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Abstract

The Hybrid High Order (HHO) method is a powerful discretization method which has only been recently applied from to non linear computational mechanics.

The Hybrid High Order method divides the domain of interest in cells of abitrary polyhedral shape, whose boundaries is called the skeleton, and introduces two kinds of degrees of freedom: the displacements in the cell and the displacements of the skeleton.

Most introductory materials to the HHO method is focus on the mathematical aspects of the methods. While those aspects are important, an approach based on physical considerations would help spreading this method to the computational mechanics and engineering communities.

This paper derives Hybrid High Order method from the classical Hu–Washizu functional.

Practical implementation of the method is discussed in depth using notations closed to the ones used in standard finite elements textbooks, highlighting the use of polyhedral cells and the use of approximation spaces based on polynomials of abritrary orders.

From the point of view of numerical performances, the elimination of the cell degrees of freedom is mandory to to reduce the size of the stiffness matrix. The standard static condensation, is presented, as well as a novel strategy called "cell equilibrium". Avantages and disadvantages of both strategies are discussed.

The resolution of axisymmetrical problems, which has seldom, if ever, been discussed in the litterature, is then presented.

Numerical examples prove the robustness of the method with regards to volumetric locking....

Keywords: Computational mechanics, Hybrid High Order method, Static condensation, Cell equilibrium algorithm, Volumetric locking, Axisymmetric modelling hypothesis

1. Introduction

The origin of DG methods dates back to the pioneering work of [**?**], where an hyperbolic formualtion is used to solve the neutron transport equation.

2. Model Problem

Paragraph 2.1 introduces the classical Hu–Washizu functional to describe the quasi-static equilibrium of a body submitted to external load and the main notations used in this paper. For the sake of simplicity, the body is assumed hyper-elastic in this section.

Paragraph **??** introduces the key idea of the HHO method, which is to divide the domain in arbitrary subdomains connected by cohesive interfaces and to apply the Hu–Washizu functional to each sub-domains.

2.1. The standard Hu–Washizu functional

Let us consider the equilibrium of a an hyperelastic solid body. The reference and current configurations of the body are denoted respectively Ω and Ω_t. The reference and current boundary of the body are denoted respectively $\partial\Omega$ and $\partial\Omega_t$.

The body is subjected to body forces $\boldsymbol{f}_v$ in Ω_t, contact loads $\boldsymbol{t}_n$ on a boundary $\partial_n\Omega_t$ and prescribed displacements $\boldsymbol{u}_d$ on the boundary $\partial_d\Omega_t$. The boundaries $\partial_n\Omega_t$ and $\partial_d\Omega_t$ form a partition of the boundary $\partial\Omega_t$, i.e. $\partial\Omega_t = \partial_d\Omega_t \cup \partial_n\Omega_t$ and $\partial_d\Omega_t \cap \partial_n\Omega_t = \emptyset$.

The three fields Hu–Washizu principle characterizes the equilibrium of the body by the minization of the following functional:

$$J =$$

where:

- $\vec{u}$ is the displacement field
- ....

2.2. Introduction of abitrary cohesive interfaces

??

Let $d \in \{1, 2, 3\}$ the euclidean dimension of the cartesian space $\mathbb{R}^d$. Let $\Omega_t \subset \mathbb{R}^d$ a solid body with boundary $\partial\Omega_t \subset \mathbb{R}^{d-1}$, that deforms in the current configuration at some time $t > 0$ under the body forces $\boldsymbol{f}_v$. It is subjected to a prescribed displacement $\boldsymbol{u}_d$ on the Dirichlet boundary $\partial_d\Omega_t$, and to a contact load $\boldsymbol{t}_n$ on the Neumann boundary $\partial_n\Omega_t$, such that $\partial\Omega_t = \partial_d\Omega_t \cup \partial_n\Omega_t$ and $\partial_d\Omega_t \cap \partial_n\Omega_t = \emptyset$.

The initial configuration of the body at time $t = 0$ (see Figure 1) is denoted $\Omega \subset \mathbb{R}^d$ with respective Dirichlet and Neumann boundaries $\partial_D\Omega$ and $\partial_N\Omega$. It is subjected to body forces $\boldsymbol{f}_V$, an imposed displacement $\boldsymbol{u}_D$ on $\partial_D\Omega$ and contact force $\boldsymbol{t}_N$ on $\partial_N\Omega$. The transformation mapping $\boldsymbol{\Phi}$ takes a point $\boldsymbol{x} \in \Omega$ from the initial configuration to $\boldsymbol{x}_t \in \Omega_t$ in the current configuration.

Let $T \subset \Omega$ an arbitrary open subset of the solid body, with boundary $\partial T \subset \mathbb{R}^{d-1}$ which is split into an eventual Dirichlet boundary $\partial_D T \subset \partial_D\Omega$ subjected to an imposed displacement $\boldsymbol{u}_D$ if T shares a boundary with $\partial_D\Omega$ and into the Neumann Boundary $\partial_N T \subset \partial_N\Omega \cup \Omega$ with contact load $\boldsymbol{t}_{\partial_N T}$ such that :

$$\boldsymbol{t}_{\partial_N T} = \begin{cases} \boldsymbol{t}_{\Omega\backslash T\to T} & \text{on } \partial_N T \cap \Omega\backslash T \\ \boldsymbol{t}_N & \text{on } \partial_N T \cap \partial_N\Omega \end{cases} \tag{1}$$

with $\boldsymbol{t}_{\Omega\backslash T\to T}$ the contact force applied by the surrounding part of the body Ω onto the subset T, and $\partial_D T \cap \partial_N T = \emptyset$.

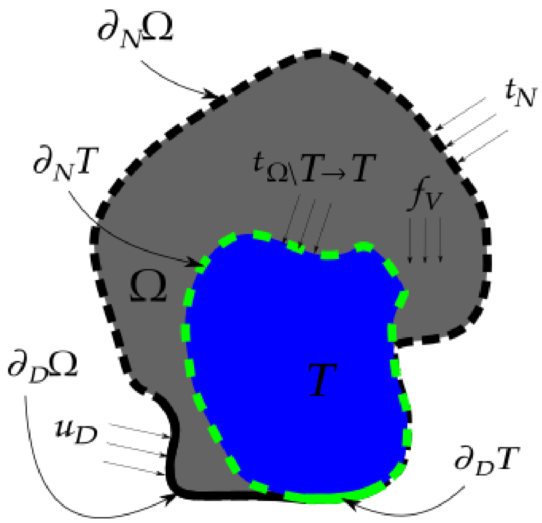

Figure 1. schematic representation of the model problem

Let $\boldsymbol{\Phi}_T$ the restriction of $\boldsymbol{\Phi}$ to T, and $\boldsymbol{u}_T \in U(T)$ the displacement field in T such that $\boldsymbol{\Phi}_T = \boldsymbol{I}_d + \boldsymbol{u}_T$ with $\boldsymbol{I}_d$ the identity function, where the notation $U(T)$ denotes the space of all kinematically admissible displacement fields in T. Let $\underset{\sim}{\boldsymbol{G}}_T \in G(T)$ the displacement gradient field in T, with $G(T)$ the space of all statically admissible displacement gradient fields in T, and $\underset{\sim}{\boldsymbol{F}}_T = \nabla\boldsymbol{\Phi}_T = \underset{\sim}{\boldsymbol{1}} + \underset{\sim}{\boldsymbol{G}}_T$ the transformation gradient, where ∇ denotes the Lagrangian nabla operator. Let $\psi_T(\underset{\sim}{\boldsymbol{F}}_T, v_{int})$ the mechanical energy potential in T that depends on the transformation gradient $\underset{\sim}{\boldsymbol{F}}_T$ and possibly on a set of internal state variables v_{int}, where ∇ denotes the Lagrangian nabla operator. Let $\underset{\sim}{\boldsymbol{P}}_T \in S(T)$ the first Piola-Kirchoff stress tensor depending on the derivative of mechanical energy potential with respect to $\underset{\sim}{\boldsymbol{F}}_T$ with

$$\frac{\partial \psi_T}{\partial \underset{\sim}{\boldsymbol{F}}_T} = \frac{\partial \psi_T}{\partial \underset{\sim}{\boldsymbol{G}}_T}\frac{\partial \underset{\sim}{\boldsymbol{G}}_T}{\partial \underset{\sim}{\boldsymbol{F}}_T} = \frac{\partial \psi_T}{\partial \underset{\sim}{\boldsymbol{G}}_T} \tag{2}$$

with $S(T)$ the space of all statically admissible stress fields in T. The equilibrium of the body T is reached for the displacement gradient field $\underset{\sim}{\boldsymbol{G}}_T \in G(T)$, the first Piola-Kirchoff stress field $\underset{\sim}{\boldsymbol{P}}_T \in S(T)$ and the displacement field $\boldsymbol{u}_T \in U(T)$ verifying $\boldsymbol{u}_T|_{\partial_D T} = \boldsymbol{u}_D$ on $\partial_D T$ minimizing the three field Hu–Washizu energy functional J_T^{HW}:

$$J_T^{HW} = \int_T \psi_T + (\nabla \boldsymbol{u}_T - \underset{\sim}{\boldsymbol{G}}_T) : \underset{\sim}{\boldsymbol{P}}_T - \int_T \boldsymbol{f}_V \cdot \boldsymbol{u}_T - \int_{\partial_N T} \boldsymbol{t}_{\partial_N T} \cdot \boldsymbol{u}_T \tag{3}$$

