Contents

1 The HHO method from the mechanical standpoint

1.1 Introduction

The Hybird High Order method (HHO) is a discontinuous discretization method, that takes root in the Discontinuous Galerkin method (DG). From the physical standpoint, DG methods ensure the continuity of the flux across interfaces, by seeking the solution element-wise, hence allowing jumps of the potential across elements. They can be seen as a generalization of Finite Volume methods, and are able to capture physically relevant discontinuities without producing spurious oscillations.

The origin of DG methods dates back to the pioneering work of [27], where an hyperbolic formualtion is used to solve the neutron transport equation. The first application of the method to elliptic problems originates in [2] where Nitsche's method [23] is used to weakly impose continuity of the flux across interfaces. In 2002, Hansbo and Larson [16] were the first to consider the Nitsche's classical DG method for nearly incompressible elasticity. They showed, theoretically and numerically, that this method is free from volumetric locking. However, the bilinear form arising from this formulation is not symmetric. A so called interior penalty term has been introduced in [34], leading to the Symmetric Interior Penalty (SIP) DG method. A first study of the method to linear elasticity has been devised by [28], where optimal error estimate has been proved. [17] generalized the Symmetric Interior Penalty method to linear elasticity. period of time, DG methods were proposed for other linear problems in solid mechanics, such as Timoshenko beams [6], Bernoulli-Euler beam and the Poisson-Kirchhoff plate [5, 14] and Reissner-Mindlin plates [1]. In the mid 2000's, the first applications of DG methods to nonlinear elasticity problems was undertaken by [33, 24], and in 2007, Ortner and Süli [25] carried out the a priori error analysis of DG methods for nonlinear elasticity.

DG methods then sollicitated a vigourus interest, mostly in fluid dynamics [29, 26] due to their local conservative property and stability in convection domniated problems. However, except some applications for instance in fracture mechanics using XFEM methods [15, 30], or gradient plasticity [13, 12] DG methods did not break through in computational solid mechanics because of their numerical cost, since nodal unknowns need be duplicated to define local basis functions in each element.

To adress this problem, in the early 2010's, [7, 32] introduced additional faces unknowns on element interfaces for linear elastic problem, hence leading to the hybridization of DG methods, or Hybridizable Discontinuous Galerkin method (HDG). By adding supplementary boundary unknowns, the authors actually allowed to eliminate original cell unknowns by a static condensation process, in order to express the global problem on faces ones only. Extension of HDG methods to non-linear elasticity were first undertaken in [31] and have then fueled intense reaserch works for various applications such as linear and non-linear convection-diffusion problems [19, 20, 21], incompressible stokes flows [21, 22] and non-linear mechanics [18].

In [8, 9], the authors introduced a higher order potential reconstruction operator in the classical HDG formulation for elliptic problems, providing a $h^{k+1}H^1$-norm convergence rate as compared to the ususal h^k-rate. This higher order term coined the name for the so called HHO method.

Recent developments of HHO methods in computational mechanics include the incompressible Stokes equations (with possibly large irrotational forces) [10], the incompressible Navier–Stokes equations [11], Biot's consolidation problem [3], and nonlinear elasticity with small deformations [4]

The difference between HHO and HDG methods is twofold: (1) the HHO reconstruction operator replaces the discrete HDG flux (a similar rewriting of an HDG method for nonlinear elastic- ity can be found in [29]), and, more importantly, (2) both HHO and HDG penalize in a least-squares sense the differ- ence between the discrete trace unknown and the trace of the discrete primal unknown (with a possibly mesh-dependent weight), but HHO uses a non-local operator over each mesh cell boundary that delivers one-order higher approximation than just penalizing pointwise the difference as in HDG. Discretization methods for linear and nonlinear elastic- ity have undergone a vigorous development over the last decade. For discontinuous Galerkin (dG) methods, we men- tion in particular [14,26,32] for linear elasticity, and [35,41] for nonlinear elasticity. HDG methods for linear elasticity have been coined in [38] (see also [13] for incompressible Stokes flows), and extensions to nonlinear elasticity can be found in [29,34,37]. Other recent developments in the last few years include, among others, Gradient Schemes for nonlinear elasticity with small deformations [22], the Virtual Element Method (VEM) for linear and nonlinear elasticity with small [3] and finite deformations [8,43], the (low-order) hybrid dG method with conforming traces for nonlinear elasticity [44], the hybridizable weakly conforming Galerkin method with nonconforming traces for linear elasticity [30], the Weak Galerkin method for linear elasticity [42], and the discon- tinuous Petrov–Galerkin method for linear elasticity [7].

1.2 Description of the model problem

Let $d \in \{1, 2\}$ the euclidean dimension of the cartesian space $\mathbb{R}^d$, and $\mathcal{R}_d$ the euclidean reference frame. Let $\Omega \subset \mathbb{R}^d$ a solid body with boundary $\partial\Omega \subset \mathbb{R}^{d-1}$, that deforms under the volumic load $\boldsymbol{f}_v$. It is subjected to a prescribed displacement $\boldsymbol{u}_d$ on the Dirichlet boundary $\partial\Omega_d$, and to a contact load $\boldsymbol{t}_n$ on the Neumann boundary $\partial\Omega_n$, such that $\partial\Omega = \partial\Omega_d \cup \partial\Omega_n$ and $\partial\Omega_d \cap \partial\Omega_n = \emptyset$.

