

## 0.1 January 21, 2026

**Plan:**

- Translate section 2 of Svidzinsky
- Establish a firm ground of accepted facts from which the BdG equations will follow.
- Understand how the approximation of the interaction is made.

**Evaluation:** I did find a way to translate section 2 of Svidzinsky, but the "accepted facts list" I generated needs improvement, and I did not get to bullet 3.

## 0.2 January 22, 2026

**Plan:**

- Do a better job identifying and understanding the necessary foundational material needed to understand the BdG derivation

### 0.2.1 Fock space and Field operators

**Physical meaning.**  $\mathcal{H}_1^{\wedge N}$  is the Hilbert space describing  $N$  identical fermions built from a one-particle Hilbert space  $\mathcal{H}_1$ . A vector in  $\mathcal{H}_1^{\wedge N}$  encodes all probability amplitudes for finding the  $N$  fermions in arbitrary one-particle states, with the fermionic exchange sign built in. Fermionic Fock space collects all such  $N$ -particle sectors into a single Hilbert space.

**Mathematical definitions.**

1. **Permutation operators.** Let  $\mathcal{H}_1$  be a complex separable Hilbert space and let  $\mathcal{H}_1^{\otimes N}$  be its  $N$ -fold tensor product. For each permutation  $\pi \in S_N$ , define a linear operator

$$P_\pi : \mathcal{H}_1^{\otimes N} \rightarrow \mathcal{H}_1^{\otimes N}$$

by its action on simple tensors,

$$P_\pi(\phi_1 \otimes \cdots \otimes \phi_N) = \phi_{\pi^{-1}(1)} \otimes \cdots \otimes \phi_{\pi^{-1}(N)},$$

extended by linearity and continuity.

2. **Antisymmetrizer and its range.** Define the antisymmetrization operator

$$A_N \equiv \frac{1}{N!} \sum_{\pi \in S_N} \text{sgn}(\pi) P_\pi.$$

The range of  $A_N$ , denoted  $\text{Ran}(A_N)$ , is

$$\text{Ran}(A_N) \equiv \{ A_N \Phi \mid \Phi \in \mathcal{H}_1^{\otimes N} \}.$$

Since  $A_N^2 = A_N$  and  $A_N^\dagger = A_N$ ,  $\text{Ran}(A_N)$  is a closed subspace of  $\mathcal{H}_1^{\otimes N}$ .

3. **Fermionic  $N$ -particle Hilbert space.** The fermionic  $N$ -particle Hilbert space is defined as

$$\mathcal{H}_1^{\wedge N} \equiv \text{Ran}(A_N).$$

Equivalently,

$$\mathcal{H}_1^{\wedge N} = \left\{ \Psi \in \mathcal{H}_1^{\otimes N} \mid P_\pi \Psi = \text{sgn}(\pi) \Psi \text{ for all } \pi \in S_N \right\}.$$

4. **Combining one-particle states.** Given  $\phi_1, \dots, \phi_N \in \mathcal{H}_1$ , the corresponding fermionic  $N$ -particle state is

$$\phi_1 \wedge \dots \wedge \phi_N \equiv A_N(\phi_1 \otimes \dots \otimes \phi_N).$$

This vector lies in  $\mathcal{H}_1^{\wedge N}$  and changes sign under exchange of any two factors.

5. **Meaning of an  $N$ -particle state.** For any  $\Psi \in \mathcal{H}_1^{\wedge N}$  and any  $\chi_1, \dots, \chi_N \in \mathcal{H}_1$ , the complex number

$$\langle \chi_1 \wedge \dots \wedge \chi_N, \Psi \rangle$$

is interpreted as the probability amplitude that the  $N$  fermions occupy the one-particle states  $\chi_1, \dots, \chi_N$ .

6. **Algebraic sum and direct sum.** Consider the algebraic sum

$$\sum_{N=0}^{\infty} \mathcal{H}_1^{\wedge N} = \left\{ \sum_{k=1}^m \Psi^{(N_k)} \mid \Psi^{(N_k)} \in \mathcal{H}_1^{\wedge N_k}, m < \infty \right\}.$$

This sum is direct because

$$\mathcal{H}_1^{\wedge N} \cap \mathcal{H}_1^{\wedge M} = \{0\} \quad \text{for } N \neq M.$$

7. **Fermionic Fock space.** The fermionic Fock space is the Hilbert-space completion of this direct sum:

$$\mathcal{F} \equiv \bigoplus_{N=0}^{\infty} \mathcal{H}_1^{\wedge N},$$

with inner product

$$\langle \Psi, \Phi \rangle = \sum_{N=0}^{\infty} \langle \Psi^{(N)}, \Phi^{(N)} \rangle_{\mathcal{H}_1^{\wedge N}}, \quad \Psi = \oplus_N \Psi^{(N)}, \quad \Phi = \oplus_N \Phi^{(N)}.$$