Deriving (3) with respect to all variables of the problem yields the weak formulation of the problem :

$$\frac{\partial J_T^{HW}}{\partial \boldsymbol{u}_T}\delta\boldsymbol{u}_T = \int_T \underset{\sim}{\boldsymbol{P}}_T : \nabla\delta\boldsymbol{u}_T - \int_T \boldsymbol{f}_V \cdot \delta\boldsymbol{u}_T - \int_{\partial T} \boldsymbol{t}_{\partial_N T} \cdot \delta\boldsymbol{u}_T|_{\partial T} \qquad \forall \delta\boldsymbol{u}_T \in U(T) \tag{4a}$$

$$\frac{\partial J_T^{HW}}{\partial \underset{\sim}{\boldsymbol{G}}_T}\delta\underset{\sim}{\boldsymbol{G}}_T = \int_T \left(\frac{\partial \psi_T}{\partial \underset{\sim}{\boldsymbol{G}}_T} - \underset{\sim}{\boldsymbol{P}}_T\right) : \delta\underset{\sim}{\boldsymbol{G}}_T \qquad \forall \delta\underset{\sim}{\boldsymbol{G}}_T \in G(T) \tag{4b}$$

$$\frac{\partial J_T^{HW}}{\partial \underset{\sim}{\boldsymbol{P}}_T}\delta\underset{\sim}{\boldsymbol{P}}_T = \int_T (\nabla\boldsymbol{u}_T - \underset{\sim}{\boldsymbol{G}}_T) : \delta\underset{\sim}{\boldsymbol{P}}_T \qquad \forall \delta\underset{\sim}{\boldsymbol{P}}_T \in S(T) \tag{4c}$$

where $\cdot\,|_{\partial T}$ denotes the trace operator on ∂T. Since the subset T is arbitrary, (4) holds true in a distributional sense, and by applying the divergence theorem on (4a), one obtains the strong formulation of problem (3) :

$$\underset{\sim}{\boldsymbol{G}}_T - \nabla\boldsymbol{u}_T = 0 \qquad \text{in } T \tag{5a}$$

$$\underset{\sim}{\boldsymbol{P}}_T - \frac{\partial \psi_T}{\partial \underset{\sim}{\boldsymbol{G}}_T} = 0 \qquad \text{in } T \tag{5b}$$

$$\nabla_X \cdot \underset{\sim}{\boldsymbol{P}}_T + \boldsymbol{f}_V = 0 \qquad \text{in } T \tag{5c}$$

$$\boldsymbol{u}_T|_{\partial_D T} = \boldsymbol{u}_D \qquad \text{on } \partial_D T \tag{5d}$$

$$\underset{\sim}{\boldsymbol{P}}_T \cdot \boldsymbol{n} = \boldsymbol{t}_{\partial_N T} \qquad \text{on } \partial_N T \tag{5e}$$

where $\boldsymbol{n}$ denotes the unit ouward normal vector on ∂T. One readliy identifies (5a) with the displacement strain relation, (5b) with the constitutive equation, (5c) with the conservation of the movment quantity, (5e) with the traction force continuity and (5d) with boundary conditions. Considering now that $\underset{\sim}{\boldsymbol{G}}_T$ and $\underset{\sim}{\boldsymbol{P}}_T$ are not unknowns of the problem, *i.e.* by explicitly elimintaing both (4b) and (4c) from (4), the problem associated to (3) simplifies as : find the displacement field $\boldsymbol{u}_T \in U(T)$ verifying $\boldsymbol{u}_T|_{\partial_D T} = \boldsymbol{u}_D$ on $\partial_D T$ that minimizes the energy functional J_T^{VW}:

$$J_T^{VW} = \int_T \psi_T - \int_T \boldsymbol{f}_V \cdot \boldsymbol{u}_T - \int_{\partial_N T} \boldsymbol{t}_{\partial_N T} \cdot \boldsymbol{u}_T \tag{6}$$

Deriving (6) with respect to the primal unknown $\boldsymbol{u}_T$ yields the weak form of the problem, which is the notorious principle of virtual work:

$$dJ_T^{VW} = \frac{\partial J_T^{VW}}{\partial \boldsymbol{u}_T}\delta\boldsymbol{u}_T = \int_T \underset{\sim}{\boldsymbol{P}}_T : \nabla\delta\boldsymbol{u}_T - \int_T \boldsymbol{f}_V \cdot \delta\boldsymbol{u}_T - \int_{\partial_N T} \boldsymbol{t}_{\partial_N T} \cdot \delta\boldsymbol{u}_T|_{\partial T} \tag{7}$$

and that relates to the strong problem defined by equation (5c), (5d), (5e) only, where (5a), (5b) are taken to be granted.

In particular, assuming $T = \Omega$, one obtains the mechanical problem to solve for the whole body Ω. Furthermore, assuming that T is made out of a partition of $N > 0$ distinct media $T_i \subset T$ with respective energy potentials ψ_{T_i}, the problem writes : for each medium T_i, find $\boldsymbol{u}_{T_i} \in U(T_i)$ verifying $\boldsymbol{u}_{T_i}|_{\partial_D T_i} = \boldsymbol{u}_D$ on $\partial_D T_i$, the displacement gradient field $\underset{\sim}{\boldsymbol{G}}_{T_i} \in G(T_i)$ and the first Piola-Kirchoff stress field $\underset{\sim}{\boldsymbol{P}}_{T_i} \in S(T_i)$, that minimize the functional

$$J_T^{HW} = \sum_{1 \leq i \leq N} \int_{T_i} \psi_{T_i} + (\nabla \boldsymbol{u}_{T_i} - \underset{\sim}{\boldsymbol{G}}_{T_i}) : \underset{\sim}{\boldsymbol{P}}_{T_i} - \int_{T_i} \boldsymbol{f}_V \cdot \boldsymbol{u}_{T_i} - \int_{\partial_N T_i \cap \partial_N T} \boldsymbol{t}_{\partial_N T} \cdot \boldsymbol{u}_{T_i} \tag{8}$$

where for all $1 \leq i \neq j \leq N$, the external forces corresponding to the traction applied by T_i onto $\partial T_j \cap \partial T_i$ and to that of T_j onto $\partial T_i \cap \partial T_j$ are direclty eliminated by continuity of the traction force across $\partial T_j \cap \partial T_i$. Assuming the fields to be continuous in T, one has : $\boldsymbol{u}_{T_i} = \boldsymbol{u}_T|_{T_i}, \underset{\sim}{\boldsymbol{G}}_{T_i} = \underset{\sim}{\boldsymbol{G}}_T|_{T_i}, \underset{\sim}{\boldsymbol{P}}_{T_i} = \underset{\sim}{\boldsymbol{P}}_T|_{T_i}$, and the problem simplifies in : find $\boldsymbol{u}_T \in U(T)$ verifying $\boldsymbol{u}_T|_{\partial_D T} = \boldsymbol{u}_D$ on $\partial_D T$, the displacement gradient field $\underset{\sim}{\boldsymbol{G}}_T \in G(T)$ and the first Piola-Kirchoff stress field $\underset{\sim}{\boldsymbol{P}}_T \in S(T)$ that minimize

$$J_T^{HW} = \sum_{1 \leq i \leq N} \int_{T_i} \psi_{T_i} + (\nabla \boldsymbol{u}_T - \underset{\sim}{\boldsymbol{G}}_T) : \underset{\sim}{\boldsymbol{P}}_T - \int_{T_i} \boldsymbol{f}_V \cdot \boldsymbol{u}_T - \int_{\partial_N T} \boldsymbol{t}_{\partial_N T} \cdot \boldsymbol{u}_T|_{\partial T} \tag{9}$$

3. Composite Demo

In order to introduce the discontinuous setting in which lies the Hybrid High Order method, let consider the body Ω to be made out of some material defined by a mechanical potential ψ_Ω. The aim of this section consists in devising the expression of the mechanical energy deriving from the HHO formulation of the mechanical model problem described in Section 2. Following the idea of a composite medium as introduced in 2, let T an arbitrary open subset in Ω, split into two distinct media; an open bulk medium $K \subset T$ with boundary $\partial K \subset \mathbb{R}^{d-1}$, and an open interface medium $I \subset T$ between the bulk K and the boundary ∂T, with boundary $\partial I = \partial K \cup \partial T$ and of some width $\ell > 0$ that is supposed to be small compared to $h_T = \max_{(\boldsymbol{x}_a, \boldsymbol{x}_b) \in T} \|\boldsymbol{x}_a - \boldsymbol{x}_b\|$ the diameter of T.

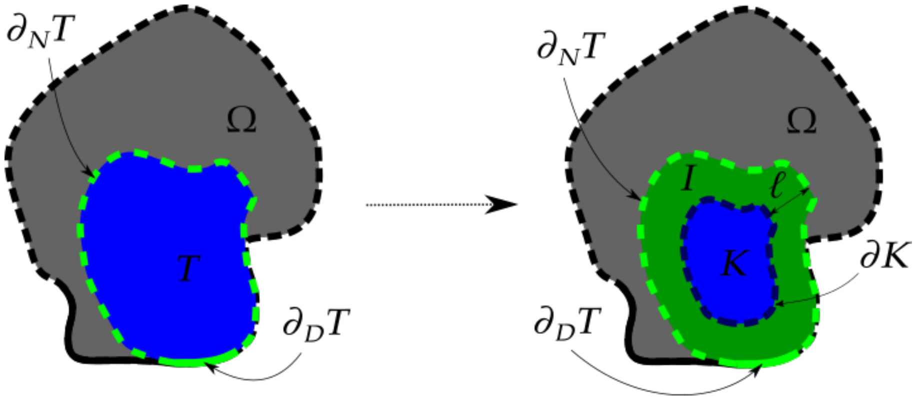

Figure 2. schematic representation of the model problem

Let the boundary ∂T move with a boundary displacement field $\boldsymbol{u}_{\partial T} \in V(\partial T)$, where $V(\partial T)$ denotes the space of kinematically admissible boundary displacements. The displacement at the boundary ∂T results from the interactions of ∂T with neighbouring media, *i.e.* from the action of $\Omega \backslash T$ onto ∂T or from some boundary condition, but let assume that the bulk K is *a priori* not influenced by the movments of ∂T; that is, it supposedly only morhphs through the action of the body load $\boldsymbol{f}_V$, producing a displacement gradient $\underset{\sim}{\boldsymbol{G}}_K \in G(K)$ and a stress $\underset{\sim}{\boldsymbol{P}}_K \in S(K)$ under the mechanical potential ψ_Ω, that are free from the influence of ∂T onto K. Hence, the bulk K is free to move away up to a rigid body motion from the rest of the body Ω at no energetical cost, which violates the conservation laws. Therefore, in order to ensure continuity of the displacement between K and ∂T, let I act as a patch, such that $\boldsymbol{u}_I \in U(I)$ the displacement in I links that of K to that of ∂T :

$$\boldsymbol{u}_I|_{\partial K} = \boldsymbol{u}_K|_{\partial K} \tag{10a}$$

$$\boldsymbol{u}_I|_{\partial T} = \boldsymbol{u}_{\partial T} \tag{10b}$$

In order to bind the behaviour of K to that of its neighbourhood through ∂T, let endow the interface I with a mechanical potential ψ_I such that it behaves like a linear elastic material of Young modulus $\beta(\ell/h_T)$ and a zero Poisson ratio :

$$\psi_I = \frac{1}{2}\beta\frac{\ell}{h_T}\nabla\boldsymbol{u}_I : \nabla\boldsymbol{u}_I \tag{11}$$

where the dimensionless ratio ℓ/h_T balances the accumulated energy with the size of the domain T. Let then $\underset{\sim}{\boldsymbol{G}}_I \in G(I)$ and $\underset{\sim}{\boldsymbol{P}}_I \in S(I)$ the displacement gradient and stress in I. Under such assumptions and using (8), the Hu–Washizu functional over T writes as :

$$J_T^{HW} = \int_K \psi_\Omega + (\nabla_X\boldsymbol{u}_K - \underset{\sim}{\boldsymbol{G}}_K) : \underset{\sim}{\boldsymbol{P}}_K + \int_I \psi_I + (\nabla_X\boldsymbol{u}_I - \underset{\sim}{\boldsymbol{G}}_I) : \underset{\sim}{\boldsymbol{P}}_I - \int_K \boldsymbol{f}_V\cdot\boldsymbol{u}_K - \int_I \boldsymbol{f}_V\cdot\boldsymbol{u}_I - \int_{\partial_N T}\boldsymbol{t}_{\partial_N T}\cdot\boldsymbol{u}_{\partial T} \tag{12}$$

