CSCI 2610 EXAM 1 CHEAT SHEET

David Luo

Logic and Proofs

Ways $p \to q$ is expressed

- "if p, then q"
- "p implies q"
- "if p, q"
- "p only if q"
- "p is sufficient for q"
- "a sufficient condition for q is p"
- "q if p"
- "q whenever p"
- "q when p"
- "q is necessary for p"
- "a necessary condition for p is q"
- "q follows from p"
- "q unless $\neg p$"

Ways $p \leftrightarrow q$ is expressed

- "p is necessary and sufficient for q"
- "if p then q, and conversely"
- "if p then q, and if q then p"
- "p iff q"

Converse, Contrapositive, and Inverse

For $p \to q$:

- $q \to p$ is the **converse**.
- $\neg p \to \neg q$ is the **inverse**.
- $\neg q \to \neg p$ is the **contrapositive**.

Logical Equivalences

Identity

$p \land \mathbf{T} \equiv p$

$p \lor \mathbf{F} \equiv p$

Domination

$p \lor \mathbf{T} \equiv \mathbf{T}$

$p \land \mathbf{F} \equiv \mathbf{F}$

Double Negation

$\neg(\neg p) \equiv p$

Commutative

$p \lor q \equiv q \lor p$

$p \land q \equiv q \land p$

Associative

$(p \lor q) \lor r \equiv p \lor (q \lor r)$

$(p \land q) \land r \equiv p \land (q \land r)$

Distributive

$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$

De Morgan's

$\neg(p \land q) \equiv \neg p \lor \neg q$

$\neg(p \lor q) \equiv \neg p \land \neg q$

Absorption

$p \lor (p \land q) \equiv p$

$p \land (p \lor q) \equiv p$

Negation

$p \lor \neg p \equiv \mathbf{T}$

$p \land \neg p \equiv \mathbf{F}$

With Conditionals

$p \to q \equiv \neg p \lor q$

$p \to q \equiv \neg q \to \neg p$

$p \lor q \equiv \neg p \to q$

$p \land q \equiv \neg(p \to \neg q)$

$\neg(p \to q) \equiv p \land \neg q$

$(p \to q) \land (p \to r) \equiv p \to (q \land r)$

$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$

$(p \to r) \land (q \to r) \equiv (p \land q) \to r$

$(p \to r) \lor (q \to r) \equiv (p \lor q) \to r$

With Biconditionals

$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Propositional Satisfiability

- **Satisfiable** if there is an assignment of truth values to variables to make proposition true
- **Unsatisfiable** otherwise. Unsatisfiable if and only if its negation is a tautology.

De Morgan's for Quantifiers

$\neg\exists x P(x) \equiv \forall x \neg P(x)$

$\neg\forall x P(x) \equiv \exists x \neg P(x)$

Rules of Inference

Modus Ponens

$$\begin{array}{rl} & p \\ & p \to q \\ \hline \therefore & q \end{array}$$

Modus Tollens

$$\begin{array}{rl} & \neg q \\ & p \to q \\ \hline \therefore & \neg p \end{array}$$

Associative

$$\begin{array}{rl} & (p \lor q) \lor r \\ \hline \therefore & p \lor (q \lor r) \end{array}$$

Commutative

$$\begin{array}{rl} & p \land q \\ \hline \therefore & q \land p \end{array}$$

Law of Biconditional Propositions

$$\begin{array}{rl} & p \to q \\ & q \to p \\ \hline \therefore & p \leftrightarrow q \end{array}$$

Exportation

$$\begin{array}{rl} & (p \land q) \to r \\ \hline \therefore & p \to (q \to r) \end{array}$$

Transposition or Contraposition Law

$$\begin{array}{rl} & p \to q \\ \hline \therefore & \neg q \to \neg p \end{array}$$

Hypothetical Syllogism

$$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$$

Material Implication

$$\begin{array}{rl} & p \to q \\ \hline \therefore & \neg p \lor q \end{array}$$

Distributive

$$\begin{array}{rl} & (p \lor q) \land r \\ \hline \therefore & (p \land r) \lor (q \land r) \end{array}$$

Absorption

$$\begin{array}{rl} & p \to q \\ \hline \therefore & p \to (p \land q) \end{array}$$

Disjunctive Syllogism

$$\begin{array}{rl} & p \lor q \\ & \neg p \\ \hline \therefore & q \end{array}$$