The initial configuration of the body (see Figure 1) is denoted $\Omega_0 \in \mathbb{R}^d$ with respective Dirichlet and Neumann boundaries $\partial\Omega_D$ and $\partial\Omega_N$. The transformation mapping $\boldsymbol{\Phi}$ takes a point $\boldsymbol{X} \in \Omega_0$ to $\boldsymbol{x} \in \Omega$, such that $\boldsymbol{x} = \boldsymbol{\Phi}(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X})$ where $\boldsymbol{u}$ denotes the displacement of the physical point. Let $\underset{\sim}{\boldsymbol{F}} = \nabla_X \boldsymbol{\Phi} = \underset{\sim}{\boldsymbol{1}} + \nabla_X \boldsymbol{u}$ the transformation gradient. The mechanical problem to solve reads, find $\boldsymbol{u}$ such that:

$$\underset{\sim}{\boldsymbol{F}} - \nabla_X \boldsymbol{u} = \underset{\sim}{\boldsymbol{1}} \qquad \text{in } \Omega_0 \tag{1a}$$

$$\underset{\sim}{\boldsymbol{P}} - \frac{\partial \psi_{mec}}{\partial \underset{\sim}{\boldsymbol{F}}} = 0 \qquad \text{in } \Omega_0 \tag{1b}$$

$$\nabla_X \cdot \underset{\sim}{\boldsymbol{P}} - \boldsymbol{f}_V = 0 \qquad \text{in } \Omega_0 \tag{1c}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \qquad \text{on } \partial_D\Omega_0 \tag{1d}$$

$$\underset{\sim}{\boldsymbol{P}} \cdot \boldsymbol{n} = \boldsymbol{t}_N \qquad \text{on } \partial_N\Omega_0 \tag{1e}$$

where ψ_{mec} denotes the mechanical energy potential, and $\underset{\sim}{\boldsymbol{P}}$ the first Piola-Kirchoff stress tensor.

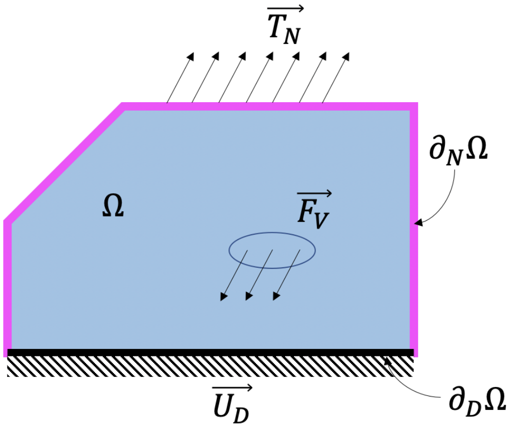


Figure 1: schematic representation of the model problem

The equilibrium of the body Ω_0 is reached for the displacement field $\boldsymbol{u} \in H^1(\Omega_0, \mathbb{R}^d)$ minimizing the energy:

$$J(\boldsymbol{u}) = \int_{\Omega_0} \psi_{mec}(\underset{\sim}{\boldsymbol{F}}) - \int_{\Omega_0} \boldsymbol{f}_V \cdot \boldsymbol{u} + \int_{\partial_N \Omega_0} \boldsymbol{t}_N \cdot \boldsymbol{u} \tag{2}$$

where $\boldsymbol{u} \in H^1(\Omega_0, \mathbb{R}^d)$, and equations (1a) and (1b) are enforced strongly. If (1a) and (1b) are considered in weak sense, one obtains the three-field Hu–Washizu functional :

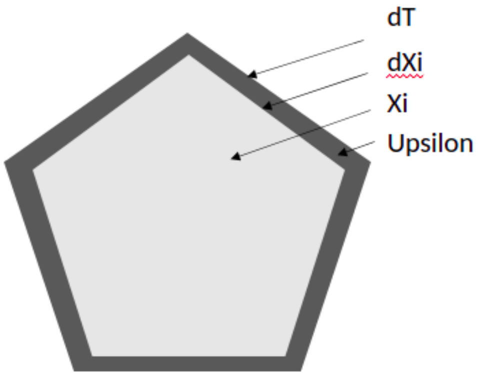
$$J(\boldsymbol{u}, \underset{\sim}{\boldsymbol{G}}, \underset{\sim}{\boldsymbol{P}}) = \int_{\Omega_0} \psi_{mec}(\underset{\sim}{\boldsymbol{F}}) + (\nabla_X \boldsymbol{u} - \underset{\sim}{\boldsymbol{G}}) : \underset{\sim}{\boldsymbol{P}} - \int_{\Omega_0} \boldsymbol{f}_V \cdot \boldsymbol{u} - \int_{\partial_N \Omega_0} \boldsymbol{t}_N \cdot \boldsymbol{u} \tag{3}$$

where $\underset{\sim}{\boldsymbol{P}} \in H^1_{\text{div}}(\Omega_0, \mathbb{R}^d)$, $\underset{\sim}{\boldsymbol{F}} \in L^2(\Omega_0, \mathbb{R}^d)$, and $\underset{\sim}{\boldsymbol{G}} := \underset{\sim}{\boldsymbol{F}} - \underset{\sim}{\boldsymbol{1}}$.