## F2. Field operators on fermionic Fock space

### F2-1 Configuration-space representation, probability interpretation, and anti-symmetry

An element  $\Psi^{(N)} \in \mathcal{H}_1^{\wedge N}$  admits a coordinate-spin representation

$$\Psi^{(N)}(x_1, \dots, x_N), \quad x_i = (\mathbf{r}_i, \sigma_i) \in \mathbb{R}^3 \times \{\uparrow, \downarrow\},$$

which is a complex-valued function on the  $N$ -particle configuration space  $(\mathbb{R}^3 \times \{\uparrow, \downarrow\})^N$ .

For any measurable region  $\Omega$  of this configuration space,

$$\int_{\Omega} |\Psi^{(N)}(x_1, \dots, x_N)|^2 dx_1 \cdots dx_N$$

is the probability of finding the  $N$  identical fermions with coordinates and spins in  $\Omega$ .

Because the fermions are physically identical,  $\Psi^{(N)}$  satisfies the antisymmetry condition

$$\Psi^{(N)}(x_{\pi(1)}, \dots, x_{\pi(N)}) = \text{sgn}(\pi) \Psi^{(N)}(x_1, \dots, x_N) \quad \text{for all } \pi \in S_N,$$

which is the coordinate-space representation of the abstract relation  $P_{\pi} \Psi^{(N)} = \text{sgn}(\pi) \Psi^{(N)}$ . This antisymmetry leaves probabilities invariant while encoding fermionic exchange at the level of amplitudes.

### F2-2 Definition of annihilation and creation operators and induced field operators

For each  $f \in L^2(\mathbb{R}^3)$  and spin  $\sigma \in \{\uparrow, \downarrow\}$ , define the annihilation operator

$$\psi_{\sigma}(f) : \mathcal{H}_1^{\wedge N} \rightarrow \mathcal{H}_1^{\wedge(N-1)}$$

by

$$(\psi_{\sigma}(f) \Psi)^{(N-1)}(x_2, \dots, x_N) = \sqrt{N} \int_{\mathbb{R}^3} d\mathbf{r} f(\mathbf{r}) \Psi^{(N)}((\mathbf{r}, \sigma), x_2, \dots, x_N),$$

and define the creation operator  $\psi_{\sigma}^{\dagger}(f)$  as the Hilbert-space adjoint of  $\psi_{\sigma}(f)$ .

The map  $f \mapsto \psi_{\sigma}(f)$  is linear in  $f$ , and the map  $f \mapsto \psi_{\sigma}^{\dagger}(f)$  is antilinear. Accordingly, there exist operator-valued distributions  $\psi_{\sigma}(\mathbf{r})$  and  $\psi_{\sigma}^{\dagger}(\mathbf{r})$  such that

$$\psi_{\sigma}(f) = \int_{\mathbb{R}^3} d\mathbf{r} f(\mathbf{r}) \psi_{\sigma}(\mathbf{r}), \quad \psi_{\sigma}^{\dagger}(f) = \int_{\mathbb{R}^3} d\mathbf{r} f^*(\mathbf{r}) \psi_{\sigma}^{\dagger}(\mathbf{r}).$$

These relations express the integral representation of the linear dependence of  $\psi_{\sigma}(f)$  and  $\psi_{\sigma}^{\dagger}(f)$  on the test function  $f$ .

### F2-3 Particle indistinguishability and slot independence

Because  $\Psi^{(N)}$  is antisymmetric, the definition of  $\psi_{\sigma}(f)$  is independent of which coordinate slot is integrated over: integrating over any one slot yields the same  $(N-1)$ -particle state up to the sign already fixed by antisymmetry. Thus  $\psi_{\sigma}(f)$  does not remove a distinguished particle but removes one fermion in the one-particle state specified by  $(f, \sigma)$ .

### F2-4 Conditional interpretation of the reduced state

For any region  $\Omega_{N-1} \subset (\mathbb{R}^3 \times \{\uparrow, \downarrow\})^{N-1}$ ,

$$\int_{\Omega_{N-1}} |(\psi_{\sigma}(f) \Psi)^{(N-1)}(x_2, \dots, x_N)|^2 dx_2 \cdots dx_N$$

is the probability density for finding the remaining  $N-1$  fermions in  $\Omega_{N-1}$ , conditional on one fermion being in the one-particle state  $(f, \sigma)$ .

