0.1 January 21, 2026

Plan:

- Translate section 2 of Svidzinsky
- Establish a firm ground of accepted facts from which the BdG equations will follow.
- Understand how the approximation of the interaction is made.

Evaluation: I did find a way to translate section 2 of Svidzinsky, but the "accepted facts list" I generated needs improvement, and I did not get to bullet 3.

0.2 January 22, 2026

Plan:

- Do a better job identifying and understanding the necessary foundational material needed to understand the BdG derivation

0.2.1 Fock space and Field operators

Physical meaning. $\mathcal{H}_1^{\wedge N}$ is the Hilbert space describing N identical fermions built from a one-particle Hilbert space $\mathcal{H}_1$. A vector in $\mathcal{H}_1^{\wedge N}$ encodes all probability amplitudes for finding the N fermions in arbitrary one-particle states, with the fermionic exchange sign built in. Fermionic Fock space collects all such N-particle sectors into a single Hilbert space.

Mathematical definitions.

1. **Permutation operators.** Let $\mathcal{H}_1$ be a complex separable Hilbert space and let $\mathcal{H}_1^{\otimes N}$ be its N-fold tensor product. For each permutation $\pi \in S_N$, define a linear operator

$$P_\pi : \mathcal{H}_1^{\otimes N} \to \mathcal{H}_1^{\otimes N}$$

by its action on simple tensors,

$$P_\pi(\phi_1 \otimes \cdots \otimes \phi_N) = \phi_{\pi^{-1}(1)} \otimes \cdots \otimes \phi_{\pi^{-1}(N)},$$

extended by linearity and continuity.

2. **Antisymmetrizer and its range.** Define the antisymmetrization operator

$$A_N \equiv \frac{1}{N!} \sum_{\pi \in S_N} \mathrm{sgn}(\pi)\, P_\pi.$$

The range of A_N, denoted $\mathrm{Ran}(A_N)$, is

$$\mathrm{Ran}(A_N) \equiv \{\, A_N \Phi \mid \Phi \in \mathcal{H}_1^{\otimes N} \,\}.$$

Since $A_N^2 = A_N$ and $A_N^\dagger = A_N$, $\mathrm{Ran}(A_N)$ is a closed subspace of $\mathcal{H}_1^{\otimes N}$.

3. **Fermionic N-particle Hilbert space.** The fermionic N-particle Hilbert space is defined as
$$\mathcal{H}_1^{\wedge N} \equiv \mathrm{Ran}(A_N).$$
Equivalently,
$$\mathcal{H}_1^{\wedge N} = \left\{ \Psi \in \mathcal{H}_1^{\otimes N} \;\middle|\; P_\pi \Psi = \mathrm{sgn}(\pi)\, \Psi \text{ for all } \pi \in S_N \right\}.$$

4. **Combining one-particle states.** Given $\phi_1, \ldots, \phi_N \in \mathcal{H}_1$, the corresponding fermionic N-particle state is
$$\phi_1 \wedge \cdots \wedge \phi_N \equiv A_N(\phi_1 \otimes \cdots \otimes \phi_N).$$
This vector lies in $\mathcal{H}_1^{\wedge N}$ and changes sign under exchange of any two factors.

5. **Meaning of an N-particle state.** For any $\Psi \in \mathcal{H}_1^{\wedge N}$ and any $\chi_1, \ldots, \chi_N \in \mathcal{H}_1$, the complex number
$$\langle \chi_1 \wedge \cdots \wedge \chi_N, \Psi \rangle$$
is interpreted as the probability amplitude that the N fermions occupy the one-particle states $\chi_1, \ldots, \chi_N$.