1.3 Composite element

Let $\mathcal{T}_h$ a triangulation of the domain Ω into a set of disjoints elements (or cells) $T_i \subset \mathbb{R}^d, 1 \leq i \leq N$, where N denotes the number of elements in the mesh. Let ∂T the boundary of an element T, composed of its faces (if $d = 3$) or edges (if $d = 2$). Let $\boldsymbol{v}_T$ the displacement field $\boldsymbol{v}$ in an element T, and $\boldsymbol{v}_{\partial T}$ that of ∂T. Assuming continuity of the displacement on the mesh, one has the natural equality :

$$\boldsymbol{v}_{\partial T} = \boldsymbol{v}_{T|\partial T} \quad \text{on } \partial T \tag{4}$$



In order to lay the ground for the description of (hybrid) discontinuous elements from mechanical arguments based on the ususal assumption of the continuity of the displacement field, let decompose T into a thin volumic region $\Upsilon \subset \mathbb{R}^d$ of width $\ell > 0$ attached to the element boundary ∂T, and into a core region Ξ, such that $T = \Xi \cup \Upsilon$. Let $\Xi = \boldsymbol{\Phi}_T(T)$ the image of T by the homotehty $\boldsymbol{\Phi}_T$ of ratio $(1 + \alpha\ell)$, with $\alpha < 0$, centered in $\boldsymbol{X}_T$ the centroid of T. Let $\partial\Xi = \boldsymbol{\Phi}_T(\partial T)$ the boundary of Ξ; since $\partial\Xi$ is an homotehy of ∂T, any point $\boldsymbol{X}_{\partial T} \in \partial T$ and $\boldsymbol{X}_{\partial\Xi} = \boldsymbol{\Phi}_T(\boldsymbol{X}_{\partial T}) \in \partial\Xi$ share the same unit outward normal $\boldsymbol{n}$. Let $\mathcal{M}_T = \{v \in L^2(\Upsilon) \mid v \cdot \boldsymbol{n} = \text{cste}\}$ the set of L^2-functions which are constant along the normal axis in Υ; for any function in $\mathcal{M}_T$, the following equality holds true:

$$\int_{\Upsilon} v \; dV = \int_{\partial T} \int_{\epsilon=0}^{\ell} v(1 + \alpha\epsilon) \; dS d\epsilon = \ell(1 + \frac{\alpha}{2}\ell) \int_{\partial T} v \; dS \tag{5}$$

Let $\boldsymbol{u}_\Upsilon \in H^1(\Upsilon, \mathbb{R}^d)$ the displacement of Υ, and $\boldsymbol{u}_\Xi \in H^1(\Xi, \mathbb{R}^d)$ that of Ξ. With a similar argument as in (4), $\boldsymbol{u}_\Upsilon$ verifies:

$$\begin{aligned} \boldsymbol{u}_{\Upsilon|\partial\Xi} &= \boldsymbol{u}_\Xi \qquad \text{on } \partial\Xi \\ \boldsymbol{u}_{\Upsilon|\partial T} &= \boldsymbol{u}_{\partial T} \qquad \text{on } \partial T \end{aligned} \tag{6}$$

Assuming the interface Υ to be thin compared to the cell volume T, such that $\ell \ll h_T$ is negligeable with respect to h_T the diameter of T, let linearize the displacement in Υ with respect to $\boldsymbol{n}$, such that:

$$\boldsymbol{u}_\Upsilon = \frac{\boldsymbol{u}_{\partial T} - \boldsymbol{u}_\Xi|_{\partial\Xi}}{\ell} \otimes \boldsymbol{n} \cdot \boldsymbol{X} + \boldsymbol{u}_\Xi|_{\partial\Xi}$$

Hence, the displacement gradient is homogeneous in Υ along $\boldsymbol{n}$ such that :

$$\nabla_X \boldsymbol{u}_\Upsilon = \frac{\boldsymbol{u}_{\partial T} - \boldsymbol{u}_\Xi|_{\partial\Xi}}{\ell} \otimes \boldsymbol{n} \tag{7}$$

Since $\partial\Xi$ and ∂T are disjoints by introducing Υ, (4) does not necessarily hold true and the difference $\boldsymbol{u}_{\partial T} - \boldsymbol{u}_\Xi|_{\partial\Xi}$ along $\boldsymbol{n}$ can have any value. Let $\boldsymbol{u}_T \in H^1(T, \mathbb{R}^d)$ the displacement in T such that:

$$\boldsymbol{u}_T = \left\{ \begin{array}{ll} \boldsymbol{u}_\Xi & \text{in } \Xi \\ \boldsymbol{u}_\Upsilon & \text{in } \Upsilon \end{array} \right. \tag{8}$$

In particular, $\boldsymbol{u}_T$ is continuous across Υ, since $\boldsymbol{u}_\Upsilon$ linearily bridges $\boldsymbol{u}_\Xi|_{\partial\Xi}$ to $\boldsymbol{u}_{\partial T}$. Similarily, let $\underset{\sim}{\boldsymbol{P}}_T \in H^1_{\text{div}}(T, \mathbb{R}^{d\times d})$ the stress in the element T, and $\underset{\sim}{\boldsymbol{G}}_T \in L^2(T, \mathbb{R}^{d\times d})$ the displacement gradient, such that they are homogeneous in Υ along $\boldsymbol{n}$ as $\ell \ll h_T$:

$$\underset{\sim}{\boldsymbol{P}}_T = \left\{ \begin{array}{ll} \underset{\sim}{\boldsymbol{P}}_\Xi & \text{in } \Xi \\ \underset{\sim}{\boldsymbol{P}}_\Xi|_{\partial\Xi} & \text{in } \Upsilon \end{array} \right. \quad \text{and} \quad \underset{\sim}{\boldsymbol{G}}_T = \left\{ \begin{array}{ll} \underset{\sim}{\boldsymbol{G}}_\Xi & \text{in } \Xi \\ \underset{\sim}{\boldsymbol{G}}_\Xi|_{\partial\Xi} & \text{in } \Upsilon \end{array} \right.$$

