_1, d'_2, \dots, d'_{n-1})$, where

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Then D is a graph score iff D' is a graph score.

- (Only if)



The set \hat{G} of all graphs on the vertex set $\{v_1, \dots, v_n\}$ in which the degree of each vertex v_i equals d_i , $i = 1, 2, \dots, n$. It will be *sufficient* to prove the following claim

Claim. The set \hat{G} contains **a** graph G_0 in which the vertex v_n is adjacent **exactly** to the **last d_n vertices**, i.e. to vertices $v_{n-d_n}, v_{n-d_n+1}, \dots, v_{n-1}$.

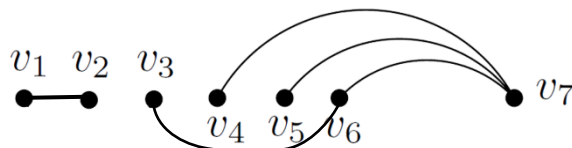
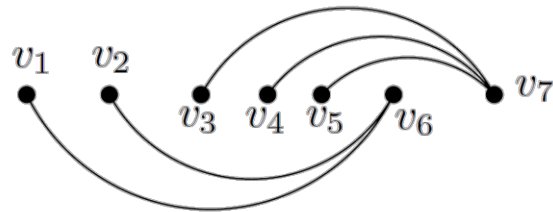
Claim. The set \hat{G} contains a graph G_0 in which the vertex v_n is adjacent *exactly* to the *last d_n vertices*, i.e. to vertices $v_{n-d_n}, v_{n-d_n+1}, \dots, v_{n-1}$.

- If $d_n = n - 1$, then any graph from \hat{G} satisfies the claim.

- O.W. $d_n < n - 1: \forall G \in \hat{G}$

$- j(G) =$

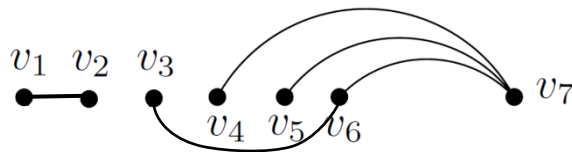
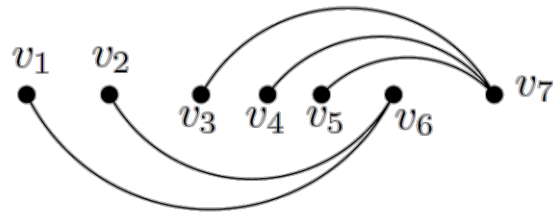
$$\text{Max} \{ j \in \{1, 2, \dots, n - 1\} \mid \{v_j, v_n\} \notin E(G) \}$$



考虑一个 \hat{G} 最大
不相交的 \hat{G} 点集 \hat{G}

Claim. The set \hat{G} contains a graph G_0 in which the vertex v_n is adjacent *exactly* to the *last d_n vertices*, i.e. to vertices $v_{n-d_n}, v_{n-d_n+1}, \dots, v_{n-1}$.

- If $d_n = n - 1$, then any graph from \hat{G} satisfies the claim.
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➤ Let G_0 be a graph in \hat{G} with *smallest* possible value of $j(G)$.

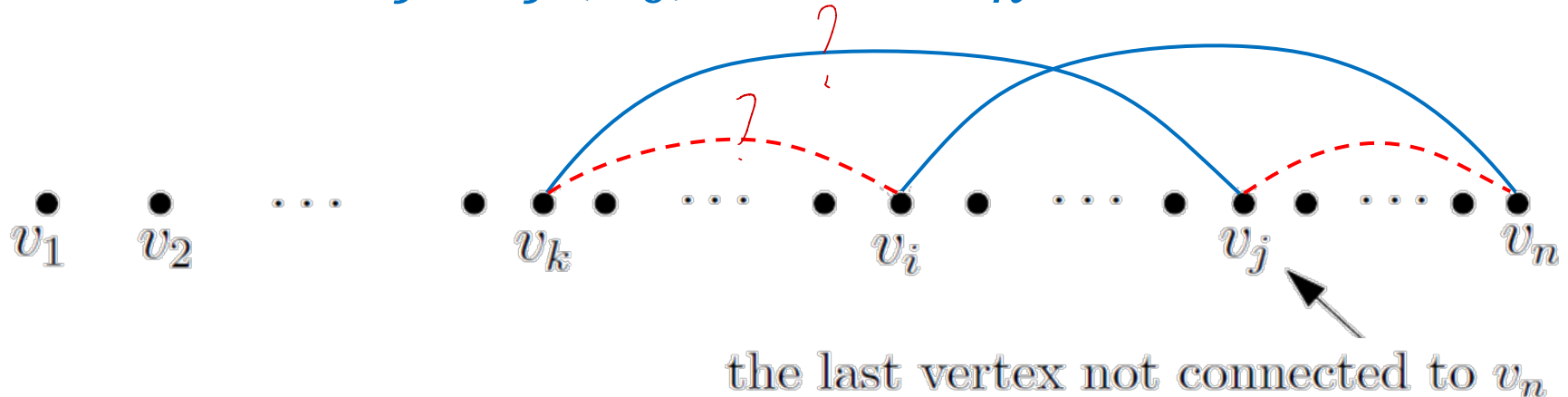
➤ Prove:

$$j(G_0) = n - d_n - 1$$

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- (Proof by contradiction) Suppose

$$j = j(G_0) > n - d_n - 1$$



$G' = (V, E')$ where

$$E' = \left(E(G_0) \setminus \left\{ \{v_i, v_n\}, \{v_j, v_k\} \right\} \right) \cup \left\{ \{v_j, v_n\}, \{v_i, v_k\} \right\}$$

The score of G' and G_0 are the same. There is a contradiction as $J(G') \leq J(G_0) - 1$.