6. **Algebraic sum and direct sum.** Consider the algebraic sum
$$\sum_{N=0}^{\infty} \mathcal{H}_1^{\wedge N} = \left\{ \sum_{k=1}^{m} \Psi^{(N_k)} \;\middle|\; \Psi^{(N_k)} \in \mathcal{H}_1^{\wedge N_k},\ m < \infty \right\}.$$
This sum is direct because
$$\mathcal{H}_1^{\wedge N} \cap \mathcal{H}_1^{\wedge M} = \{0\} \quad \text{for } N \neq M.$$

7. **Fermionic Fock space.** The fermionic Fock space is the Hilbert-space completion of this direct sum:
$$\mathcal{F} \equiv \bigoplus_{N=0}^{\infty} \mathcal{H}_1^{\wedge N},$$
with inner product
$$\langle \Psi, \Phi \rangle = \sum_{N=0}^{\infty} \langle \Psi^{(N)}, \Phi^{(N)} \rangle_{\mathcal{H}_1^{\wedge N}}, \quad \Psi = \oplus_N \Psi^{(N)},\ \Phi = \oplus_N \Phi^{(N)}.$$

F2. Field operators on fermionic Fock space

F2-1 Configuration-space representation, probability interpretation, and antisymmetry

An element $\Psi^{(N)} \in \mathcal{H}_1^{\wedge N}$ admits a coordinate–spin representation
$$\Psi^{(N)}(x_1, \ldots, x_N), \qquad x_i = (\mathbf{r}_i, \sigma_i) \in \mathbb{R}^3 \times \{\uparrow, \downarrow\},$$

which is a complex-valued function on the N-particle configuration space $(\mathbb{R}^3 \times \{\uparrow, \downarrow\})^N$.

For any measurable region Ω of this configuration space,

$$\int_\Omega |\Psi^{(N)}(x_1, \ldots, x_N)|^2 \, dx_1 \cdots dx_N$$

is the probability of finding the N identical fermions with coordinates and spins in Ω.

Because the fermions are physically identical, $\Psi^{(N)}$ satisfies the antisymmetry condition

$$\Psi^{(N)}(x_{\pi(1)}, \ldots, x_{\pi(N)}) = \mathrm{sgn}(\pi) \, \Psi^{(N)}(x_1, \ldots, x_N) \quad \text{for all } \pi \in S_N,$$

which is the coordinate-space representation of the abstract relation $P_\pi \Psi^{(N)} = \mathrm{sgn}(\pi)\Psi^{(N)}$. This antisymmetry leaves probabilities invariant while encoding fermionic exchange at the level of amplitudes.

F2-2 Definition of annihilation and creation operators and induced field operators

For each $f \in L^2(\mathbb{R}^3)$ and spin $\sigma \in \{\uparrow, \downarrow\}$, define the annihilation operator

$$\psi_\sigma(f) : \mathcal{H}_1^{\wedge N} \to \mathcal{H}_1^{\wedge(N-1)}$$

by

$$(\psi_\sigma(f)\Psi)^{(N-1)}(x_2, \ldots, x_N) = \sqrt{N} \int_{\mathbb{R}^3} d\mathbf{r} \, f(\mathbf{r}) \, \Psi^{(N)}\big((\mathbf{r}, \sigma), x_2, \ldots, x_N\big),$$

and define the creation operator $\psi_\sigma^\dagger(f)$ as the Hilbert-space adjoint of $\psi_\sigma(f)$.

Question 0.2.1. Why is the creation operator the adjoint of the annihilation operator?

The map $f \mapsto \psi_\sigma(f)$ is linear in f, and the map $f \mapsto \psi_\sigma^\dagger(f)$ is antilinear. Accordingly, there exist operator-valued distributions $\psi_\sigma(\mathbf{r})$ and $\psi_\sigma^\dagger(\mathbf{r})$ such that

$$\psi_\sigma(f) = \int_{\mathbb{R}^3} d\mathbf{r} \, f^*(\mathbf{r}) \, \psi_\sigma(\mathbf{r}), \qquad \psi_\sigma^\dagger(f) = \int_{\mathbb{R}^3} d\mathbf{r} \, f(\mathbf{r}) \, \psi_\sigma^\dagger(\mathbf{r}).$$

These relations express the integral representation of the linear dependence of $\psi_\sigma(f)$ and $\psi_\sigma^\dagger(f)$ on the test function f.