3.1. Interface description

Let Ξ_T the homotethy of ratio $(1+\alpha\ell)$ and center $\boldsymbol{x}_T$ the centroid of T, with $-1/\ell < \alpha < 0$ such that K (respectively ∂K) is the image of T (respectively ∂T) by Ξ_T. Since ∂K is an homotethy of ∂T, any point $\boldsymbol{x}_{\partial T} \in \partial T$ and $\boldsymbol{x}_{\partial K} = \Xi_T(\boldsymbol{x}_{\partial T}) \in \partial K$ share the same unit outward normal $\boldsymbol{n}$. Assuming the interface I to be thin compared to the cell volume T, let linearize the displacement in I with respect to $\boldsymbol{n}$, such that :

$$\boldsymbol{u}_I(\boldsymbol{x}) = \frac{\boldsymbol{u}_{\partial T}(\boldsymbol{x}) - \boldsymbol{u}_K|_{\partial K}(\boldsymbol{x})}{\ell}\otimes\boldsymbol{n}\cdot(\boldsymbol{x} - \boldsymbol{m}_{\partial K}) + \boldsymbol{u}_K|_{\partial K} \tag{13}$$

where $\boldsymbol{m}_{\partial K} = \min_{\boldsymbol{x}_{\partial K}}\|\boldsymbol{x}_{\partial K} - \boldsymbol{x}\|$. Furthermore, let assume that the interface is thin enough such that $\underset{\sim}{\boldsymbol{P}}_I$ is constant along the $\boldsymbol{n}$ direction in I. By continuity of the traction force across ∂K, the following equality holds true :

$$(\underset{\sim}{\boldsymbol{P}}_I - \underset{\sim}{\boldsymbol{P}}_K|_{\partial K})\cdot\boldsymbol{n} = 0 \quad \text{in} \quad I \tag{14}$$

Using (14) and (13), one can write the internale contribution in I as a term depending on the bulk and boundary displacement :

$$\begin{aligned} J_I^{HW_{\text{int}}} :=& \int_I \psi_I + (\nabla\boldsymbol{u}_I - \underset{\sim}{\boldsymbol{G}}_I) : \underset{\sim}{\boldsymbol{P}}_I \\ =&(1-\frac{\alpha}{2}\ell)\int_{\partial K}\frac{\beta}{2h_T}\|\boldsymbol{u}_{\partial T} - \boldsymbol{u}_K|_{\partial K}\|^2 + (1-\frac{\alpha}{2}\ell)\int_{\partial K}(\boldsymbol{u}_{\partial T} - \boldsymbol{u}_K|_{\partial K})\otimes\boldsymbol{n} : \underset{\sim}{\boldsymbol{P}}_K|_{\partial K} - \int_I \underset{\sim}{\boldsymbol{G}}_I : \underset{\sim}{\boldsymbol{P}}_I \end{aligned} \tag{15}$$

Development 3.1 (Interafce simplification). *Let $C_I = \{v \in L^2(I) \mid v\cdot\boldsymbol{n} = cste\}$ the set of L^2-functions which are constant along the normal axis in I. For any function in C_I, the following equality holds true:*

$$\int_I v\,dV = \int_{\partial K}\int_{\epsilon=0}^{\ell} v(1-\alpha\epsilon)\,dS\,d\epsilon = \ell(1-\frac{\alpha}{2}\ell)\int_{\partial K} v\,dS \tag{16}$$

Noticing that $\nabla \boldsymbol{u}_I \in C_I$, one has :

$$
\begin{aligned}
\int_I \psi_I =& \ell(1-\frac{\alpha}{2}\ell)\int_{\partial K}\frac{1}{2}\beta\frac{\ell}{h_T}\nabla\boldsymbol{u}_I : \nabla\boldsymbol{u}_I \\
=& \ell(1-\frac{\alpha}{2}\ell)\int_{\partial K}\frac{\beta}{2\ell h_T}(\boldsymbol{u}_{\partial T}-\boldsymbol{u}_K|_{\partial K})\otimes\boldsymbol{n} : (\boldsymbol{u}_{\partial T}-\boldsymbol{u}_K|_{\partial K})\otimes\boldsymbol{n} \\
=& \ell(1-\frac{\alpha}{2}\ell)\int_{\partial K}\frac{\beta}{2\ell h_T}\sum_{i,j}(u_{\partial T\,i}-u_{K\,i}|_{\partial K})^2 {n_j}^2 \\
=& \ell(1-\frac{\alpha}{2}\ell)\int_{\partial K}\frac{\beta}{2\ell h_T}\sum_j {n_j}^2\sum_i(u_{\partial T\,i}-u_{K\,i}|_{\partial K})^2 \\
=& \ell(1-\frac{\alpha}{2}\ell)\int_{\partial K}\frac{\beta}{2\ell h_T}\sum_i(u_{\partial T\,i}-u_{K\,i}|_{\partial K})^2 \\
=& \ell(1-\frac{\alpha}{2}\ell)\int_{\partial K}\frac{\beta}{2\ell h_T}\|\boldsymbol{u}_{\partial T}-\boldsymbol{u}_K|_{\partial K}\|^2 \\
=& (1-\frac{\alpha}{2}\ell)\int_{\partial K}\frac{\beta}{2h_T}\|\boldsymbol{u}_{\partial T}-\boldsymbol{u}_K|_{\partial K}\|^2
\end{aligned} \tag{17}
$$

Moreover, for $\underset{\sim}{\boldsymbol{P}}_I$ in C_I :

$$
\begin{aligned}
\int_I \nabla\boldsymbol{u}_I : \underset{\sim}{\boldsymbol{P}}_I =& \ell(1-\frac{\alpha}{2}\ell)\int_{\partial K}\nabla\boldsymbol{u}_I : \underset{\sim}{\boldsymbol{P}}_I \\
=& \ell(1-\frac{\alpha}{2}\ell)\int_{\partial K}\frac{1}{\ell}(\boldsymbol{u}_{\partial T}-\boldsymbol{u}_K|_{\partial K})\otimes\boldsymbol{n} : \underset{\sim}{\boldsymbol{P}}_K|_{\partial K} \\
=& \ell(1-\frac{\alpha}{2}\ell)\int_{\partial K}\frac{1}{\ell}\sum_{i,j}(u_{\partial T\,i}-u_{K\,i}|_{\partial K})n_j P_{K\,ij}|_{\partial K} \\
=& \ell(1-\frac{\alpha}{2}\ell)\int_{\partial K}\frac{1}{\ell}(\boldsymbol{u}_{\partial T}-\boldsymbol{u}_K|_{\partial K})\cdot\underset{\sim}{\boldsymbol{P}}_K|_{\partial K}\cdot\boldsymbol{n} \\
=& (1-\frac{\alpha}{2}\ell)\int_{\partial K}(\boldsymbol{u}_{\partial T}-\boldsymbol{u}_K|_{\partial K})\cdot\underset{\sim}{\boldsymbol{P}}_K|_{\partial K}\cdot\boldsymbol{n}
\end{aligned} \tag{18}
$$

And Finally :

$$
J_I = (1-\frac{\alpha}{2}\ell)\int_{\partial K}\frac{\beta}{2h_T}\|\boldsymbol{u}_{\partial T}-\boldsymbol{u}_K|_{\partial K}\|^2 + (1-\frac{\alpha}{2}\ell)\int_{\partial K}(\boldsymbol{u}_{\partial T}-\boldsymbol{u}_K|_{\partial K})\cdot\underset{\sim}{\boldsymbol{P}}_K|_{\partial K}\cdot\boldsymbol{n} - \int_I \underset{\sim}{\boldsymbol{G}}_I : \underset{\sim}{\boldsymbol{P}}_I \tag{19}
$$

Injecting (15) in (12) :

$$
\begin{aligned}
J_T =& \int_K \psi_\Omega + (\nabla\boldsymbol{u}_K - \underset{\sim}{\boldsymbol{G}}_K) : \underset{\sim}{\boldsymbol{P}}_K + (1-\frac{\alpha}{2}\ell)\int_{\partial K}(\boldsymbol{u}_{\partial T}-\boldsymbol{u}_K|_{\partial K})\cdot\underset{\sim}{\boldsymbol{P}}_K|_{\partial K}\cdot\boldsymbol{n} \\
&+ (1-\frac{\alpha}{2}\ell)\int_{\partial K}\frac{\beta}{2h_T}\|\boldsymbol{u}_{\partial T}-\boldsymbol{u}_K|_{\partial K}\|^2 - \int_I \underset{\sim}{\boldsymbol{G}}_I : \underset{\sim}{\boldsymbol{P}}_I - \int_K \boldsymbol{f}_V\cdot\boldsymbol{u}_K - \int_I \boldsymbol{f}_V\cdot\boldsymbol{u}_I - \int_{\partial_N T}\boldsymbol{t}_{\partial_N T}\cdot\boldsymbol{u}_{\partial T}
\end{aligned} \tag{20}
$$

Since ℓ is arbitrary, let $\ell \to 0$, the interface region vanishes such that $I = \emptyset, K = T$ and $\partial K = \partial T$, and the expression of the Hu–Washizu functional over the region T writes:

$$
\begin{aligned}
J_T =& \int_T \psi_\Omega + (\nabla\boldsymbol{u}_T - \underset{\sim}{\boldsymbol{G}}_T) : \underset{\sim}{\boldsymbol{P}}_T + \int_{\partial T}(\boldsymbol{u}_{\partial T}-\boldsymbol{u}_T|_{\partial T})\cdot\underset{\sim}{\boldsymbol{P}}_T|_{\partial T}\cdot\boldsymbol{n} + \int_{\partial T}\frac{\beta}{2h_T}\|\boldsymbol{u}_{\partial T}-\boldsymbol{u}_T|_{\partial T}\|^2 \\
&- \int_T \boldsymbol{f}_V\cdot\boldsymbol{u}_T - \int_{\partial_N T}\boldsymbol{t}_{\partial_N T}\cdot\boldsymbol{u}_{\partial T}
\end{aligned} \tag{21}
$$

