0.1 January 21, 2026

Plan:

- Translate section 2 of Svidzinsky
- Establish a firm ground of accepted facts from which the BdG equations will follow.
- Understand how the approximation of the interaction is made.

I. Mathematical and Structural Foundations

1. **Tensor product structure.** The many-body Hilbert space is constructed as a graded tensor product of one-particle Hilbert spaces, with fermionic antisymmetrization imposed.

   Question 0.1.1. What is a tensor product of spaces? What does it mean that fermionic antisymmetrization is imposed? I understand that "swapping particles" must result in negation of the wave function, but what is the mathematical definition of "swapping particles" - I recall that with two particles, it is realized by negating the coordinate system, but am not sure how this extends to the tensor product and many particles.

   Answer 0.1.1.1. I re-consulted my linear algebra textbook for the formal definition, and physically, the tensor product of states ϕ and τ is the state of two particles where particle 1 is in state ϕ and particle 2 is in state τ.
   This is a tensor product of $\mathcal{H}$ with itself, so swapping particles is simply permuting factors in the tensor product. Hence "anticommutativity" of the $\otimes$ operator.

   Question 0.1.2. $\psi(r_1, r_2)$ is shorthand for $\psi_1(r) \otimes \psi_2(r)$? But is then $\psi(r_1, r_2)$ no longer a map from R^3 to $\mathbb{C}$, but rather from $\mathbb{R}^6$ (not considering the spinor)? Is the "ansatz" wave function $\tilde{\psi}$ a map from some huge configuration space, or just R^3? I guess if ψ represents a state for which the particle number is uncertain, then it would not even be certain what ψ is a map from. Perhaps the interpretation of wave functions as functions breaks down under the application of the tensor product, and meaning is only extracted through field operators.

2. **Operator-valued distributions.** Field operators $\psi(\mathbf{r})$ and $\psi^\dagger(\mathbf{r})$ are operator-valued distributions. Products at coincident points are only meaningful under integrals or suitable regularization.

   Question 0.1.3. Are coincident points just identical points?

II. Second Quantization: Formal Axioms

4. **Fermionic Fock space.** The fermionic Fock space is constructed from the one-particle Hilbert space. The vacuum state $|0\rangle$ is annihilated by all annihilation operators.

   Question 0.1.4. How is it constructed?

5. **Canonical anticommutation relations (CAR).**

$$\{\psi_\alpha(\mathbf{r}), \psi^\dagger_\beta(\mathbf{r}')\} = \delta_{\alpha\beta}\delta(\mathbf{r} - \mathbf{r}'), \qquad \{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0.$$

Question 0.1.5. These are related to the fermionic particle swapping negation rules right? How?

6. **Operator correspondence.** One-body operators correspond to bilinear forms in field operators; two-body operators correspond to quartic forms. Normal ordering is defined relative to the vacuum.

   Question 0.1.6. What is normal ordering?

7. **Density and number operators.**

$$\hat{n}(\mathbf{r}) = \sum_{\alpha} \psi_{\alpha}^{\dagger}(\mathbf{r})\psi_{\alpha}(\mathbf{r}).$$

   This is taken as an operator identity.

   Question 0.1.7. $\int_V \hat{n}(\mathbf{r})\mathrm{d}\mathbf{r}$ is the expectation value for the number of particles in volume V?

III. Microscopic Hamiltonian Assumptions

8. **Electronic Hamiltonian.** The Hamiltonian consists of kinetic energy, an external potential, and an instantaneous two-body interaction. Spin indices are included explicitly; spin–orbit coupling is absent unless stated.

9. **Attractive interaction channel.** Only the interaction channel relevant for pairing is retained. This is a controlled truncation, not an identity.

IV. Statistical Mechanics Layer

10. **Grand canonical ensemble.** The system is described in the grand canonical ensemble with chemical potential μ included in the Hamiltonian.

11. **Thermodynamic limit.** The limit $V \to \infty$ is taken prior to symmetry considerations, allowing inequivalent representations of the CAR.

    Question 0.1.8. What is V, and what are inequivalent representations of the CAR?

12. **Expectation values.** Expectation values $\langle \cdot \rangle$ denote ensemble averages in a chosen equilibrium state and yield c-numbers.

    Question 0.1.9. What is a c-number?

V. Mean-Field / Saddle-Point Commitment

13. **Mean-field factorization.** Quartic operator products may be approximated by bilinear terms plus c-number fields via a saddle-point approximation.

    Question 0.1.10. What is a saddle=point approximation, and why is this an axiom? Doesn't Svidzinsky go through this approximation somewhat carefully?

14. **Order parameter definition.**

$$\Delta(\mathbf{r}, \mathbf{r}') \equiv -V(\mathbf{r} - \mathbf{r}')\langle \psi_\downarrow(\mathbf{r}')\psi_\uparrow(\mathbf{r})\rangle.$$

This is a definition.

Question 0.1.11. We're working over momentum states, not position states, so should these $\mathbf{r}$ be replaced by $\mathbf{k}$. If not, what does $V(\mathbf{r} - \mathbf{r}')$ mean physically, considering particles are not localized?

15. **Self-consistency condition.** The mean field must equal the expectation value computed in the resulting quadratic Hamiltonian.

Question 0.1.12. Define the mean field, expectation value of what?

VI. Symmetry-Breaking Commitments

16. **Broken $U(1)$ symmetry.** Particle-number conservation is not enforced at the level of the effective Hamiltonian.

Question 0.1.13. Does this mean that the Hamiltonian can change the number of particles in the system? What would this even mean physically?

17. **Anomalous averages.** Expectation values of the form $\langle \psi\psi \rangle$ are permitted and nonzero in the chosen symmetry-broken sector.

Question 0.1.14. What is a symmetry-broken sector? I assume ψ are the field operators. Why might these not be permitted? Permitted is a rather heuristic term.

18. **Gauge fixing.** A definite global phase of the order parameter is implicitly fixed.

Question 0.1.15. What is a global phase, and what is an order parameter?

VII. Quadratic Fermionic Hamiltonians

19. **Quadratic Hamiltonians.** Any fermionic Hamiltonian quadratic in field operators admits exact diagonalization.

20. **Bogoliubov transformation.** Diagonalization is achieved via a linear canonical transformation preserving the CAR.

21. **Nambu formalism.** Doubling of degrees of freedom is a bookkeeping device and introduces no new physical assumptions.

0.1.1 Understanding initial interaction equation

What is the approximatio

$$H_{\text{int}} = g \int \psi_{\uparrow}^{+}(\mathbf{r})\, \psi_{\downarrow}^{+}(\mathbf{r})\, \psi_{\downarrow}(\mathbf{r})\, \psi_{\uparrow}(\mathbf{r})\, d\mathbf{r}. \tag{2.3}$$