F2-3 Particle indistinguishability and slot independence

Because $\Psi^{(N)}$ is antisymmetric, the definition of $\psi_\sigma(f)$ is independent of which coordinate slot is integrated over: integrating over any one slot yields the same $(N-1)$-particle state up to the sign already fixed by antisymmetry. Thus $\psi_\sigma(f)$ does not remove a distinguished particle but removes one fermion in the one-particle state specified by (f, σ).

F2-4 Conditional interpretation of the reduced state

For any region $\Omega_{N-1} \subset (\mathbb{R}^3 \times \{\uparrow,\downarrow\})^{N-1}$,

$$\int_{\Omega_{N-1}} |(\psi_\sigma(f)\Psi)^{(N-1)}(x_2,\dots,x_N)|^2\, dx_2 \cdots dx_N$$

is the probability density for finding the remaining $N-1$ fermions in Ω_{N-1}, conditional on one fermion being in the one-particle state (f,σ).

F2-5 Derivation of the mode expansion of $\psi(f)$

Assume that the annihilation operator

$$f \;\mapsto\; \psi(f)$$

is a linear map from the one-particle Hilbert space H_1 to operators on fermionic Fock space.

Let $\{\psi_k\} \subset H_1$ be an orthonormal basis of H_1, and define the corresponding mode operators by

$$c_k \;\equiv\; \psi(\psi_k).$$

Any $f \in H_1$ admits the expansion

$$f(x) = \sum_k \psi_k(x)\,\langle \psi_k, f\rangle.$$

Applying the linear map $\psi(\cdot)$ to this expansion and using linearity,

$$\psi(f) = \psi\left(\sum_k \langle \psi_k, f\rangle\,\psi_k\right) = \sum_k \langle \psi_k, f\rangle\,\psi(\psi_k) = \sum_k \langle \psi_k, f\rangle\, c_k.$$

The inner product $\langle \psi_k, f\rangle$ can be written in coordinate representation as

$$\langle \psi_k, f\rangle = \int dx\, \psi_k^*(x)\, f(x).$$

Substituting this expression yields

$$\psi(f) = \int dx\, f^*(x)\,\Big(\sum_k \psi_k(x)\, c_k\Big).$$

F2-6 Canonical anticommutation relations

The field operators satisfy the canonical anticommutation relations

$$\{\psi_\sigma(\mathbf{r}), \psi_{\sigma'}^\dagger(\mathbf{r}')\} = \delta_{\sigma\sigma'}\,\delta(\mathbf{r}-\mathbf{r}'), \qquad \{\psi_\sigma(\mathbf{r}), \psi_{\sigma'}(\mathbf{r}')\} = 0, \qquad \{\psi_\sigma^\dagger(\mathbf{r}), \psi_{\sigma'}^\dagger(\mathbf{r}')\} = 0,$$

to be interpreted after smearing with test functions.

Equivalently, for smeared operators,

$$\{\psi_\sigma(f), \psi_{\sigma'}^\dagger(g)\} = \delta_{\sigma\sigma'}\,\langle f, g\rangle_{\mathcal{H}_1},$$

where

$$\langle f, g\rangle_{\mathcal{H}_1} = \int_{\mathbb{R}^3} d\mathbf{r}\; f^*(\mathbf{r})g(\mathbf{r})$$

is the inner product on the one-particle Hilbert space.

This relation expresses the consistency condition that creating a fermion in a one-particle state and then testing for its presence returns precisely the overlap of those one-particle states, and nothing else.

F2-7 Derivation of the one-body Hamiltonian in field-operator form

Let h_0 be a linear operator on the one-particle Hilbert space H_1. Fix an orthonormal basis $\{\psi_k\} \subset H_1$, and define mode operators

$$c_k \equiv \psi(\psi_k).$$

Define the matrix elements of h_0 in this basis by

$$h_{kl} \;\equiv\; \langle \psi_k,\, h_0\psi_l\rangle_{H_1}.$$

The corresponding second-quantized one-body Hamiltonian is defined by

$$H_0 \;\equiv\; \sum_{k,l} h_{kl}\, c_k^\dagger c_l.$$

Using the mode expansions

$$\psi(x) = \sum_l \psi_l(x)\, c_l, \qquad \psi^\dagger(x) = \sum_k \psi_k^*(x)\, c_k^\dagger,$$

one computes

$$\int dx\, \psi^\dagger(x)\,(h_0\psi)(x) = \int dx\, \left(\sum_k \psi_k^*(x) c_k^\dagger\right)\left(h_0 \sum_l \psi_l(x) c_l\right).$$