The Hu–Washizu functional over the element T writes:

$$J_T(\boldsymbol{u}_T, \underset{\sim}{\boldsymbol{G}}_T, \underset{\sim}{\boldsymbol{P}}_T) = \int_T \psi_T + (\nabla_X \boldsymbol{u}_T - \underset{\sim}{\boldsymbol{G}}_T) : \underset{\sim}{\boldsymbol{P}}_T - \int_T \boldsymbol{f}_V \cdot \boldsymbol{u}_T - \int_{\partial_N T} \boldsymbol{t}_N \cdot \boldsymbol{u}_{\partial T} \tag{9}$$

specifying J_T over both Ξ and Υ gives:

$$J_T(\boldsymbol{u}_T, \underset{\sim}{\boldsymbol{G}}_T, \underset{\sim}{\boldsymbol{P}}_T) = J_\Xi(\boldsymbol{u}_\Xi, \underset{\sim}{\boldsymbol{G}}_T, \underset{\sim}{\boldsymbol{P}}_T) + J_\Upsilon(\boldsymbol{u}_\Xi, \boldsymbol{u}_{\partial T}, \underset{\sim}{\boldsymbol{G}}_T, \underset{\sim}{\boldsymbol{P}}_T) \tag{10}$$

with :

$$J_\Xi(\boldsymbol{u}_\Xi, \underset{\sim}{\boldsymbol{G}}_T, \underset{\sim}{\boldsymbol{P}}_T) = \int_\Xi \psi_\Xi + (\nabla_X \boldsymbol{u}_\Xi - \underset{\sim}{\boldsymbol{G}}_T) : \underset{\sim}{\boldsymbol{P}}_T - \int_\Xi \boldsymbol{f}_V \cdot \boldsymbol{u}_\Xi \tag{11}$$

where ψ_Ξ describes the mechanical energy in Ξ, and:

$$J_\Upsilon(\boldsymbol{u}_\Xi, \boldsymbol{u}_{\partial T}, \underset{\sim}{\boldsymbol{G}}_T, \underset{\sim}{\boldsymbol{P}}_T) = \int_\Upsilon \psi_\Upsilon + (\nabla_X \boldsymbol{u}_\Upsilon - \underset{\sim}{\boldsymbol{G}}_T) : \underset{\sim}{\boldsymbol{P}}_T - \int_{\partial_N T} \boldsymbol{t}_N \cdot \boldsymbol{u}_{\partial T} \tag{12}$$

In particlaur, we assumed that the volumetric load is concentrated in the core part of the element Ξ, and neglegted in the boundary part. Moreover, let endow the interface Υ with a linear elastic behaviour, such that :

$$\psi_\Upsilon = \frac{1}{2}\beta \frac{\ell}{h_T} \nabla_X \boldsymbol{u}_\Upsilon : \nabla_X \boldsymbol{u}_\Upsilon \tag{13}$$

where the parameter β is the Young modulus of the membrane, and the dimensionless ratio ℓ/h_T balances the accumulated energy with the size of the element. Using (7) and (5) for $\psi_\Upsilon \in \mathcal{M}_T$ one can write the expression of the mechanical energy in the membrane as a term only depending on the boundary :

$$\int_\Upsilon \psi_\Upsilon = (1 + \frac{\alpha}{2}\ell) \int_{\partial T} \frac{\beta}{2h_T} \|\boldsymbol{u}_{\partial T} - \boldsymbol{u}_\Xi|_{\partial \Xi}\|^2 \tag{14}$$

using a similar argument for the second volumetric term in (12), one has:

$$\int_\Upsilon (\nabla_X \boldsymbol{u}_\Upsilon - \underset{\sim}{\boldsymbol{G}}_T) : \underset{\sim}{\boldsymbol{P}}_T = (1 + \frac{\alpha}{2}\ell)(\int_{\partial T} \underset{\sim}{\boldsymbol{P}}_T \cdot \boldsymbol{n} \cdot (\boldsymbol{u}_{\partial T} - \boldsymbol{u}_\Xi|_{\partial \Xi}) - \ell \int_{\partial T} \underset{\sim}{\boldsymbol{G}}_T : \underset{\sim}{\boldsymbol{P}}_T) \tag{15}$$

hence :

$$\begin{aligned} J_T(\boldsymbol{u}_\Xi, \boldsymbol{u}_{\partial T}, \underset{\sim}{\boldsymbol{G}}_T, \underset{\sim}{\boldsymbol{P}}_T) = & \int_\Xi \psi_\Xi + (\nabla_X \boldsymbol{u}_\Xi - \underset{\sim}{\boldsymbol{G}}_T) : \underset{\sim}{\boldsymbol{P}}_T \\ & + (1 + \frac{\alpha}{2}\ell) \int_{\partial T} \underset{\sim}{\boldsymbol{P}}_T \cdot \boldsymbol{n} \cdot (\boldsymbol{u}_{\partial T} - \boldsymbol{u}_\Xi|_{\partial \Xi}) \\ & + (1 + \frac{\alpha}{2}\ell) \int_{\partial T} \frac{\beta}{2h_T} \|\boldsymbol{u}_{\partial T} - \boldsymbol{u}_\Xi|_{\partial \Xi}\|^2 \\ & - (\ell + \frac{\alpha}{2}\ell^2) \int_\Upsilon \underset{\sim}{\boldsymbol{G}}_T : \underset{\sim}{\boldsymbol{P}}_T \\ & - \int_\Xi \boldsymbol{f}_V \cdot \boldsymbol{u}_\Xi - \int_{\partial_N T} \boldsymbol{t}_N \cdot \boldsymbol{u}_{\partial T} \end{aligned} \tag{16}$$