Assuming that the displacement is continuous at the boundary ∂T such that $\boldsymbol{u}_{\partial T}$ is the trace of the cell displacement $\boldsymbol{u}_T$ on ∂T and $\boldsymbol{u}_{\partial T} - \boldsymbol{u}_T|_{\partial T} = 0$, one recovers the usual expression of the Hu–Washizu integral over the element for the three variables $(\boldsymbol{u}_T, \underset{\sim}{\boldsymbol{G}}_T, \underset{\sim}{\boldsymbol{P}}_T)$. However, if one considers that $\boldsymbol{u}_{\partial T}$ and $\boldsymbol{u}_T$ are disticnt variables, *i.e.* that the boundary ∂T is able to move from the cell T such that the displacement across ∂T is discontinuous, J_T writes as a function of the four variables $(\boldsymbol{u}_T, \boldsymbol{u}_{\partial T}, \underset{\sim}{\boldsymbol{G}}_T, \underset{\sim}{\boldsymbol{P}}_T)$. Such an assumtion relates to the concept of hybridization of the displacement unknown, which is at the foundation of the Hybrid Discontinuous Galerkin method and of the HHO method. Besides, replacing $\boldsymbol{u}_{\partial T}$ by $\boldsymbol{u}_{T'}$ for any neighbourging region T' to T defines the Discontinuous Galerkin method, where only the core unknown $\boldsymbol{u}_T$ is considered, and the displacement jumps depends on neighbouring regions. Differentiating J_T over each of these variables, and introducing the explicit traction force $\boldsymbol{\theta}_{\partial T} = \underset{\sim}{\boldsymbol{P}}_T|_{\partial T} \cdot \boldsymbol{n} + (\beta/h_T)(\boldsymbol{u}_{\partial T} - \boldsymbol{u}_T|_{\partial T})$ one obtains the system

$$\frac{\partial J_T}{\partial \boldsymbol{u}_T}\delta \boldsymbol{u}_T = \int_T \underset{\sim}{\boldsymbol{P}}_T : \nabla \delta \boldsymbol{u}_T - \int_T \boldsymbol{f}_V \cdot \delta \boldsymbol{u}_T - \int_{\partial T} \boldsymbol{\theta}_{\partial T} \cdot \delta \boldsymbol{u}_T|_{\partial T} \qquad \forall \delta \boldsymbol{u}_T \in U(T) \tag{22a}$$

$$\frac{\partial J_T}{\partial \boldsymbol{u}_{\partial T}}\delta \boldsymbol{u}_{\partial T} = \int_{\partial_N T} (\boldsymbol{\theta}_{\partial T} - \boldsymbol{t}_{\partial_N T}) \cdot \delta \boldsymbol{u}_{\partial T} \qquad \forall \delta \boldsymbol{u}_{\partial T} \in V(\partial T) \tag{22b}$$

$$\frac{\partial J_T}{\partial \underset{\sim}{\boldsymbol{G}}_T}\delta \underset{\sim}{\boldsymbol{G}}_T = \int_T (\frac{\partial \psi_\Omega}{\partial \underset{\sim}{\boldsymbol{G}}_T} - \underset{\sim}{\boldsymbol{P}}_T) : \delta \underset{\sim}{\boldsymbol{G}}_T \qquad \forall \delta \underset{\sim}{\boldsymbol{G}}_T \in G(T) \tag{22c}$$

$$\frac{\partial J_T}{\partial \underset{\sim}{\boldsymbol{P}}_T}\delta \underset{\sim}{\boldsymbol{P}}_T = \int_T (\nabla \boldsymbol{u}_T - \underset{\sim}{\boldsymbol{G}}_T) : \delta \underset{\sim}{\boldsymbol{P}}_T + \int_{\partial T} (\boldsymbol{u}_{\partial T} - \boldsymbol{u}_T|_{\partial T}) \cdot \delta \underset{\sim}{\boldsymbol{P}}_T|_{\partial T} \cdot \boldsymbol{n} \qquad \forall \delta \underset{\sim}{\boldsymbol{P}}_T \in S(T) \tag{22d}$$

The set of equation (22) fully defines the framework for Discontinuous Galerkin methods. In particular, (22a) is the expression of the principle of virtual works in T, where the explicit traction force $\boldsymbol{\theta}_{\partial T}$ replaces the usual expression $\underset{\sim}{\boldsymbol{P}}_T \cdot \boldsymbol{n}$ in the external contribution, and (22b) denotes a supplementary equation to the usual continuous problem as seen in (4), to account for the continuity of the flux $\boldsymbol{\theta}_{\partial T}$ across the cell boundary. This feature consitutes one of the key assets of non-conformal method; indeed, by defining a richer flux than in the ususal continuous framework, that aslo depends on the displacement jump, one allows for the latter to act as a Lagrange multiplyier in order to fulfil the flux continuity requriement on ∂T. The tradeoff for this condition to hold true consists in loosening the displacement continuity condition through the introduction of the displacement jump at the boundary. (22c) accounts for the constitutive equation in a weak sense, and (22d) defines the equation of an enhanced gradient gradient field, that does not reduces to the projection of $\nabla \boldsymbol{u}_T$ onto $G(T)$ as in (4c), since it is enriched by a boundary component that depends on the displacement jump, which is at the origin of the robustness of non-conformal methods to volumetric locking.

Instead of seeking the four fields explicitly, by noticing that minimization of (22d) defines a linear problem with any displacement pair $(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}) \in U(T) \times V(\partial T)$ such that there is a unique $\underset{\sim}{\boldsymbol{G}}_T$ minimizing (22d), and that minimization of (22c) is linear with the derivative of ψ_Ω with respect to $\underset{\sim}{\boldsymbol{G}}_T$, one can eliminate (22c) and (22d) from the system, by considering the simplifed functional

$$J_T = \int_T \psi_\Omega + \int_{\partial T} \frac{\beta}{2h_T} \|\boldsymbol{u}_{\partial T} - \boldsymbol{u}_T|_{\partial T}\|^2 - \int_T \boldsymbol{f}_V \cdot \boldsymbol{u}_T - \int_{\partial_N T} \boldsymbol{t}_{\partial_N T} \cdot \boldsymbol{u}_{\partial T} \tag{23}$$

where (22d) results in the definition of the reconstructed gradient $\underset{\sim}{\boldsymbol{G}}_T(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T})$ associated with any displacement pair $(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}) \in U(T) \times V(\partial T)$ that solves

$$\int_T \underset{\sim}{\boldsymbol{G}}_T(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}) : \underset{\sim}{\boldsymbol{\tau}}_T = \int_T \nabla \boldsymbol{v}_T : \underset{\sim}{\boldsymbol{\tau}}_T + \int_{\partial T} (\boldsymbol{v}_{\partial T} - \boldsymbol{v}_T|_{\partial T}) \cdot \underset{\sim}{\boldsymbol{\tau}}_T|_{\partial T} \cdot \boldsymbol{n} \quad \forall \underset{\sim}{\boldsymbol{\tau}}_T \in S(T) \tag{24}$$

and the stress $\underset{\sim}{\boldsymbol{P}}_T(\underset{\sim}{\boldsymbol{G}}_T(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}))$ defines as the projection onto $G(T)$ of the derivative of ψ_Ω with respect to $\underset{\sim}{\boldsymbol{G}}_T(\boldsymbol{u}_T, \boldsymbol{u}_{\partial T})$ for any displacement pair $(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}) \in U(T) \times V(\partial T)$

$$\int_T \underset{\sim}{\boldsymbol{P}}_T(\underset{\sim}{\boldsymbol{G}}_T(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T})) : \underset{\sim}{\boldsymbol{\gamma}}_T = \int_T \frac{\partial \psi_\Omega}{\partial \underset{\sim}{\boldsymbol{G}}_T(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T})} : \underset{\sim}{\boldsymbol{\gamma}}_T \quad \forall \underset{\sim}{\boldsymbol{\gamma}}_T \in G(T) \tag{25}$$

where one notices that the equality holds in a weak sense if $S(T) \subset G(T)$.

Development 3.2 (Elliptic projection). *Let $U^h(T) \subset U(T)$ and $U^\perp(T) \subset U(T)$ such that $U(T) = U^h(T) \oplus U^\perp(T)$, and set $\boldsymbol{u}_T = \boldsymbol{u}_T^h + \boldsymbol{u}_T^\perp$ with $\boldsymbol{u}_T^h \in U^h(T)$ and $\boldsymbol{u}_T^\perp \in U^\perp(T)$ the orthogonal projections of $\boldsymbol{u}_T$ onto $U^h(T)$ and $U^\perp(T)$*

respectively. Let $V^h(\partial T) \subset V(\partial T)$ and $\boldsymbol{u}^h_{\partial T} \in V^h(\partial T)$ the orthogonal projection of $\boldsymbol{u}_T$ onto $V^h(\partial T)$. The orthogonal projection of $\boldsymbol{u}_T$ onto $U^h(T) \times V^h(\partial T)$ is then the displacement pair $(\boldsymbol{u}^h_T, \boldsymbol{u}^h_{\partial T})$. Let $S^h(T) = \{\underset{\sim}{\tau} \in S(T) \mid \nabla \cdot \underset{\sim}{\tau}^h_T \in U^h(T) \mid \underset{\sim}{\tau}^h_T|_{\partial T} \cdot \boldsymbol{n} \in V^h(\partial T)\}$, and $\underset{\sim}{\boldsymbol{G}}^h_T \in S^h(T)$ the solution of (24) in $U^h(T) \times V^h(\partial T)$ such that

$$\int_T \underset{\sim}{\boldsymbol{G}}^h_T(\boldsymbol{u}^h_T, \boldsymbol{u}^h_{\partial T}) : \underset{\sim}{\tau}^h_T = \int_T \nabla \boldsymbol{u}^h_T : \underset{\sim}{\tau}^h_T + \int_{\partial T} (\boldsymbol{u}^h_{\partial T} - \boldsymbol{u}^h_T|_{\partial T}) \cdot \underset{\sim}{\tau}^h_T|_{\partial T} \cdot \boldsymbol{n} \qquad \forall \underset{\sim}{\tau}^h_T \in S^h(T) \tag{26}$$

using the fact that $\boldsymbol{u}^h_{\partial T}$ is the projection of $\boldsymbol{u}_T$ onto $V^h(\partial T)$ and that $\underset{\sim}{\tau}|_{\partial T} \cdot \boldsymbol{n} \in V^h(\partial T)$:

$$\begin{aligned} \int_T \underset{\sim}{\boldsymbol{G}}^h_T(\boldsymbol{u}^h_T, \boldsymbol{u}^h_{\partial T}) : \underset{\sim}{\tau}^h_T &= \int_T \nabla \boldsymbol{u}^h_T : \underset{\sim}{\tau}^h_T + \int_{\partial T} (\boldsymbol{u}_T|_{\partial T} - \boldsymbol{u}^h_T|_{\partial T}) \cdot \underset{\sim}{\tau}^h_T|_{\partial T} \cdot \boldsymbol{n} && \forall \underset{\sim}{\tau}^h_T \in S^h(T) \\ &= \int_T \nabla \boldsymbol{u}^h_T : \underset{\sim}{\tau}^h_T + \int_{\partial T} \boldsymbol{u}^\perp_T|_{\partial T} \cdot \underset{\sim}{\tau}^h_T|_{\partial T} \cdot \boldsymbol{n} && \forall \underset{\sim}{\tau}^h_T \in S^h(T) \end{aligned} \tag{27}$$

using the divergence theorem and the fact that $\nabla \cdot \underset{\sim}{\tau}^h_T \in U^h(T)$, one has :

$$\int_T \nabla \boldsymbol{u}^\perp_T : \underset{\sim}{\tau}^h_T = \int_{\partial T} \boldsymbol{u}^\perp_T|_{\partial T} \cdot \underset{\sim}{\tau}^h_T|_{\partial T} \cdot \boldsymbol{n} \tag{28}$$

such that :

$$\begin{aligned} \int_T \underset{\sim}{\boldsymbol{G}}^h_T(\boldsymbol{u}^h_T, \boldsymbol{u}^h_{\partial T}) : \underset{\sim}{\tau}^h_T &= \int_T \nabla \boldsymbol{u}^h_T : \underset{\sim}{\tau}^h_T + \int_T \nabla \boldsymbol{u}^\perp_T : \underset{\sim}{\tau}^h_T && \forall \underset{\sim}{\tau}^h_T \in S^h(T) \\ &= \int_T \nabla \boldsymbol{u}_T : \underset{\sim}{\tau}^h_T && \forall \underset{\sim}{\tau}^h_T \in S^h(T) \end{aligned} \tag{29}$$

which proves that $\underset{\sim}{\boldsymbol{G}}^h_T(\boldsymbol{u}^h_T, \boldsymbol{u}^h_{\partial T})$ is the orthogonal projection of $\nabla \boldsymbol{u}_T$ onto $S^h(T)$. IL FAUDRAIT MONTRER QUE $S^h(T)$ EST SUFFISANT POUR EMPECHER LE LOCKING DANS $U^h(T) \times V^h(\partial T)$?? ET FAIRE LE LIEN ENTRE $S^h(T)$ et $G^h(T)$. Trouver aussi une justification pour $S(T) \subset G(T)$, la contrainte est plus régulière que le gradient, ce qui semble vrai (avec les décompositions sphérique dévaitoriques par exemples, etc)

The problem in primal form amounts to find the displacement pair $(\boldsymbol{u}_T, \boldsymbol{u}_{\partial T}) \in U(T) \times V(\partial T)$ verifying $\boldsymbol{u}_{\partial T} = \boldsymbol{u}_D$ on $\partial_D T$, such that for all kinematically admissible displacements pairs $(\delta \boldsymbol{u}_T, \delta \boldsymbol{u}_{\partial T}) \in U(T) \times V(\partial T)$, the functional (23) is minimal :

$$dJ_T = \delta J^{\text{int}}_T - \delta J^{\text{ext}}_T = 0 \tag{30}$$

with

$$\delta J^{\text{int}}_T = \int_T \underset{\sim}{\boldsymbol{P}}_T(\underset{\sim}{\boldsymbol{G}}_T(\boldsymbol{u}_T, \boldsymbol{u}_{\partial T})) : \underset{\sim}{\boldsymbol{G}}_T(\delta \boldsymbol{u}_T, \delta \boldsymbol{u}_{\partial T}) + \int_{\partial T} (\beta / h_T) \boldsymbol{Z}_{\partial T}(\boldsymbol{u}_T, \boldsymbol{u}_{\partial T}) \cdot \boldsymbol{Z}_{\partial T}(\delta \boldsymbol{u}_T, \delta \boldsymbol{u}_{\partial T}) \tag{31a}$$

$$\delta J^{\text{ext}}_T = \int_{\partial_N T} \boldsymbol{t}_{\partial_N T} \cdot \delta \boldsymbol{u}_{\partial T} + \int_T \boldsymbol{f}_V \cdot \delta \boldsymbol{u}_T \tag{31b}$$

where we introduced the jump function $\boldsymbol{Z}_{\partial T}$:

$$\boldsymbol{Z}_{\partial T}(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}) = \boldsymbol{v}_{\partial T} - \boldsymbol{v}_T|_{\partial T} \quad \forall (\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}) \in U(T) \times V(\partial T) \tag{32}$$

In particular, one can readliy see the resemblance of (31) with the ususal formulation of the principle of virtual works, where the so called reconstructed displacement gradient $\underset{\sim}{\boldsymbol{G}}_T(\boldsymbol{u}_T, \boldsymbol{u}_{\partial T})$ plays the role of the usual displacement Lagrangian gradient $\nabla \boldsymbol{u}_T$, and where an additional term corresponding to a traction energy on the boundary has been added to account for the penalization of the displacement jump on ∂T through $\boldsymbol{Z}_{\partial T}$. This term is discribed in the literature as the stabilization term, as it prevents rigid body motions to be solutions of the problem.

4. Discretization

In the idea of (8), let consider that Ω is made out of a composition of N_T distinct open polyhedral subsets with planar faces $T_i \subset \Omega$ called cells or elements, collected in the set $\mathcal{T}(\Omega) = \{T_i \subset \Omega \mid 1 \leq i \leq N_T\}$, called a triangulation or a mesh of the domain Ω as is usual with the standard Finite Element method. Considering that each of these cells behave like a composite material as defined ABOVE, such that they depend on both a bulk and boundary displacement unknowns, where two elements sharing a boundary also share the same boundary unknown, *i.e.* for any $(T, T') \in \mathcal{T}(\Omega)$ such that $\partial T \cap \partial T' \neq \emptyset$, there is $\boldsymbol{u}_{\partial T \cap \partial T'} = \boldsymbol{u}_{\partial T' \cap \partial T}$, one naturally introduces $\mathcal{F}(\Omega) = \{F_i \subset \Omega \mid 1 \leq i \leq N_F\}$ the skeleton of the mesh, collecting all element faces F_i in the mesh, where N_F denotes the numfer of faces in the mesh. A face $F \subset \mathbb{R}^{d-1}$ is a closed subset of Ω, and either there are two cells T and T' such that $F = \partial T \cap \partial T'$ (F is then an interior face), or there is a single cell T such that $F = \partial T \cap \partial\Omega$ (F is then an exterior face). Let $\mathcal{F}^i(\Omega)$ denote the set of interior faces, and $\mathcal{F}^e(\Omega)$ that of exterior ones. $\mathcal{F}^e(\Omega)$ is partitioned into $\mathcal{F}^e_D(\Omega) = \{F \in \mathcal{F}^e(\Omega) \mid F \subset \partial_D\Omega\}$ the set of exterior faces imposed to prescribed Dirichlet boundary conditions, and into $\mathcal{F}^e_N(\Omega) = \{F \in \mathcal{F}^e(\Omega) \mid F \subset \partial_N\Omega\}$ the set of exterior faces imposed to prescribed Neumann boundary conditions. Similarily, for any element $T \in \mathcal{T}$, let $\mathcal{F}(T) = \{F \in \mathcal{F} \mid F \subset \partial T\}$ the set of faces composing the boundary of T. The composition of both $\mathcal{T}(\Omega)$ and $\mathcal{F}(\Omega)$ forms the hybrid mesh $\bar{\mathcal{T}}(\Omega) = \{\mathcal{T}(\Omega), \mathcal{F}(\Omega)\}$.

Contrary to the standard finite element method that consists in seeking a global solution in a regular enough space over the whole mesh, we consider here a much broader space that consists in $U(\mathcal{T}) = \prod_{T \in \mathcal{T}(\Omega)} U(T)$ the collection of all regular enough displacements element-wise, such that displacement jumps are actually possible across elements. Similarily, we consider $V(\mathcal{F}) = \prod_{F \in \mathcal{F}(\Omega)} V(F)$ the space of all regular enough displacements face-wise for the skeleton unknown, such that the global solution space $U(\bar{\mathcal{T}}) = U(\mathcal{T}) \times V(\mathcal{F})$ for the whole problem is simply the assembly of all element and face spaces in the mesh.

The global mechanical problem of Ω reads : find the global displacement unknown pair $(\boldsymbol{u}_{\mathcal{T}}, \boldsymbol{u}_{\mathcal{F}}) \in U(\mathcal{T}) \times V(\mathcal{F})$ verifying $\boldsymbol{u}_{\mathcal{F}}|_{\partial_D\Omega} = \boldsymbol{u}_D$ on $\partial_D\Omega$ such that

$$\delta J^{\text{int}}_{\mathcal{T}} - \delta J^{\text{ext}}_{\mathcal{T}} = 0 \tag{33}$$

with

$$\delta J^{\text{int}}_{\mathcal{T}} = \sum_{T \in \mathcal{T}(\Omega)} \int_T \underset{\sim}{\boldsymbol{P}}_T(\underset{\sim}{\boldsymbol{G}}_T(\boldsymbol{u}_T, \boldsymbol{u}_{\partial T})) : \underset{\sim}{\boldsymbol{G}}_T(\delta\boldsymbol{u}_T, \delta\boldsymbol{u}_{\partial T}) + \int_{\partial T} (\beta/h_T) \boldsymbol{Z}_{\partial T}(\boldsymbol{u}_T, \boldsymbol{u}_{\partial T}) \cdot \boldsymbol{Z}_{\partial T}(\delta\boldsymbol{u}_T, \delta\boldsymbol{u}_{\partial T}) \tag{34a}$$

$$\delta J^{\text{ext}}_{\mathcal{T}} = \sum_{F \in \mathcal{F}^e_N(\Omega)} \int_F \boldsymbol{t}_N \cdot \delta\boldsymbol{u}_F + \sum_{T \in \mathcal{T}(\Omega)} \int_T \boldsymbol{f}_V \cdot \delta\boldsymbol{u}_T \tag{34b}$$

where for each element $T \in \mathcal{T}$, the boundary displacement field $\boldsymbol{v}_{\partial T}$ is such that $\boldsymbol{v}_{\partial T} = \boldsymbol{v}_F$ on F for every $F \in \mathcal{F}(T)$, and the displacement space of its boundary $V(\partial T)$ is the collection of displacement spaces in faces composing it such that $V(\partial T) = \prod_{F \in \mathcal{F}(T)} V(F)$.