Since h_0 acts only on the x-dependence of $\psi_l(x)$, this becomes

$$\int dx\, \psi^\dagger(x)\,(h_0\psi)(x) = \sum_{k,l}\left(\int dx\, \psi_k^*(x)\,(h_0\psi_l)(x)\right) c_k^\dagger c_l = \sum_{k,l} h_{kl}\, c_k^\dagger c_l.$$

Therefore,

$$H_0 = \sum_{k,l} \langle \psi_k, h_0\psi_l\rangle\, c_k^\dagger c_l = \int dx\, \psi^\dagger(x)\, h_0\, \psi(x).$$

F2-8. One-body Hamiltonian: $\int dx\, \psi^\dagger(x)\, h_0\, \psi(x)$ equals $\sum_{i=1}^N h_0^{(i)}$ on $\mathcal{H}_1^{\wedge N}$

Let h_0 be a one-particle operator on $\mathcal{H}_1$ (e.g. a differential operator in the coordinate representation). Define the second-quantized one-body operator on fermionic Fock space by

$$\widehat{H}_0 \;\equiv\; \int dx\; \psi^\dagger(x)\, h_0\, \psi(x), \qquad x = (\mathbf{r}, \sigma), \quad dx = \sum_\sigma d^3\mathbf{r},$$

interpreted distributionally via smearing as in F2-2/F2-5.

We show that the restriction of $\widehat{H}_0$ to the N-particle sector $\mathcal{H}_1^{\wedge N}$ agrees with the first-quantized operator

$$H_0^{(N)} \;=\; \sum_{i=1}^N h_0^{(i)},$$

where $h_0^{(i)}$ denotes h_0 acting on the i-th coordinate x_i of $\Psi^{(N)}(x_1, \dots, x_N)$.

Let $\Phi^{(N)}, \Psi^{(N)} \in \mathcal{H}_1^{\wedge N}$. Consider the matrix element

$$\langle \Phi^{(N)}, \widehat{H}_0\, \Psi^{(N)} \rangle.$$

Using adjoints and that h_0 is a one-particle operator acting on the x-variable,

$$\langle \Phi^{(N)}, \widehat{H}_0\, \Psi^{(N)} \rangle = \int dx\; \big\langle \psi(x)\Phi^{(N)},\; h_0\, \psi(x)\Psi^{(N)} \big\rangle_{\mathcal{H}_1^{\wedge(N-1)}}.$$

By the definition of the annihilation operator in coordinate representation (F2-2),

$$(\psi(x)\Psi^{(N)})^{(N-1)}(x_2, \dots, x_N) = \sqrt{N}\, \Psi^{(N)}(x, x_2, \dots, x_N),$$

and similarly for $\Phi^{(N)}$. Substituting this into the previous line and writing the $(N-1)$-particle inner product as an integral over $x_2, \dots, x_N$ gives

$$\langle \Phi^{(N)}, \widehat{H}_0\, \Psi^{(N)} \rangle = N \int dx\, dx_2 \cdots dx_N\; \Phi^{(N)}(x, x_2, \dots, x_N)^*\; \big(h_0^{(x)} \Psi^{(N)}\big)(x, x_2, \dots, x_N),$$

where $h_0^{(x)}$ means: apply the one-particle operator h_0 to the x-dependence of $\Psi^{(N)}(x, x_2, \dots, x_N)$, holding $x_2, \dots, x_N$ fixed.