making $\ell \to 0$, the interface region vanishes such that $\Upsilon \to \emptyset$ and $\Xi \to T$, yielding the expression of the Hu–Washizu functional over the element:

$$
\begin{aligned}
J_T(\boldsymbol{u}_T, \boldsymbol{u}_{\partial T}, \underset{\sim}{\boldsymbol{G}}_T, \underset{\sim}{\boldsymbol{P}}_T) = & \int_T \psi_T + (\nabla_X \boldsymbol{u}_T - \underset{\sim}{\boldsymbol{G}}_T) : \underset{\sim}{\boldsymbol{P}}_T \\
& + \int_{\partial T} \underset{\sim}{\boldsymbol{P}}_T \cdot \boldsymbol{n} \cdot (\boldsymbol{u}_{\partial T} - \boldsymbol{u}_T|_{\partial T}) \\
& + \int_{\partial T} \frac{\beta}{2h_T} \|\boldsymbol{u}_{\partial T} - \boldsymbol{u}_T|_{\partial T}\|^2 \\
& - \int_T \boldsymbol{f}_V \cdot \boldsymbol{u}_T - \int_{\partial_N T} \boldsymbol{t}_N \cdot \boldsymbol{u}_{\partial T}
\end{aligned}
\tag{17}
$$

Assuming that the displacement is continuous at the boundary ∂T such that (4) holds true, then one recovers the usual expression of the Hu–Washizu integral over the element. However, if one considers that $\boldsymbol{u}_{\partial T}$ and $\boldsymbol{u}_T$ are disticnt variables, the functional writes as a function of the four variables $J(\boldsymbol{u}_T, \boldsymbol{u}_{\partial T}, \underset{\sim}{\boldsymbol{G}}_T, \underset{\sim}{\boldsymbol{P}}_T)$. Differentiating over each variable, and introducing the numerical flux $\boldsymbol{\theta}_{\partial T} = \underset{\sim}{\boldsymbol{P}}_T \cdot \boldsymbol{n} + (\beta/h_T)(\boldsymbol{u}_{\partial T} - \boldsymbol{u}_T|_{\partial T})$ one obtains the system:

$$
\frac{\partial J}{\partial \boldsymbol{u}_T}\delta \boldsymbol{u}_T = \int_T (\underset{\sim}{\boldsymbol{P}} : \nabla_X \delta \boldsymbol{u}_T - \boldsymbol{f}_V) \cdot \delta \boldsymbol{u}_T \qquad \forall \delta \boldsymbol{u}_T \in H^1(T, \mathbb{R}^d) \tag{18a}
$$

$$
- \int_{\partial T} \boldsymbol{\theta}_{\partial T} \cdot \delta \boldsymbol{u}_T \tag{18b}
$$

$$
\frac{\partial J}{\partial \boldsymbol{u}_{\partial T}}\delta \boldsymbol{u}_{\partial T} = \int_{\partial T} (\boldsymbol{\theta}_{\partial T} - \boldsymbol{t}_N) \cdot \delta \boldsymbol{u}_{\partial T} \qquad \forall \delta \boldsymbol{u}_{\partial T} \in H^1(\partial T, \mathbb{R}^d) \tag{18c}
$$

$$
\frac{\partial J}{\partial \underset{\sim}{\boldsymbol{G}}_T}\delta \underset{\sim}{\boldsymbol{G}}_T = \int_T (\frac{\partial \psi_T}{\partial \underset{\sim}{\boldsymbol{G}}_T} - \underset{\sim}{\boldsymbol{P}}_T) : \delta \underset{\sim}{\boldsymbol{G}}_T \qquad \forall \delta \underset{\sim}{\boldsymbol{G}}_T \in L^2(T, \mathbb{R}^{d\times d}) \tag{18d}
$$

$$
\frac{\partial J}{\partial \underset{\sim}{\boldsymbol{P}}_T}\delta \underset{\sim}{\boldsymbol{P}}_T = \int_T (\nabla_X \boldsymbol{u}_T - \underset{\sim}{\boldsymbol{G}}_T) : \delta \underset{\sim}{\boldsymbol{P}}_T \qquad \forall \delta \underset{\sim}{\boldsymbol{P}}_T \in H^1_{\text{div}}(T, \mathbb{R}^{d\times d}) \tag{18e}
$$

$$
+ \int_{\partial T} (\boldsymbol{u}_{\partial T} - \boldsymbol{u}_T|_{\partial T}) \otimes \boldsymbol{n} : \delta \underset{\sim}{\boldsymbol{P}}_T \tag{18f}
$$