A polynomial approximation of the global solution is then sought in a subspace of $U(\mathcal{T}) \times V(\mathcal{F})$, and for each element $T \in \mathcal{T}$, we denote $U^h(T) \subset U(T)$ the approximation displacement space in the cell, and $V^h(\partial T) \subset V(\partial T)$ that on the boundary. Moreover, though they are not primal unknowns of the problem, one need to define $G^h(T) \subset G(T)$ the space used to build the discrete reconstructed gradient and $S^h(T) \subset S(T)$ that chosen for the discrete stress :

$$\begin{aligned}
U^h(T) &= P^l(T, \mathbb{R}^d) \\
V^h(\partial T) &= P^k(\partial T, \mathbb{R}^d) \\
G^h(T) &= P^k(T, \mathbb{R}^{d \times d}) \\
S^h(T) &= P^k(T, \mathbb{R}^{d \times d})
\end{aligned}$$

where the cell displacement polynomial order l might be chosen different from the face displacement order k such that $k - 1 \leq l \leq k + 1$. Accounting for the possible different polynomial order bteween the cell and faces, one can specify a discrete jump function in a natural way such that it delivers the displacement difference pointwise for any displacement pair $(\boldsymbol{v}^l_T, \boldsymbol{v}^k_{\partial T}) \in U^h(T) \times V^h(\partial T)$:

$$\boldsymbol{Z}^{HDG}_{\partial T}(\boldsymbol{v}^l_T, \boldsymbol{v}^k_{\partial T}) = \Pi^k_{\partial T}(\boldsymbol{v}^k_{\partial T} - \boldsymbol{v}^l_T|_{\partial T}) \tag{35}$$

Where $\Pi^k_{\partial T}$ denotes the orthogonal projector onto $V^h(\partial T)$. This straightforward discrete jump function is at the origin of the Hybrid Discontinuous Galerkin method, and grants a convergence of order k in the energy norm. A richer discrete jump function $\boldsymbol{Z}^{HHO}_{\partial T}$ prviding a convergence of order $k+1$ in the energy norm was introduced in [**?**], hence giving the method its name, such that :

$$\boldsymbol{Z}^{HHO}_{\partial T}(\boldsymbol{v}^l_T, \boldsymbol{v}^k_{\partial T}) = \Pi^k_{\partial T}(\boldsymbol{v}^k_{\partial T} - \boldsymbol{v}^l_T|_{\partial T} - ((\boldsymbol{I}^{k+1}_T - \Pi^k_T)(\boldsymbol{D}^{k+1}_T))|_{\partial T}) \tag{36}$$

where Π^k_T is the projector onto $P^k(T, \mathbb{R}^d)$, and $\boldsymbol{D}^{k+1}_T \in D(T)$ denotes a higher order discrete displacement in $D(T) = P^{k+1}(T, \mathbb{R}^d)$ solving the following linear problem with any displacement pair $(\boldsymbol{v}^l_T, \boldsymbol{v}^k_{\partial T}) \in U^h(T) \times V^h(\partial T)$:

$$\int_T \nabla \boldsymbol{D}^{k+1}_T : \nabla \boldsymbol{w}^{k+1}_T = \int_T \nabla \boldsymbol{v}^l_T : \nabla \boldsymbol{w}^{k+1}_T + \int_{\partial T} (\boldsymbol{v}^k_{\partial T} - \boldsymbol{v}^l_T) \cdot \nabla \boldsymbol{w}^{k+1}_T \cdot \boldsymbol{n} \qquad \forall \boldsymbol{w}^{k+1}_T \in P^{k+1}(T, \mathbb{R}^d) \tag{37a}$$

$$\int_T \boldsymbol{D}^{k+1}_T = \int_T \boldsymbol{v}^l_T \tag{37b}$$

With obvious notations, let $U^h(\mathcal{T}) = \prod_{T \in \mathcal{T}(\Omega)} U^h(T)$ the global discrete cell displacement space, $V^h(\mathcal{F}) = \prod_{F \in \mathcal{F}(\Omega)} V^h(F)$ the global discrete face displacement space, and $U^h(\bar{\mathcal{T}}) = U^h(\mathcal{T}) \times V^h(\mathcal{F})$ the global unknwon approximation space. The global problem in discrete form finally writes : find the global displacement unknown pair $(\boldsymbol{u}^l_{\mathcal{T}}, \boldsymbol{u}^k_{\mathcal{F}}) \in U^h(\mathcal{T}) \times V^h(\mathcal{F})$ verifying $\boldsymbol{u}^k_{\mathcal{F}}|_{\partial_D \Omega} = \boldsymbol{u}_D$ on $\partial_D \Omega$ such that

$$\delta J^{\text{int}}_{\mathcal{T}} - \delta J^{\text{ext}}_{\mathcal{T}} = 0 \tag{38}$$

with

$$\delta J^{\text{int}}_{\mathcal{T}} = \sum_{T \in \mathcal{T}(\Omega)} \int_T \underset{\sim}{\boldsymbol{P}}^k_T(\underset{\sim}{\boldsymbol{G}}^k_T(\boldsymbol{u}^l_T, \boldsymbol{u}^k_{\partial T})) : \underset{\sim}{\boldsymbol{G}}^k_T(\delta \boldsymbol{u}^l_T, \delta \boldsymbol{u}^k_{\partial T}) + \int_{\partial T} (\beta / h_T) \boldsymbol{Z}^{HHO}_{\partial T}(\boldsymbol{u}^l_T, \boldsymbol{u}^k_{\partial T}) \cdot \boldsymbol{Z}^{HHO}_{\partial T}(\delta \boldsymbol{u}^l_T, \delta \boldsymbol{u}^k_{\partial T}) \tag{39a}$$

$$\delta J^{\text{ext}}_{\mathcal{T}} = \sum_{F \in \mathcal{F}^e_N(\Omega)} \int_F \boldsymbol{t}_N \cdot \delta \boldsymbol{u}^k_F + \sum_{T \in \mathcal{T}(\Omega)} \int_T \boldsymbol{f}_V \cdot \delta \boldsymbol{u}^l_T \tag{39b}$$

5. Axisymmetric

In the following section, we devise a Hybrid High order method for an axisymmetric framework. The cartesian space is expressed in cylindrical coordinates and a point $\boldsymbol{x} \in \Omega$ has coordinates $\boldsymbol{x} = (r, z, \theta)$ where r denotes the radial component, z the ordonal one, and θ is the angular componant describing a revolution around the axis $r = 0$. By cylindrical symmetry, the angular displacement u_θ is supposed to be zero, and both components u_r and u_z do not depend on the angular coordinate θ. Adopting notations introduced in section ABOVE, let T an open subset of $\Omega \subset \mathbb{R}^2$ in the (r, z) plane with cell displacement $\boldsymbol{u}_T \in U(T)$ and boundary displacement $\boldsymbol{u}_{\partial T} \in V(\partial T)$. The partial derivatives of $\boldsymbol{u}_T$ with respect to the cylindrical coordinates are given by :

$$\forall i, j \in \{r, z\}, u_{Ti,j} = \frac{\partial u_{Ti}}{\partial j} \quad \text{and} \quad u_{T\theta,\theta} = \frac{u_{Tr}}{r} \tag{40}$$

In order to express a Hybrid High Order method in such a framework and owing to the assumtptions on the displacement and its gradient, the definition of the reconstructed gradient (24) needs be modified accordingly, and the angular componenent $G_{T\theta\theta}$ does not defines by the same equation as those in the other directions. In particular, for any displacement pair $(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}) \in U(T) \times V(\partial T)$, the componenent $G_{T\theta\theta}(v_{Tr}, v_{\partial Tr})$ solves

$$\int_T 2\pi r G_{T\theta\theta}(v_{Tr}, v_{\partial Tr}) \tau_{T\theta\theta} = \int_T 2\pi r \frac{u_{Tr}}{r} \tau_{T\theta\theta} = \int_T 2\pi u_{Tr} \tau_{T\theta\theta} \quad \forall \underset{\sim}{\tau}_T \in S(T) \tag{41}$$

whereas in the radial and ordonal directions, *i.e.* $\forall i, j \in \{r, z\}$, the expression given in (24) is retrieved, and the component $G_{Tij}(v_{Ti}, v_{\partial Ti})$ solves :

$$\int_T 2\pi r G_{Tij}(v_{Ti}, v_{\partial Ti}) \tau_{Tij} = \int_T 2\pi r \frac{\partial u_{Ti}}{\partial j} \tau_{ij} - \int_{\partial T} 2\pi r (u_{\partial Ti} - u_{Ti}|_{\partial T}) \tau_{Tij}|_{\partial T} n_j \quad \forall \underset{\sim}{\tau}_T \in S(T) \tag{42}$$

As for the reconstructed gradient, the higher order potential term needed to define the HHO jump function nedds also be reconsidered such that $\forall \boldsymbol{w}_T \in \mathbb{P}^{k+1}(T, \mathbb{R}^2)$, the radial component w_{Tr} solves

$$\begin{aligned}\int_T 2\pi r(\sum_{i\in\{r,z\}} \frac{\partial D_{Tr}}{\partial i}\frac{\partial w_{Tr}}{\partial i} + \frac{D_{Tr}}{r}\frac{w_{Tr}}{r}) = \int_T 2\pi r(\sum_{i\in\{r,z\}} \frac{\partial u_{Tr}}{\partial i}\frac{\partial w_{Tr}}{\partial i} + \frac{u_{Tr}}{r}\frac{w_{Tr}}{r}) \\ + \int_{\partial T} 2\pi r \sum_{i\in\{r,z\}} (u_{\partial Tr} - u_{Tr}|_{\partial T})\frac{\partial w_{Tr}}{\partial i}|_{\partial T} n_i\end{aligned} \tag{43}$$

and the ordonal component D_{Tz} solves :

$$\int_T 2\pi r \sum_{i\in\{r,z\}} \frac{\partial D_{Tz}}{\partial i}\frac{\partial w_{Tz}}{\partial i} = \int_T 2\pi r \sum_{i\in\{r,z\}} \frac{\partial u_{Tz}}{\partial i}\frac{\partial w_{Tz}}{\partial i} - \int_{\partial T} 2\pi r \sum_{i\in\{r,z\}} (u_{\partial Tz} - u_{Tz}|_{\partial T})\frac{\partial w_{Tz}}{\partial i}|_{\partial T} n_i \tag{44a}$$

$$\int_T 2\pi r D_{Tz} = \int_T 2\pi r u_{Tz} \tag{44b}$$

In particular, one notices that the mean value condition is not needed on the radial componenent of the higher order displacement since the left hand side of the system described by (43) depends direclty on the displacement unknown and not only on itrs gradient as in (44).