Question 0.2.2. In order for this to be well-defined then, the value of

Relabeling x as x_1,

$$\langle \Phi^{(N)}, \widehat{H}_0\, \Psi^{(N)} \rangle = N \int dx_1 \cdots dx_N\; \Phi^{(N)}(x_1, \dots, x_N)^*\; \big(h_0^{(1)} \Psi^{(N)}\big)(x_1, \dots, x_N).$$

Because $\Phi^{(N)}$ and $\Psi^{(N)}$ are antisymmetric, the integral with h_0 acting on the first slot has the same value as the integral with h_0 acting on any slot i. Summing over $i = 1, \dots, N$ therefore produces the factor N, i.e.

$$N \int dx_1 \cdots dx_N\; \Phi^{(N)*}\, (h_0^{(1)} \Psi^{(N)}) = \sum_{i=1}^N \int dx_1 \cdots dx_N\; \Phi^{(N)*}\, (h_0^{(i)} \Psi^{(N)}).$$

Hence

$$\langle \Phi^{(N)}, \widehat{H}_0\, \Psi^{(N)} \rangle = \left\langle \Phi^{(N)}, \left(\sum_{i=1}^{N} h_0^{(i)} \right) \Psi^{(N)} \right\rangle.$$

Since this holds for arbitrary $\Phi^{(N)}$, the restriction of $\widehat{H}_0$ to $\mathcal{H}_1^{\wedge N}$ equals $\sum_{i=1}^{N} h_0^{(i)}$ as an operator on the N-particle sector.

F2-9. Two-body interaction in a mode basis

Let $\{\varphi_i(x)\}$ be an orthonormal basis of the one-particle Hilbert space $\mathcal{H}_1$, where $x = (\mathbf{r}, \sigma)$, and let

$$\psi(x) = \sum_i \varphi_i(x)\, c_i, \qquad \psi^\dagger(x) = \sum_i \varphi_i^*(x)\, c_i^\dagger$$

be the corresponding mode expansion of the field operators.

Consider a general two-body interaction kernel $V(x_1, x_2)$, symmetric under exchange of its arguments. Define the interaction matrix elements in the mode basis by

$$V_{ijkl} \;\equiv\; \int dx_1\, dx_2\; \varphi_i^*(x_1)\, \varphi_j^*(x_2)\, V(x_1, x_2)\, \varphi_k(x_1)\, \varphi_l(x_2).$$

These quantities are the matrix elements of the two-body operator V between antisymmetrized two-particle states constructed from the one-particle basis $\{\varphi_i\}$.

With this definition, the second-quantized two-body interaction operator can be written in mode form as

$$\widehat{H}_{\text{int}} = \frac{1}{2} \sum_{ijkl} V_{ijkl}\, c_i^\dagger c_j^\dagger c_l c_k.$$

The factor $\frac{1}{2}$ compensates for the double counting of particle pairs in the summation over mode indices and corresponds to the restriction $i \neq j$ in the first-quantized expression $\frac{1}{2} \sum_{i \neq j} V(x_i, x_j)$.

This expression is purely algebraic once the kernel $V(x_1, x_2)$ and the one-particle basis are fixed; no approximation is involved at this stage. The equivalence between this mode-space form and the coordinate-space field-operator form is established by explicit comparison of matrix elements (see F2-10).

F2-10. Equivalence of the two-body interaction in first and second quantization

We show that the second-quantized operator

$$\widehat{H}_{\text{int}} = \frac{1}{2} \int dx_1\, dx_2\; \psi^\dagger(x_1) \psi^\dagger(x_2)\, V(x_1, x_2)\, \psi(x_2) \psi(x_1) \tag{1}$$

acts on the N-particle sector $\mathcal{H}_1^{\wedge N}$ as the first-quantized interaction

$$(H_{\text{int}}^{(N)} \Psi^{(N)})(x_1, \ldots, x_N) = \frac{1}{2} \sum_{i \neq j} V(x_i, x_j)\, \Psi^{(N)}(x_1, \ldots, x_N).$$

To avoid ill-defined pointwise products, $\widehat{H}_{\rm int}$ is interpreted distributionally via smearing, as in F2-2 and F2-5. We prove equality by comparing matrix elements on the N-particle sector.

