`PytHHOn3D` documentation

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1 The HHO method

1.1 Principle

The HHO method is a so-called "non-conformal method" (as opposed to conformal ones, among which is the Galerkin method). The main feature of non-conformal method lies in the foundation of the definition of the problem (**??**)

The HHO method, as well as the FE method, consists in discretizing (**??**) both geometrically and functionally in order to actually compute approximated solutions that are not analytically reachable.

Contrary to the FE method, the HHO method does not consider finding the solution in $H^1(\Omega;\mathbb{R}^d)$, but not in a richer space, namely the broken Sobolev space. Given, a mesh $\mathcal{T}_h(\Omega)$ of Ω (see 1.2 for further description of what a mesh is within the framework of the HHO method), one defines the broken Sobolev space in the following way :

Definition : **Broken Sobolev space**

$$H^1(\mathcal{T}_h;\mathbb{R}^d) = \left\{ \underline{\boldsymbol{u}} \in L^2(\Omega;\mathbb{R}^d) \;\middle|\; \forall T \in \mathcal{T}_h(\Omega), \underline{\boldsymbol{u}}|_T \in H^1(T;\mathbb{R}^d) \right\} \tag{1}$$

In other words, the broken Sobolev space is that of all piece-wise Sobolev space in every element K in $\mathcal{T}_h(\Omega)$: hence, discontinuities or jumps across elements are allowed.

> **Remark** : **Mechanical interpretation of the broken Sobolev space**
> In mechanical terms, considerting the broken Sobolev space for the displacement $\underline{\boldsymbol{u}}$

1.2 Admissible meshes

As mentioned above, contrary to the FE method, the HHO method separates the notions of mesh and that of polynomial basis (or shape function in the FE method) : the latter are not intrinsically linked to the mesh as it is the case with the FE method (see section 1.3), which allows to consider a larger range of admssible meshes than those spanned by the FE method in the framework of the HHO method :

Definition : **Admissible meshes**

A mesh $\mathcal{T}_h(\Omega)$ over Ω is said to be admissible if it is is a finite collection of **nonempty disjoint open convex polytopes** T with **planar faces**, such that $\Omega = \cup_{T\in\mathcal{T}_h(\Omega)} T$, and :

$$h = \sup_{T\in\mathcal{T}_h(\Omega)} \{h_T\} \text{ where } h_T \text{ denotes the diameter of } T \tag{2}$$

Since any polytopal domain is considered, elements in the HHO method can directly take the shape of any grain in a crystal for instance, giving a direct geometrical and physical interpretation to the mesh.

> **Exemple** : **Admissible meshes**
> EXEMPLE DE MESH

1.3 Polynomial basis

As mentioned above, the polynomial basis of $P_h^k(\mathcal{T}_h;\mathbb{R}^d)$ considered in the HHO method is not the Lagrange polynomial basis, but the monomial basis, which is the assembly of monomials in $P_h^k(\mathcal{T}_h;\mathbb{R}^d)$ (*i.e.* the assembly of all $x \mapsto x^\alpha$ where α defines the power of the monomial). In addition, the monomial is sclaed with respect to the element in which it acts, so that values taken by the unknown at a given point in Ω are not influenced by the mesh topolgy.

In order to properly define such a basis, we define the following exponents sets :

Definition : **Scaled monomial exponents**

Let $k \geq 0$ and $d \geq 1$ two integers. One defines the exponents vector set $\boldsymbol{\alpha}(d,k)$:

$$\boldsymbol{\alpha}(d,k) = \left\{ \boldsymbol{\alpha}_j(d) \;\middle|\; 0 \leq j \leq k \right\} \quad \text{and} \quad \boldsymbol{\alpha}_k(d) = \left\{ \underline{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_d) \in \mathbb{N}^d \;\middle|\; \textstyle\sum_{1\leq i\leq d} \boldsymbol{\alpha}_i = k \right\} \tag{3}$$

> **Exemple** : **Scaled monomial exponents**
> For $k = 3$ and $d = 1$, $\boldsymbol{\alpha}_3(1) = \{(3)\}$ and $\boldsymbol{\alpha}(1,3) = \{(0),(1),(2),(3)\}$
> For $k = 2$ and $d = 2$, $\boldsymbol{\alpha}_2(2) = \{(0,2),(1,1),(2,0)\}$ and $\boldsymbol{\alpha}(2,2) = \{(0,0),(0,1),(1,0),(0,2),(1,1),(2,0)\}$

$\boldsymbol{\alpha}_k(d)$ and $\boldsymbol{\alpha}(d,k)$ define exponents vectors sets that entirely define the size of the polynomial basis to be used. As mentioned above, the chosen polynomial basis is scaled, hence we define the scaled monomial basis with respect to a given domain in which it acts :

Definition : **Scaled monomial basis**

Let $1 \leq d \leq 3$ and $k \geq 0$ two integers. Let $D \subset \mathbb{R}^d$ a closed domain of volume $\boldsymbol{v}_D$ and of barycenter $\underline{\boldsymbol{x}}_D$. The (scaled) monomial basis $\mathcal{B}_{sm}(k,d)$ of polynbomials of order k in D writes as :

$$\mathcal{B}_{sm}(k,d) = \left\{ \underline{\boldsymbol{x}} \mapsto \prod_{1\leq i\leq d} \left(\frac{\boldsymbol{x}_i - \boldsymbol{x}_{iD}}{\boldsymbol{v}_D} \right)^{\boldsymbol{\alpha}_i} \;\middle|\; \underline{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_d) \in \boldsymbol{\alpha}(d,k) \right\} \tag{4}$$

In particular, the dimension N_k^d of $\mathcal{B}_{sm}(k,d)$ is $N_d^k = \binom{d+k}{k}$

1.4 HHO elements

References