By explicitly eliminating (18d) and (18e) from the system, one obtains the problem in primal form: find $(\boldsymbol{u}_T, \boldsymbol{u}_{\partial T}) \in H^1(T, \mathbb{R}^d) \times H^1(\partial T, \mathbb{R}^d)$, such that for all $(\delta\boldsymbol{u}_T, \delta\boldsymbol{u}_{\partial T}) \in H^1(T, \mathbb{R}^d) \times H^1(\partial T, \mathbb{R}^d)$

$$
dJ = \frac{\partial J}{\partial \boldsymbol{u}_T}\delta \boldsymbol{u}_T + \frac{\partial J}{\partial \boldsymbol{u}_{\partial T}}\delta \boldsymbol{u}_{\partial T} = 0 \tag{19}
$$

injetcing (18b) and (18c) :

$$
\begin{aligned}
dJ = & \int_T \underset{\sim}{\boldsymbol{P}}_T : \nabla_X \delta \boldsymbol{u}_T \\
& + \int_{\partial T} \underset{\sim}{\boldsymbol{P}}_T \cdot \boldsymbol{n} \cdot (\delta \boldsymbol{u}_{\partial T} - \delta \boldsymbol{u}_T|_{\partial T}) \\
& + \int_{\partial T} (\beta/h_T)(\boldsymbol{u}_{\partial T} - \boldsymbol{u}_T|_{\partial T}) \cdot (\delta \boldsymbol{u}_{\partial T} - \delta \boldsymbol{u}_T|_{\partial T}) \\
& - \int_{\partial T} \boldsymbol{t}_N \cdot \delta \boldsymbol{u}_{\partial T} - \int_T \boldsymbol{f}_V \cdot \delta \boldsymbol{u}_T = 0
\end{aligned}
\tag{20}
$$

using both (18d) and (18e) :

$$
\begin{aligned}
dJ = &\int_T \frac{\partial \psi_T}{\partial \underset{\sim}{\boldsymbol{G}}_T} : \delta \underset{\sim}{\boldsymbol{G}}_T \\
&+ \int_{\partial T} (\beta / h_T)(\boldsymbol{u}_{\partial T} - \boldsymbol{u}_T|_{\partial T}) \cdot (\delta \boldsymbol{u}_{\partial T} - \delta \boldsymbol{u}_T|_{\partial T}) \\
&- \int_{\partial T} \boldsymbol{t}_N \cdot \delta \boldsymbol{u}_{\partial T} - \int_T \boldsymbol{f}_V \cdot \delta \boldsymbol{u}_T = 0
\end{aligned} \tag{21}
$$

where $\delta \underset{\sim}{\boldsymbol{G}}_T$ (respectively $\underset{\sim}{\boldsymbol{G}}_T$) solves (18e) for the unknowns set $(\delta \boldsymbol{u}_T, \delta \boldsymbol{u}_{\partial T})$ (respectively $(\boldsymbol{u}_T, \boldsymbol{u}_{\partial T})$)

Le problème (**??**) discrétisé consiste à chercher l'inconnue $(\boldsymbol{u}_T^l, \boldsymbol{u}_{\partial T}^k)$ dans l'espace des polynômes $P^l(T, \mathbb{R}^d) \times P^k(\partial T, \mathbb{R}^d)$ d'ordre respectivement l et k tels que $k > 0$ avec $k - 1 \leq l \leq k + 1$, et les champs de gradients de déplacement $\underset{\sim}{\boldsymbol{G}}_T^k$ et de contraintes $\underset{\sim}{\boldsymbol{P}}_T^k$ dans $P^k(T, \mathbb{R}^{d \times d})$. On définit la force de traction discrète $\boldsymbol{\theta}_{\partial T}^{HHO} = \underset{\sim}{\boldsymbol{P}}_T^k \cdot \boldsymbol{n} + (\beta_{mec}/h_T) \boldsymbol{S}_{\partial T}^{k*}$ telle que $\boldsymbol{S}_{\partial T}^{k*}$ est l'opérateur adjoint de l'opérateur de stabilisation $\boldsymbol{S}_{\partial T}^k$ définit par:

$$
\boldsymbol{S}_{\partial T}^k(\boldsymbol{v}_T^l, \boldsymbol{v}_{\partial T}^k) = \Pi_{\partial T}^k(\boldsymbol{v}_{\partial T}^k - \boldsymbol{v}_T^l - (\boldsymbol{1} - \Pi_T^k)\boldsymbol{D}_T^{k+1}) \tag{22}
$$

où $\Pi_{\partial T}^k$ et Π_T^k sont les projecteurs orthogonaux au sens L^2 sur $P^k(\partial T, \mathbb{R}^d)$ et $P^k(T, \mathbb{R}^d)$ respectivement, et le champ de déplacement $\boldsymbol{D}_T^{k+1} \in P^{k+1}(T, \mathbb{R}^d)$ est solution du problème (23):

$$
\begin{aligned}
\int_T (\nabla_X \boldsymbol{D}_T^{k+1} - \nabla_X \boldsymbol{u}_T^l) : \nabla_X \boldsymbol{w}^{k+1} = \int_{\partial T} (\boldsymbol{u}_{\partial T}^k - \boldsymbol{u}_T^l) \cdot \nabla_X \boldsymbol{w}^{k+1} \boldsymbol{n} \quad \forall \boldsymbol{w}^{k+1} \in P^{k+1}(T, \mathbb{R}^d) \\
\int_T \boldsymbol{D}_T^{k+1} = \int_T \boldsymbol{u}_T^l
\end{aligned} \tag{23}
$$

