

0.1 January 21, 2026

Plan:

- Translate section 2 of Svidzinsky
- Establish a firm ground of accepted facts from which the BdG equations will follow.
- Understand how the approximation of the interaction is made.

Evaluation: I did find a way to translate section 2 of Svidzinsky, but the "accepted facts list" I generated needs improvement, and I did not get to bullet 3.

0.2 January 22, 2026

Plan:

- Do a better job identifying and understanding the necessary foundational material needed to understand the BdG derivation

0.2.1 Fock space and Field operators

Physical meaning. $\mathcal{H}_1^{\wedge N}$ is the Hilbert space describing N identical fermions built from a one-particle Hilbert space \mathcal{H}_1 . A vector in $\mathcal{H}_1^{\wedge N}$ encodes all probability amplitudes for finding the N fermions in arbitrary one-particle states, with the fermionic exchange sign built in. Fermionic Fock space collects all such N -particle sectors into a single Hilbert space.

Mathematical definitions.

1. **Permutation operators.** Let \mathcal{H}_1 be a complex separable Hilbert space and let $\mathcal{H}_1^{\otimes N}$ be its N -fold tensor product. For each permutation $\pi \in S_N$, define a linear operator

$$P_\pi : \mathcal{H}_1^{\otimes N} \rightarrow \mathcal{H}_1^{\otimes N}$$

by its action on simple tensors,

$$P_\pi(\phi_1 \otimes \cdots \otimes \phi_N) = \phi_{\pi^{-1}(1)} \otimes \cdots \otimes \phi_{\pi^{-1}(N)},$$

extended by linearity and continuity.

2. **Antisymmetrizer and its range.** Define the antisymmetrization operator

$$A_N \equiv \frac{1}{N!} \sum_{\pi \in S_N} \text{sgn}(\pi) P_\pi.$$

The range of A_N , denoted $\text{Ran}(A_N)$, is

$$\text{Ran}(A_N) \equiv \{ A_N \Phi \mid \Phi \in \mathcal{H}_1^{\otimes N} \}.$$

Since $A_N^2 = A_N$ and $A_N^\dagger = A_N$, $\text{Ran}(A_N)$ is a closed subspace of $\mathcal{H}_1^{\otimes N}$.

3. **Fermionic N -particle Hilbert space.** The fermionic N -particle Hilbert space is defined as

$$\mathcal{H}_1^{\wedge N} \equiv \text{Ran}(A_N).$$

Equivalently,

$$\mathcal{H}_1^{\wedge N} = \left\{ \Psi \in \mathcal{H}_1^{\otimes N} \mid P_\pi \Psi = \text{sgn}(\pi) \Psi \text{ for all } \pi \in S_N \right\}.$$

4. **Combining one-particle states.** Given $\phi_1, \dots, \phi_N \in \mathcal{H}_1$, the corresponding fermionic N -particle state is

$$\phi_1 \wedge \dots \wedge \phi_N \equiv A_N(\phi_1 \otimes \dots \otimes \phi_N).$$

This vector lies in $\mathcal{H}_1^{\wedge N}$ and changes sign under exchange of any two factors.

5. **Meaning of an N -particle state.** For any $\Psi \in \mathcal{H}_1^{\wedge N}$ and any $\chi_1, \dots, \chi_N \in \mathcal{H}_1$, the complex number

$$\langle \chi_1 \wedge \dots \wedge \chi_N, \Psi \rangle$$

is interpreted as the probability amplitude that the N fermions occupy the one-particle states χ_1, \dots, χ_N .

6. **Algebraic sum and direct sum.** Consider the algebraic sum

$$\sum_{N=0}^{\infty} \mathcal{H}_1^{\wedge N} = \left\{ \sum_{k=1}^m \Psi^{(N_k)} \mid \Psi^{(N_k)} \in \mathcal{H}_1^{\wedge N_k}, m < \infty \right\}.$$

This sum is direct because

$$\mathcal{H}_1^{\wedge N} \cap \mathcal{H}_1^{\wedge M} = \{0\} \quad \text{for } N \neq M.$$

7. **Fermionic Fock space.** The fermionic Fock space is the Hilbert-space completion of this direct sum:

$$\mathcal{F} \equiv \bigoplus_{N=0}^{\infty} \mathcal{H}_1^{\wedge N},$$

with inner product

$$\langle \Psi, \Phi \rangle = \sum_{N=0}^{\infty} \langle \Psi^{(N)}, \Phi^{(N)} \rangle_{\mathcal{H}_1^{\wedge N}}, \quad \Psi = \oplus_N \Psi^{(N)}, \quad \Phi = \oplus_N \Phi^{(N)}.$$

