Prob1: 1. 证明:

$$\sum_{j=0}^{n-r} C_{r+j}^{r} = C_{n+1}^{r+1}$$

由定理 $C_n^k = C_{n-1}^{k} + C_{n-1}^{k-1}$

可知: 右边 $= C_n^{r+1} + C_n^{r}$

$$= C_n^{r} + (C_{n-1}^{r+1} + C_{n-1}^{r})$$

$$= C_n^{r} + C_{n-1}^{r} + C_{n-2}^{r+1} + C_{n-2}^{r}$$

$$= \cdots = C_n^{r} + C_{n-1}^{r} + C_{n-2}^{r} + C_{n-3}^{r}$$

$$+ \cdots + C_{r+1}^{r} + C_{r+1}^{r+1}$$

$$= C_n^r + C_{n-1}^r + \cdots + C_{r+1}^r + C_r^r$$

= 左边

2、证明：$\sum_{k=0}^{n} C_{m+k-1}^{k} = C_{n+m}^{n}$

左边 $= C_{m-1}^0 + C_m^1 + C_{m+1}^2 + \cdots + C_{n+m-3}^{n-2}$

$$+ C_{n+m-2}^{n-1} + C_{n+m-1}^{n}$$

右边 $= C_{n+m}^n = C_{n+m-1}^{n-1} + C_{n+m-1}^{n}$

$$= C_{n+m-1}^n + C_{n+m-2}^{n-1} + C_{n+m-2}^{n-2}$$

$$= C_{n+m-1}^{n} + C_{n+m-2}^{n-1} + C_{n+m-3}^{n-2} + C_{n+m-3}^{n-3}$$

$$= C_{n+m-1}^{n} + C_{n+m-2}^{n-1} + C_{n+m-3}^{n-2} + \cdots$$

$$+ C_{m}^{1} + C_{m}^{0}$$

$$= C_{n+m-1}^{n} + C_{n+m-2}^{n-1} + \cdots + C_{m}^{1} + C_{m-1}^{0}$$

= 左边，得证

prob2: ~~解~~: $C_{k}^{m} \cdot C_{n}^{k}$

$$= \frac{k!}{(k-m)!\,m!} \cdot \frac{n!}{(n-k)!\,k!}$$

$$= \frac{n!}{m!\,(k-m)!\,(n-k)!}$$

$$= \frac{n!}{(n-m)!\,m!} \cdot \frac{(n-m)!}{(n-k)!\,(k-m)!}$$

$$= C_n^m \cdot C_{n-m}^{n-k}$$

$$\therefore \sum_{k=m}^{n} \binom{k}{m} \binom{n}{k}$$

$$= C_n^m \cdot \sum_{k=m}^{n} C_{n-m}^{n-k}$$

$$= C_n^m \cdot 2^{n-m}$$

prob3: 1. $\sum_{k=1}^{n} C_k^m \frac{1}{k}$

$$= \sum_{k=1}^{n} \frac{k!}{(k-m)!\,m!} \frac{1}{k}$$

$$= \sum_{k=1}^{n} \frac{(k-1)!}{(k-m)!\,m!}$$

$$= \sum_{k=1}^{n} \frac{1}{m} \cdot \frac{(k-1)!}{(k-m)!\,(m-1)!}$$

$$= \sum_{k=1}^{n} \frac{1}{m} C_{k-1}^{m-1} = \frac{1}{m} \sum_{k=0}^{n-1} C_k^{m-1}$$

$$= \frac{1}{m} C_n^m$$ (prob1的结论)

