0.1 January 21, 2026

Plan:

- Translate section 2 of Svidzinsky
- Establish a firm ground of accepted facts from which the BdG equations will follow.
- Understand how the approximation of the interaction is made.

Evaluation: I did find a way to translate section 2 of Svidzinsky, but the "accepted facts list" I generated needs improvement, and I did not get to bullet 3.

0.2 January 22, 2026

Plan:

- Do a better job identifying and understanding the necessary foundational material needed to understand the BdG derivation

0.2.1 Fock space and Field operators

Physical meaning. $\mathcal{H}_1^{\wedge N}$ is the Hilbert space describing N identical fermions built from a one-particle Hilbert space $\mathcal{H}_1$. A vector in $\mathcal{H}_1^{\wedge N}$ encodes all probability amplitudes for finding the N fermions in arbitrary one-particle states, with the fermionic exchange sign built in. Fermionic Fock space collects all such N-particle sectors into a single Hilbert space.

Mathematical definitions.

1. **Permutation operators.** Let $\mathcal{H}_1$ be a complex separable Hilbert space and let $\mathcal{H}_1^{\otimes N}$ be its N-fold tensor product. For each permutation $\pi \in S_N$, define a linear operator

$$P_\pi : \mathcal{H}_1^{\otimes N} \to \mathcal{H}_1^{\otimes N}$$

by its action on simple tensors,

$$P_\pi(\phi_1 \otimes \cdots \otimes \phi_N) = \phi_{\pi^{-1}(1)} \otimes \cdots \otimes \phi_{\pi^{-1}(N)},$$

extended by linearity and continuity.

2. **Antisymmetrizer and its range.** Define the antisymmetrization operator

$$A_N \equiv \frac{1}{N!} \sum_{\pi \in S_N} \operatorname{sgn}(\pi)\, P_\pi.$$

The range of A_N, denoted $\operatorname{Ran}(A_N)$, is

$$\operatorname{Ran}(A_N) \equiv \{\, A_N \Phi \mid \Phi \in \mathcal{H}_1^{\otimes N} \,\}.$$

Since $A_N^2 = A_N$ and $A_N^\dagger = A_N$, $\operatorname{Ran}(A_N)$ is a closed subspace of $\mathcal{H}_1^{\otimes N}$.

3. **Fermionic N-particle Hilbert space.** The fermionic N-particle Hilbert space is defined as
$$\mathcal{H}_1^{\wedge N} \equiv \mathrm{Ran}(A_N).$$
Equivalently,
$$\mathcal{H}_1^{\wedge N} = \left\{ \Psi \in \mathcal{H}_1^{\otimes N} \;\middle|\; P_\pi \Psi = \mathrm{sgn}(\pi)\, \Psi \text{ for all } \pi \in S_N \right\}.$$

4. **Combining one-particle states.** Given $\phi_1, \dots, \phi_N \in \mathcal{H}_1$, the corresponding fermionic N-particle state is
$$\phi_1 \wedge \cdots \wedge \phi_N \equiv A_N(\phi_1 \otimes \cdots \otimes \phi_N).$$
This vector lies in $\mathcal{H}_1^{\wedge N}$ and changes sign under exchange of any two factors.

5. **Meaning of an N-particle state.** For any $\Psi \in \mathcal{H}_1^{\wedge N}$ and any $\chi_1, \dots, \chi_N \in \mathcal{H}_1$, the complex number
$$\langle \chi_1 \wedge \cdots \wedge \chi_N, \Psi \rangle$$
is interpreted as the probability amplitude that the N fermions occupy the one-particle states $\chi_1, \dots, \chi_N$.

6. **Algebraic sum and direct sum.** Consider the algebraic sum
$$\sum_{N=0}^{\infty} \mathcal{H}_1^{\wedge N} = \left\{ \sum_{k=1}^{m} \Psi^{(N_k)} \;\middle|\; \Psi^{(N_k)} \in \mathcal{H}_1^{\wedge N_k},\ m < \infty \right\}.$$
This sum is direct because
$$\mathcal{H}_1^{\wedge N} \cap \mathcal{H}_1^{\wedge M} = \{0\} \quad \text{for } N \neq M.$$

7. **Fermionic Fock space.** The fermionic Fock space is the Hilbert-space completion of this direct sum:
$$\mathcal{F} \equiv \bigoplus_{N=0}^{\infty} \mathcal{H}_1^{\wedge N},$$
with inner product
$$\langle \Psi, \Phi \rangle = \sum_{N=0}^{\infty} \langle \Psi^{(N)}, \Phi^{(N)} \rangle_{\mathcal{H}_1^{\wedge N}}, \quad \Psi = \oplus_N \Psi^{(N)},\ \Phi = \oplus_N \Phi^{(N)}.$$

F2. Field operators on fermionic Fock space

F2-1 Configuration-space representation, probability interpretation, and antisymmetry

An element $\Psi^{(N)} \in \mathcal{H}_1^{\wedge N}$ admits a coordinate–spin representation
$$\Psi^{(N)}(x_1, \dots, x_N), \qquad x_i = (\mathbf{r}_i, \sigma_i) \in \mathbb{R}^3 \times \{\uparrow, \downarrow\},$$

which is a complex-valued function on the N-particle configuration space $(\mathbb{R}^3 \times \{\uparrow,\downarrow\})^N$.