F2. Field operators on fermionic Fock space

F2-1 Configuration-space representation, probability interpretation, and anti-symmetry

An element $\Psi^{(N)} \in \mathcal{H}_1^{\wedge N}$ admits a coordinate-spin representation

$$\Psi^{(N)}(x_1, \dots, x_N), \quad x_i = (\mathbf{r}_i, \sigma_i) \in \mathbb{R}^3 \times \{\uparrow, \downarrow\},$$

which is a complex-valued function on the N -particle configuration space $(\mathbb{R}^3 \times \{\uparrow, \downarrow\})^N$.

For any measurable region Ω of this configuration space,

$$\int_{\Omega} |\Psi^{(N)}(x_1, \dots, x_N)|^2 dx_1 \cdots dx_N$$

is the probability of finding the N identical fermions with coordinates and spins in Ω .

Because the fermions are physically identical, $\Psi^{(N)}$ satisfies the antisymmetry condition

$$\Psi^{(N)}(x_{\pi(1)}, \dots, x_{\pi(N)}) = \text{sgn}(\pi) \Psi^{(N)}(x_1, \dots, x_N) \quad \text{for all } \pi \in S_N,$$

which is the coordinate-space representation of the abstract relation $P_{\pi} \Psi^{(N)} = \text{sgn}(\pi) \Psi^{(N)}$. This antisymmetry leaves probabilities invariant while encoding fermionic exchange at the level of amplitudes.

F2-2 Definition of annihilation and creation operators and induced field operators

For each $f \in L^2(\mathbb{R}^3)$ and spin $\sigma \in \{\uparrow, \downarrow\}$, define the annihilation operator

$$\psi_{\sigma}(f) : \mathcal{H}_1^{\wedge N} \rightarrow \mathcal{H}_1^{\wedge(N-1)}$$

by

$$(\psi_{\sigma}(f) \Psi)^{(N-1)}(x_2, \dots, x_N) = \sqrt{N} \int_{\mathbb{R}^3} d\mathbf{r} f(\mathbf{r}) \Psi^{(N)}((\mathbf{r}, \sigma), x_2, \dots, x_N),$$

and define the creation operator $\psi_{\sigma}^{\dagger}(f)$ as the Hilbert-space adjoint of $\psi_{\sigma}(f)$.

Question 0.2.1. Why is the creation operator the adjoint of the annihilation operator?

The map $f \mapsto \psi_{\sigma}(f)$ is linear in f , and the map $f \mapsto \psi_{\sigma}^{\dagger}(f)$ is antilinear. Accordingly, there exist operator-valued distributions $\psi_{\sigma}(\mathbf{r})$ and $\psi_{\sigma}^{\dagger}(\mathbf{r})$ such that

$$\psi_{\sigma}(f) = \int_{\mathbb{R}^3} d\mathbf{r} f^*(\mathbf{r}) \psi_{\sigma}(\mathbf{r}), \quad \psi_{\sigma}^{\dagger}(f) = \int_{\mathbb{R}^3} d\mathbf{r} f(\mathbf{r}) \psi_{\sigma}^{\dagger}(\mathbf{r}).$$

These relations express the integral representation of the linear dependence of $\psi_{\sigma}(f)$ and $\psi_{\sigma}^{\dagger}(f)$ on the test function f .

F2-3 Particle indistinguishability and slot independence

Because $\Psi^{(N)}$ is antisymmetric, the definition of $\psi_{\sigma}(f)$ is independent of which coordinate slot is integrated over: integrating over any one slot yields the same $(N-1)$ -particle state up to the sign already fixed by antisymmetry. Thus $\psi_{\sigma}(f)$ does not remove a distinguished particle but removes one fermion in the one-particle state specified by (f, σ) .

F2-4 Conditional interpretation of the reduced state

For any region $\Omega_{N-1} \subset (\mathbb{R}^3 \times \{\uparrow, \downarrow\})^{N-1}$,

$$\int_{\Omega_{N-1}} |(\psi_\sigma(f)\Psi)^{(N-1)}(x_2, \dots, x_N)|^2 dx_2 \cdots dx_N$$

is the probability density for finding the remaining $N-1$ fermions in Ω_{N-1} , conditional on one fermion being in the one-particle state (f, σ) .