2. $\sum_{k=0}^{n} C_k^m \, k$

$$= \sum_{k=0}^{n} \frac{(m+1)(k+1)}{m+1} C_k^m - \sum_{k=0}^{n} C_k^m$$

$$= (m+1) \sum_{k=0}^{n} \frac{k!\,(k+1)}{(k-m)!\,m!\,(m+1)}$$

$$- C_{n+1}^{m+1}$$

$$= (m+1) \sum_{k=0}^{n} \frac{(k+1)!}{(k-m)!\,(m+1)!} - C_{n+1}^{m+1}$$

$$= (m+1) \cdot \sum_{k=0}^{n} C_{k+1}^{m+1} - C_{n+1}^{m+1}$$

$$= (m+1) \cdot C_{n+2}^{m+2} - C_{n+1}^{m+1}$$

Prob4: (a): 代 $r=2$ 矩

$$C_{n+1}^{3} = C_2^2 + C_3^2 + C_4^2 + \cdots$$

$$+ C_{n-1}^{2} + C_n^2$$

$$= \frac{2\times1}{2} + \frac{3\times2}{2} + \frac{4\times3}{2}$$

$$+ \cdots + \frac{(n-1)(n-2)}{2} + \frac{n(n-1)}{2}$$

$$\therefore 2C_{n+1}^{3} = 2\times1 + 3\times2 + \cdots + n(n-1)$$

即 $\sum_{j=2}^{n} j(j-1) = 2C_{n+1}^{3} = \frac{(n+1)n\cdot(n-1)}{3}$

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{1} i^2 + \sum_{i=2}^{n} i^2$$

$$= 1 + \sum_{i=2}^{n} \left[i(i-1) + i \right]$$

$$= 1 + \frac{(n+1)\,n\,(n-1)}{3} + \sum_{i=2}^{n} i$$

$$= 1 + \frac{(n+1)\,n\,(n-1)}{3} + 2 + 3 + \cdots + n$$

$$= 1 + \frac{(n+1)\,n\,(n-1)}{3} + \frac{(2+n)(n-1)}{2}$$

$$= 1 + \frac{1}{3}(n^3 - n) + \frac{n^2 + n - 2}{2}$$

$$= \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{n}{6}$$

(b) 令 $r=3$, 知 C_{n+1}^{4}

$$= C_3^3 + C_4^3 + C_5^3 + \cdots + C_{n-1}^3 + C_n^3$$

$$= \frac{3\times2\times1}{6} + \frac{4\times3\times2}{6} + \frac{5\times4\times3}{6} + \cdots + \frac{(n-1)(n-2)(n-3)}{6} + \frac{n(n-1)(n-2)}{6}$$

$$\therefore 3\times2\times1 + 4\times3\times2 + \cdots + \frac{n(n-1)(n-2)}{6}$$

$$= 6\,\frac{(n+1)n(n-1)(n-2)}{24} = \sum_{i=3}^{n} i(i-1)(i-2)$$

即 $\sum_{j=3}^{n} j^3-3j^2+2j = \frac{(n+1)n(n-1)(n-2)}{4}$

$\therefore \sum_{j=1}^{n} j^3 = \sum_{j=1}^{2} j^3 + \sum_{j=3}^{n} j^3$

$= 9 + \sum_{j=3}^{n} \left[(j^3-3j^2+2j)+3j^2-2j\right]$

$= 9 + \frac{(n+1)n(n-1)(n-2)}{4}$

$+ 3\left(\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{n}{6}-5\right) - 2\left[3+4+\dots+n\right]$

$= 9 + \frac{n^4-2n^3-n^2+2n}{4} + n^3+\frac{3}{2}n^2+\frac{n}{2}-15$

$-2\cdot\frac{(3+n)(n-2)}{2} = \frac{n^4+2n^3+n^2}{4}$

Prob5: 定义域: $\{1,2,\dots,n\}$

值域: $\{1,2,\cdots n\}$

分析: 当确定了值域中每个元素被映射的次数后, 函数便唯一确定了

设 x_i 为 i 被映射的次数

则有方程 $x_1+x_2+\cdots+x_n=n$

求出其非负解的个数即可得出结论

由定理: $x_1+x_2+\cdots+x_r=n$ 有

C_{n+r-1}^{r-1} 个非负解

$\because x_1 + x_2 + \cdots + x_n = n$

有 $C_{2n-1}^{n-1} = \dfrac{(2n-1)!}{(n-1)!\,n!}$ 个解

即有 $\dfrac{(2n-1)!}{(n-1)!\,n!}$ 个解