Moreover, since in cylindrical coordinates, all integrals depend on the radial componenent r, there is a singularity at $r = 0$ for boundary integrals on faces located on the symmetry axis, and from a geometrical standpoint, these faces lose a dimension; a face that is not located on the symmetry axis behaves like a shell by revolution of the (r, z) plane, whereas one attached to the axis reduces to a beam that is only allowed to move and morph in the z direction. On a more algebraic note, the problem as such is ill-defined, since building the jump function involves inverting a mass matrix in $V^h(\partial T)$ to define the projector $\Pi^k_{\partial T}$. Therefore, a face on the axis is swelled by a small radius ϱ such that it becomes a cylinder with same dimensions as the others (see Figure 3)

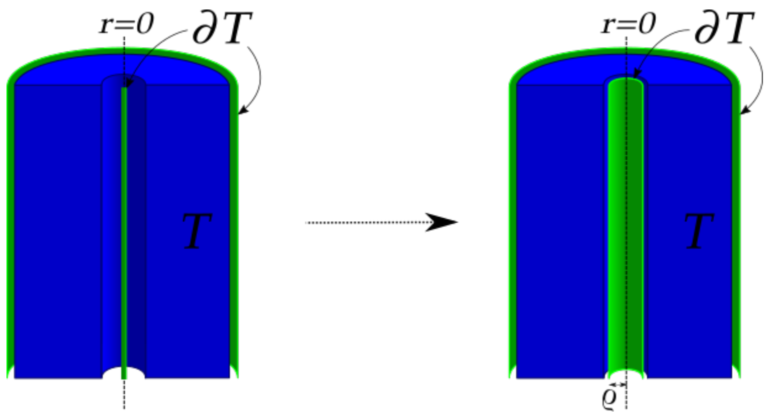

Figure 3. schematic representation of the model problem

6. Implementation

$$\left\{\underset{\sim}{\boldsymbol{G}}^k_T(\boldsymbol{v}_T, \boldsymbol{v}_{\partial T})\right\}(\boldsymbol{x}_q) = \begin{bmatrix} B_T & B_{\partial T}\end{bmatrix}(\boldsymbol{x}_q) \cdot \begin{Bmatrix} \boldsymbol{v}_T \\ \boldsymbol{v}_{\partial T}\end{Bmatrix} \quad \forall (\boldsymbol{v}_T, \boldsymbol{v}_{\partial T}) \in U(T) \times V(\partial T) \tag{45}$$

$$\int_{\partial T} (\beta/h_T) \boldsymbol{Z}^{HHO}_{\partial T}(\boldsymbol{u}^l_T, \boldsymbol{u}^k_{\partial T}) \cdot \boldsymbol{Z}^{HHO}_{\partial T}(\delta\boldsymbol{u}^l_T, \delta\boldsymbol{u}^k_{\partial T}) = \beta \begin{Bmatrix} \boldsymbol{u}^l_T \\ \boldsymbol{u}^k_{\partial T}\end{Bmatrix}^{\mathrm{t}} \cdot \begin{bmatrix} Z_{TT} & Z_{T\partial T} \\ Z_{\partial TT} & Z_{\partial T\partial T}\end{bmatrix} \cdot \begin{Bmatrix} \delta\boldsymbol{u}^l_T \\ \delta\boldsymbol{u}^k_{\partial T}\end{Bmatrix} \tag{46}$$

$$\left\{F_T^{int}(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m})\right\} = \sum_Q \begin{bmatrix} B_T & B_{\partial T} \end{bmatrix}^t (\boldsymbol{x}_q) \cdot \left\{\underset{\sim}{\boldsymbol{P}}_T^k(\underset{\sim}{\boldsymbol{G}}_T^k(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m}))\right\}(\boldsymbol{x}_q) + \beta \begin{bmatrix} Z_{TT} & Z_{T\partial T} \\ Z_{\partial TT} & Z_{\partial T\partial T} \end{bmatrix} \cdot \begin{Bmatrix} \boldsymbol{u}_T^{l,m} \\ \boldsymbol{u}_{\partial T}^{k,m} \end{Bmatrix} \tag{47}$$

$$\left\{F_T^{ext}\right\} = \begin{Bmatrix} \boldsymbol{f}_V \\ \boldsymbol{t}_N \end{Bmatrix} \tag{48}$$

$$\left[K_T^{tan}(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m})\right] = \sum_Q \begin{bmatrix} B_T & B_{\partial T} \end{bmatrix}^t (\boldsymbol{x}_q) \cdot \left[\underset{\approx}{\boldsymbol{A}}(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m})\right](\boldsymbol{x}_q) \cdot \begin{bmatrix} B_T & B_{\partial T} \end{bmatrix}(\boldsymbol{x}_q) \tag{49}$$

$$\left[\underset{\approx}{\boldsymbol{A}}(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m})\right] = \frac{\partial \underset{\sim}{\boldsymbol{P}}_T^k(\underset{\sim}{\boldsymbol{G}}_T^k(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m}))}{\partial \underset{\sim}{\boldsymbol{G}}_T^k(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m})} \tag{50}$$

$$\left[K_T^{tan}(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m})\right] = \begin{bmatrix} K_{TT}(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m}) & K_{T\partial T}(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m}) \\ K_{\partial TT}(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m}) & K_{\partial T\partial T}(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m}) \end{bmatrix} \tag{51}$$

$$\left[K_T^{tan}(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m})\right] \cdot \begin{Bmatrix} \delta\boldsymbol{u}_T^l \\ \delta\boldsymbol{u}_{\partial T}^k \end{Bmatrix} = \left\{F_T^{int}(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m})\right\} - \left\{F_T^{ext}\right\} \tag{52}$$

$$\left[K_T^{tan}(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m})\right]_c \cdot \left\{\delta\boldsymbol{u}_{\partial T}^k\right\} = \left\{F_T^{int}(\boldsymbol{u}_T^{l,m},\boldsymbol{u}_{\partial T}^{k,m})\right\}_c - \left\{F_T^{ext}\right\}_c \tag{53}$$

6.1. Résolution par condensation statique

Le problème global incrémental (??) est alors l'assemblage des systèmes élémentaires condensés (??), dont la résolution consiste en un algorithme de Newton sur l'incrément des inconnues de faces uniquement. Ce schéma de résolution par condensation statique dont on donne le principe Figure 4 exploite la relation linéaire entre l'incrément des inconnues de cellules et celui des faces.

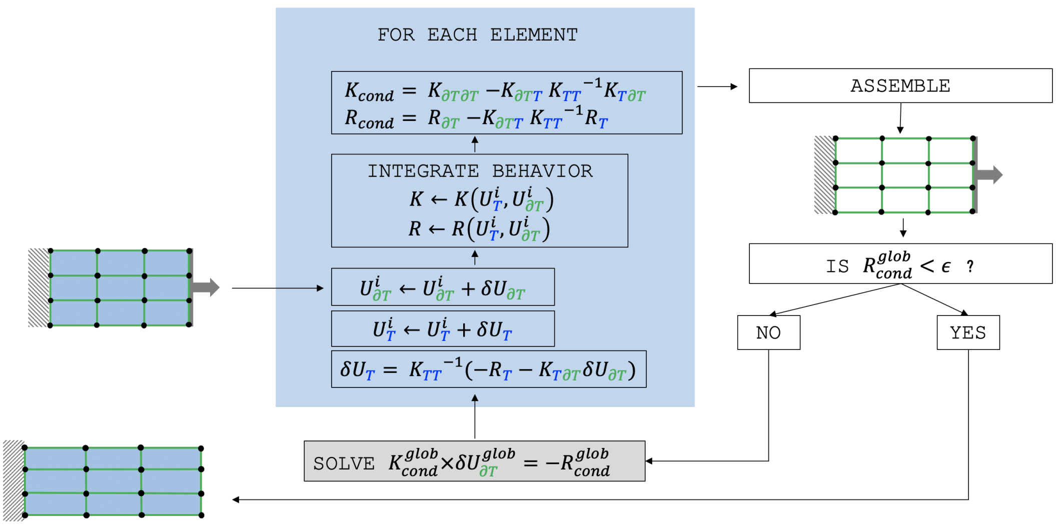

Figure 4. Description schématique de l'algorithme de condensation statique

6.2. *Résolution par équilibre de cellule*

Nous proposons une alternative à l'algorithme de résolution par condensation statique, postulant une relation implicite entre l'incrément des inconnues de cellule est celui des faces et consistant à résoudre localement un système non-linéaire sur l'incrément de cellule à incrément de faces fixé, afin de vérifier l'équilibre de la cellule avec ses faces à chaque itération du problème global. Ce nouveau schéma de résolution est décrit Figure 5, où on note i une itération de Newton pour la résolution du problème global sur l'ensemble des inconnues de faces, et j une itération de Newton pour la résolution du problème local sur les inconnues de cellule dans un élément T:

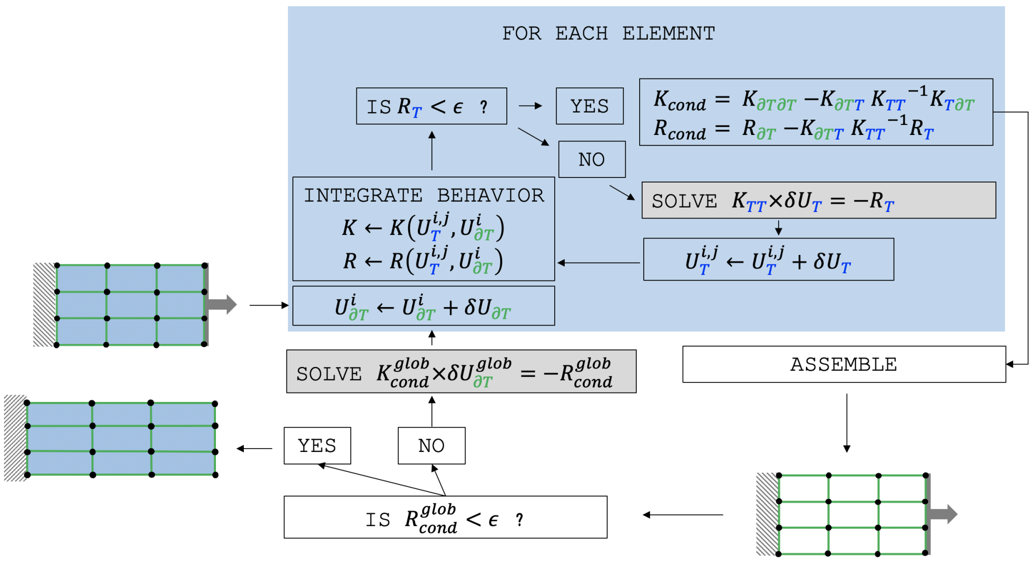

Figure 5. Description schématique de l'algorithme d'équilibre de cellule

7. Numerical exemples

The goal of this section is to evaluate the proposed HHO method on two and three-dimensional benchmarks from the literature: (i) a necking of a 2D rectangular bar subjected to uniaxial extension, (ii) a quasi-incompressible sphere under internal pressure. We compare our results to the analytical solution whenever available or to numerical results obtained using the industrial open-source FEM software code aster. In this case, we consider a linear, respectively, quadratic, cG formulation, referred to as Q1, respectively, T2 or Q2, when full integration is used, or, Q2-RI when reduced integration is used, depending on the mesh, and a three-field mixed formulation in which the unknowns are the displacement, the pressure, and the volumetric strain fields referred to as UPG 6 ; in the UPG method, the displacement field is quadratic, whereas both the pressure and the volumetric strain fields are linear. The conforming Q1, T2, and Q2 methods with full integration, contrary to the Q2-RI method with reduced integration in most of the situations, are known to present volumetric locking due to plastic incompressibility, whereas the UPG method is known to be robust but costly. Numerical results obtained using the UPG method are used as a reference solution whenever an analytical solution is not available. The nonlinear isotropic plasticity model with a von Mises yield criterion described in Section ABOVE is used for the test cases. For the first three test cases, strain-hardening plasticity is considered with the following material parameters: Young modulus E = 206.9 GPa, Poisson ratio nu = 0.29, hardening parameter H = 129.2 MPa, initial yield stress sig = 450 MPa, infinite yield stress sigunf = 715 MPa, and saturation parameter δ = 16.93. For the fourth case, perfect plasticity is considered with the following material parameters: Young modulus E = 28.85 MPa, Poisson ratio nu = 0.499, hardening parameter H = 0 MPa, initial and infinite yield stresses 6 MPa, and saturation parameter 0. Moreover, for the two-dimensional test cases (i) and (ii), we assume additionally a plane strain condition. In the numerical experiments reported in this section, the stabilization parameter is taken to be 1, and all the quadratures use positive weights. In particular, for the HHO method, we employ a quadrature of order k Q =

2k for the behavior cell integration. We employ the notation HHO(k, l) when using face polynomials of order k and cell polynomials of order l. In Section 5, we perform further numerical investigations to test other aspects of HHO methods such as the support of general meshes with possibly nonconforming interfaces, the possibility of considering the lowest-order case k = 0, and the dependence on the stabilization parameter β.

7.1. Necking of a 2D rectangular bar

In this first benchmark, we consider a 2D rectangular bar with an initial imperfection. The bar is subjected to uniaxial extension. This example has been studied previously by many authors as a necking problem 3,5,7,8,22 and can be used to test the robustness of the different methods. The bar has a length of 53.334 mm and a variable width from an initial width value of 12.826 mm at the top to a width of 12.595 mm at the center of the bar to create a geometric imperfection. A vertical displacement u y = 5mm is imposed at both ends, as shown in Figure 2A. For symmetry reasons, only one-quarter of the bar is discretized, and the mesh is composed of 400 quadrangles (see Figure 2B). The load-displacement curve is plotted in Figure 2C. We observe that, except for Q1, all the other methods give very similar results. Moreover, the equivalent plastic strain p, respectively, the trace of the Cauchy stress tensor sigma, are shown in Figure 3, respectively, in Figure 4, at the quadrature points on the final configuration. A sign of locking is the presence of strong oscillations in the trace of the Cauchy stress tensor sigma. We notice that the cG formulations Q1 and Q2 lock, contrary to the HHO, Q2-RI, and UPG methods that deliver similar results. We remark, however, that the results for HHO(1;1), HHO(1;2), and Q2-RI are slightly less smooth than for HHO(2;2), HHO(2;3), and UPG. The reason is that, on a fixed mesh, the three former methods have less quadrature points than the three latter ones (see Table 1) (HHO(2;2), HHO(2;3), and UPG have the same number of quadrature points). Therefore, the stress is evaluated using less points in HHO(1;1), HHO(1;2), and Q2-RI. It is sufficient to refine the mesh or to increase the order of the quadrature by two in HHO(1;1) and HHO(1;2) to retrieve similar results to those for the three other methods (not shown for brevity).

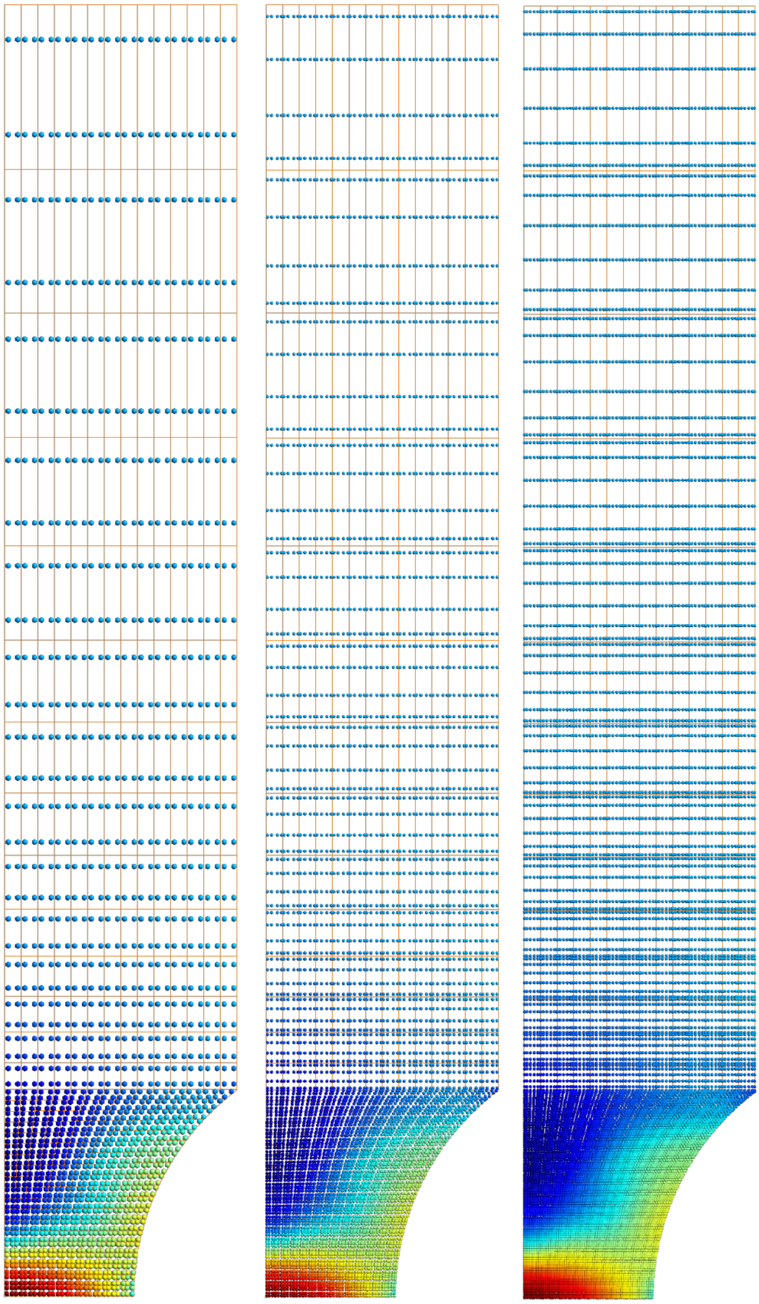

Figure 6. schematic representation of the model problem

7.2. *Quasi-incompressible sphere under internal pressure*

This last benchmark 6 consists of a quasi-incompressible sphere under internal pressure for which an analytical solution is known when the entire sphere has reached a plastic state. This benchmark is particularly challenging compared to the previous ones since we consider here perfect plasticity. The sphere has an inner radius R in = 0.8 mm and an outer radius R out = 1 mm. An internal radial pressure P is imposed. For symmetry reasons, only one-eighth of the sphere is discretized, and the mesh is composed of 1580 tetrahedra (see Figure 10A). The simulation is performed until the limit load corresponding to an internal pressure 2.54 MPa is reached. The equivalent plastic strain p is plotted for HHO(1;2) in Figure 10B, and the trace of the Cauchy stress tensor simga is compared for HHO, UPG, and T2 methods in Figure 11 at all the quadrature points on the final configuration for the limit load. We notice that the quadratic element T2 locks, whereas HHO and UPG do not present any sign of locking and produce results that are very close to the analytical solution. However, the trace of the Cauchy stress tensor simga is slightly more dispersed around the analytical solution for HHO(2;2) and HHO(2;3) than for HHO(1;1) and HHO(1;2) near the outer boundary. For this test case, we do not expect that HHO(2;2) and HHO(2;3) will deliver more accurate solutions than HHO(1;1) and HHO(1;2) since the geometry is discretized using tetrahedra with planar faces. We next investigate the influence of the quadrature order k Q on the accuracy of the solution. The trace of the Cauchy stress tensor simga is compared for HHO(1;1), HHO(2;2), and UPG methods in Figure 12 at all the quadrature points on the final configuration for the limit load, and for a quadrature order k Q higher than the one employed in Figure 11 (HHO(1;2) and HHO(2;3) give similar results and are not shown for brevity). We remark that, when we increase the quadrature order, UPG locks for quasi-incompressible finite deformations, whereas HHO does not lock, and the results are (only) a bit more dispersed around the analytical solution. Moreover, HHO(2;2) is less sensitive than HHO(1;1) to the choice of the quadrature order k Q . Note that this problem is not present for HHO methods with small deformations. Furthermore, this sensitivity to the quadrature order seems to be absent for finite deformations when the elastic deformations are compressible (the plastic deformations are still incompressible). To illustrate this claim, we perform the same simulations as before but for a compressible material. The Poisson ratio is taken now as nu = 0.3 (recall that we used nu = 0.499 in the quasi-incompressible case), whereas the other material parameters are unchanged. Unfortunately, an analytical solution is no longer available in the compressible case. We compare again the trace of the Cauchy stress tensor simga for HHO(1;1), HHO(2;2), and UPG methods in Figure 13 at all the quadrature points on the final configuration and for different quadrature orders k Q . We observe a quite marginal dependence on the quadrature order for HHO methods (as in the quasi-incompressible case), whereas the UPG method still locks if the order of the quadrature is increased. Moreover, in the compressible case, HHO(2;2) gives a more accurate solution than HHO(1;1).