F2-5 Canonical anticommutation relations

The field operators satisfy the canonical anticommutation relations

$$\{\psi_\sigma(\mathbf{r}), \psi_{\sigma'}^\dagger(\mathbf{r}')\} = \delta_{\sigma\sigma'} \delta(\mathbf{r} - \mathbf{r}'), \quad \{\psi_\sigma(\mathbf{r}), \psi_{\sigma'}(\mathbf{r}')\} = 0, \quad \{\psi_\sigma^\dagger(\mathbf{r}), \psi_{\sigma'}^\dagger(\mathbf{r}')\} = 0,$$

to be interpreted after smearing with test functions.

Equivalently, for smeared operators,

$$\{\psi_\sigma(f), \psi_{\sigma'}^\dagger(g)\} = \delta_{\sigma\sigma'} \langle f, g \rangle_{\mathcal{H}_1},$$

where

$$\langle f, g \rangle_{\mathcal{H}_1} = \int_{\mathbb{R}^3} d\mathbf{r} f^*(\mathbf{r})g(\mathbf{r})$$

is the inner product on the one-particle Hilbert space.

This relation expresses the consistency condition that creating a fermion in a one-particle state and then testing for its presence returns precisely the overlap of those one-particle states, and nothing else.

F2-6 Derivation of the mode expansion of $\psi(f)$

Assume that the annihilation operator

$$f \mapsto \psi(f)$$

is a linear map from the one-particle Hilbert space H_1 to operators on fermionic Fock space.

Let $\{\psi_k\} \subset H_1$ be an orthonormal basis of H_1 , and define the corresponding mode operators by

$$c_k \equiv \psi(\psi_k).$$

Any $f \in H_1$ admits the expansion

$$f(x) = \sum_k \psi_k(x) \langle \psi_k, f \rangle.$$

Applying the linear map $\psi(\cdot)$ to this expansion and using linearity,

$$\psi(f) = \psi\left(\sum_k \langle \psi_k, f \rangle \psi_k\right) = \sum_k \langle \psi_k, f \rangle \psi(\psi_k) = \sum_k \langle \psi_k, f \rangle c_k.$$

The inner product $\langle \psi_k, f \rangle$ can be written in coordinate representation as

$$\langle \psi_k, f \rangle = \int dx \psi_k^*(x) f(x).$$

Substituting this expression yields

$$\psi(f) = \int dx f^*(x) \left(\sum_k \psi_k(x) c_k \right).$$

F2-7 Derivation of the one-body Hamiltonian in field-operator form

Let h_0 be a linear operator on the one-particle Hilbert space H_1 . Fix an orthonormal basis $\{\psi_k\} \subset H_1$, and define mode operators

$$c_k \equiv \psi(\psi_k).$$

Define the matrix elements of h_0 in this basis by

$$h_{kl} \equiv \langle \psi_k, h_0 \psi_l \rangle_{H_1}.$$

The corresponding second-quantized one-body Hamiltonian is defined by

$$H_0 \equiv \sum_{k,l} h_{kl} c_k^\dagger c_l.$$

Using the mode expansions

$$\psi(x) = \sum_l \psi_l(x) c_l, \quad \psi^\dagger(x) = \sum_k \psi_k^*(x) c_k^\dagger,$$

one computes

$$\int dx \psi^\dagger(x) (h_0 \psi)(x) = \int dx \left(\sum_k \psi_k^*(x) c_k^\dagger \right) \left(h_0 \sum_l \psi_l(x) c_l \right).$$

Since h_0 acts only on the x -dependence of $\psi_l(x)$, this becomes

$$\int dx \psi^\dagger(x) (h_0 \psi)(x) = \sum_{k,l} \left(\int dx \psi_k^*(x) (h_0 \psi_l)(x) \right) c_k^\dagger c_l = \sum_{k,l} h_{kl} c_k^\dagger c_l.$$

Therefore,

$$H_0 = \sum_{k,l} \langle \psi_k, h_0 \psi_l \rangle c_k^\dagger c_l = \int dx \psi^\dagger(x) h_0 \psi(x).$$

F2-8. One-body Hamiltonian: $\int dx \psi^\dagger(x) h_0 \psi(x)$ equals $\sum_{i=1}^N h_0^{(i)}$ on $\mathcal{H}_1^{\wedge N}$

Let h_0 be a one-particle operator on \mathcal{H}_1 (e.g. a differential operator in the coordinate representation). Define the second-quantized one-body operator on fermionic Fock space by

$$\hat{H}_0 \equiv \int dx \psi^\dagger(x) h_0 \psi(x), \quad x = (\mathbf{r}, \sigma), \quad dx = \sum_{\sigma} d^3 \mathbf{r},$$

interpreted distributionally via smearing as in F2-2/F2-5.