Let $\Phi^{(N)}, \Psi^{(N)} \in \mathcal{H}_1^{\wedge N}$. Consider

$$\langle \Phi^{(N)}, \widehat{H}_{\rm int} \Psi^{(N)} \rangle.$$

Using adjoints and the fact that $V(x_1, x_2)$ is a c-number kernel,

$$\langle \Phi^{(N)}, \widehat{H}_{\rm int} \Psi^{(N)} \rangle = \frac{1}{2} \int dx_1\, dx_2\, \langle \psi(x_2)\psi(x_1)\Phi^{(N)},\, V(x_1, x_2)\, \psi(x_2)\psi(x_1)\Psi^{(N)} \rangle_{\mathcal{H}_1^{\wedge(N-2)}}.$$

From the definition of the annihilation operator (F2-2),

$$(\psi(x_2)\psi(x_1)\Psi^{(N)})^{(N-2)}(x_3, \ldots, x_N) = \sqrt{N(N-1)}\, \Psi^{(N)}(x_1, x_2, x_3, \ldots, x_N),$$

and similarly for $\Phi^{(N)}$.

Substituting this into the previous expression and performing the $(N-2)$-particle inner product yields

$$\begin{aligned} &\langle \Phi^{(N)}, \widehat{H}_{\rm int} \Psi^{(N)} \rangle = \\ &\frac{1}{2} N(N-1) \int dx_1\, dx_2\, dx_3 \cdots dx_N\ \Phi^{(N)}(x_1, \ldots, x_N)^*\, V(x_1, x_2)\, \Psi^{(N)}(x_1, \ldots, x_N). \end{aligned}$$

Because $\Phi^{(N)}$ and $\Psi^{(N)}$ are antisymmetric, the integral with V acting on the specific ordered pair $(1, 2)$ has the same value as the integral with V acting on any ordered pair of distinct slots (i, j). There are exactly $N(N-1)$ such ordered pairs. Hence,

$$\begin{aligned} &N(N-1) \int dx_1 \cdots dx_N\ \Phi^{(N)*}(x_1, \ldots, x_N)\, V(x_1, x_2)\, \Psi^{(N)}(x_1, \ldots, x_N) = \\ &\sum_{i \neq j} \int dx_1 \cdots dx_N\ \Phi^{(N)*}(x_1, \ldots, x_N)\, V(x_i, x_j)\, \Psi^{(N)}(x_1, \ldots, x_N). \end{aligned}$$

Combining the previous expressions gives

$$\langle \Phi^{(N)}, \widehat{H}_{\rm int} \Psi^{(N)} \rangle = \left\langle \Phi^{(N)}, \left(\frac{1}{2} \sum_{i \neq j} V(x_i, x_j) \right) \Psi^{(N)} \right\rangle.$$

Since this holds for arbitrary $\Phi^{(N)}$, the operators agree on $\mathcal{H}_1^{\wedge N}$. Thus $\widehat{H}_{\rm int}$ is the second-quantized representative of the two-body interaction.

F2-11. Special case of two body interaction with $V(x, x') = g_{\alpha,\beta}\delta(\mathbf{r} - \mathbf{r}')$

Let $\mathbf{r_j}$ be the position coordinate of x_j (which includes spin), and $g_{\alpha,\beta} = \begin{cases} g & \alpha \neq \beta \\ 0 & \alpha = \beta \end{cases}$.

Then

$$\widehat{H}_{\rm int} = \frac{1}{2} \int dx_1\, dx_2\ \psi^\dagger(x_1)\psi^\dagger(x_2)\, g_{\alpha\beta}\delta(\mathbf{r}_1 - \mathbf{r}_2)\, \psi(x_2)\psi(x_1).$$

Since the delta function projects this integral onto the subspace where $\mathbf{r}_1 = \mathbf{r}_2$,

$$\frac{1}{2}g\sum_{\alpha\neq\beta}\int \mathrm{d}\mathbf{r}\ \psi^\dagger_\alpha(\mathbf{r})\psi^\dagger_\beta(\mathbf{r})\psi_\beta(\mathbf{r})\psi_\alpha(\mathbf{r})$$

0.3 January 23, 2026

Tasks

- Finish studying foundational field operator and second quantization material
- Begin studying first part of Svidzinsky derivation of approximation for interaction.