### F2-5 Canonical anticommutation relations

The field operators satisfy the canonical anticommutation relations

$$\{\psi_\sigma(\mathbf{r}), \psi_{\sigma'}^\dagger(\mathbf{r}')\} = \delta_{\sigma\sigma'} \delta(\mathbf{r} - \mathbf{r}'), \quad \{\psi_\sigma(\mathbf{r}), \psi_{\sigma'}(\mathbf{r}')\} = 0, \quad \{\psi_\sigma^\dagger(\mathbf{r}), \psi_{\sigma'}^\dagger(\mathbf{r}')\} = 0,$$

to be interpreted after smearing with test functions.

Equivalently, for smeared operators,

$$\{\psi_\sigma(f), \psi_{\sigma'}^\dagger(g)\} = \delta_{\sigma\sigma'} \langle f, g \rangle_{\mathcal{H}_1},$$

where

$$\langle f, g \rangle_{\mathcal{H}_1} = \int_{\mathbb{R}^3} d\mathbf{r} f^*(\mathbf{r}) g(\mathbf{r})$$

is the inner product on the one-particle Hilbert space.

This relation expresses the consistency condition that creating a fermion in a one-particle state and then testing for its presence returns precisely the overlap of those one-particle states, and nothing else.

### F2-6 Derivation of the mode expansion of $\psi(f)$

Assume that the annihilation operator

$$f \mapsto \psi(f)$$

is a linear map from the one-particle Hilbert space  $H_1$  to operators on fermionic Fock space.

Let  $\{\psi_k\} \subset H_1$  be an orthonormal basis of  $H_1$ , and define the corresponding mode operators by

$$c_k \equiv \psi(\psi_k).$$

Any  $f \in H_1$  admits the expansion

$$f(x) = \sum_k \psi_k(x) \langle \psi_k, f \rangle.$$

Applying the linear map  $\psi(\cdot)$  to this expansion and using linearity,

$$\psi(f) = \psi\left(\sum_k \langle \psi_k, f \rangle \psi_k\right) = \sum_k \langle \psi_k, f \rangle \psi(\psi_k) = \sum_k \langle \psi_k, f \rangle c_k.$$

The inner product  $\langle \psi_k, f \rangle$  can be written in coordinate representation as

$$\langle \psi_k, f \rangle = \int dx \psi_k^*(x) f(x).$$

Substituting this expression yields

$$\psi(f) = \int dx f(x) \left( \sum_k \psi_k(x) c_k \right).$$

**F2-7 (Derivation of the one-body Hamiltonian in field-operator form).**

Let  $h_0$  be a linear operator on the one-particle Hilbert space  $H_1$ . Fix an orthonormal basis  $\{\psi_k\} \subset H_1$ , and define mode operators

$$c_k \equiv \psi(\psi_k).$$

Define the matrix elements of  $h_0$  in this basis by

$$h_{kl} \equiv \langle \psi_k, h_0 \psi_l \rangle_{H_1}.$$

The corresponding second-quantized one-body Hamiltonian is defined by

$$H_0 \equiv \sum_{k,l} h_{kl} c_k^\dagger c_l.$$

Using the mode expansions

$$\psi(x) = \sum_l \psi_l(x) c_l, \quad \psi^\dagger(x) = \sum_k \psi_k^*(x) c_k^\dagger,$$

one computes

$$\int dx \psi^\dagger(x) (h_0 \psi)(x) = \int dx \left( \sum_k \psi_k^*(x) c_k^\dagger \right) \left( h_0 \sum_l \psi_l(x) c_l \right).$$

Since  $h_0$  acts only on the  $x$ -dependence of  $\psi_l(x)$ , this becomes

$$\int dx \psi^\dagger(x) (h_0 \psi)(x) = \sum_{k,l} \left( \int dx \psi_k^*(x) (h_0 \psi_l)(x) \right) c_k^\dagger c_l = \sum_{k,l} h_{kl} c_k^\dagger c_l.$$

Therefore,

$$H_0 = \sum_{k,l} \langle \psi_k, h_0 \psi_l \rangle c_k^\dagger c_l = \int dx \psi^\dagger(x) h_0 \psi(x).$$

### 0.2.2 Understanding initial interaction equation

What is the approximatio

$$H_{\text{int}} = g \int \psi_{\uparrow}^{\dagger}(\mathbf{r}) \psi_{\downarrow}^{\dagger}(\mathbf{r}) \psi_{\downarrow}(\mathbf{r}) \psi_{\uparrow}(\mathbf{r}) d\mathbf{r}. \quad (2.3)$$