We show that the restriction of \hat{H}_0 to the N -particle sector $\mathcal{H}_1^{\wedge N}$ agrees with the first-quantized operator

$$H_0^{(N)} = \sum_{i=1}^N h_0^{(i)},$$

where $h_0^{(i)}$ denotes h_0 acting on the i -th coordinate x_i of $\Psi^{(N)}(x_1, \dots, x_N)$.

Let $\Phi^{(N)}, \Psi^{(N)} \in \mathcal{H}_1^{\wedge N}$. Consider the matrix element

$$\langle \Phi^{(N)}, \hat{H}_0 \Psi^{(N)} \rangle.$$

Using adjoints and that h_0 is a one-particle operator acting on the x -variable,

$$\langle \Phi^{(N)}, \hat{H}_0 \Psi^{(N)} \rangle = \int dx \langle \psi(x) \Phi^{(N)}, h_0 \psi(x) \Psi^{(N)} \rangle_{\mathcal{H}_1^{\wedge(N-1)}}.$$

By the definition of the annihilation operator in coordinate representation (F2-2),

$$(\psi(x) \Psi^{(N)})^{(N-1)}(x_2, \dots, x_N) = \sqrt{N} \Psi^{(N)}(x, x_2, \dots, x_N),$$

and similarly for $\Phi^{(N)}$. Substituting this into the previous line and writing the $(N-1)$ -particle inner product as an integral over x_2, \dots, x_N gives

$$\langle \Phi^{(N)}, \hat{H}_0 \Psi^{(N)} \rangle = N \int dx dx_2 \cdots dx_N \Phi^{(N)}(x, x_2, \dots, x_N)^* (h_0^{(x)} \Psi^{(N)})(x, x_2, \dots, x_N),$$

where $h_0^{(x)}$ means: apply the one-particle operator h_0 to the x -dependence of $\Psi^{(N)}(x, x_2, \dots, x_N)$, holding x_2, \dots, x_N fixed.

Question 0.2.2. In order for this to be well-defined then, the value of

Relabeling x as x_1 ,

$$\langle \Phi^{(N)}, \hat{H}_0 \Psi^{(N)} \rangle = N \int dx_1 \cdots dx_N \Phi^{(N)}(x_1, \dots, x_N)^* (h_0^{(1)} \Psi^{(N)})(x_1, \dots, x_N).$$

Because $\Phi^{(N)}$ and $\Psi^{(N)}$ are antisymmetric, the integral with h_0 acting on the first slot has the same value as the integral with h_0 acting on any slot i . Summing over $i = 1, \dots, N$ therefore produces the factor N , i.e.

$$N \int dx_1 \cdots dx_N \Phi^{(N)*} (h_0^{(1)} \Psi^{(N)}) = \sum_{i=1}^N \int dx_1 \cdots dx_N \Phi^{(N)*} (h_0^{(i)} \Psi^{(N)}).$$

Hence

$$\langle \Phi^{(N)}, \hat{H}_0 \Psi^{(N)} \rangle = \left\langle \Phi^{(N)}, \left(\sum_{i=1}^N h_0^{(i)} \right) \Psi^{(N)} \right\rangle.$$

Since this holds for arbitrary $\Phi^{(N)}$, the restriction of \hat{H}_0 to $\mathcal{H}_1^{\wedge N}$ equals $\sum_{i=1}^N h_0^{(i)}$ as an operator on the N -particle sector.

F2-10. Equivalence of the two-body interaction in first and second quantization

We show that the second-quantized operator

$$\hat{H}_{\text{int}} = \frac{1}{2} \int dx_1 dx_2 \psi^\dagger(x_1) \psi^\dagger(x_2) V(x_1, x_2) \psi(x_2) \psi(x_1) \quad (1)$$

acts on the N -particle sector $\mathcal{H}_1^{\wedge N}$ as the first-quantized interaction

$$(H_{\text{int}}^{(N)} \Psi^{(N)})(x_1, \dots, x_N) = \frac{1}{2} \sum_{i \neq j} V(x_i, x_j) \Psi^{(N)}(x_1, \dots, x_N).$$

To avoid ill-defined pointwise products, \hat{H}_{int} is interpreted distributionally via smearing, as in F2-2 and F2-5. We prove equality by comparing matrix elements on the N -particle sector.