D'un point de vue numérique, on calcule dans une étape de pré-traitement l'opérateur de stabilisation $[S] : (\boldsymbol{v}_T^l, \boldsymbol{v}_{\partial T}^k) \to \boldsymbol{S}_{\partial T}^k$ défini par (22) et l'opérateur de dérivation $[B] : (\boldsymbol{v}_T^l, \boldsymbol{v}_{\partial T}^k) \to \underset{\sim}{\boldsymbol{G}}_T^k$ défini par la formulation discrète de (**??**), de sorte que le problème discrétisé local (**??**) ne dépend plus que de l'inconnue primale $(\boldsymbol{u}_T^l, \boldsymbol{u}_{\partial T}^k)$ vérifiant $\forall (\boldsymbol{v}_T^l, \boldsymbol{v}_{\partial T}^k) \in P^l(T, \mathbb{R}^d) \times P^k(\partial T, \mathbb{R}^d)$:

$$
\int_T \underset{\sim}{\boldsymbol{P}}_T^k : \underset{\sim}{\boldsymbol{G}}_T^k + \int_{\partial_T} \frac{\beta_{mec}}{h_T} \boldsymbol{S}_{\partial T}^k(\boldsymbol{u}_T^l, \boldsymbol{u}_{\partial T}^k) \cdot \boldsymbol{S}_{\partial T}^k(\boldsymbol{v}_T^l, \boldsymbol{v}_{\partial T}^k) = \int_\Omega \boldsymbol{f}_V \cdot \boldsymbol{v}_T^l + \int_{\partial_T} \boldsymbol{t}_N \cdot \boldsymbol{v}_{\partial T}^k \tag{24}
$$

où les contraintes $\underset{\sim}{\boldsymbol{P}}_T^k$ sont calculées aux points de quadrature par intégration de la loi de comportement. Le principe des travaux virtuels discret à l'échelle de la structure vérifie donc $\forall (\boldsymbol{v}_{\mathcal{T}}^l, \boldsymbol{v}_{\mathcal{F}}^k) \in P^l(\mathcal{T}, \mathbb{R}^d) \times P^k(\mathcal{F}, \mathbb{R}^d)$:

$$
\begin{aligned}
\sum_{T \in \mathcal{T}(\Omega_0)} \int_T \underset{\sim}{\boldsymbol{P}}_T^k : \underset{\sim}{\boldsymbol{G}}_T^k + \int_{\partial_T} \frac{\beta_{mec}}{h_T} \boldsymbol{S}_{\partial T}^k(\boldsymbol{u}_T^l, \boldsymbol{u}_{\partial T}^k) \cdot \boldsymbol{S}_{\partial T}^k(\boldsymbol{v}_T^l, \boldsymbol{v}_{\partial T}^k) = \sum_{T \in \mathcal{T}(\Omega_0)} \int_\Omega \boldsymbol{f}_V \cdot \boldsymbol{v}_T^l \\
+ \sum_{\partial T \in \mathcal{F}_N(\Omega_0)} \int_{\partial_T} \boldsymbol{t}_N \cdot \boldsymbol{v}_{\partial T}^k
\end{aligned} \tag{25}
$$

References

[1] Douglas N. Arnold, Franco Brezzi, and L. Donatella Marini. A Family of Discontinuous Galerkin Finite Elements for the Reissner–Mindlin Plate. *Journal of Scientific Computing*, 22(1):25–45, June 2005.

[2] Ivo Babuska. The Finite Element Method with Penalty. *Mathematics of Computation*, 27(122):221, April 1973.

[3] Daniele Boffi, Michele Botti, and Daniele A. Di Pietro. A Nonconforming High-Order Method for the Biot Problem on General Meshes. *SIAM Journal on Scientific Computing*, 38(3):A1508–A1537, January 2016.

[4] Michele Botti, Daniele A. Di Pietro, and Pierre Sochala. A Hybrid High-Order Method for Nonlinear Elasticity. *SIAM Journal on Numerical Analysis*, 55(6):2687–2717, January 2017.

[5] Susanne C. Brenner and Li-yeng Sung. Balancing domain decomposition for nonconforming plate elements. *Numerische Mathematik*, 83(1):25–52, July 1999.

[6] Fatih Celiker, Bernardo Cockburn, and Henryk K. Stolarski. Locking-Free Optimal Discontinuous Galerkin Methods for Timoshenko Beams. *SIAM Journal on Numerical Analysis*, 44(6):2297–2325, 2006.

[7] Bernardo Cockburn, Jayadeep Gopalakrishnan, and Raytcho Lazarov. Unified Hybridization of Discontinuous Galerkin, Mixed, and Continuous Galerkin Methods for Second Order Elliptic Problems. *SIAM Journal on Numerical Analysis*, 47(2):1319–1365, January 2009.

[8] Daniele A. Di Pietro and Alexandre Ern. A hybrid high-order locking-free method for linear elasticity on general meshes. *Computer Methods in Applied Mechanics and Engineering*, 283:1–21, January 2015.

[9] Daniele A. Di Pietro, Alexandre Ern, and Simon Lemaire. An Arbitrary-Order and Compact-Stencil Discretization of Diffusion on General Meshes Based on Local Reconstruction Operators. *Computational Methods in Applied Mathematics*, 14(4):461–472, October 2014.

[10] Daniele A. Di Pietro, Alexandre Ern, Alexander Linke, and Friedhelm Schieweck. A discontinuous skeletal method for the viscosity-dependent Stokes problem. *Computer Methods in Applied Mechanics and Engineering*, 306:175–195, July 2016.

[11] Daniele A. Di Pietro and Stella Krell. A Hybrid High-Order Method for the Steady Incompressible Navier–Stokes Problem. *Journal of Scientific Computing*, 74(3):1677–1705, March 2018.