Addition

$$\begin{array}{rl} & p \\ \hline \therefore & p \lor q \end{array}$$

Simplification

$$\begin{array}{rl} & p \land q \\ \hline \therefore & p \end{array}$$

Conjunction

$$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \land q \end{array}$$

Double Negation

$$\begin{array}{rl} & p \\ \hline \therefore & \neg\neg p \end{array}$$

Disjunctive Simplification

$$\begin{array}{rl} & p \lor p \\ \hline \therefore & p \end{array}$$

Resolution

$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \hline \therefore q \lor r \end{array}$$

Quantifier Inferences

Universal Instantiation (UI)

$$\begin{array}{l} \forall x P(x) \\ \hline \therefore P(c) \end{array}$$

Universal Generalization (UG)

$$\begin{array}{l} P(c) \text{ for an arbitrary } c \\ \hline \therefore \forall x P(x) \end{array}$$

Existential Instantiation (EI)

$$\begin{array}{l} \exists x P(x) \\ \hline \therefore P(c) \text{ for some element } c \end{array}$$

Existential Generalization (EG)

$$\begin{array}{l} P(c) \text{ for some element } c \\ \hline \therefore \exists x P(x) \end{array}$$

Types of Proofs

For $p \to q$:

Direct

Assume p is true, then show q must also be true.

Contraposition

Assume $\neg q$ is true, then show $\neg p$ must also be true.

Contradiction

Show that $\neg p \to q$ is invalid (i.e. assume $\neg p$ and derive a contradiction such as $r \land \neg r$).

Case / WLOG

For statements in the form

$$(p_1 \lor p_2 \lor \ldots \lor p_n) \to q$$

use tautology

$$[(p_1 \lor p_2 \lor \ldots \lor p_n) \to q] \leftrightarrow [(p_1 \to q) \land (p_2 \to q) \land \ldots \land (p_n \to q)]$$

i.e. show that for all possible cases of the proposition, it is true.

Note: cases must be such that they apply for all cases, i.e. *without loss of generality* (WLOG).

Universally Quantified Assertions

For form $\forall x P(x)$, assume x is arbitrary member of domain and show that $P(x)$ must be true. Using **UG** it follows that $\forall P(x)$.

Existence Proofs

For form $\exists x P(x)$, find an explicit value c, for which $P(c)$ is true. Using **EG** it follows that $\exists x P(x)$.

Nonconstructive Existence Proofs

Assume no c exists that makes $P(c)$ true and derive a contradiction.

Counterexample

Basically find an example that makes a proposition false.

Uniqueness Proofs

There exists a unique x such that $P(x)$ - $\exists! x P(x)$.

Two parts:

- *Existence*: show that the $\exists$ part is true.
- *Uniqueness*: show that if $y \neq x$, then y does not have the property.

Boolean Algebra

Basics

3 Operators:

- $+$ Sum (OR)
- $*$ Product (AND)
- $\overline{\text{complement}}$ (NOT)

Normal Forms

Disjunctive Normal Form

- An *OR* of *ANDs*
- sum of products

EX:

$F(x, y) = x + y$

WITH TRUTH TABLE

x	y	$x+y$	Minterm
0	0	0	$\overline{x}\overline{y}$
0	1	1	$\overline{x}y$
1	0	1	$x\overline{y}$
1	1	1	xy

Add together all minterms for which the output is 1.

$$F(x, y) = \overline{x}y + x\overline{y} + xy$$

WITH BOOLEAN IDENTITIES

$$\begin{aligned} F(x,y) &= x + y \\ &= x1 + 1y && \text{identity} \\ &= x(y + \overline{y}) + (x + \overline{x})y && \text{unit property} \\ &= xy + x\overline{y} + xy + \overline{x}y && \text{distributive} \\ &= xy + x\overline{y} + \overline{x}y && \text{idempotent} \end{aligned}$$

Conjunctive Normal Form

- An *AND* of *ORs*
- product of sums

EX:

$F(x, y) = (x + y)\overline{(xy)}$

WITH TRUTH TABLE

x	y	$(x+y)\overline{(xy)}$	Maxterm
0	0	0	$x + y$
0	1	1	$x + \overline{y}$
1	0	1	$\overline{x} + y$
1	1	0	$\overline{x} + \overline{y}$

Multiply together all maxterms for which the output is 0.

$$F(x, y) = (x + y)(\overline{x} + \overline{y})$$

WITH BOOLEAN IDENTITIES

$$\begin{aligned} F(x,y) &= (x + y)\overline{(xy)} \\ &= (x + y)(\overline{x} + \overline{y}) && \text{De Morgan's} \end{aligned}$$