Let $\Phi^{(N)}, \Psi^{(N)} \in \mathcal{H}_1^{\wedge N}$. Consider

$$\langle \Phi^{(N)}, \hat{H}_{\text{int}} \Psi^{(N)} \rangle.$$

Using adjoints and the fact that $V(x_1, x_2)$ is a c-number kernel,

$$\langle \Phi^{(N)}, \hat{H}_{\text{int}} \Psi^{(N)} \rangle = \frac{1}{2} \int dx_1 dx_2 \langle \psi(x_2) \psi(x_1) \Phi^{(N)}, V(x_1, x_2) \psi(x_2) \psi(x_1) \Psi^{(N)} \rangle_{\mathcal{H}_1^{\wedge(N-2)}}.$$

From the definition of the annihilation operator (F2-2),

$$(\psi(x_2) \psi(x_1) \Psi^{(N)})^{(N-2)}(x_3, \dots, x_N) = \sqrt{N(N-1)} \Psi^{(N)}(x_1, x_2, x_3, \dots, x_N),$$

and similarly for $\Phi^{(N)}$.

Substituting this into the previous expression and performing the $(N-2)$ -particle inner product yields

$$\begin{aligned} \langle \Phi^{(N)}, \hat{H}_{\text{int}} \Psi^{(N)} \rangle &= \\ \frac{1}{2} N(N-1) \int dx_1 dx_2 dx_3 \cdots dx_N \Phi^{(N)*}(x_1, \dots, x_N) V(x_1, x_2) \Psi^{(N)}(x_1, \dots, x_N). \end{aligned}$$

Because $\Phi^{(N)}$ and $\Psi^{(N)}$ are antisymmetric, the integral with V acting on the specific ordered pair $(1, 2)$ has the same value as the integral with V acting on any ordered pair of distinct slots (i, j) . There are exactly $N(N-1)$ such ordered pairs. Hence,

$$\begin{aligned} N(N-1) \int dx_1 \cdots dx_N \Phi^{(N)*}(x_1, \dots, x_N) V(x_1, x_2) \Psi^{(N)}(x_1, \dots, x_N) &= \\ \sum_{i \neq j} \int dx_1 \cdots dx_N \Phi^{(N)*}(x_1, \dots, x_N) V(x_i, x_j) \Psi^{(N)}(x_1, \dots, x_N). \end{aligned}$$

Combining the previous expressions gives

$$\langle \Phi^{(N)}, \hat{H}_{\text{int}} \Psi^{(N)} \rangle = \left\langle \Phi^{(N)}, \left(\frac{1}{2} \sum_{i \neq j} V(x_i, x_j) \right) \Psi^{(N)} \right\rangle.$$

Since this holds for arbitrary $\Phi^{(N)}$, the operators agree on $\mathcal{H}_1^{\wedge N}$. Thus \hat{H}_{int} is the second-quantized representative of the two-body interaction.

F2-11. Special case of two body interaction with $V(x, x') = g_{\alpha, \beta} \delta(\mathbf{r} - \mathbf{r}')$

Let \mathbf{r}_j be the position coordinate of x_j (which includes spin), and $g_{\alpha, \beta} = \begin{cases} g & \alpha \neq \beta \\ 0 & \alpha = \beta \end{cases}$.

Then

$$\hat{H}_{\text{int}} = \frac{1}{2} \int dx_1 dx_2 \psi^\dagger(x_1) \psi^\dagger(x_2) g_{\alpha\beta} \delta(\mathbf{r}_1 - \mathbf{r}_2) \psi(x_2) \psi(x_1).$$

Since the delta function projects this integral onto the subspace where $\mathbf{r}_1 = \mathbf{r}_2$,

$$\frac{1}{2} g \sum_{\alpha \neq \beta} \int d\mathbf{r} \psi_\alpha^\dagger(\mathbf{r}) \psi_\beta^\dagger(\mathbf{r}) \psi_\beta(\mathbf{r}) \psi_\alpha(\mathbf{r})$$

0.2.2 Understanding initial interaction equation

What is the approximatio

$$H_{\text{int}} = g \int \psi_{\uparrow}^{\dagger}(\mathbf{r}) \psi_{\downarrow}^{\dagger}(\mathbf{r}) \psi_{\downarrow}(\mathbf{r}) \psi_{\uparrow}(\mathbf{r}) d\mathbf{r}. \quad (2.3)$$