[12] J. K. Djoko, F. Ebobisse, A. T. McBride, and B. D. Reddy. A discontinuous Galerkin formulation for classical and gradient plasticity. Part 2: Algorithms and numerical analysis. *Computer Methods in Applied Mechanics and Engineering*, 197(1):1–21, December 2007.

[13] J. K. Djoko, F. Ebobisse, A. T. McBride, and B. D. Reddy. A discontinuous Galerkin formulation for classical and gradient plasticity – Part 1: Formulation and analysis. *Computer Methods in Applied Mechanics and Engineering*, 196(37):3881–3897, August 2007.

[14] G. Engel, K. Garikipati, T.J.R. Hughes, M.G. Larson, L. Mazzei, and R.L. Taylor. Continuous/discontinuous finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity. *Computer Methods in Applied Mechanics and Engineering*, 191(34):3669–3750, July 2002.

[15] Robert Gracie, Hongwu Wang, and Ted Belytschko. Blending in the extended finite element method by discontinuous Galerkin and assumed strain methods. *International Journal for Numerical Methods in Engineering*, 74(11):1645–1669, 2008. _eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/nme.2217.

[16] Peter Hansbo and Mats G. Larson. Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method. *Computer Methods in Applied Mechanics and Engineering*, 191(17-18):1895–1908, February 2002.

[17] Adrian Lew, Patrizio Neff, Deborah Sulsky, and Michael Ortiz. Optimal BV estimates for a discontinuous Galerkin method for linear elasticity. *Applied Mathematics Research eXpress*, 2004(3):73, 2004.

[18] N.C. Nguyen and J. Peraire. Hybridizable discontinuous Galerkin methods for partial differential equations in continuum mechanics. *Journal of Computational Physics*, 231(18):5955–5988, July 2012.

[19] N.C. Nguyen, J. Peraire, and B. Cockburn. An implicit high-order hybridizable discontinuous Galerkin method for linear convection–diffusion equations. *Journal of Computational Physics*, 228(9):3232–3254, May 2009.

[20] N.C. Nguyen, J. Peraire, and B. Cockburn. An implicit high-order hybridizable discontinuous Galerkin method for nonlinear convection–diffusion equations. *Journal of Computational Physics*, 228(23):8841–8855, December 2009.

[21] N.C. Nguyen, J. Peraire, and B. Cockburn. A hybridizable discontinuous Galerkin method for Stokes flow. *Computer Methods in Applied Mechanics and Engineering*, 199(9-12):582–597, January 2010.

[22] N.C. Nguyen, J. Peraire, and B. Cockburn. An implicit high-order hybridizable discontinuous Galerkin method for the incompressible Navier–Stokes equations. *Journal of Computational Physics*, 230(4):1147–1170, February 2011.

[23] Von J Nitsche. Uber ein Yariationsprinzip zur Liisung yon Dirichlet-Problemen bei Verwendung yon Teilr˜iumen, die keinen Randbedingungen unterworfen sind. page 7, 1970.

[24] L. Noels and R. Radovitzky. A general discontinuous Galerkin method for finite hyperelasticity. Formulation and numerical applications. *International Journal for Numerical Methods in Engineering*, 68(1):64–97, October 2006.

[25] Christoph Ortner and Endre Süli. Discontinuous Galerkin Finite Element Approximation of Nonlinear Second-Order Elliptic and Hyperbolic Systems. *SIAM Journal on Numerical Analysis*, 45(4):1370–1397, January 2007. Publisher: Society for Industrial and Applied Mathematics.

[26] P.-O. Persson, J. Bonet, and J. Peraire. Discontinuous Galerkin solution of the Navier–Stokes equations on deformable domains. *Computer Methods in Applied Mechanics and Engineering*, 198(17-20):1585–1595, April 2009.

[27] W.H. Reed and T.R. Hill. Triangular mesh methods for the neutron transport equation. Technical report, United States, 1973. LA-UR–73-479 INIS Reference Number: 4080130.

[28] Béatrice Rivière and Mary F. Wheeler. Optimal Error Estimates for Discontinuous Galerkin Methods Applied to Linear Elasticity Problems. *Comput. Math. Appl*, 46:141–163, 2000.

[29] Khosro Shahbazi, Paul F. Fischer, and C. Ross Ethier. A high-order discontinuous Galerkin method for the unsteady incompressible Navier–Stokes equations. *Journal of Computational Physics*, 222(1):391–407, March 2007.

[30] Yongxing Shen and Adrian Lew. Stability and convergence proofs for a discontinuous-Galerkin-based extended finite element method for fracture mechanics. *Computer Methods in Applied Mechanics and Engineering*, 199(37):2360–2382, August 2010.

[31] S.-C. Soon. *Hybridizable discontinuous Galerkin method for solid mechanics*. PhD thesis, University of Minnesota, 2008.

[32] S.-C. Soon, B. Cockburn, and Henryk K. Stolarski. A hybridizable discontinuous Galerkin method for linear elasticity: AN HDG METHOD. *International Journal for Numerical Methods in Engineering*, 80(8):1058–1092, November 2009.

[33] A. Ten Eyck and A. Lew. Discontinuous Galerkin methods for non-linear elasticity. *International Journal for Numerical Methods in Engineering*, 67(9):1204–1243, August 2006.

[34] Mary Fanett Wheeler. An Elliptic Collocation-Finite Element Method with Interior Penalties. *SIAM Journal on Numerical Analysis*, 15(1):152–161, February 1978.