For any measurable region Ω of this configuration space,

$$\int_\Omega |\Psi^{(N)}(x_1,\ldots,x_N)|^2 \, dx_1 \cdots dx_N$$

is the probability of finding the N identical fermions with coordinates and spins in Ω.

Because the fermions are physically identical, $\Psi^{(N)}$ satisfies the antisymmetry condition

$$\Psi^{(N)}(x_{\pi(1)},\ldots,x_{\pi(N)}) = \operatorname{sgn}(\pi)\, \Psi^{(N)}(x_1,\ldots,x_N) \quad \text{for all } \pi \in S_N,$$

which is the coordinate-space representation of the abstract relation $P_\pi \Psi^{(N)} = \operatorname{sgn}(\pi)\Psi^{(N)}$. This antisymmetry leaves probabilities invariant while encoding fermionic exchange at the level of amplitudes.

F2-2 Definition of annihilation and creation operators and induced field operators

For each $f \in L^2(\mathbb{R}^3)$ and spin $\sigma \in \{\uparrow,\downarrow\}$, define the annihilation operator

$$\psi_\sigma(f) : \mathcal{H}_1^{\wedge N} \to \mathcal{H}_1^{\wedge(N-1)}$$

by

$$(\psi_\sigma(f)\Psi)^{(N-1)}(x_2,\ldots,x_N) = \sqrt{N} \int_{\mathbb{R}^3} d\mathbf{r}\; f(\mathbf{r})\, \Psi^{(N)}\big((\mathbf{r},\sigma), x_2,\ldots,x_N\big),$$

and define the creation operator $\psi_\sigma^\dagger(f)$ as the Hilbert-space adjoint of $\psi_\sigma(f)$.

The map $f \mapsto \psi_\sigma(f)$ is linear in f, and the map $f \mapsto \psi_\sigma^\dagger(f)$ is antilinear. Accordingly, there exist operator-valued distributions $\psi_\sigma(\mathbf{r})$ and $\psi_\sigma^\dagger(\mathbf{r})$ such that

$$\psi_\sigma(f) = \int_{\mathbb{R}^3} d\mathbf{r}\; f(\mathbf{r})\, \psi_\sigma(\mathbf{r}), \qquad \psi_\sigma^\dagger(f) = \int_{\mathbb{R}^3} d\mathbf{r}\; f^*(\mathbf{r})\, \psi_\sigma^\dagger(\mathbf{r}).$$

These relations express the integral representation of the linear dependence of $\psi_\sigma(f)$ and $\psi_\sigma^\dagger(f)$ on the test function f.

F2-3 Particle indistinguishability and slot independence

Because $\Psi^{(N)}$ is antisymmetric, the definition of $\psi_\sigma(f)$ is independent of which coordinate slot is integrated over: integrating over any one slot yields the same $(N-1)$-particle state up to the sign already fixed by antisymmetry. Thus $\psi_\sigma(f)$ does not remove a distinguished particle but removes one fermion in the one-particle state specified by (f,σ).

F2-4 Conditional interpretation of the reduced state

For any region $\Omega_{N-1} \subset (\mathbb{R}^3 \times \{\uparrow,\downarrow\})^{N-1}$,

$$\int_{\Omega_{N-1}} |(\psi_\sigma(f)\Psi)^{(N-1)}(x_2,\ldots,x_N)|^2 \, dx_2 \cdots dx_N$$

is the probability density for finding the remaining $N-1$ fermions in Ω_{N-1}, conditional on one fermion being in the one-particle state (f,σ).

F2-5 Canonical anticommutation relations

The field operators satisfy the canonical anticommutation relations

$$\{\psi_\sigma(\mathbf{r}), \psi^\dagger_{\sigma'}(\mathbf{r}')\} = \delta_{\sigma\sigma'}\,\delta(\mathbf{r}-\mathbf{r}'), \qquad \{\psi_\sigma(\mathbf{r}), \psi_{\sigma'}(\mathbf{r}')\} = 0, \qquad \{\psi^\dagger_\sigma(\mathbf{r}), \psi^\dagger_{\sigma'}(\mathbf{r}')\} = 0,$$

to be interpreted after smearing with test functions.

Equivalently, for smeared operators,

$$\{\psi_\sigma(f), \psi^\dagger_{\sigma'}(g)\} = \delta_{\sigma\sigma'}\,\langle f, g\rangle_{\mathcal{H}_1},$$

where

$$\langle f, g\rangle_{\mathcal{H}_1} = \int_{\mathbb{R}^3} d\mathbf{r}\; f^*(\mathbf{r})g(\mathbf{r})$$

is the inner product on the one-particle Hilbert space.

This relation expresses the consistency condition that creating a fermion in a one-particle state and then testing for its presence returns precisely the overlap of those one-particle states, and nothing else.

0.2.2 Understanding initial interaction equation

What is the approximatio

$$H_{\text{int}} = g \int \psi_{\uparrow}^{+}(\mathbf{r})\, \psi_{\downarrow}^{+}(\mathbf{r})\, \psi_{\downarrow}(\mathbf{r})\, \psi_{\uparrow}(\mathbf{r})\, d\mathbf{r}. \tag{2.3}$$